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Introduction

1.1 About This Book

This text is the successor to *Finite Element Computational Fluid Mechanics* published in 1983. It thoroughly organizes and documents the subsequent three decades of progress in weak form theory derivation of optimal performance CFD algorithms for the infamous Navier–Stokes (NS) nonlinear partial differential equation (PDE) systems. The text content addresses the complete range of NS and filtered NS (for addressing turbulence) PDE systems in the incompressible fluid-thermal sciences. Appendix B extends subject NS content to a weak form algorithm addressing hypersonic shock layer aerothermodynamics.

As perspective color and dynamic computer graphics are support imperatives for CFD *a posteriori* data assimilation, and hence interpretation, www.wiley.com/go/baker/GalerkinCFD renders available the full color graphics content absent herein. The website also contains detailed academic course lecture content at advanced graduate levels in support of outreach and theory exposure/implementation.

Weak form theory is the mathematically elegant process for generating approximate solutions to nonlinear NS PDE systems. Theoretical formalities are always conducted in the continuum, and only after such musings are completed are space and time discretization decisions made. This final step is a matter of choice, with a finite element (FE) spatial semi-discretization retaining use of calculus and vector field theory throughout conversion to terminal computable form. This choice enables implementing weak form theory precision into an optimal performance compute engine, eliminating any need for heurism.

The text tenor assumes that the reader remembers some calculus and is adequately versed in fluid mechanics and heat and mass transport at a post-baccalaureate level. It further assumes that this individual is neither comfortable with nor adept at formal mathematical manipulations. Therefore, text fluid mechanics subject exposure sequentially enables just-in-time exposure to essential mathematical concepts and methodology, in progressively addressing more detailed NS PDE systems and closure formulations.

Potential flow enables elementary weak form theory exposure, with subsequent theorization modifications becoming progressively more involved in addressing NS pathological nonlinearity. The exposure process is sequentially supported by *a posteriori* data from...
precisely designed computational experiments, enabling quantitative validation of theory predictions of accuracy, convergence, stability and error estimation/distribution which, in concert, lead to confirmation of optimal mesh solution existence.

Text content firmly quantifies the practice preference for an FE semi-discrete spatial implementation. The apparent simplicity of finite volume (FV) and finite difference (FD) discretizations engendered the FV/FD commercial CFD code legacy practice. However, as documented herein, FV/FD spatial discretizations constitute non-Galerkin weak form decisions leading to nonlinear schemes via heuristic arguments. This is totally obviated in converting FE algorithms to computable syntax using calculus and vector field theory. This aspect hopefully further prompts the reader’s interest in acquiring knowledge of these elegant practice aspects, such that assimilating FE constructs proves to be worth the effort.

The progression within each chapter, hence throughout the text, sequentially addresses more detailed fluid/thermal NS PDE systems, each chapter building on prior material. The elegant uniformity of weak form theory facilitates this approach with mathematical formalities never requiring an ad hoc scheme decision. In his reflections on teaching the finite element method Bruce Irons is quoted, “Most people, mathematicians apart, abhor abstraction.” Booker T. Washington concurred, “An ounce of application is worth a ton of abstraction.” These precepts guide the development and exposition strategies in this text, with abstraction never taking precedence over developing a firm engineering-based theoretical exposure.

Summarizing, modified continuous Galerkin weak formulations for fluid/thermal sciences CFD generate practical computational algorithms fully validated as optimal in performance as predicted by a rich theory. Conception and practice goals always lead to the theoretical exposition, to convince the reader that its comprehension is a worthwhile goal, paying the requisite dividend.

