Chapter Objectives

In this chapter methods for solving simultaneous equations analytically and graphically are introduced. These methods are then applied to the analysis of equilibrium in the goods, labour and money markets and consumer and producer surplus. At the end of this chapter, using linear functions, you should be able to:

- Solve two equations in two unknowns and illustrate the solution graphically
- Distinguish between unique solutions, no solutions and infinitely many solutions
- Solve three equations in three unknowns
- Calculate the equilibrium price and quantity in the goods market and illustrate the solution graphically
- Analyse and illustrate graphically the effect of intervention in the goods market (price ceilings and price floors)
- Analyse and illustrate graphically the effect of taxes and subsidies in the goods market
- Calculate and illustrate graphically break-even, profit and loss
- Calculate consumer and producer surplus
- Calculate the equilibrium conditions for the national income model and illustrate equilibrium graphically
- Calculate and illustrate graphically the equilibrium values of national income and interest rates based in the IS-LM model
- Use Excel to plot all of the above.
3.1 Solving Simultaneous Linear Equations

At the end of this section you should be familiar with:

- How to solve two simultaneous equations in two unknowns using:
  - (a) Algebra
  - (b) Graphical methods.
- Determining when two equations in two unknowns have:
  - (a) A unique solution
  - (b) No solution
  - (c) Infinitely many solutions.
- Solving three simultaneous equations in three unknowns.

In many applications, both practical and theoretical, there will be several equations with several variables or unknowns. These are referred to generally as simultaneous equations. This section introduces some methods to solve simultaneous linear equations.

Reminder

A solution of an equation in an unknown, say \( x \), is simply the value for \( x \) for which the left-hand side (LHS) of the equation is equal to the right-hand side (RHS).

For example, consider the equation \( x + 4 = 6 \).
- The solution of this equation is \( x = 2 \).
- \( x = 2 \) is the only value of \( x \) for which the LHS = RHS.
- The statement ‘\( x = 2 \) satisfies the equation’ is another way of saying that \( x = 2 \) is a solution.

The solution, therefore, of a set of simultaneous equations is a set of values for the variables which satisfy all the equations.

3.1.1 Two equations in two unknowns

A standard method for solving two linear equations in two unknowns is outlined in Worked Examples 3.1 to 3.3.

WORKED EXAMPLE 3.1
SOLVING SIMULTANEOUS EQUATIONS 1

Given the simultaneous equations

\[
\begin{align*}
x + 3y &= 4 \\
-x + 2y &= 6
\end{align*}
\]

(a) Solve for \( x \) and \( y \) algebraically.
(b) Solve for \( x \) and \( y \) graphically.
SIMULTANEOUS EQUATIONS

SOLUTION

(a) Method: Eliminate $x$ from the system of equations by adding equations (1) and (2). The two equations reduce to a single equation in which the only unknown is $y$. Solve for $y$, then substitute the value of $y$ into either of the original equations and solve for $x$.

Step 1: Adding equations (1) and (2):

\[
\begin{align*}
  x + 3y &= 4 \\
- x + 2y &= 6 \\
\hline
  0 + 5y &= 10
\end{align*}
\]

Adding

Solving for $y$:

\[y = \frac{10}{5} = 2\]

Step 2: Solve for $x$ by substituting $y = 2$ into either equation (1) or equation (2):

\[
\begin{align*}
  - x + 2(2) &= 6 \\
  - x &= 6 - 4 \\
  x &= -2
\end{align*}
\]

Substituting $y = 2$ into equation (2)

Step 3: Check the solution $x = -2$, $y = 2$ by substituting these values into equations (1) and (2) and confirm that both equations balance.

- Substitute $x = -2$ and $y = 2$ into equation (1):

\[
\begin{align*}
  x + 3y &= 4 \\
  (-2) + 3(2) &= 4 \\
  -2 + 6 &= 4
\end{align*}
\]

Substituting in $x = -2$ and $y = 2$

\[4 = 4\] so equation (1) balances and $x = -2, y = 2$ is a solution

- Substitute $x = -2$ and $y = 2$ into equation (2):

\[
\begin{align*}
  - x + 2y &= 6 \\
  -(-2) + 2(2) &= 6 \\
  2 + 4 &= 6
\end{align*}
\]

Substituting in $x = -2$ and $y = 2$

\[6 = 6\] so equation (2) balances and $x = -2, y = 2$ is a solution

Since the point $(-2, 2)$ satisfies equations (1) and (2), then this point is at the point of intersection of the lines represented by equations (1) and (2) as shown in Figure 3.1.

(b) The two lines are plotted in Figure 3.1. The point of intersection is the solution. The coordinates of this point are $x = -2$ and $y = 2$. In this case, it is a unique solution: that is, the lines intersect at only one point. This point is on the first line, so it satisfies equation (1), and also on the second line, so it satisfies equation (2).
WORKED EXAMPLE 3.2
SOLVING SIMULTANEOUS EQUATIONS 2

Given the simultaneous equations

\[ 2x + 3y = 12.5 \]  \hspace{1cm} (1)
\[ -x + 2y = 6 \]  \hspace{1cm} (2)

(a) Solve for \( x \) and \( y \) algebraically.
(b) Solve for \( x \) and \( y \) graphically.

SOLUTION

(a) In this example, neither the \( x \)- nor the \( y \)-terms are the same. However, all terms on both sides of any equation may be multiplied by a constant without affecting the solution of the equation. So, if equation (2) is multiplied by 2, the \( x \)-terms in both equations will be the same with opposite signs. Then, proceed as in Worked Example 3.1.

Step 1: Eliminate \( x \) from the system of equations by multiplying equation (2) by 2 and then add the equations; \( 2x \) and \(-2x \) cancel to leave a single equation in one unknown, \( y \):

\[
\begin{align*}
2x + 3y &= 12.5 & \text{(1) as given} \\
-2x + 4y &= 12 & \text{(2) } \times 2 \\
0 + 7y &= 24.5 & \text{adding the two equations} \\
y &= \frac{24.5}{7} = 3.5 & \text{solving for } y
\end{align*}
\]

Step 2: Solve for the value of \( x \) by substituting \( y = 3.5 \) into either equation (1) or equation (2):

\[-x + 2(3.5) = 6 \hspace{1cm} \text{substituting } y = 3.5 \text{ into (2) to find the value of } x \]
\[-x = 6 - 7 \]
\[-x = -1 \rightarrow x = 1 \]
Simultaneous Equations

Step 3: Check the solution \( x = 1, y = 3.5 \), by substituting these values into equations (1) and (2) and confirm that both equations balance.

- Substitute \( x = 1 \) and \( y = 3.5 \) into equation (1):
  \[
  2x + 3y = 12.5
  \]
  \[
  2(1) + 3(3.5) = 12.5 \quad \text{substituting in } x = 1 \text{ and } y = 3.5
  \]
  \[
  2 + 10.5 = 12.5
  \]
  \[
  12.5 = 12.5 \quad \text{so (1) balances and } x = 1, \ y = 3.5 \text{ is a solution}
  \]

- Substitute \( x = 1 \) and \( y = 3.5 \) into equation (2):
  \[
  -x + 2y = 6
  \]
  \[
  -(1) + 2(3.5) = 6 \quad \text{substituting in } x = 1 \text{ and } y = 3.5
  \]
  \[
  -1 + 7 = 6
  \]
  \[
  6 = 6 \quad \text{so (2) balances and } x = 1, \ y = 3.5 \text{ is a solution}
  \]

Therefore, the solution of equations (1) and (2) is at the point of intersection of the lines represented by equations (1) and (2), as shown in Figure 3.2.

(b) The two lines are plotted in Figure 3.2. The point of intersection is the solution. The coordinates of this point are \( x = 1 \) and \( y = 3.5 \). In this case it is a unique solution: that is, the lines intersect at only one point. This point is on the first line, so it satisfies equation (1), and also on the second line, so it satisfies equation (2).

3.1.2 Solve simultaneous equations by methods of elimination and substitution

The method of elimination

In Worked Examples 3.1 and 3.2 the simultaneous equations were solved by adding a multiple of one equation to the other in order to eliminate one of the variables. Hence the method is called the 'method of elimination'.
The method of substitution

Alternatively, from either equation derive an expression for one variable in terms of the other: substitute this expression into the other equation and solve. This is illustrated for the equations given in Worked Example 3.1 as follows:

\[
\begin{align*}
2x + 3y &= 12.5 \\
-x + 2y &= 6
\end{align*}
\]

From (2), \(x = 2y - 6\) is the expression for \(x\) in terms of \(y\) (\(x\) is the subject of the formula).

Substitute \(2y - 6\) for \(x\) in equation (1)

\[
\begin{align*}
2x + 3y &= 12.5 \\
\Rightarrow 2(2y - 6) + 3y &= 12.5 \\
4y - 12 + 3y &= 12.5 \\
7y &= 24.5 \\
y &= 3.5
\end{align*}
\]

Substitute \(y = 3.5\) into any of the equations to solve for \(x\). For example,

\[
x = 2y - 6 \rightarrow x = 2(3.5) - 6 = 1
\]

Note: The method of substitution is particularly suitable for solving simultaneous equations where one equation is linear and the other non-linear. See Chapter 4.

WORKED EXAMPLE 3.3
SOLVING SIMULTANEOUS EQUATIONS 3

Given the simultaneous equations

\[
\begin{align*}
2x + 3y &= 0.75 \\
5x + 2y &= 6
\end{align*}
\]

Solve for \(x\) and \(y\) algebraically.

**SOLUTION**

In these two equations, neither the \(x\)- nor the \(y\)-terms are the same. If equation (1) is multiplied by 2 and equation (2) is multiplied by \(-3\), the \(y\)-terms in both equations will be the same with opposite signs. Then proceed as in Worked Example 3.1 above.

**Step 1:** Eliminate \(y\)-terms from the system of equations:

\[
\begin{align*}
4x + 6y &= 1.5 \quad (1) \times 2 \\
-15x - 6y &= 18 \quad (2) \times -3 \\
\hline
-11x &= -16.5 \\
\hline
x &= \frac{-16.5}{-11} = 1.5 \quad \text{solving for } x
\end{align*}
\]
SIMULTANEOUS EQUATIONS

Step 2: Solve for \( y \) by substituting \( x = 1.5 \) into either equation (1) or equation (2):

\[
2(1.5) + 3y = 0.75 \quad \text{substituting } x = 1.5 \text{ into equation (1)}
\]

\[
3y = 0.75 - 3
\]

\[
y = -2.25
\]

Step 3: Checking the solution: \( x = 1.5, \ y = -\frac{2.25}{3} \) is left as an exercise for the reader.

