INDEX

Symbols

Ideal point, 125
Extended Euclidean plane, 343, 377
Inversive plane, 386
Projective plane, 386

[A B C D]
Symbol for area of quadrilateral A B C D, 134

[ X Y Z]
Symbol for area of △ X Y Z, 61
≡ Symbol for congruence, 7

G(P, r) Circle with center P and radius r, 7

~ Symbol for similarity, 35, 60

l ⊥ m Lines l and m are perpendicular, 38
l || m Lines l and m are parallel, 12

A
Abelian group, 240
Adjacent angles, 12
Altitudes, 46
Analysis figure, 32
Angle Bisector Theorem, 71
Angle bisectors, 25
Characterization, 25
Concurrent, 44
Internal, 44
Angle-Side Inequality, 10

Angles
Congruent, 6
Directed, 209
Undirected, 5
Vertically opposite, 6

Antiquity, three problems of, 159
Apollonius’ Theorem, 68
Arbelos Theorem, 355
Archimedean tiling, 325

Area
Additivity property, 134
Bounded figures, 139
Circle, 140
Fundamental unit, 134
Invariance property, 134
Parallelogram, 5, 135
Polygon, 134
Postulates, 134
Ratios, 146
Rectangle, 134
Square, 135
Trapezoid, 137
Triangle, 5
Unit square, 134
Area of a triangle
Base-altitude formula, 136

I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky
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Heron's Formula, 151
Inradius formula, 136
Arithmetic progression, 377
Arms of an angle, 25

B
Basic constructions, 29
Bijection, 212
Brianchon's Theorem, 394, 409
Butterfly Theorem, 93

C
Caito tiling, 317
Cancellation laws, 272
Cayley table, 285
Center of inversion, 340
Center of perspective, 445
Center of reciprocation, 386
Center of rotation, 210
Center of similitude, 293
Centroid
Triangle, 50
Ceva's Theorem, 97
Attitudes concurrent, 102
Applications, 99
Extended plane, 127, 129
Extended version, 124
Internal angle bisectors concurrent, 101
Medians concurrent, 101
Proof, 116
To compute ratio or distance, 102
Cevian, 97, 127
Cevian product, 98
Circle
Tangent line, 43
Circle of Apollonius, 197, 378
Inversion, 384
Circle of inversion, 340
Circle of reciprocation, 386
Circumcenter, 43
Circumcircle, 43
Circumcircle of a triangle, 19
Closest point to a line, 25
Coaxial triangles, 107, 453
Collinear points, 74, 98
Compass-only constructions, 339, 362
Complete
Quadrangle, 402
Quadrilateral, 402, 438
Composition of isometries, 214
Concurrent lines, 41, 97
Concyclic quadrilateral, 18
Cone
Oblique circular, 457
Right circular, 457
Two-napped, 457
Congruency conditions, 7
ASA, 7
HSS, 22
SSA, 21
SAS, 7
SSA', 22
SSS, 7
Congruent
Angles, 6
Circles, 6
Figures, 6
Line segments, 6
Lines, 6
Rays, 6
Triangles, 6
Congruent figures, 217
Conic section
Apollonian definition, 457
Cartesian definition, 406
Diracrix, 404
Eccentricity, 405
Focus, 404
Conjugate
Lines, 395
Points, 395
Constructible locus, 32
Constructible numbers, 164
Constructing a segment
Length \( \sqrt{ab} \), 163
Length \( p/q \), 162
Length \( pq \), 163
Construction of regular polygons, 166
Central angle, 167
Decagon, 168
Gauss' Theorem, 170
Pentagon, 168, 201
Vertex angle, 167
Construction problem
Analysis figure, 78, 80, 81
Construction, 78, 80, 81
Justification, 79, 81, 82
Construction problems, 28, 50, 75, 295
Compass
Operations, 28
Straightedge
Operations, 28
Constructions
Compass only, 339
Poncelet-Steiner, 442
Straightedge only, 435
Converse of Isosceles Triangle Theorem, 9
Converse of Midline Theorem, 66
Converse of Pythagoras' Theorem, 68
Converse of Thales' Theorem, 16
Convex
  Hexagon, 154
  Pentagon, 157
  Polygon, 138
  Quadrilateral, 155
  Cupola triangles, 106, 453
  Pole, 106
  Corresponding angles, 12
Converse ratios, 411
  Applications, 422
  Desargues' Two Triangle Theorem, 425
  Pappus' Theorem, 430
  Pascal's Mystic Hexagon Theorem, 429
Crossbar Theorem, 100
Cyclic quadrilateral
  Circumcircle, 18

D
Dart, 29
Desargues line, 107
Desargues' Two Triangle Theorem, 106, 453, 455
  Proof, 107
Dihedral group of order 2n, 276
Dilatation, 304
Direct isometry, 241
  Identity, 247
  Rotations, 247
  Translations, 247
Directed angles, 209
Directed distance, 85, 95, 411
  Properties, 96, 411
Directed ratio, 95
Directed segment, 291
Distance from a point to a line, 25
Distortion Theorem, 372
Dual
  Figure, 394
  Lines, 392
  Points, 392
  Statement, 394
  Translation table, 394
Dual of Desargues's Two Triangle Theorem, 395
Dual of Pascal's Mystic Hexagon Theorem, 394
Duality, 392