1.2 The Navier–Stokes Conservation Principles System

Computational fluid-thermal system simulation involves seeking a solution to the nonlinear PDE systems generated from the basic conservation observations in engineering mechanics. In the lagrangian (point mass) perspective, these principles state

\[
\begin{align*}
\text{conservation of mass:} & \quad dM = 0, M = \Sigma m_i \\
\text{Newton’s second law:} & \quad d\mathbf{P} = \Sigma \mathbf{F}, \mathbf{P} = M\mathbf{V} \\
\text{thermodynamics, first law:} & \quad dE = dQ - dW \\
\text{thermodynamic process:} & \quad dS \geq 0
\end{align*}
\]

In (1.1) \( m_i \) denotes a point mass, \( M \) is total mass of a particle system, \( \mathbf{V} \) is velocity of that system and \( \mathbf{F} \) denotes applied (external) forces. Equations (1.3–1.4) are statements of the first and second law of thermodynamics where \( E \) is system total energy, \( Q \) is heat added, \( W \) is work done by the system and \( S \) is entropy.

Practical CFD applications almost never involve addressing the conservation principles in lagrangian form. Instead, the transition to the continuum (eulerian) description is made, wherein one assumes that there exist so many mass points per characteristic volume \( V \) that
a density function $\rho$ can be defined

$$\rho(x, t) = \lim_{V \to 0} \frac{1}{V} \sum_{i} m_i$$  \hspace{1cm} (1.5)$$

One then identifies a control volume CV, with bounding control surface CS, Figure 1.1, and transforms the conservation principles from lagrangian to eulerian viewpoint via Reynolds transport theorem

$$d() \Rightarrow D() \equiv \frac{\partial}{\partial t} \int_{cv} (\cdot) d\tau + \oint_{cs} (\cdot) V \cdot \hat{n} \ d\sigma$$  \hspace{1cm} (1.6)$$

Thus is produced a precise mathematical statement of the conservation principles for continuum descriptions as a system of integro-differential equations

$$DM = \frac{\partial}{\partial t} \int_{cv} \rho d\tau + \oint_{cs} \rho V \cdot \hat{n} \ d\sigma = 0$$  \hspace{1cm} (1.7)$$

$$DP \Rightarrow \frac{\partial}{\partial t} \int_{cv} \rho V d\tau + \oint_{cs} V \rho V \cdot \hat{n} \ d\sigma = \int_{cv} \rho B d\tau + \int_{cs} T d\sigma$$  \hspace{1cm} (1.8)$$

$$DE \Rightarrow \frac{\partial}{\partial t} \int_{cv} \rho e d\tau + \oint_{cs} (e + p/\rho) V \cdot \hat{n} \ d\sigma = \int_{cv} s d\tau + \int_{cs} (W - q \cdot \hat{n}) d\sigma$$  \hspace{1cm} (1.9)$$

Note the eulerian “filling in” of the right hand sides for DP and DE with $\sum F \Rightarrow$ body forces $B +$ surface tractions $T,$ and $dQ - dW \Rightarrow$ heat added $s,$ bounding surface heat efflux $q \cdot \hat{n}$ and work done $W.$

From (1.7–1.9), one easily develops the PDE statements of direct use for CFD formulations by assuming that the control volume CV is stationary, followed by invoking the divergence theorem for the identified surface integrals. For example for (1.7)

$$\oint_{cs} \rho V \cdot \hat{n} \ d\sigma = \int_{cv} \nabla \cdot \rho V d\tau$$  \hspace{1cm} (1.10)$$

where $\nabla$ is the gradient (vector) differential operator.

Via the divergence theorem the integro-differential equation system (1.7–1.9) is uniformly re-expressed as integrals vanishing identically on the CV. Such expressions can

![Figure 1.1 Control volume for Reynolds transport theorem](image-url)
hold in general if and only if \(( \text{iff} )\) the \textit{integrand} vanishes identically, whereupon \(DM, DP\) and \(DE\) morph to the \textit{nonlinear} PDE system

\[
\begin{align*}
DM: & \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \\
DP: & \quad \frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u} \mathbf{u} = \rho \mathbf{g} + \nabla \mathbf{T} \\
DE: & \quad \frac{\partial p}{\partial t} + \nabla \cdot (p \rho + \rho \mathbf{u}) = s - \nabla \cdot \mathbf{q}
\end{align*}
\] (1.11–1.13)

Herein the velocity vector label \(\mathbf{v}\) in the preceding equations is replaced with the more conventional symbol \(\mathbf{u}\).