3.1.3 Unique, infinitely many and no solutions of simultaneous equations

A set of simultaneous equations may have:

- A unique solution.
- No solution.
- Infinitely many solutions.

**Unique solution**

This occurs when a set of equations has one set of values which satisfy all equations. See Worked Examples 3.1 to 3.3.

**No solution**

This occurs when a set of equations has no set of values which satisfy all equations.

**WORKED EXAMPLE 3.4**

SIMULTANEOUS EQUATIONS WITH NO SOLUTION

Given the simultaneous equations:

\[
\begin{align*}
y &= 1 + x \\
y &= 2 + x
\end{align*}
\]

(a) Solve for \( x \) and \( y \) algebraically.

(b) Solve for \( x \) and \( y \) graphically.

**SOLUTION**

(a)

\[
\begin{align*}
y &= 1 + x \quad (1) \\
y &= 2 + x \quad (2) \\
0 &= -1 \quad (1) - (2)
\end{align*}
\]

\( 0 = -1 \) is not possible, therefore, there is no solution. Even from a purely practical point of view, you can see that there is no way that these equations can both be true. How can \( y \) be equal to \((1 + x)\) and \((2 + x)\) at the same time?
Note: A false statement (or a contradiction) like \(0 = -1\) indicates a set of equations with no solution.

(b) The two equations are plotted in Figure 3.3. The lines will never meet since they are parallel and thus will never have a point (solution) in common.

\[
y = 2 + x
\]

\[
y = 1 + x
\]

Figure 3.3  No solution

**Infinitely many solutions**

A set of equations has infinitely many solutions when there is an infinite number of sets of values that satisfy all equations.

**WORKED EXAMPLE 3.5**

**SIMULTANEOUS EQUATIONS WITH INFINITELY MANY SOLUTIONS**

Given the simultaneous equations

\[
y = 2 - x \tag{1}
\]

\[
2y = 4 - 2x \tag{2}
\]

(a) Solve for \(x\) and \(y\) algebraically.

(b) Solve for \(x\) and \(y\) graphically.
**Simultaneous Equations**

**Solution**

(a) When equation (2) is divided by 2, the result is exactly the same as equation (1), since

\[ \frac{2y}{2} = \frac{4}{2} - \frac{2x}{2} \rightarrow y = 2 - x \]

So, equations (1) and (2) are the same! There is only one equation in two unknowns. If \( x \) is given any value, the corresponding \( y \)-value can be calculated.

\[ y = 2 - x \text{ and } 2y = 4 - 2x \]

For example, when

\[ x = 2 \rightarrow y = 2 - 2 = 0 \]
\[ x = 3 \rightarrow y = 2 - 3 = -1 \]
\[ x = 5 \rightarrow y = 2 - 5 = -3 \text{ etc.} \]

There is an infinite number of \((x, y)\) pairs which satisfy equations (1) and (2).

(b) Equations (1) and (2) are plotted in Figure 3.4. Note that these equations represent coincident lines, therefore every point on one line is also a point on the other line. Since a line has infinitely many points, there are infinitely many solutions or points in common.

![Figure 3.4 Infinitely many solutions](image)

3.1.4 Three simultaneous equations in three unknowns

The methods used above to solve two equations in two unknowns may be extended to three equations in three unknowns, four equations in four unknowns, etc. The strategy is to eliminate one of the
variables first by adding multiples of equations to other equations and hence reducing the problem to two equations in two unknowns. This is best demonstrated by Worked Example 3.6.

**WORKED EXAMPLE 3.6**

**SOLVE THREE EQUATIONS IN THREE UNKNOWNS**

Solve the equations

\[
\begin{align*}
2x + y - z &= 4 \quad (1) \\
x + y - z &= 3 \quad (2) \\
2x + 2y + z &= 12 \quad (3)
\end{align*}
\]

**SOLUTION**

The simplest approach is to add equation (3) to equation (1), and hence eliminate \( z \), giving an equation in \( x \) and \( y \). Then add equation (3) to equation (2), eliminating \( z \) again, giving another equation in \( x \) and \( y \).

\[
\begin{align*}
2x + y - z &= 4 \quad (1) \\
2x + 2y + z &= 12 \quad (3) \\
4x + 3y + 0 &= 16 \quad (4) \text{ adding equations (1) and (2)} \\
x + y - z &= 3 \quad (2) \\
2x + 2y + z &= 12 \quad (3) \\
3x + 3y + 0 &= 15 \quad (5) \text{ adding equations (2) and (3)}
\end{align*}
\]

Equations (4) and (5) are the usual two equations in two unknowns, so solve for \( x \) and \( y \). Then solve for \( z \) later.

\[
\begin{align*}
4x + 3y &= 16 \quad (4) \\
-3x - 3y &= -15 \quad (6) \text{ equation (5) multiplied by } -1 \\
x &= 1 \quad \text{ adding equations (4) and (6)}
\end{align*}
\]

So, \( x = 1 \). Substitute \( x = 1 \) into equations (4), (5) or (6) to solve for \( y \).

Substituting \( x = 1 \) into equation (4) gives \( 4(1) + 3y = 16 \rightarrow y = 4 \).

Finally, find \( z \) by substituting \( x = 1 \), \( y = 4 \) into any of the equations (1), (2) or (3).

For example, substituting into equation (2),

\[
1 + 4 - z = 3 \rightarrow z = 2
\]

Therefore, the values which satisfy all three equations (1), (2) and (3) are \( x = 1 \), \( y = 4 \), \( z = 2 \).
Simultaneous Equations

Progress Exercises 3.1

Simultaneous Equations: Solving for Two and Three Unknowns

Solve the following simultaneous equations:

1. \[ y = x \]
   \[ y = 3 - x \]

2. \[ x + y = 10 \]
   \[ x - y = 4 \]

3. \[ x + y = 19 \]
   \[ x - 8y = 10 \]

4. \[ 5x + 2y = 11 \]
   \[ 3x + 3y = 15 \]

5. \[ 3y + 2x = 5 \]
   \[ 4y - x = 3 \]

6. \[ 2y - x = 12 \]
   \[ y = 2x - 3 \]

7. \[ 2x - 4y = -1 \]
   \[ 3y - 2x = 12 \]

8. \[ x + 2y = 10 \]
   \[ y = -0.5x + 5 \]

9. \[ 3x - 2y = 12 \]
   \[ 2y - 3x = 11.2 \]

10. \[ 3x - 2y = 15 \]
    \[ 15x - 45 = 6y \]

11. \[ 2x - 5y = 7 \]
    \[ 2 = 3x - 2.5y \]

12. \[ 5 + 2P = 6Q \]
    \[ 5P + 8Q = 25 \]

13. \[ 5 + 2P = 6Q \]
    \[ 5P + 8Q = 25 \]

14. \[ x - y + z = 0 \]
    \[ 2y - 2z = 2 \]
    \[ -x + 2y + 2z = 29 \]

15. \[ P_1 - 3P_2 = 0 \]
    \[ 5P_2 - P_3 = 10 \]
    \[ P_1 + P_2 + P_3 = 8 \]

16. \[ 3x - 2y + 1 = 0 \]
    \[ 0.5x + 2.5y - 14 = 0 \]

17. \[ 4x - 3y + 1 = 13 \]
    \[ 0.5x + y - 3 = -1 \]

18. \[ \frac{5}{2}q - 3p = \frac{7}{2} \]
    \[ 3p = 3(q - 3) \]

19. \[ P = \frac{18 - 10Q}{5} \]
    \[ 2 = \frac{3Q + 5P}{2} \]

20. \[ \frac{3(x - 16) + y}{5} = 0 \]
    \[ \frac{5y + 10x}{5} = 20 \]

21. \[ 2x + y + 2z = 4 \]
    \[ 3x + z = 2 + y \]
    \[ x + 2y + 4z + 1 = 0 \]

22. In questions 16 to 20, plot the graphs to confirm your answer.

3.2 Equilibrium and Break-even

At the end of this section you should be familiar with:

- Equilibrium in the goods market and labour market
- Price controls and government intervention in various markets
- Market equilibrium for substitute and complementary goods
- Taxes, subsidies and their distribution between producer and consumer
- Break-even analysis.

The method of simultaneous equations is now applied to the determination of equilibrium conditions in various markets; for example, the goods, labour and money markets. In addition, situations are considered that prevent the occurrence of equilibrium. Furthermore, the analysis also considers factors that change the state of equilibrium from one position to another.
CHAPTER 3

Note: A state of equilibrium within a model is a situation that is characterised by a lack of tendency to change.

3.2.1 Equilibrium in the goods and labour markets

Goods market equilibrium

Goods market equilibrium (market equilibrium) occurs when the quantity demanded \( Q_d \) by consumers and the quantity supplied \( Q_s \) by producers of a good or service are equal. Equivalently, market equilibrium occurs when the price that a consumer is willing to pay \( P_d \) is equal to the price that a producer is willing to accept \( P_s \). The equilibrium condition, therefore, is expressed as

\[
Q_d = Q_s \quad \text{and} \quad P_d = P_s
\]  

(3.1)

Note: In equilibrium problems, once the equilibrium condition is stated, \( Q \) and \( P \) are used to refer to the equilibrium quantity and price respectively.

WORKED EXAMPLE 3.7
GOODS MARKET EQUILIBRIUM

The demand and supply functions for a good are given as

- Demand function: \( P_d = 100 - 0.5Q_d \) \hspace{1cm} (3.2)
- Supply function: \( P_s = 10 + 0.5Q_s \) \hspace{1cm} (3.3)

Calculate the equilibrium price and quantity algebraically and graphically.