E
Ellipse, 404, 407
Equidistant, 24
Equivalence relation, 64
Euclidean constructions, 160
  Tools, 159
    Modern compass, 160
    Straightedge, 159
Euclidean geometry, 4
Euclidean plane, 126
Euclidean transformations, 207
  Glide reflections, 214
  Reflections, 210
  Rotations, 210
  Translations, 211
Euler line, 190, 302
Euler's Inequality, 301
Extended Euclidean plane, 126
Exterior angle, 14
Exterior Angle Inequality, 14, 22
Exterior Angle Theorem, 14
External angle bisectors, 38, 44, 71, 132
Extremal problem, 223
Eyeball Theorem, 90

F
Fagnano's problem, 266
Fermat numbers, 169
Fermat point, 223
Fermat prime, 169
Finding the inverse
  Compass method, 341
  Perpendicular diameter method, 342
  Tangent method, 342
Fixed point, 210, 257
  Projectivity, 450
Frieze group, 285
Function, 211
  Bijection, 212
  Image, 211
  Injection, 212
  Many-to-one, 211
  Mapping, 211
  One-to-one, 211, 212
  Onto, 212
  Preimage, 211
  Surjection, 212
Fundamental Theorem of Projective Geometry, 449

G
Gardner, Martin, 82
Gauss' Theorem, 170, 203
Gergonne point of a triangle, 131
  Glide reflection, 214, 257
Golden ratio, 169
Group, 240, 271
  Conjugation, 273
Cyclic, 276
Dihedral, 276
Identity element, 272
Inverse element, 272
Multiplication, 272
Of isometries, 240
Of transformations, 240
Operation, 240
Order of an element, 275

Group of symmetries
Polygon, 283
Rectangle, 279
Square, 278

H
Halfplane, 16
Halfturn, 238
Harmonic conjugates, 375, 412
Inverses, 382
Harmonic progression, 377
Hexagon, 109
Nonsimple, 109
Simple, 109
Homothetic
Figures, 295
Image, 295
Homothety, 79, 290
In constructions, 295
In proofs, 300
Properties preserved by, 292
Hyperbola, 404, 407
Hypotenuse, 3

I
Ideal line, 125
Ideal point
Inversive plane, 343
Ideal points, 415
Extended Euclidean plane, 125, 343
Projective plane, 125, 386
Identity mapping, 212
Implication
Conclusion, 23
Converse, 9
Counterexample, 23
Hypothesis, 23
Impossible Euclidean constructions
Doubling the cube, 160
Squaring the circle, 160
Trisecting a general angle, 161
Impossible straightedge constructions
Center of a given circle, 441
Midpoint of a segment, 436
Incenter of a triangle, 44
Incircle of a triangle, 35
Inequalities in proofs, 26
Injection, 212
Inscribed angles, 16
Interior angle, 5
Internal angle bisectors, 38, 71
Invariant point, 210
Inverses, 340
Inversion, 339
Angles, 352
Circles, 346
Distances, 348
Effect on Euclidean properties, 345
Lines, 346
Inversive plane, 343
Involution, 241
Irrational numbers, 134
Isometry, 213
Isosceles triangle, 8
Isosceles Triangle Theorem, 8

J
Johnson’s Theorem, 177

K
Kie, 29

L
Law of cosines, 152
Law of sines, 148
Lengths of internal angle bisectors, 72
Lengths of medians, 69
Leonardo’s Theorem, 279
Lexicographical order, 321
Lily pad problem, 82
Line at infinity, 125
Lines, 5
Locus of a point, 31
Longfellow, Henry Wadsworth, 82

M
Magnification factor, 60
Mapping, 211
Function, 211
Martin Gardner, 332
Measure of an angle, 5
Medians concurrent
Area proof, 142
Ceva’s Theorem proof, 101
Menelaus pattern, 114, 115
Menelaus point, 97
Menelaus’ Theorem, 97, 103
Angle bisectors, 104
Applications, 103
Einstein’s area proof, 123
INDEX

Einstein's ugly proof, 123
Extended plane, 129
Extended version, 129, 130
Proof, 119
Proof of Pappus' Theorem, 112
Proof of Simson's Theorem, 104
To compute ratio or distance, 114
Method of loci, 31
Medial Theorem, 49
Miquel point, 172
Miquel's Theorem, 171, 202
Mohr's problem, 365
Mohr-Mascheroni Theorem, 367
Morley's Theorem, 178

N
Nagel point of a triangle, 131
Nine-point circle, 185
Center, 191
Feuerbach's Theorem, 186
Radius, 192
Nonconstructive numbers, 166
Nonoverlapping figures, 133

O
Obtuse triangle
Circumcircle and 9-point circle, 402
Open Jaw Inequality, 27
Opposite isometry, 241
Opposite or alternate angles, 12
Opposite or alternate exterior angles, 12
Ordinary line, 125
Ordinary points, 125
Ordinary triangle, 127, 130
Orthogonal circles, 353, 385
Overlapping figures, 133