It remains to simplify (1.11–1.13) for constant density \(\rho_0\) and to identify \textit{constitutive closure} for traction vector \(\mathbf{T}\) and heat flux vector \(\mathbf{q}\). For laminar flow \(\mathbf{T}\) contains pressure and a fluid viscosity hypothesis involving the Stokes strain rate tensor. For constant density \(\rho_0\) and multiplied through by \(\nabla\) the resultant vector statement is

\[
\nabla \mathbf{T} = -\nabla p + \nabla \cdot \mu \nabla \mathbf{u}
\] (1.14)

where \(p\) is pressure and \(\mu\) is fluid absolute viscosity. The Fourier conduction hypothesis for heat flux vector \(\mathbf{q}\) is

\[
\nabla \cdot \mathbf{q} = -\nabla \cdot k \nabla T
\] (1.15)

where \(k\) is fluid thermal conductivity and \(T\) is temperature.

Substituting these closure \textit{models} and enforcing that density and specific heat are assumed constant converts (1.11–1.13) to the very familiar textbook appearance of Navier–Stokes. Herein the \textit{homogeneous} form preference leads to the subject incompressible NS PDE system

\[
\begin{align*}
DM: & \quad \nabla \cdot \mathbf{u} = 0 \\
DP: & \quad \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u} \mathbf{u} + \rho_0 \nabla p - \nabla \cdot \nu \nabla \mathbf{u} + (\rho / \rho_0) \mathbf{g} = 0 \\
DE: & \quad \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{u} \mathbf{T} - \nabla \cdot \kappa \nabla \mathbf{T} - s / \rho_0 c_p = 0
\end{align*}
\] (1.16–1.18)

In (1.17), \(\nu = \mu / \rho_0\) is fluid kinematic viscosity with density assumed as the constant \(\rho_0\) except for thermally induced impact in the gravity body force term in (1.17). Finally, in (1.18) \(\kappa = k / \rho c_p\) is fluid \textit{thermal diffusivity}.

Thermo-fluid system performance is thus characterized by a balance between unsteadiness and convective and diffusive processes. This identification is precisely established by non-dimensionalizing (1.16–1.18). The reference time, length and velocity scales are \(\tau, L,\) and \(U\), respectively, with the (potential) temperature scale definition \(\Theta = (T - T_{\text{min}}) / (T_{\text{max}} - T_{\text{min}})\). Then implementing the Boussinesq buoyancy model for the gravity body
force, the non-D incompressible NS PDE system for thermal-laminar flow with mass transport is

\[
DM: \quad \nabla \cdot \mathbf{u} = 0 \tag{1.19}
\]

\[
DP: \quad St \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u} \mathbf{u} + \nabla P - \frac{1}{Re} \nabla \cdot \nabla \mathbf{u} + \frac{Gr}{Re^2} \Theta \mathbf{g} = 0 \tag{1.20}
\]

\[
DE: \quad St \frac{\partial \Theta}{\partial t} + \nabla \cdot \mathbf{u} \Theta - \frac{1}{RePr} \nabla \cdot \nabla \Theta - s_\Theta = 0 \tag{1.21}
\]

\[
DY: \quad St \frac{\partial Y}{\partial t} + \nabla \cdot \mathbf{u} Y - \frac{1}{ReSc} \nabla \cdot \nabla Y - s_Y = 0 \tag{1.22}
\]

The unknowns in PDE system (1.19–1.22) are the non-D velocity vector \( \mathbf{u} \), kinematic pressure \( P \), temperature \( \Theta \) and mass fraction \( Y \). No special notation emphasizes that these are non-D, which will always be the case. These variables as a group are hereinafter referenced as the NS PDE system state variable symbolized as the column matrix \( \{ q(x,t) \} = \{ \mathbf{u}, P, \Theta, Y \}^T \).