SOLUTION

Market equilibrium occurs when \( Q_d = Q_s \) and \( P_d = P_s \). Since the functions are written in the form \( P = f(Q) \) with \( P \) as the only variable on the LHS of each equation, it is easier to equate prices, thereby reducing the system to an equation in \( Q \) only; hence, solve for \( Q \):

\[
P_d = P_s
\]

\[
100 - 0.5Q = 10 + 0.5Q \quad \text{equating the RHS of equations (3.2) and (3.3)}
\]

\[
100 - 10 = 0.5Q + 0.5Q
\]

\[
90 = Q \quad \text{equilibrium quantity}
\]

Now solve for the equilibrium price by substituting \( Q = 90 \) into either equation (3.2) or (3.3):

\[
P = 100 - 0.5(90) \quad \text{substituting } Q = 90 \text{ into equation (3.2)}
\]

\[
P = 55 \quad \text{equilibrium price}
\]
SIMULTANEOUS EQUATIONS

Check the solution, \( Q = 90, P = 55 \), by substituting these values into either equation (3.2) or (3.3). This exercise is left to the reader.

Figure 3.5 illustrates market equilibrium at point \( E_0 \) with equilibrium quantity, 90, and equilibrium price, £55. The consumer pays £55 for the good which is also the price that the producer receives for the good. There are no taxes (what a wonderful thought!).

\[
P = 10 + 0.5Q
\]

\[
P = 100 - 0.5Q
\]

Labour market equilibrium

Labour market equilibrium occurs when the labour demanded (\( L_d \)) by firms is equal to the labour supplied (\( L_s \)) by workers or, equivalently, when the wage that a firm is willing to offer (\( w_s \)) is equal to the wage that workers are willing to accept (\( w_d \)). Labour market equilibrium, therefore, is expressed as

\[
L_d = L_s \quad \text{and} \quad w_d = w_s \quad (3.4)
\]

Again, in solving for labour market equilibrium, once the equilibrium condition is stated, \( L \) and \( w \) refer to the equilibrium number of labour units and the equilibrium wage, respectively.

WORKED EXAMPLE 3.8

LABOUR MARKET EQUILIBRIUM

The labour demand and supply functions are given as

Labour demand function: \( w_d = 9 - 0.6L_d \quad (3.5) \)

Labour supply function: \( w_s = 2 + 0.4L_s \quad (3.6) \)

Calculate the equilibrium wage and equilibrium number of workers algebraically and graphically. (In this example 1 worker \( \equiv 1 \) unit of labour.)

SOLUTION

Labour market equilibrium occurs when \( L_d = L_s \) and \( w_d = w_s \). Since the functions are written in the form \( w = f (L) \), equate wages, thereby reducing the system to an equation in \( L \) only;
hence, solve for \( L \):

\[
\begin{align*}
\frac{w_1}{w_s} &= 9 - 0.6L = 2 + 0.4L \\
equating\ equations\ (3.5)\ and\ (3.6) &
\end{align*}
\]

\( 9 - 2 = L \)

\( 7 = L \)

equilibrium number of workers

Now solve for \( w \) by substituting \( L = 7 \) into either equation (3.5) or (3.6):

\[
\begin{align*}
w &= 9 - 0.6(7) \quad \text{substituting} \ L = 7 \ \text{into equation (3.5)} \\
w &= 4.8 \quad \text{equilibrium wage}
\end{align*}
\]

Figure 3.6 illustrates labour market equilibrium at point \( E_0 \) with equilibrium number of workers, 7, and equilibrium wage, £4.80. Each worker receives £4.80 per hour for his or her labour services, which is also the wage that the firm is willing to pay.

3.2.2 Price controls and government intervention in various markets

In reality, markets may fail to achieve market equilibrium due to a number of factors: for example, the intervention of governments or the existence of firms with monopoly power. Government intervention in the market through the use of price controls is now analysed.

**Price ceilings**

Price ceilings are used by governments in cases where they believe that the equilibrium price is too high for the consumer to pay. Thus, price ceilings operate below market equilibrium and are aimed at protecting consumers. Price ceilings are also known as maximum price controls, where the price is not allowed to go above the maximum or 'ceiling' price (for example, rent controls or maximum price orders).
WORKED EXAMPLE 3.9
GOODS MARKET EQUILIBRIUM AND PRICE CEILINGS

The demand and supply functions for a good are given by

Demand function: \( P_d = 100 - 0.5Q_d \) \hspace{1cm} (3.7)

Supply function: \( P_s = 10 + 0.5Q_s \) \hspace{1cm} (3.8)

(a) Analyse the effect of the introduction of a price ceiling of £40 in this market.
(b) Calculate the profit made by black marketeers if a black market operated in this market.

SOLUTION

(a) The demand and supply functions are the same as those in Worked Example 3.7 where the equilibrium price and quantity were £55 and 90 units, respectively. The price ceiling of £40 is below the equilibrium price of £55. Its effect is analysed by comparing the levels of quantity demanded and supplied at \( P = £40 \).

The quantity demanded at \( P = £40 \) is
\[
P_d = 100 - 0.5Q_d \quad \text{equation (3.7)}
\]
Substituting \( P = £40 \):
\[
40 = 100 - 0.5Q_d
\]
Solving for \( Q_d \):
\[
0.5Q_d = 60 \quad \Rightarrow \quad Q_d = 120
\]

The quantity supplied at \( P = £40 \) is
\[
P_s = 10 + 0.5Q_s \quad \text{equation (3.8)}
\]
Substituting \( P = £40 \):
\[
40 = 10 + 0.5Q_s
\]
Solving for \( Q_s \):
\[
-0.5Q_s = -30 \quad \Rightarrow \quad Q_s = 60
\]

Since the quantity demanded \( (Q_d = 120) \) is greater than the quantity supplied \( (Q_s = 60) \), there is an excess demand \( (XD) \) of: \( XD = Q_d - Q_s = 120 - 60 = 60 \). This is also referred to as a shortage in the market. It is illustrated in Figure 3.7.

(b) The existence of price ceilings often leads to the establishment of black markets where goods are sold illegally at prices above the legal maximum. Black marketeers would buy the 60 units supplied at the controlled price of £40 per unit. However, as there is a shortage of goods, consumers are willing to pay a higher price for these 60 units. The price that consumers are willing to pay is calculated from the demand function for \( Q = 60 \). Substitute \( Q = 60 \) into the demand function:
\[
P_d = 100 - 0.5Q_d
\]
Substituting \( Q = 60 \):
\[
P_d = 100 - 0.5(60) = 100 - 30 = 70
\]

So, \( P_d = 70 \) is the price consumers are willing to pay.
Therefore, black marketeers buy the 60 units at the maximum price of £40 per unit, costing them $60 \times £40 = £2400$, and then sell these 60 units at £70 per unit, generating revenue of $60 \times £70 = £4200$. Their profit ($\pi$) is the difference between revenue and costs:

$$
\pi = TR - TC = (70 \times 60) - (40 \times 60) = 4200 - 2400 = 1800
$$

This is illustrated as the shaded area in Figure 3.7.

**Price floors**

Price floors are used by governments in cases where they believe that the equilibrium price is too low for the producer to receive. Thus, price floors operate above market equilibrium and are aimed at protecting producers. Price floors are also known as minimum prices, where the price is not allowed to go below the minimum or ‘floor’ price (for example, the Common Agricultural Policy (CAP) in the European Union and minimum wage laws).

**WORKED EXAMPLE 3.10**

**LABOUR MARKET EQUILIBRIUM AND PRICE FLOORS**

Given the labour demand and supply functions as

- Labour demand function: $w_d = 9 - 0.6L_d$ \hspace{1cm} (3.9)
- Labour supply function: $w_s = 2 + 0.4L_s$ \hspace{1cm} (3.10)

analyse the effect on the labour market if the government introduces a minimum wage law of £6 per hour.
SIMULTANEOUS EQUATIONS

SOLUTION

The labour demand and supply functions are the same as those in Worked Example 3.8 where the equilibrium wage and units of labour were £4.80 per hour and 7 labour units, respectively. The minimum wage law (price floor) of £6 is above market equilibrium. Its effect is analysed by comparing the levels of labour demanded and supplied at $w = 6$.

Labour demanded at $w = 6$ is

- $w_d = 9 - 0.6L_d$ equation (3.9)
- $6 = 9 - 0.6L_d$ substituting $w = 6$
- $0.6L_d = 3$
- $L_d = 5$

Labour supplied at $w = 6$ is

- $w_s = 2 + 0.4L_s$ equation (3.10)
- $6 = 2 + 0.4L_s$ substituting $w = 6$
- $4 = 0.4L_s$
- $10 = L_s$

Since labour supplied ($L_s = 10$) is greater than labour demanded ($L_d = 5$), there is an excess supply of labour: $XS = L_s - L_d = 10 - 5 = 5$. This is also referred to as a surplus, that is, there is unemployment in the labour market. The graphical illustration of this result is left to the reader.

PROGRESS EXERCISES 3.2

Determination of Equilibrium for Linear Functions

1. (a) Determine the equation of the line which has a slope $m = 1.5$, and which passes through the point $x = 4, y = 12$.
   (b) A supplier is known to supply a quantity of goods $Q$ when the market price $P$ is 25 per unit. If the quantity supplied increases by 3 for each unit increase in price, determine the equation of the supply function in the form $Q = f(P)$.

2. The demand and supply functions for fashion rings are given by the equations

   Demand function: $P_d = 800 - 2Q$
   Supply function: $P_s = -40 + 8Q$

   (a) Calculate the equilibrium quantity and price.
   (b) Calculate the level of excess supply ($Q_s - Q_d$) for rings when $P = 720$.
   (c) Find the level of excess demand ($Q_d - Q_s$) when $P = 560$.

3. The demand and supply functions for a product are given by the equations

   Demand function: $P_d = 400 - 5Q$
   Supply function: $P_s = 3Q + 24$

   (a) Calculate the equilibrium price and quantity.
   (b) Plot the graphs of the demand and supply functions, hence confirm the answer in (a).
4. A company hires out cabin cruisers on a daily basis. The demand and supply functions are given by the equations:

\[ Q_d = 920 - 8P \quad \text{and} \quad Q_s = -120 + 2P \]

(a) Calculate the equilibrium price and quantity algebraically and graphically.
(b) Calculate the level of excess demand \((Q_d - Q_s)\) when \(P = 90\).

5. The demand and supply functions for a good (jeans) are given by:

Demand function: \( P_d = 50 - 3Q \)
Supply function: \( P_s = 14 + 1.5Q \)

where \(P\) is the price of a pair of jeans; \(Q\) is the number of pairs of jeans.
(a) Calculate the equilibrium price and quantity.
(b) Calculate the level of excess supply \((Q_s - Q_d)\) when \(P = 38\).