P
Pappus' Ancient Theorem, 355
Pappus' Theorem, 112, 454
Parabola, 404, 407
Parallel Axiom, 4
Parallel cevians, 131
Parallel lines, 4
Parallel postulate, 4
Parallel projections, 62
Parallelogram, 5, 46
Pascal line, 110
Pascal's Mystic Hexagon Theorem, 109, 408
Pasch's Axiom, 62
Pencil of lines, 390
Cross ratio, 417
Pentagram, 135
Pentominoes, 332
Perimeter of a triangle, 131
Perpendicular, 24
Foot of, 25
Perpendicular bisectors, 24
Characterization, 24
Concurrent, 41
Perspective transformation, 445
Perspectivity, 445
Parallel, 443
Platonic tiling, 325
Playfair's axiom, 4
Point divides a segment
Externally, 99
Internally, 99
Points C and D divide AB harmonically, 375
Points at infinity, 125, 345
Polar circle, 400
Polar of a point, 386
Pole of a line, 387
Polygon, 133
Regular, 166
Polygonal region, 133
Polygons
Decomposition into triangular regions, 138
Poncelet-Steiner constructions, 442
Power of a point, 82, 85
Principle of Duality, 394
Product
Translation and rotation, 259
Two rotations, 262
Two translations, 258
Product of reflections, 245
Projecting a line to infinity, 453
Projective 3-space, 445
Projective geometry, 126
Fundamental Theorem of, 449
Projective plane, 126, 383, 445
Projectivities in 2-space, 447
Projectivity, 446
Properties preserved, 446
Proportionality constant, 60
Proportionals, 314
Ptolemy's Theorem, 351
Pythagoras' Theorem, 24, 67
Area proof, 137
Pythagorean relation, 3

Q
Quadrilateral, 6, 17
Cyclic, 18
Dart, 29
Inscribed circle, 20
Kite, 29
Rectangle, 23
Rhombus, 29
Simple
Interior diagonal, 17
Sum of interior angles, 17
Quaternion group, 288

R
Radius of inversion, 340
Range of points, 390
Rational numbers, 134
Rays, 5
Reciprocal of a circle, 403
Reciprocation, 385
Reciprocal Theorem, 388
Rectangle, 46
Reflection in a point, 238
Reflectional symmetry, 208
Reflections, 210
Reflex angle, 5
Regular dodecagon, 330
Regular polygon
Octagon, 153
Pentagon, 156
Regular star polygons, 167
Rhombus, 29, 46
Right bisector, 24
Right triangle, 3
Hypotenuse, 22
Rotations, 210
Routh's Theorem, 146

S
Self-conjugate, 395
Self-dual, 395
Self-polar triangles, 399
Semiperimeter, 156
Signed distance, 83, 95
Similar
Figures, 295
Triangles, 59
Similarity, 291
Homoethy, 290
Similarity conditions, 65
AAA, 65
AA, 65
sAs, 65
sss, 65
Simple quadrilateral
Cyclic
Sum of opposite angles, 18
Simson line, 20
Simson's Theorem, 20, 104
Square, 46
Steiner chain of circles, 359
Steiner's Porism, 356
Steiner-Lehmus Theorem, 193
Stewart's Theorem, 69
Straightedge
Admissible operations, 435
Straightedge-only constructions, 383, 436
Harmonic conjugates, 440
Sum of angles in a triangle, 3, 13
Surjection, 212

T
Tangent line to a circle, 43
Tessellations, 313
Thales' Locus, 51
Thales' Theorem, 14, 52
Thomsen's Relation
Halfturns, 266
Reflections, 266
Three Problems of Antiquity, 159, 166
Tiling the plane
Convex hexagons, 318
Cut and merge method, 328
Edge-to-edge, 319
Edge-to-vertex, 316
Hexagons, 316
Pentagons, 316
Rectangles, 315
Regular, 325
Regular polygons, 319
Vertex sequence of a point, 319
Semiregular, 325
Triangles, 315
Vertex-to-vertex, 316
Tilings, 313
k-hedral, 314
Diheiral, 314
Monohedral, 314
Order k, 314
Triheiral, 314
Trace-and-fit method, 217
Transformation, 211
Composition, 214
Equality, 237
Halfturn, 238
Inverse, 213
Involution, 241
Isometry, 213
Translations, 80, 211
Transversal, 12, 97
Triangle
Altitudes, 48
Centroid, 50
Circumcenter, 43
Circumcircle, 19, 42
Equilateral, 156
Excenter, 45
Excircle, 45
Fermat point, 223
Gergonne point, 131
Incenter, 44
Incircle, 35, 44
Mediatrix, 48
Nagel point, 131
Napoleon, 268
Obtuse, 399
Ordinary, 130
Orthocenter, 48
Perimeter, 131
Self-polar, 399
Transversal, 129
Triangle Inequality, 8
Triangulation, 138

U
Undirected angles, 5

V
Vertex sequence of a point, 319

W
Willson, John, 479