The definitions for Stanton, Reynolds, Grashoff, Prandtl and Schmidt numbers are conventional as \( St \equiv \tau UL \), \( Re \equiv UL/\nu \), \( Gr \equiv g\beta \Delta T L^3/\nu^2 \), \( Pr \equiv \rho_0 c_p k/\nu \), and \( Sc \equiv D/\nu \), where \( D \) is the binary diffusion coefficient. Additionally \( \Delta T \equiv (T_{max} - T_{min}) \), \( \beta \equiv 1/T_{abs} \) and \( P \equiv p/\rho_0 \). \( St \) is defined unity, \( \tau \equiv L/U \), except when addressing flowfields exhibiting harmonic oscillation, and the Peclet number \( Pe \equiv RePr \) is the common replacement in \( DE \).

### 1.3 Navier–Stokes PDE System Manipulations

The NS PDE system (1.19–1.22) is universally accepted as an accurate descriptor of fluid-thermal phenomena for all Reynolds numbers \( Re \). However, it is also universally recognized that for \( Re \geq O(\sim E+04) \), where \( O(\bullet) \) signifies order, the resultant NS flowfields will be characterized as turbulent.

The CFD simulation procedure that addresses the expressed NS PDE system for all \( Re \) is called direct numerical simulation (DNS). Even with massive computer resources the DNS approach to solution of practical NS problem statements is contraindicated, cf. Dubois et al. (1999). The DNS approach is not addressed herein, although these algorithms do enjoy an identical weak form theoretical basis.

Instead, generating computational simulation algorithms for NS PDE statements for practical \( Re \) requires manipulations of (1.19–1.22). In the CFD community, the operation of time averaging generates the Reynolds-averaged NS (RaNS) PDE system. The alternative is spatial filtering via convolution with a filter function that produces the large eddy simulation (LES) NS PDE system. A union of the two has been termed very large eddy simulation (VLES), or a RaNS-LES hybrid termed detached eddy simulation (DES).

In each instance the mathematical manipulations introduce a priori unknown variables into the resultant PDE system state variable due to the nonlinear convection terms in (1.20–1.22). Time averaging resolves \( \{ q(x,t) \} = \{ \mathbf{u}, p, \Theta, Y \}^T \) into the time-independent steady component and the fluctuation (in time) thereabout. In tensor index notation the resolution statement for velocity vector \( \mathbf{u} \) is

\[
\mathbf{u}(x, t) \Rightarrow u_i(x_k, t) = \bar{u}_i(x_k) + u'_i(x_k, t) \tag{1.23}
\]
Time averaging the convection term in (1.20) produces
\[ \overline{u_j u_i} = \overline{u_j u_i} + \overline{u' j u_i} \]  \quad (1.24)
that is, the tensor product of time averaged velocity plus the time average of the tensor product of velocity fluctuations about the steady average.

The second term in (1.24) is called the RaNS Reynolds stress tensor, a model for which must be constructed to close the RaNS PDE system state variable. Similar operations on DE and DY produce mean convection term products plus the fluctuating product Reynolds vectors
\[ \overline{u_j \Theta} = \overline{u_j \Theta} + \overline{u' j \Theta'} \quad \text{and} \quad \overline{u_j Y} = \overline{u_j Y} + \overline{u' j Y'} \]  \quad (1.25)
which must also be modeled to achieve closure.

The specifics of RaNS closure model development are derived in Chapter 4, then further detailed in Chapters 8 and Appendix B. It is sufficient here to note RaNS closure models emulate the force-flux form of the Stokes and Fourier fluid-dependent constitutive closures (1.14–1.15). The correlation coefficient becomes a turbulent eddy viscosity, \( \nu' \), extended to turbulent heat/mass flux vector closure via turbulent Pr and Sc number assumptions.