6. (See question 5)
(a) Calculate the level of excess demand \((Q_d - Q_s)\) when \(P = 20\).
(b) Calculate the profit made on the black market if a maximum price of £20 per pair of jeans is imposed.

7. The demand and supply functions for labour are given by:

Labour demand function: \( w_d = 70 - 4L \)
Labour supply function: \( w_s = 10 + 2L \)

(a) Calculate the equilibrium number of workers employed and the equilibrium wage per hour.
(b) Calculate the excess demand for labour \((L_d - L_s)\) when \(w = 20\).
(c) Calculate the excess supply for labour \((L_s - L_d)\) when \(w = 40\).

3.2.3 Market equilibrium for substitute and complementary goods

Complementary goods are goods that are consumed together (for example, cars and petrol; computer hardware and computer software). One good cannot function without the other. On the other hand, substitute goods are consumed instead of each other (for example, coffee versus tea; bus versus train on given routes).

The general demand function is now written as

\[ Q = f(P, P_s, P_c) \]

that is, the quantity demanded of a good is a function of the price of the good itself and the prices of those goods that are substitutes and complements to it.

Note: In this case, \(P_s\) refers to the price of substitute goods, not to be confused with \(P_s\) which is used to refer to the supply price of a good.
SIMULTANEOUS EQUATIONS

Consider two goods, X and Y. The demand function for good X is written differently depending on whether good Y is a substitute to X or a complement to X.

**X and Y are substitutes**

\[ Q_X = a - bP_X + dP_Y \]

**Note:** the positive sign before \( dP_Y \)

There is a positive relationship between the quantity demanded of good X and the price of good Y, since

\[ Q_X = (a + dP_Y) - bP_X \]

Therefore, as \( P_Y \) increases, so does \( Q_X \).

Similarly, the demand function for good Y is

\[ Q_Y = \alpha + \beta P_Y - \delta P_X \]

**Example:** if train fares increase, individuals will reduce their demand for train journeys and increase their demand for bus journeys.

**X and Y are complements**

\[ Q_X = a - bP_X - dP_Y \]

**Note:** the negative sign before \( dP_Y \)

There is a negative relationship between the quantity demanded of good X and the price of good Y, since

\[ Q_X = (a - dP_Y) - bP_X \]

Therefore, as \( P_Y \) increases, \( Q_X \) decreases.

Similarly, the demand function for good Y is

\[ Q_Y = \alpha - \beta P_Y - \delta P_X \]

**Example:** if car prices increase, individuals will reduce their demand for cars and consequently the demand for petrol decreases.

---

**WORKED EXAMPLE 3.11**

EQUILIBRIUM FOR TWO SUBSTITUTE GOODS

Find the equilibrium price and quantity for two substitute goods X and Y given their respective demand and supply equations as:

\[ Q_{dx} = 82 - 3P_X + P_Y \quad (3.11) \]

\[ Q_{sx} = -5 + 15P_X \quad (3.12) \]

\[ Q_{dy} = 92 + 2P_X - 4P_Y \quad (3.13) \]

\[ Q_{sy} = -6 + 32P_Y \quad (3.14) \]

**SOLUTION**

The equilibrium condition for this two-goods market is

\[ Q_{dx} = Q_{sx} \quad \text{and} \quad Q_{dy} = Q_{sy} \]

Therefore, the equilibrium prices and quantities are calculated as follows:

\[ 82 - 3P_X + P_Y = -5 + 15P_X \quad \text{equating equations (3.11) and (3.12)} \]

\[ -18P_X + P_Y = -87 \quad \text{simplifying} \quad (3.15) \]

and

\[ 92 + 2P_X - 4P_Y = -6 + 32P_Y \quad \text{equating equations (3.13) and (3.14)} \]

\[ 2P_X - 36P_Y = -98 \quad \text{simplifying} \quad (3.16) \]

Equations (3.15) and (3.16) are two equations in two unknowns, \( P_X \) and \( P_Y \).
Therefore, solve these simultaneous equations for the equilibrium prices, \( P_X \) and \( P_Y \):

\[
\begin{align*}
-18P_X + P_Y &= -87 & \text{equation (3.15)} \\
18P_X - 324P_Y &= -882 & \text{equation (3.16) multiplied by 9} \\
-323P_Y &= -969 \\
P_Y &= 3
\end{align*}
\]

Solve for \( P_X \) by substituting \( P_Y = 3 \) into either equation (3.15) or equation (3.16):

\[
\begin{align*}
-18P_X + 3 &= -87 & \text{substituting } P_Y = 3 \text{ into equation (3.15)} \\
-18P_X &= -90 \\
P_X &= 5
\end{align*}
\]

Now, solve for \( Q_X \) and \( Q_Y \)

Solve for \( Q_X \) by substituting \( P_X = 5 \) and \( P_Y = 3 \) into either equation (3.11) or equation (3.12) as appropriate:

\[
\begin{align*}
Q_X &= -5 + 15P_X & \text{using equation (3.12)} \\
Q_X &= -5 + 15(5) & \text{substituting } P_X = 5 \\
Q_X &= 70
\end{align*}
\]

Solve for \( Q_Y \) by substituting \( P_Y = 3 \) and \( P_X = 5 \) into either equation (3.13) or equation (3.14) as appropriate:

\[
\begin{align*}
Q_Y &= -6 + 32P_Y & \text{using equation (3.14)} \\
Q_Y &= -6 + 32(3) & \text{substituting } P_Y = 3 \\
Q_Y &= 90
\end{align*}
\]

The equilibrium prices and quantities in this two-goods market are

\[
P_X = 5, \quad Q_X = 70, \quad P_Y = 3, \quad Q_Y = 90
\]

### 3.2.4 Taxes, subsidies and their distribution

Taxes and subsidies are another example of government intervention in the market. A tax on a good is known as an indirect tax. Indirect taxes may be:

- A fixed amount per unit of output (excise tax); for example, the tax imposed on petrol and alcohol. This will translate the supply function vertically upwards by the amount of the tax.
- A percentage of the price of the good; for example, value added tax. This will change the slope of the supply function. The slope will become steeper since a given percentage tax will be a larger absolute amount the higher the price.

#### Fixed tax per unit of output

When a tax is imposed on a good, two issues of concern arise:

- How does the imposition of the tax affect the equilibrium price and quantity of the good?
- What is the distribution (incidence) of the tax; that is, what percentage of the tax is paid by consumers and producers, respectively?
SIMULTANEOUS EQUATIONS

In these calculations:

- The consumer always pays the equilibrium price.
- The supplier receives the equilibrium price minus the tax.

WORKED EXAMPLE 3.12
TAXES AND THEIR DISTRIBUTION

Find animated worked examples at www.wiley.com/college/bradley

The demand and supply functions for a good are given as

Demand function: \( P_d = 100 - 0.5Q_d \) \([3.17]\)

Supply function: \( P_s = 10 + 0.5Q_s \) \([3.18]\)

(a) Calculate the equilibrium price and quantity.
(b) Assume that the government imposes a fixed tax of £6 per unit sold.

(i) Write down the equation of the supply function, adjusted for tax.
(ii) Find the new equilibrium price and quantity algebraically and graphically.
(iii) Outline the distribution of the tax, that is, calculate the tax paid by the consumer and the producer.

SOLUTION

(a) The equilibrium quantity and price are 90 units and £55, respectively.

(b) The tax of £6 per unit sold means that the effective price received by the producer is \( P_s - 6 \). The equation of the supply function adjusted for tax is

\[
P_s - 6 = 10 + 0.5Q_s \\
P_s = 16 + 0.5Q_s
\]

\([3.19]\)

The supply function is translated vertically upwards by 6 units (with a corresponding horizontal leftward shift). This is illustrated in Figure 3.8 as a line parallel to the original supply function.
Remember

(i) Translations, Chapter 2.

(ii) The new equilibrium price and quantity are calculated by equating the original demand function, equation (3.17), and the supply function adjusted for tax, equation (3.19):

\[ P_d = P_s \]
\[ 100 - 0.5Q = 16 + 0.5Q \]
\[ Q = 84 \]

Substitute the new equilibrium quantity, \( Q = 84 \), into either equation (3.17) or equation (3.19) and solve for the new equilibrium price:

\[ P = 100 - 0.5(84) \]
\[ P = 58 \]

The point (84, 58) is shown as point \( E_1 \) in Figure 3.8.

(iii) The consumer always pays the equilibrium price, therefore the consumer pays £58, an increase of £3 on the original equilibrium price with no tax, which was £55. This means that the consumer pays 50% of the tax. The producer receives the new equilibrium price, minus the tax, so the producer receives £58 – £6 = £52, a reduction of £3 on the original equilibrium price of £55. This also means that the producer pays 50% of the tax.

In this example, the tax is evenly distributed between the consumer and producer. The reason for the 50:50 distribution is due to the fact that the slope of the demand function is equal to the slope of the supply function (ignoring signs). This suggests that changes in the slope of either the demand or supply functions will alter this distribution.

Figure 3.8 Goods market equilibrium and taxes
SIMULTANEOUS EQUATIONS

Subsidies and their distribution

Similar ideas may be analysed with respect to subsidies and their distribution. In the case of subsidies, one would be interested in analysing how the benefit of the subsidy is distributed between the producer and consumer.

In the analysis of subsidies, a number of important points need to be highlighted:

- A subsidy per unit sold will translate the supply function vertically downwards, that is, the price received by the producer is \( P + \text{subsidy} \).
- The equilibrium price will decrease (the consumer pays the new lower equilibrium price).
- The price that the producer receives is the new equilibrium price plus the subsidy.
- The equilibrium quantity increases.

WORKED EXAMPLE 3.13

SUBSIDIES AND THEIR DISTRIBUTION

The demand and supply functions for a good (£ per ton of potatoes) are given as

Demand function:

\[
P_d = 450 - 2Q_d
\]

Supply function:

\[
P_s = 100 + 5Q_s
\]

(a) Calculate the equilibrium price and quantity.
(b) The government provides a subsidy of £70 per unit (ton) sold:
(i) Write down the equation of the supply function, adjusted for the subsidy.
(ii) Find the new equilibrium price and quantity algebraically and graphically.
(iii) Outline the distribution of the subsidy, that is, calculate how much of the subsidy is received by the consumer and the supplier.