The non-D turbulent eddy viscosity defines the turbulent Reynolds number
\[ \text{Re}' \equiv \nu' / \nu \]  \quad (1.26)
and the non-D RaNS PDE system alternative to laminar NS PDEs (1.19–1.22), assuming unit Stanton number is

\begin{align*}
DM : & \quad \nabla \cdot \overline{u} = 0 \\
DP : & \quad \frac{\partial \overline{u}}{\partial t} + \nabla \cdot \overline{uu} + \nabla P - \frac{1}{\text{Re}} \nabla \cdot \left[ (1 + \text{Re}') \nabla \overline{u} \right] + \frac{\text{Gr}}{\text{Re}^2} \overline{\Theta} = 0 \\
DE : & \quad \frac{\partial \overline{\Theta}}{\partial t} + \nabla \cdot \overline{u \Theta} - \frac{1}{\text{Re}} \nabla \cdot \left[ \left( \frac{1}{\text{Pr}} + \frac{\text{Re}'}{\text{Pr}'^2} \right) \nabla \overline{\Theta} \right] - \overline{\Theta} = 0 \\
DY : & \quad \frac{\partial \overline{Y}}{\partial t} + \nabla \cdot \overline{u Y} - \frac{1}{\text{Re}} \nabla \cdot \left[ \left( \frac{1}{\text{Sc}} + \frac{\text{Re}'}{\text{Sc}'^2} \right) \nabla \overline{Y} \right] - \overline{Y} = 0
\end{align*}

The time averaging alternative of spatial filtering employs the mathematical operation of convolution. In one dimension for velocity scalar component \( u \) the space filtered velocity definition is
\[ \overline{u}(x, t) = \int_{-\infty}^{\infty} g(y) u(x - y, t) dy \]  \quad (1.31)
where \( g(y) \) denotes the filter function. The filtered velocity remains time dependent, and in \( n \) dimensions the CFD literature notation for spatial filtering
\[ \overline{u_i}(x_j, t) = g_\delta * u_i(x_j, t) \]  \quad (1.32)
is symbolized by * with \( \delta \) denoting the measure (diameter) of the filter function \( g \).
Spatial filtering the NS PDE system (1.19–1.22) produces the LES PDE system addressed in Chapter 9. Following convolution, Fourier transformation and deconvolution spatial filtering of the convection term in (1.20) generates the a priori unknown stress tensor quadruple

\[ \overline{\mu_i\mu_i} = \overline{\mu_i\mu_i} + \overline{\mu_i\mu_i} + \overline{\mu_i\mu_i} \]

(1.33)

Generating closure models for the LES PDE system typically involves consequential approximations in (1.33). For example, adding and subtracting the filtered velocity tensor product in (1.33) produces the triple decomposition approximation

\[ \overline{\mu_i\mu_i} = \overline{\mu_i\mu_i} + (\overline{\mu_i\mu_i} - \overline{\mu_i\mu_i}) + (\overline{\mu_i\mu_i} + \overline{\mu_i\mu_i}) + \overline{\mu_i\mu_i} \]

(1.34)

for \( L_{ij} \) termed the Leonard stress, \( C_{ij} \) the cross stress and \( R_{ij} \) the Reynolds subfilter scale tensors.

LES theory states that the latter accounts for energetic dissipation at the unresolved scale threshold, the spatial scale defined by filter measure \( \delta \). This is the LES theory equivalent of viscous dissipation in the unfiltered NS system at molecular scale. The legacy published sub-grid scale (SGS) tensor models again are of force-flux mathematical form, (1.27–1.30), with many variations, Piomelli (1999). As with time averaging, spatial filtering the NS DE and DY PDEs generates the companion unknown filtered thermal and mass vector quadruples.

An NS PDE system manipulation particularly pertinent to aerodynamics CFD applications alters the steady form of (1.19–1.22) assuming the velocity vector field is unidirectional. The end results are the famous boundary layer (BL) PDE system, and the \( n \)-dimensional generalization parabolic Navier–Stokes (PNS) PDE system. Both systems possess initial-value character in the direction of dominant flow. Importantly, both have largely supported development and validation of Reynolds stress tensor/vector closure models for the RaNS PDE system. The compressible turbulent PNS (PRaNS) PDE system is applicable to hypersonic external shock layer aerothermodynamics, detailed in Appendix B.

### 1.4 Weak Form Overview

The incompressible NS PDEs, also the model closed BL, PNS, RaNS and LES PDE systems, are elliptic boundary value (EBV) with selective initial-value character. Weak form theory is a thoroughly formal process for constructing approximate solutions to I-EBV/EBV PDE systems. The mathematical hooker in these PDEs is that each contains the differential constraint of DM, (1.19), which requires that the velocity field be divergence-free. This is the fundamental theoretical issue in identifying PDE systems with boundary conditions (BCs) that are well-posed, fully detailed in subsequent chapters.

The fundamental axiom of weak form theory is that one indeed seeks to construct an approximate solution. By very definition, a PDE solution is a function of space (and time perhaps) distributed continuously and smoothly, or possessing a finite number of finite discontinuities, over the PDE domain and on its boundaries. Therefore, the prime requirement for a weak form CFD algorithm is to clearly identify the candidate approximate solution.
This is completely distinct from historical FD/FV CFD approaches which generate a union of stencils via Taylor series approximations to PDE derivatives rather than stating the sought-for solution.

So, the starting point for a weak form construction is to identify a set of functions, called the trial space, endowed with properties appropriate for supporting an approximate solution. With certified existence of a trial space, the following questions come to mind:

- How good (accurate) an approximate solution can be supported by the selected trial space?
- How does the trial space supporting a finite element (FE) approximation differ from that for a finite difference (FD) scheme or a finite volume (FV) integral construction?
- Can these trial spaces be identical, and, if not, what are the distinguishing issues?
- Bottom line: is the error in the approximation related to a specific trial space selection?

Weak form theory possesses an elegant formalism for defining approximate solution error, developed in thoroughness in the next few chapters. The premier realization is that approximation error is a function(!) distributed over the PDE domain and its boundaries, as are the approximate and the exact NS solutions, with the latter of course never known. The key precept of weak form theory is to establish an integral constraint on error which requires definition of another set of functions termed the test space, against which the approximation error can be “tested.” In English:

\[
\text{weak form theory formalizes the ingredients of approximate solution construction in terms of a trial space, a test space and the rendering of the resultant approximation error an extremum in an appropriate integral measure called a norm}
\]

Weak form theory practice is no more complicated than implementing this statement. Thereby, one must immediately enquire whether there exists an optimal test space, such that the approximation error is the smallest possible for any trial space selection. Again weak form theory provides the answer:

\[
\text{Weak form theory predicts the approximation error is extremized, in practice minimized, when the trial and test spaces contain the identical members, which is termed the Galerkin criterion}
\]

Weak form theoretical musings always occur in the continuum and fully utilize calculus and vector field theory for generality with precision. Of pertinence, the weak form continuum construction holds for the PDE + BCs analytical solution as well as any approximation! Once the theory statement is formed, the remaining decision is trial space selection, hence identical test space, for error extremization. The key trial space requirement is for members to possess differentiability sufficient to enable integrals of their PDE derivatives to exist. This is really no problem, so the completion issue is forming the integrals that the weak form generates.
The continuum trial space contains functions spanning the global extent of the PDE domain. Finding suitable functions that one can integrate is nigh impossible (the challenge in DNS), hence the solution is to discretize the PDE domain and its boundaries, and hence identify much smaller subsets of the global trial (and test) space. Underlying is interpolation theory with net result a discrete approximation trial space basis. Trial space bases possess support only in the generic discretization cell, the union (non-overlapping sum) of which constitutes the computational mesh. This process admits full geometric generality, accurate evaluation of weak form integrals and a rigorous path to progressively more accurate formulations. Of ultimate importance is that it supports analytical formation of nonlinear algebraic matrix statements amenable to computing.