SOLUTION

(a) The solution to this part is given over to the reader. Show that the equilibrium quantity and price are 50 units and £350, respectively.
(b) (i) With a subsidy of £70 per unit sold, the producer receives \( P_s + 70 \). The equation of the supply function adjusted for subsidy is

\[
P_s + 70 = 100 + 5Q
\]

The supply function is translated vertically downwards by 70 units. This is illustrated in Figure 3.9 as a line parallel to the original supply function.
(ii) The new equilibrium price and quantity are calculated by equating the original demand function, equation (3.20), and the supply function adjusted for the subsidy, equation (3.22):

\[
P_d = (P_s + \text{subsidy})
450 - 2Q = 30 + 5Q \quad \text{equating equations (3.20) and (3.22)}
\]

\[
Q = 60
\]

Substitute the new equilibrium quantity \(Q = 60\) into either equation (3.20) or equation (3.22) and solve for the new equilibrium price:

\[
P = 450 - 2Q
P = 450 - 2(60) \quad \text{substituting } Q = 60 \text{ into equation (3.20)}
\]

\[
P = 330
\]

The point \((P = 330, Q = 60)\) is shown as point \(E_1\) in Figure 3.9.

(iii) The consumer always pays the equilibrium price, therefore, the consumer pays \(\£330\), a decrease of \(\£20\) on the equilibrium price with no subsidy (\(\£350\)). This means that the consumer receives 20/70 of the subsidy. The producer receives the equilibrium price, plus the subsidy, so the producer receives \(\£330 + \£70 = \£400\), an increase of \(\£50\) on the original price of \(\£350\). The producer receives 50/70 of the subsidy.

In this case, the subsidy is not evenly distributed between the consumer and producer; the producer receives a greater fraction of the subsidy than the consumer. The reason? The slope of the supply function is greater than the slope of the demand function (ignoring signs).

**Distribution of taxes/subsidies**

The fraction of the tax/subsidy that the consumer pays/receives is given by the equation

\[
\frac{|m_d|}{|m_d| + |m_s|}
\]
SIMULTANEOUS EQUATIONS

The fraction of the tax/subsidy that the producer pays/receives is given by the equation

\[
\frac{|m_s|}{|m_d| + |m_s|}
\]

where \(m_d\) and \(m_s\) are the slopes of the demand and supply functions, respectively. See Appendix to this chapter for proof of these formulae.

3.2.5 Break-even analysis

The break-even point for a good occurs when total revenue is equal to total cost.

WORKED EXAMPLE 3.14
CALCULATING THE BREAK-EVEN POINT

The total revenue and total cost functions are given as follows:

\[
\begin{align*}
TR &= 3Q \\
TC &= 10 + 2Q
\end{align*}
\]  \hspace{1cm} (3.23) \hspace{1cm} (3.24)

(a) Calculate the equilibrium quantity algebraically and graphically at the break-even point.
(b) Calculate the value of total revenue and total cost at the break-even point.

SOLUTION

(a) The break-even point is algebraically solved by equating total revenue, equation (3.23), and total cost, equation (3.24):

\[
3Q = 10 + 2Q \\
Q = 10
\]

The equilibrium quantity at the break-even point is \(Q = 10\). This is illustrated in Figure 3.10.
(b) The value of total revenue and total cost at the break-even point is calculated by substituting \(Q = 10\) into the respective revenue and cost functions:

\[
\begin{align*}
TR &= 3Q = 3(10) = 30 \\
TC &= 10 + 2Q = 10 + 2(10) = 30
\end{align*}
\]

At \(Q = 10\), \(TR = TC = 30\).
CHAPTER 3

Figure 3.10 Break-even point

PROGRESS EXERCISES 3.3

Equilibrium, Break-even

1. The following demand and supply functions for a safari holiday package are

Demand function: \( Q = 81 - 0.05P \)

Supply function: \( Q = -24 + 0.025P \)

(a) Calculate the equilibrium price and quantity, algebraically and graphically.

(b) Graph the supply and demand function, showing the equilibrium.

2. A perfectly competitive firm producing lamps has fixed costs of \( €1000 \) per week; each lamp costs \( €15 \) to produce and is sold at \( €35 \).

(a) Calculate the break-even quantity.

(b) Does the firm make a profit or loss when: (i) 500 lamps; (ii) 1000 lamps are produced and sold?

(c) Confirm the answers to (a) and (b) graphically.

3. A craftsman has fixed costs of \( €90 \) and a cost of \( €3 \) for each bracelet he produces. Bracelets are sold at \( €6 \) each.

(a) Calculate the break-even quantity.

(b) Confirm your answers graphically.

4. A canteen has fixed costs of \( £1500 \) per week. Meals cost \( £5 \) each and are sold at a fixed price of \( £9 \) each.

(a) Calculate the break-even quantity.

(b) Confirm your answers graphically.

5. (See question 1)

The government imposes a tax of \( £120 \) on each safari holiday.

(a) Write down the equation of the supply function adjusted for tax, hence graph it on the diagram in 1(b).

(b) Calculate the equilibrium price and quantity when the tax is imposed.

(c) Outline the distribution of the tax, that is, calculate the tax paid by the consumer and by the travel agent.
6. The demand and supply functions for two complementary products X and Y, pitching wedges and putters (for pitch and putt), respectively, are given as:

\[ Q_{dX} = 190 - 2P_X - 2P_Y \]
\[ Q_{dY} = 240 - 2P_X - 4P_Y \]
\[ Q_{sX} = -10 + 2P_X \]
\[ Q_{sY} = -40 + P_Y \]

Find the equilibrium price and quantity for each good.

7. The demand and supply functions for golf lessons at Greens Club are

Demand function: \( P = 200 - 5Q \)
Supply function: \( P = 92 + 4Q \)

(a) Calculate the equilibrium price and quantity algebraically and graphically.
(b) The government imposes a tax of £9 per lesson:
   (i) Write down the equation of the supply function adjusted for tax, hence graph it on the same diagram as in part (a).
   (ii) Calculate the equilibrium price and quantity when the tax is imposed.
   (iii) Outline the distribution of the tax, that is, how much of the tax is paid by the customer and the club (supplier).

8. The demand function for a perfectly competitive firm (same price charged for each good) is given as \( P = £30 \). The firm has fixed costs of £200 and variable costs of £5 per unit sold.

(a) Calculate the equilibrium quantity at the break-even point.
(b) Calculate the value of total revenue and total cost at the break-even point.

9. The demand and supply functions for free-range Christmas turkeys are given by the equations:

\( P_d = 80 - 0.4Q_d \) and \( P_s = 20 + 0.4Q_s \)

(a) Calculate the equilibrium price and quantity.
(b) If the government provides a subsidy of £4 per bird:
   (i) Rewrite the equation of the supply function to include the subsidy.
   (ii) Calculate the new equilibrium price and quantity.
   (iii) Outline the distribution of the subsidy, that is, how much of the subsidy is received by the customer and by the supplier.

10. A firm which makes travel alarm clocks has a total cost function \( TC = 800 + 0.2Q \).

(a) If the price of the clock is £6.6, write down the equation of the total revenue function. Calculate the number of clocks which must be made and sold to break even.
(b) When the firm charges a price \( P \) for each alarm clock the break-even point is \( Q = 160 \). Write down the equation for break-even, hence calculate the price charged per clock.
(c) Graph the total revenue functions (a) and (b) with the total cost on the same diagram, showing each break-even point.

11. The supply and demand functions for complementary goods (jeans and shirts) are given by the equations:

\[ P_{dX} = 100 - 5Q_X - Q_Y \]
\[ P_{dY} = 240 - 10Q_X - 8Q_Y \]
\[ P_{sX} = 50 + Q_X \]
\[ P_{sY} = 40 + 2Q_Y \]

Find the equilibrium price and quantity for each good.
3.3 Consumer and Producer Surplus

At the end of this section you should be familiar with:

- The meaning of consumer and producer surplus
- How to measure consumer and producer surplus.

3.3.1 Consumer and producer surplus

**Consumer surplus (CS)**

This is the difference between the expenditure a consumer is willing to make on successive units of a good from $Q = 0$ to $Q = Q_0$ and the actual amount spent on $Q_0$ units of the good at the market price of $P_0$ per unit. To explain how consumer surplus is calculated geometrically consider the demand function, $P = 100 - 0.5Q$, which is graphed in Figure 3.11.

To calculate consumer surplus when the market price is 55, proceed as follows. When the market price per unit is 55 the consumer will purchase 90 units, since, according to the demand function, $55 = 100 - 0.5Q \rightarrow 0.5Q = 100 - 55 = 45 \rightarrow Q = 90$. The consumer, therefore, spends a total of $P \times Q = (55)(90) = £4950$. This is equivalent to the area of the rectangle $0P_0E_0Q_0$ (since area = length \times breadth = $P \times Q$).

The consumer pays the same price, 55, for each of the 90 units purchased; however, he or she is willing to pay more than 55 for each of the units preceding the 90th when the good was scarcer, $Q < 90$. (These higher prices, which the consumer is willing to pay, may be calculated from the demand function.) The total amount which the consumer is willing to pay for the first 90 units is given by the area under the demand function between $Q = 0$ and $Q = 90$, that is, area $0AE_0Q_0$.

Since consumer surplus is the difference between the amount that the consumer is willing to pay and the amount that the consumer actually pays, then

$$CS = 0ABE_0Q_0 - 0P_0E_0Q_0 = AP_0E_0$$

$$= 0.5(90)(55) = 2475 \quad \text{(See Area of triangles, below)}$$

![Figure 3.11 Consumer surplus](image)
SIMULTANEOUS EQUATIONS

Area $AP_0E_0$ is a benefit to the consumer as the amount which the consumer is willing to pay exceeds the amount which is actually paid.

**Producer surplus (PS)**

This is the difference between the revenue the producer receives for $Q_0$ units of a good when the market price is $P_0$ per unit and the revenue that the producer was willing to accept for successive units of the good from $Q = 0$ to $Q = Q_0$. To explain how producer surplus is calculated geometrically, consider the supply function, $P = 10 + 0.5Q$, which is graphed in Figure 3.12.

To calculate producer surplus when the market price is 55 per unit, proceed as follows. When the market price per unit is 55 the producer will supply 90 units since

$$P = 10 + 0.5Q \rightarrow 55 = 10 + 0.5Q \rightarrow 45 = 0.5Q \rightarrow 90 = Q$$

So, the producer receives 55 for each of the 90 units, giving a total revenue of $P \times Q = (55)(90) = £4950$. This is equivalent to the area of rectangle $0P_0E_0Q_0$.