Summarizing, weak form theory involves clear organization of the sequence of decisions required for approximate solution error extremization, prior to definition of any specific discrete trial space basis. Thus, any given discrete solution methodology, specifically FE, FD or FV, becomes clearly identifiable among its peers by the sequence of decisions exposed in the weak formulation process. This text develops the subject in thoroughness, across a broad spectrum of fluid-thermal NS and manipulated NS PDE systems for which an optimal performance CFD algorithm is sought.

1.5 A Brief History of Finite Element CFD

Finite difference methods in CFD were first reported in the late 1920s, Courant et al. (1928), with fundamental theoretical developments emerging from the Courant Institute following World War II, Lax (1954), Lax and Wendroff (1960). Thereafter, many contributions to CFD emerged from the Los Alamos Scientific Laboratory (LASL), Amsden and Harlow (1970), Harlow (1971), coincident with the Imperial College team’s development of the “SIMPLE” algorithm, Gosman et al. (1969). The NASA Ames Research Center (ARC) picked up the lead on compressible aerodynamics CFD, MacCormack (1969), Beam and Warming (1976), with timing coincident with CFD maturing to an international research topic with hundreds of contributors.

The progenitor of weak form theory is the finite element method developed in practice in the late 1950s by aeronautical engineers to analyze aircraft structural components. Exploratory musings preceded this; for example Hrenikoff (1941) developed an elasticity solution for torsion problems based on triangles, Courant (1943) developed a variational formulation for problems in vibrations. Turner et al. (1956) first derived the stiffness matrix for truss and beam analysis, and Clough (1960) coined the term finite element. Argyris (1963) published the first monograph detailing a precise mathematical foundation for the engineer’s newly emerging finite element analysis capability.

The finite element method’s first application to the non-structural problem of unsteady heat conduction required convolution, Zienkiewicz and Cheung (1965). A formal addressing of the wider problem class in nonlinear mechanics followed, Oden (1972). As finite element structural theory and methodology matured, the mechanistic engineering precepts became replaced by a rich mathematical basis founded in the variational calculus and Rayleigh–Ritz methods, Rayleigh (1877), Ritz (1909). This classic theory base for finite element structural analyses grew rapidly, including many fundamental contributions, Babuska and Aziz (1972), Ciarlet and Raviart (1972), Aubin (1972), Lions and Magenes (1972), Strang and Fix (1973) and Oden and Reddy (1976).
The direct extension of classic variational mechanics to CFD algorithm construction for fluid/thermal descriptions is not possible. The impediment is the conservation principle eulerian reference frame, which renders momentum conservation \( DP \) explicitly nonlinear. For this reason, at least, pioneering CFD procedures employed replacement of derivatives by divided differences, that is, finite differences, Richtmyer and Morton (1967), Roache (1972).

The FD successor finite volume CFD developments involved direct integration of each PDE over cells of a domain discretization, followed by the divergence theorem which exposed cell face fluxes. These were evaluated via finite difference quotients coupled with an approximate enforcement for DM, Patankar (1980). The particle-in-cell method, Evans and Harlow (1957), also employed a cell flux concept and used pseudo-lagrangian particle distributions for approximate satisfaction of the DM constraint.

Most FD/FV quotient-based CFD algorithms were discovered recoverable as specific criteria selections within the discrete weighted residuals (WR) framework, with Finlayson (1972) the pioneering exposition. In WR theory, approximation error was constrained by requiring local integrals containing “weights” to vanish. Within the class, the collocation method weights were the Dirac delta, which exactly reproduced classic FD quotients. A finite volume algorithm was retrieved for a constant weight. Generalizing the weights to functions, and defining them identical to the discrete trial space basis reproduced the Galerkin method, named after the (non-discrete, non-CFD) procedure of B.G. Galerkin (1915). Finally, defining the weights to be the PDE operator itself reproduced the least squares method.

Pioneering FE CFD algorithms employed various WR criteria. Oden (1969, 1972) was among the first to derive the basic theoretical analog for the NS PDE system. Using a Galerkin WR FE formulation, Baker (1971, 1973, 1974) reported two-dimensional compressible and incompressible flow simulations with recirculation regions. Olson (1972) detailed a pseudo-variational (hence Galerkin) FE algorithm for the streamfunction biharmonic PDE equivalent of the two-dimensional NS PDE system.


From the mid 1970s through the 1980s burgeoning interest in weak form FE CFD algorithm research and applications sparked annual international conferences. This resulted in the Wiley conference monograph series Finite Elements in Fluids, with Gallagher, Oden and Zienkiewicz principal editors, Gallagher et al. (1975–1988). Publication of FE discrete CFD algorithms for NS systems moved to textbooks. Finite Element Computational Fluid Mechanics, Baker (1983), was the first topical treatise to specifically include asymptotic convergence theory validation for RaNS systems. (Therein weak form never appeared, owing to the author’s opinion that engineers didn’t do weak things!)

It’s quite alarming looking into these early texts and viewing solution graphics computed on incredibly coarse meshes! Much larger scale PC computing emerged in the 1990s, as the FE CFD textbook litany moved forward. First prize for size (over 1000 pages!) goes to Gresho and Sani (1998) for *Incompressible Flow and the Finite Element Method*, which contained incredible mathematical detail for laminar isothermal incompressible flows only! Reddy and Gartling (2001) published *The Finite Element Method in Heat Transfer and Fluid Dynamics*, with applications to low-Re non-Newtonian fluids.


### 1.6 A Brief Summary

So, does the CFD community need another textbook on weak form FE-implemented algorithms? I believe the answer is yes, as the broad duplication of historical, small Re constructions reported can now be replaced by a single comprehensive weak form theory text with precise attention to performance estimation and validation. This current knowledge base is applicable across the complete spectrum of NS, BL, PNS, RaNS, LES and PRaNS PDE systems.

Weak form theory, of necessity linear, predicts that the solution associated with the stationary coordinate of the Galerkin criterion is optimal. Its failure to address dispersive error instability and phase accuracy compromising issues associated with pioneering Galerkin NS constructions is now fully resolved. Via weak form manipulations based on limiting independent coordinate-derived Taylor series, differential term additions to the classic text NS PDE statements generate modified (mPDE) systems. The analysis framework ultimately leads to a precise theory recovering essentially all prior FD/FV/FE constructions, but most importantly theoretical prediction of an optimal construction.

Hence, CFD algorithms for all NS, BL, PNS, RaNS, LES and PRaNS mPDE systems herein are based on this theoretically sound optimal modified continuous Galerkin weak
form CFD theory. The key required confirmation of this assertion, as detailed herein, is that linear weak form theory is the accurate predictor of nonlinear NS PDE/mPDE CFD algorithm performance. Had this not occurred, this text would not have been written!

To summarize, weak form theory enables removal of essentially all elements of mystery surrounding algorithm constructions in NS fluid-thermal CFD. Comparison to legacy and/or current discrete CFD formulations is direct, simply by selecting the non-Galerkin, non-augmented PDE weak form criteria appropriate to reproducing that algorithm. This text is the culmination of the author’s decades-held premise that the (multiple) hundreds of published CFD algorithms do not produce a corresponding number of linearly independent constructions! That the cogent approach to the subject can shed light uniformly across the spectrum was my academic research objective. This text is the result, hopefully written in a manner making the exposé enjoyable and rewarding. Bon voyage!

References