The producer, however, is willing to supply each unit, up to the 90th unit, at prices less than 55. (These lower prices may be calculated from the equation of the supply function.) The revenue the producer is willing to accept for units below the 90th unit is given by the area under the supply function between $Q = 0$ and $Q = 90$, that is, area $0BE_0Q_0$.

Since producer surplus is the difference between the revenue the producer receives at $Q = 90$ and the revenue that the producer was willing to accept for units supplied up to the 90th, then

$$PS = 0P_0E_0Q_0 - 0BE_0Q_0 = BP_0E_0$$

$$= 0.5(90)(45) = 2025$$  (See Area of triangles, below)

Therefore, area $BP_0E_0$ is a benefit to the producer.

![Figure 3.12 Producer surplus](image-url)
CHAPTER 3

Figure 3.13 Area of triangle $= 0.5 \times \text{area of rectangle} = 0.5 \times (b \times h)$

**Total surplus (\(TS\))**

This is the sum of consumer and producer surplus.

**Area of triangles:** The consumer and producer surplus are each represented by the area of a triangle. The area of a triangle is half the area of the rectangle formed by the two sides which meet at right angles. This is usually referred to as ‘half the length of the base \((b)\) multiplied by the length of the perpendicular height \((h)\)’, as illustrated in Figure 3.13.

**WORKED EXAMPLE 3.15**

**CONSUMER AND PRODUCER SURPLUS AT MARKET EQUILIBRIUM**

The demand and supply functions of a good (shirts) are given as

- **Demand function:** \(P = 60 - 0.6Q\)
- **Supply function:** \(P = 20 + 0.2Q\)

(a) Calculate the equilibrium price and quantity for shirts algebraically and graphically.

(b) Calculate the values of consumer and producer surplus at market equilibrium. Illustrate CS and PS on the graph in (a).

(c) What is the value of total surplus?

**SOLUTION**

(a) The algebraic solution to this part is given over to the reader. Show that the equilibrium quantity and price of shirts are 50 units and £30, respectively. The graphical solution is the point \(E_0\) illustrated in Figure 3.14.
SIMULTANEOUS EQUATIONS

Figure 3.14 Consumer and producer surplus

(b) Consumer and producer surplus at market equilibrium are calculated as follows:

At $P = 30$, $Q = 50$, $CS = \triangle AP_0E_0 = 0.5 \times 50 \times 30 = 750$

At $P = 30$, $Q = 50$, $PS = \triangle BP_0E_0 = 0.5 \times 50 \times 10 = 250$

(c) Total surplus is the sum of consumer and producer surplus; therefore

$$TS = CS + PS = 750 + 250 = 1000$$

PROGRESS EXERCISES 3.4

CS/PS for Linear Functions

1. (a) Define (i) consumer surplus, (ii) producer surplus, (iii) total surplus, at the equilibrium price $P_0$. Illustrate all three surpluses graphically.

(b) Define (i) consumer surplus, (ii) producer surplus, (iii) total surplus, at price $P_A$, which is below the equilibrium price $P_0$. Illustrate all three surpluses graphically.

2. The demand and supply functions for seats on a certain weekend bus route are given by

Demand function: $P = 58 - 0.2Q$

Supply function: $P = 4 + 0.1Q$

(a) Calculate the equilibrium price and quantity. Plot the demand and supply functions and illustrate consumer and producer surplus at equilibrium.

(b) Calculate:

(i) The amount consumers pay for bus journeys at equilibrium.

(ii) The amount consumers are willing to pay for bus journeys up to equilibrium.

(iii) The consumer surplus ($CS$); hence, show that $CS = (ii) - (i)$.

(c) Calculate:

(i) The amount the producer (bus company) receives for bus journeys at equilibrium.

(ii) The amount the producer is willing to accept for bus journeys up to equilibrium.

(iii) The producer surplus ($PS$); hence, show that $PS = (i) - (ii)$. 
CHAPTER 3

3. The demand and supply functions for a product (helicopter rides) are given by

\[
\text{Demand function: } Q = 50 - 0.1P \\
\text{Supply function: } Q = -10 + 0.1P
\]

(a) Calculate the equilibrium price and quantity. Plot the demand and supply functions in the form \( P = g(Q) \). Illustrate graphically the consumer and producer surplus at equilibrium.

(b) Calculate the consumer surplus at equilibrium.

(c) Calculate the producer surplus at equilibrium.

(d) Calculate the total surplus at equilibrium.

4. (See question 3) The price per helicopter ride decreases to £250.

(a) Calculate the number of helicopter rides demanded at the reduced price £250.

(b) Calculate the consumer surplus at £250, graphically illustrating your answer. What is the change in \( CS \) as a result of the price reduction?

5. (See question 3) The price per helicopter ride decreases to £250.

(a) Calculate the number of helicopter rides supplied at the reduced price £250.

(b) Calculate the producer surplus at £250, graphically illustrating your answer. What is the change in \( PS \) as a result of the price reduction?

6. The demand and supply functions for a product are

\[
\text{Demand function: } P_d = 200 - 4Q \\
\text{Supply function: } P_s = 50 + Q
\]

(a) Find the equilibrium price and quantity.

(b) Plot the demand and supply functions. Illustrate graphically the consumer (\( CS \)) and producer surplus (\( PS \)) at equilibrium.

(c) Calculate the consumer surplus at equilibrium.

(d) Calculate the producer surplus at equilibrium.

(e) Calculate the total surplus at equilibrium.

7. The demand and supply functions for a product are

\[
\text{Demand function: } P_d = 255 - 4Q \\
\text{Supply function: } P_s = 25 + 7.5Q
\]

(a) Confirm that the equilibrium point is at \( Q = 20, P = 175 \).

(b) Calculate the consumer surplus at equilibrium.

(c) Calculate the producer surplus at equilibrium.

(d) Illustrate the \( CS \) and the \( PS \) graphically.

8. The demand and supply functions for a product are

\[
\text{Demand function: } P = 120 - 3Q \\
\text{Supply function: } P = 24 + 5Q
\]

(a) Calculate the equilibrium price and quantity.

(b) Graph the demand and supply functions, illustrating the equilibrium point.

(c) Calculate the consumer and producer surplus at equilibrium.

(d) What is the value of the total surplus?
3.4 The National Income Model and the *IS-LM* Model

At the end of this section you should be familiar with:

- The national income model: national income equilibrium and expenditure multipliers
- The *IS-LM* model: determination of equilibrium national income and interest rates

3.4.1 National income model

National income, *Y*, is the total income generated within an economy from all productive activity over a given period of time, usually one year. Equilibrium national income occurs when aggregate national income, *Y*, is equal to aggregate planned expenditure, *E*, that is,

\[ Y = E \]  

(3.25)

**Note:** In the discussion which follows it is assumed that all expenditure is planned expenditure.

Aggregate expenditure, *E*, is the sum of households' consumption expenditure, *C*; firms' investment expenditure, *I*; government expenditure, *G*; foreign expenditure on domestic exports, *X*; minus domestic expenditure on imports, *M*, that is,

\[ E = C + I + G + X - M \]  

(3.26)

**Note:** Expenditure on imports is income lost to the economy, hence the minus sign.

Therefore, substituting equation (3.26) into equation (3.25) gives the equation for equilibrium national income:

\[ Y = C + I + G + X - M \]  

(3.27)

That is, equilibrium national income exists when total income is equal to total expenditure.

**Note:** Aggregate expenditure on goods and services (*E*) is one method of measuring national income. Alternatively, national income may be measured by aggregating total income received by firms and individuals (total income) or aggregating total production (total output). For the purposes of this text, these differences in measuring national income may be ignored.

**Steps for deriving the equilibrium level of national income**

**Step 1:** Express expenditure in terms of income, *Y*: *E = f(Y)*.

**Step 2:** Substitute expenditure, expressed as a function of *Y*, into the RHS of the equilibrium condition, *Y = E*. Solve the equilibrium equation for the equilibrium level of national income, *Y*.

**Graphical solution:** The point of intersection of the equilibrium condition, *Y = E* (the 45° line through the origin), and the expenditure equation, *E = C + I + G + X - M*, gives the equilibrium level of national income.

**Note:** When graphing the national income equations, *Y* is plotted on the horizontal axis. As a reminder: ‘*E* on the vertical, *Y* on the horizontal’, all equations will be written in the form *E = f(Y)*.
CHAPTER 3

Equilibrium level of national income when \( E = C + I \)

Initially, the model assumes the existence of only two economic agents, households and firms, operating in a closed economy (no foreign sector) with no government sector and no inflation. Households’ consumption expenditure, \( C \), is modelled by the equation \( C = C_0 + bY \), where \( C_0 \) is autonomous consumption, that is, consumption which does not depend on income. \( b \) \((0 < b < 1)\) is called the marginal propensity to consume. \( b = MPC = \frac{\Delta C}{\Delta Y} \) measures the change in consumption per unit increase in income. A firm’s investment expenditure is autonomous, \( I = I_0 \).

WORKED EXAMPLE 3.16
EQUILIBRIUM NATIONAL INCOME WHEN \( E = C + I \)

In a two-sector economy, autonomous consumption expenditure, \( C_0 = £50m \), autonomous investment expenditure, \( I_0 = £100m \), and \( b = 0.5 \).

(a) Determine (i) the equilibrium level of national income, \( Y_e \), and (ii) the equilibrium level of consumption, \( C_e \), algebraically.

(b) Plot the consumption function, \( C = C_0 + bY \), the expenditure function, \( E = C + I_0 \), and the equilibrium condition, \( Y = E \), on the same diagram. Hence, determine the equilibrium level of national income, \( Y_e \), and the equilibrium level of consumption, \( C_e \).

(c) Given that \( Y = C + S \), determine the equilibrium level of savings. Plot the savings function. Plot the investment function on the same diagram. Comment.

SOLUTION

(a) (i) **Step 1:** Households’ consumption expenditure and firms’ investment expenditure are the only components of aggregate expenditure, therefore

\[ E = C + I = C_0 + bY + I_0 = 50 + 0.5Y + 100 = 150 + 0.5Y \]

**Step 2:** At equilibrium, \( Y = E \) (equation 3.25), therefore:

\[ Y = 150 + 0.5Y \]
\[ 0.5Y = 150 \]
\[ Y = \frac{150}{0.5} \]
\[ Y_e = \frac{1}{0.5} \times 150 = 300 \]

Example:
\[
\begin{align*}
Y & = 150 + 0.5Y \\
Y - 0.5Y & = 150 \\
0.5Y & = 150 \\
Y & = \frac{150}{0.5} \\
Y_e & = \frac{1}{0.5} \times 150 = 300
\end{align*}
\]

In general:
\[
\begin{align*}
Y & = (C_0 + bY) + I_0 \\
Y - bY & = C_0 + I_0 \\
Y(1 - b) & = C_0 + I_0 \\
Y & = \frac{C_0 + I_0}{1 - b} \\
Y_e & = \frac{1}{1 - b} (C_0 + I_0) \quad (3.28)
\end{align*}
\]

The equilibrium level of national income \( Y_e = 300 \).
(ii) When the equilibrium level of income has been found, the equilibrium level of consumption is calculated directly from the consumption function,

\[ C_e = C_0 + bY_e = 50 + 0.5(300) = 50 + 150 = 200 \]  

\[ \text{(3.29)} \]

Figure 3.15  Equilibrium national income with consumption and investment

(b) The consumption function \( C = 50 + 0.5Y \), the expenditure function \( E = 150 + 0.5Y \) and the equilibrium condition \( Y = E \) are plotted in Figure 3.15, with \( E \) plotted vertically and \( Y \) plotted horizontally. The equilibrium condition, \( Y = E \), is represented by a 45° line from the origin, provided the number scale is the same on both axes. Graphically, equilibrium national income \( Y_e \) is illustrated in Figure 3.15(a) with the equilibrium point occurring at the point of intersection of the expenditure equation and the line \( Y = E \). The point of intersection is at \( Y = 300 = E \). Graphically, the equilibrium level of consumption, \( C_e \), is at the point of intersection of the consumption function and the vertical line \( Y_e = 300 \).

c) Since \( C_e = 200 \), the equilibrium level of savings \( S_e = Y_e - C_e = 300 - 200 = 100 \). The savings function \( S = Y - (C_0 + bY) = -C_0 + (1 - b)Y = -50 + 0.5Y \) is plotted in Figure 3.15(b). Graphically, investment expenditure is illustrated by a horizontal line in Figure 3.15(b). Notice that, at equilibrium national income, savings is equal to investment,

\[ S_e = I \]

Therefore, in this example, one can also say that equilibrium national income occurs when savings (leakages) are equal to investment (injections).
CHAPTER 3

PROGRESS EXERCISES 3.5

The National Income Model

1. Given the condition for equilibrium national income \( Y = E \), and the expenditure equation \( E = C \), where \( C = C_0 + bY \):
   (a) Describe the constants \( C_0 \) and \( b \). Are there any restrictions on the range of values which \( b \) can assume?
   (b) Find an expression for the equilibrium level of income (the reduced form).
   (c) Deduce how the equilibrium level of income changes as:
      (i) \( b \) increases.
      (ii) \( b \) decreases.

2. Given the national income model \( Y = E; E = C + I \) where \( C = 280 + 0.6Y, I_0 = 80 \):
   (a) Write down the value of the intercept and slope of the expenditure equation.
   (b) Graph the equilibrium equation and the expenditure equation on the same diagram; hence, determine the equilibrium level of income \( Y_e \) (i) graphically, (ii) algebraically.
   (c) How will the equilibrium level of income change if the marginal propensity to consume increases to 0.9?

3. Given the national income model \( Y = E; E = C + I \) where \( C = 280 + 0.6Y_d, I_0 = 80, T = 0.2Y \) (that is, \( t = 0.2 \)):
   (a) Write down the value of the intercept and slope of the expenditure equation.
   (b) Graph the equilibrium equation and the expenditure equation on the same diagram; hence, determine the equilibrium level of income \( Y_e \) (i) graphically, (ii) algebraically.
   (c) If the marginal propensity to consume increases to 0.9, how will:
      (i) The expenditure equation change?
      (ii) The equilibrium level of income change?

4. Assuming that the initial level of national income is £800m, calculate the new level of national income when \( b = 0.6 \) and \( \Delta I = £500m \).

Go to the website for the following additional material that accompanies Chapter 3:

- Expenditure multiplier
  Table 3.1 Relationship between the expenditure multiplier and MPC, \( b \)
  Worked Example 3.17 Effect of changes in MPC and \( I_0 \) on \( Y_e \)
- Government expenditure and taxation: \( E = C + I + G \)
- Expenditure multiplier with taxes
  Worked Example 3.18 Equilibrium national income and effect of taxation
  Figure 3.16 Equilibrium national income and effect of taxation
- Foreign trade
  Worked Example 3.19 Expenditure multiplier with imports and trade balance
  Table 3.2 Summary of national income model
SIMULTANEOUS EQUATIONS

Progress Exercises 3.5, questions 5, 6, 7
Solutions to questions 5, 6, 7
3.4.2 IS-LM model: determination of equilibrium national income and interest rates

- IS schedule
- LM schedule
- Equilibrium national income and equilibrium interest rates
  Worked Example 3.20 IS-LM analysis
  Figure 3.17 Equilibrium income and interest rates
  Progress Exercises 3.6 IS-LM analysis
  Solutions to Progress Exercises 3.6

3.5 Excel for Simultaneous Linear Equations

If you found Excel useful for plotting graphs in Chapter 2, you should find it equally useful in this chapter as the same skills are required in graph plotting. However, since this chapter is mainly concerned with finding solutions to simultaneous equations, you will need to show the solution, that is, the point of intersection of the lines on the graph. To ensure that the solution is shown on the graph, solve the equations algebraically first. Once the solution is known, the axis may then be scaled by choosing a range of \( x \)-values which includes the \( x \)-value of the solution.

Alternatively, if the equations are not easy to solve algebraically then a table of \( y \)-values calculated for each of the simultaneous equations \( y_1 = f(x) \), \( y_2 = g(x) \) may be used to detect whether the solution \((y_1 = y_2)\) is included in the table. If \( y_1 > y_2 \) at the start of the table, the first function is greater than the second; if \( y_1 < y_2 \) at the end of the table, the first function is less than the second, or vice versa, then the solution, \( y_1 = y_2 \), is included.

WORKED EXAMPLE 3.21
COST, REVENUE, BREAK-EVEN, PER UNIT TAX WITH EXCEL

A firm receives £2.5 per unit for a particular good. The fixed costs incurred are £44 while each unit produced costs £1.4.

(a) Write down the equations for (i) total revenue, and (ii) total cost.

(b) Calculate the break-even point algebraically.

(c) If the government imposes a tax of £0.70 per unit, recalculate the break-even point. Show the graphical solutions to parts (b) and (c) on the same diagram (using Excel).

SOLUTION

(a) (i) \( TR = P \times Q = 2.5Q \)
     (ii) \( TC = FC + VC = 44 + 1.4Q \)
CHAPTER 3

(b) Break-even occurs when $TR = TC$:

- Algebraically:

\[
2.5Q = 44 + 1.4Q \\
1.1Q = 44 \\
Q = 40
\]

When $Q = 40$, then $TR = TC = 100$.

(c) If a tax per unit is imposed, either the total revenue function or the total cost function may be adjusted for the tax as follows:

- Algebraically:

  The net revenue per unit is (price – tax): $TR = (2.5 - 0.7)Q = 1.8Q$.

  Break-even is at $TR = TC \rightarrow 1.8Q = 44 + 1.4Q \rightarrow Q = 110$

- Graphically:

  To show the break-even points on a graph, choose values of $Q$ such as $Q = 0$ to $Q = 160$. Set up the table of points in Excel and plot the graph as shown in Figure 3.18. (Since the graph is a straight line, a minimum of two points is required.)

WORKED EXAMPLE 3.22

DISTRIBUTION OF TAX WITH EXCEL

(a) Show graphically that the equilibrium price and quantity for the demand and supply functions given by the pairs of equations (i) and (ii) is the same.

\[
\begin{align*}
(i) \quad P_d &= 120 - 2Q \\
P_s &= 10 + 2Q \\
(ii) \quad P_d &= 120 - 2Q \\
P_s &= 37.5 + Q
\end{align*}
\]
SIMULTANEOUS EQUATIONS

(b) If a tax of 20 is imposed on each unit produced, recalculate the equilibrium price for (i) and (ii) above, hence determine the distribution of the tax. Show the distribution of tax graphically.

(c) Can you deduce a general rule describing how the distribution of the tax changes according to the function with the flatter slope?

SOLUTION

(a) The table of points and the graphs of equations (i) and (ii) are shown in Figure 3.19. The equilibrium point for each pair is the same, \( Q = 27.5, P = 65 \).

<table>
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<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
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</tr>
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<td>10</td>
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<td>60</td>
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<td>50</td>
</tr>
<tr>
<td></td>
<td>P_s</td>
<td>37.5</td>
<td>47.5</td>
<td>57.5</td>
<td>67.5</td>
<td>77.5</td>
</tr>
</tbody>
</table>

Figure 3.19 Market equilibrium

(b) When a tax of 20 is imposed, the price the supplier receives is the (original price – tax); therefore, replace \( P \) in the supply functions by \( (P – 20) \).

(i) With the tax, the set of equations is now

\[
P_d = 120 - 2Q,
\]

\[
P_s - 20 = 10 + 2Q \rightarrow P_s = 30 + 2Q
\]

Solve for equilibrium at \( Q = 22.5, P = 75 \).

The consumer (who always pays the equilibrium price) pays \( (P_{e(tax)} - P_e) = 75 - 65 = 10 \) more than before the tax. The producer receives \( (P_{e(tax)} - tax) = 75 - 20 = 55 \). This is 10 units less than before the tax was imposed. So when the slopes of the demand and supply functions are equal the distribution of tax is 50:50.

A table of values is set up to plot this pair of graphs with the untaxed supply function. The range of \( Q \)-values is selected so that the graph focuses on the original and the new equilibrium points as shown in Figure 3.20.
(ii) With the tax, the set of equations is now

\[
P_d = 120 - 2Q \\
P_s - 20 = 37.5 + Q \rightarrow P_s = 57.5 + Q
\]

Solve for equilibrium at \( Q = 20.83, P = 78.33 \).

The consumer pays \((P_{e(tax)} - P_e) = 78.33 - 65 = 13.33\) more than before the tax. The producer receives \((P_{e(tax)} - \text{tax}) = 78.33 - 20 = 58.33\). This is 6.67 units less than before the tax was imposed. As expected, the function whose slope is greater in magnitude pays the greater share of the tax.

A table of values is set up to plot this pair of graphs with the untaxed supply function. The range of \( Q \)-values is selected so that the graph focuses on the original and the new equilibrium points as shown in Figure 3.21.
SIMULTANEOUS EQUATIONS

(c)

Remember

The distribution of tax for linear functions is given as

\[
\text{Consumer pays } \frac{|m_s|}{|m_d| + |m_s|} \times \text{tax} \quad \text{Producer pays } \frac{|m_d|}{|m_d| + |m_s|} \times \text{tax}
\]

See Appendix to chapter. The distribution of tax is graphically illustrated in Figures 3.20 and 3.21.

PROGRESS EXERCISES 3.7

Excel or Otherwise

1. Solve the following simultaneous equations (i) algebraically, (ii) graphically with Excel.

   \(\begin{align*}
   (a) & \quad x + y - 3 = 0 \\
   (b) & \quad 3.8P - 0.75Q = 12 \\
   & \quad 2x - y = 15 \\
   & \quad P = 5Q - 6
   \end{align*}\)

2. Set up a table in Excel to find the approximate solution to the following equations:

   \[
   P - 8Q = 120 \\
   3Q - 1.5P + 270 = 0
   \]

   Hence, determine the solution:
   (a) algebraically, (b) graphically.

3. Given the demand and supply functions,

   \(P = 124 - 4.5Q\) and \(Q = -16.5 + 0.5P\), respectively.
   (a) Find the equilibrium point (i) graphically, (ii) algebraically.
   (b) If a tax of 30 per unit is imposed, calculate:
       (i) the equilibrium point, (ii) the distribution of tax.
       Show the distribution of tax graphically.

4. On the same diagram plot:

   (a) (i) the 45° line \(Y = E\), (ii) \(E = C + I\), where \(C = 125 + 0.65Y\) and \(I = 20\).
   (b) (i) the 45° line \(Y = E\), (ii) \(E = C + I\), where \(C = 125 + 0.65Y_d\), \(I = 20\), \(T = 0.2Y\).
   Hence, determine the equilibrium level of income (i) graphically, (ii) algebraically.
3.6 Summary

**Mathematics**

The solution of a set of simultaneous equations is the values of $x$ and $y$ which satisfy all equations.

(a) To solve the equations algebraically, eliminate all but one variable, solve for this one variable, then solve for the other(s).

(b) To solve the equations graphically, plot the graphs. The solution is given by the coordinates of the point of intersection of the graphs.

Simultaneous equations may have:

(i) A unique solution.

(ii) No solution.

(iii) Infinitely many solutions.

**Applications**

- **Goods market equilibrium:** $Q_d = Q_s$ and $P_d = P_s$
- **Labour demand:** $w_d = a - bL$: a negative relationship between the number of labour units and the wage rate (price per unit).
- **Labour supply:** $w_s = c + dL$: a positive relationship between the number of labour units and the wage rate (price per unit).
- **Labour market equilibrium:** $L_d = L_s$ and $w_d = w_s$
- **Equilibrium for complementary and substitute goods:**
  
  when $Q_{dX} = Q_{sX}$ and $Q_{dY} = Q_{sY}$, $P_{dX} = P_{sX}$ and $P_{dY} = P_{sY}$: $X$ and $Y$ are substitutes $\rightarrow Q_X = a - bP_X + dP_Y$; $X$ and $Y$ are complements $\rightarrow Q_X = a - bP_X - dP_Y$.

- **Tax and its distribution between consumer and producer:**
  
  When a tax per unit is imposed the price to the supplier is $(P - \text{tax})$

  
  
  $P_s = c + dQ \rightarrow P_s - \text{tax} = c + dQ$

  
  The fraction of tax paid by the consumer and producer, respectively:

  $\frac{|m_d|}{|m_d| + |m_s|}$ (consumer), $\frac{|m_s|}{|m_d| + |m_s|}$ (producer)

- **Break-even**: $TR = TC$.
- **Consumer surplus**: (CS) is the difference between the expenditure a consumer is willing to make on successive units of a good from $Q = 0$ to $Q = Q_0$ and the actual amount spent on $Q_0$ units of the good at the market price $P_0$ per unit.
- **Producer surplus**: (PS) is the difference between the revenue the producer receives for $Q_0$ units of a good when the market price is $P_0$ per unit, and the revenue that the producer was willing to accept for successive units of the good from $Q = 0$ to $Q = Q_0$.

  Revise the effect of price increases and decreases on CS and PS.

- **National income model**: Equilibrium exists when income ($Y$) = expenditure ($E$).
SIMULTANEOUS EQUATIONS

Expenditure may consist of (i) consumption: \( C = C_0 + bY \), less tax, (ii) investment, \( I_0 \), (iii) government expenditure, \( G_0 \), (iv) exports, \( X_0 \), (v) less imports, \( M = M_0 + mY \), hence \( E = C + I + G + X - M \).

To find the level of income (\( Y_e \)) at which equilibrium exists, solve the equation \( Y = E \) for \( Y \). The solution is \( Y_e \). Hence, equilibrium consumption and taxation: \( C_e = C_0 + bY_e \); \( T_e = tY_e \).

Revise the reduced expressions for \( Y_e \), with multipliers. These are formulae from which \( T_e \) may be calculated directly for various standard national income models.

- **IS-LM model**: Equilibrium in the goods market when \( Y = E \) (national income), but consider investment as a function of interest rate: \( I = I_0 - dr \), thus

\[
Y = \frac{1}{1 - b(1-t)} \cdot (C_0 + I_0 - dr + G_0)
\]

hence the equation: \( r = f(Y) \). This is the IS schedule.

Equilibrium in the money market when money supply = money demand.

Money demand: \( M_d = M_d^T + M_d^F + M_d^S = L_1 + L_2 = kY + (a - hr) \)

Money supply: \( M_s = M_0 \)

Equilibrium, \( M_e = M_d \Rightarrow M_0 = kY + a - hr \)

This equation may also be written as \( r = g(Y) \). This is the LM schedule. The goods and money markets are simultaneously in equilibrium for the values of \( r \) and \( Y \) which satisfy the simultaneous IS and LM equations: for example,

- **IS schedule**: \( r = 32 - 0.014Y \)
- **LM schedule**: \( r = -2.0 + 0.0025Y \)

- **Excel**: Useful as described in Chapter 2. In this chapter, points of intersection and equilibrium points may be viewed. It should prove helpful for problems on the national income model, to view the effect of various sectors on the equilibrium level of income.

---

**Appendix**

- **Distribution of tax paid by the consumer and producer**

See Figure 3.22 where \( |m_d| \) and \( |m_s| \) represent the magnitudes of the slopes of the demand and supply functions respectively.
CHAPTER 3

Figure 3.22 Distribution of tax

Consumer pays \((P_e2 - P_e1)\) of the tax: \(t_c\)
Supplier (producer) pays \((P_e1 - P)\) of the tax: \(t_s\)

\[
|m_d| = \frac{t_c}{x} \quad \Rightarrow \quad x = \frac{t_c}{|m_d|} \\
|m_s| = \frac{t_s}{x} \quad \Rightarrow \quad x = \frac{t_s}{|m_s|}
\]

Therefore

\[
t_c |m_s| = t_s |m_d|
\]

\[
t_c |m_s| = (t - t_c) |m_d| \quad \text{where} \quad t_c + t_s = t
\]

\[
t_c (|m_s| + |m_d|) = t |m_d|
\]

\[
t_c = \frac{|m_d|}{|m_s| + |m_d|} \quad t_s = \frac{|m_s|}{|m_s| + |m_d|}
\]

TEST EXERCISES 3

1. Solve the following simultaneous equations (i) algebraically, (ii) graphically.

   \[
   \begin{align*}
   (a) \quad 2x + y &= 12 \\
   (b) \quad 3.5P - Q &= 12 \\
   x - y &= 15 \\
   P &= 5Q - 6
   \end{align*}
   \]

2. Given the demand and supply functions, \(P = 50 - 1.5Q\) and \(Q = -11 + 0.5P\), respectively:
   (a) Find the equilibrium point (i) graphically, (ii) algebraically.
   (b) If a tax of 15.05 per unit is imposed calculate:
       (i) the equilibrium point, (ii) the distribution of tax.
   Show the distribution of tax graphically.
SIMULTANEOUS EQUATIONS

3. On the same diagram plot:
   (a) the 45° line \( Y = E, E = C + I \), where \( C = 100 + 0.8Y \) and \( I = 50 \);
   (b) the 45° line \( Y = E, E = C + I \), where \( C = 100 + 0.8Y_d, I = 50, T = 0.2Y \).
   Determine the equilibrium level of income for (a) and (b): (i) graphically, (ii) algebraically.
   Comment on the effect of taxation.

4. (See question 3) Determine the equilibrium levels of consumption and taxation (i) graphically, (ii) algebraically.

5. The demand and supply functions for a good are given by the equations
   \[ P = 80 - 2Q \] and \[ P = 20 + 4Q \]
   respectively.
   (a) Calculate the equilibrium price and quantity.
   (b) Calculate the consumer and producer surplus at equilibrium.

6. (See question 5) The supplier pays an excise tax of 12 per unit sold.
   (a) Write down the equation for the supply function when the excise tax is imposed.
   (b) Calculate the equilibrium price and quantity.
   (c) Calculate the tax paid by (i) the consumer, (ii) the producer.
   (d) Calculate the consumer and producer surplus at equilibrium.

7. The demand and supply functions for complementary goods \( X \) and \( Y \) are given by the equations:
   \[ Q_{d,X} = 200 - 4P_X - 4P_Y \quad Q_{s,X} = -65 + 6P_X \]
   \[ Q_{d,Y} = 80 - P_X - P_Y \quad Q_{s,Y} = -20 + 6P_Y \]
   respectively. Calculate the equilibrium price and quantity for each good.

8. Solve the simultaneous equations
   \[ x + y + z = 9 \]
   \[ 2x - y + 4z = 19 \]
   \[ 3x + 6y - z = 16 \]