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Fractional Brownian motion and related processes

1.1 Introduction

In his study of long-term storage capacity and design of reservoirs based on investigations of river water levels along the Nile, Hurst (1951) observed a phenomenon which is invariant to changes in scale. Such a scale-invariant phenomenon was recently observed in problems connected with traffic patterns of packet flows in high-speed data networks such as the Internet (cf. Leland et al. (1994), Willinger et al. (2003), Norros (2003)) and in financial data (Willinger et al. (1999)). Lamperti (1962) introduced a class of stochastic processes known as semi-stable processes with the property that, if an infinite sequence of contractions of the time and space scales of the process yield a limiting process, then the limiting process is semi-stable. Mandelbrot (1982) termed these processes as self-similar and studied applications of these models to understand scale-invariant phenomena. Long-range dependence is related to the concept of self-similarity for a stochastic process in that the increments of a self-similar process with stationary increments exhibit long-range dependence under some conditions. A long-range dependence pattern is also observed in macroeconomics and finance (cf. Henry and Zafforoni (2003)). A fairly recent monograph by Doukhan et al. (2003) discusses the theory and applications of long-range dependence. Before we discuss modeling of processes with long-range dependence, let us look at the consequences of long-range dependence phenomena. A brief survey of self-similar processes, fractional Brownian motion and statistical inference is given in Prakasa Rao (2004d).

Suppose \{X_i, 1 \leq i \leq n\} are independent and identically distributed (i.i.d.) random variables with mean \(\mu\) and variance \(\sigma^2\). It is well known that the sample mean \(\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i\) is an unbiased estimator of the mean \(\mu\) and its variance
is \( \sigma^2/n \) which is proportional to \( n^{-1} \). In his work on yields of agricultural experiments, Smith (1938) studied mean yield \( \bar{X}_n \) as a function of the number \( n \) of plots and observed that \( \text{Var}(\bar{X}_n) \) is proportional to \( n^{-a} \) where \( 0 < a < 1 \). If \( a = 0.4 \), then approximately 100,000 observations are needed to achieve the same accuracy of \( \bar{X}_n \) as from 100 i.i.d. observations. In other words, the presence of long-range dependence plays a major role in estimation and prediction problems. Long-range dependence phenomena are said to occur in a stationary time series \( \{X_n, n \geq 0\} \) if \( \text{Cov}(X_0, X_n) \) of the time series tends to zero as \( n \to \infty \) and yet the condition

\[
\sum_{n=0}^{\infty} |\text{Cov}(X_0, X_n)| = \infty
\] (1.1)

holds. In other words, the covariance between \( X_0 \) and \( X_n \) tends to zero as \( n \to \infty \) but so slowly that their sums diverge.

1.2 Self-similar processes

A real-valued stochastic process \( Z = \{Z(t), -\infty < t < \infty\} \) is said to be self-similar with index \( H > 0 \) if, for any \( a > 0 \),

\[
\mathcal{L}([Z(at), -\infty < t < \infty]) = \mathcal{L}([a^H Z(t), -\infty < t < \infty])
\] (1.2)

where \( \mathcal{L} \) denotes the class of all finite-dimensional distributions and the equality indicates the equality of the finite-dimensional distributions of the process on the right of Equation (1.2) with the corresponding finite-dimensional distributions of the process on the left of Equation (1.2). The index \( H \) is called the scaling exponent or the fractal index or the Hurst index of the process. If \( H \) is the scaling exponent of a self-similar process \( Z \), then the process \( Z \) is called an \( H \)-self-similar process or \( H \)-ss process for short. A process \( Z \) is said to be degenerate if \( P(Z(t) = 0) = 1 \) for all \( t \in \mathbb{R} \). Hereafter, we write \( X \overset{\Delta}{=} Y \) to indicate that the random variables \( X \) and \( Y \) have the same probability distribution.

**Proposition 1.1:** A non-degenerate \( H \)-ss process \( Z \) cannot be a stationary process.

**Proof:** Suppose the process \( Z \) is a stationary process. Since the process \( Z \) is non degenerate, there exists \( t_0 \in \mathbb{R} \) such that \( Z(t_0) \neq 0 \) with positive probability and, for all \( a > 0 \),

\[
Z(t_0) \overset{\Delta}{=} Z(at_0) \overset{\Delta}{=} a^H Z(t_0)
\]

by stationarity and self-similarity of the process \( Z \). Let \( a \to \infty \). Then the family of random variables on the right diverge with positive probability, whereas the random variable on the left is finite with probability one, leading to a contradiction. Hence the process \( Z \) is not a stationary process.
**Proposition 1.2:** Suppose that \( \{Z(t), -\infty < t < \infty\} \) is an \( H \)-ss process. Define

\[
Y(t) = e^{-tH}Z(e^t), \quad -\infty < t < \infty.
\]

Then the process \( \{Y(t), -\infty < t < \infty\} \) is a stationary process.

**Proof:** Let \( k \geq 1 \) be an integer. For \( a_i, 1 \leq j \leq k \) real and \( h \in R \),

\[
\sum_{j=1}^{k} a_j Y(t_j + h) \Delta = \sum_{j=1}^{k} a_j e^{-(t_j + h)H} Z(e^{t_j + h})
\]

\[
\overset{\Delta}{=} \sum_{j=1}^{k} a_j e^{-(t_j + h)H} e^{hH} Z(e^{t_j})
\]

(by self-similarity of the process \( Z \))

\[
\overset{\Delta}{=} \sum_{j=1}^{k} a_j e^{-t_j H} Z(e^{t_j})
\]

\[
\overset{\Delta}{=} \sum_{j=1}^{k} a_j Y(t_j).
\]

Since the above relation holds for every \( (a_1, \ldots, a_k) \in R^k \), an application of the Cramér–Wold theorem shows that the finite-dimensional distribution of the random vector \( \{Y(t_1 + h), \ldots, Y(t_k + h)\} \) is the same as that of the random vector \( \{Y(t_1), \ldots, Y(t_k)\} \). Since this property holds for all \( t_i, 1 \leq i \leq k, k \geq 1 \) and for all \( h \) real, it follows that the process \( Y = \{Y(t), -\infty < t < \infty\} \) is a stationary process.

The transformation defined by (1.3) is called the Lamperti transformation. By retracing the arguments given in the proof of Proposition 1.2, the following result can be proved.

**Proposition 1.3:** Suppose \( \{Y(t), -\infty < t < \infty\} \) is a stationary process. Let \( X(t) = t^H Y(\log t) \) for \( t > 0 \). Then \( \{X(t), t > 0\} \) is an \( H \)-ss process.

**Proposition 1.4:** Suppose that a process \( \{Z(t), -\infty < t < \infty\} \) is a second-order process, that is, \( E[Z^2(t)] < \infty \) for all \( t \in R \), and it is an \( H \)-ss process with stationary increments, that is,

\[
Z(t + h) - Z(t) \overset{\Delta}{=} Z(h) - Z(0)
\]

for \( t \in R, h \in R \). Let \( \sigma^2 = Var(Z(1)) \). Then the following properties hold:

(i) \( Z(0) = 0 \) a.s.

(ii) If \( H \neq 1 \), then \( E(Z(t)) = 0, -\infty < t < \infty \).
(iii) $Z(-t) \overset{\Delta}{=} -Z(t)$, $-\infty < t < \infty$.

(iv) $E(Z^2(t)) = |t|^{2H} E(Z^2(1))$, $-\infty < t < \infty$.

(v) Suppose $H \neq 1$. Then

$$
\text{Cov}(Z(t), Z(s)) = (\sigma^2/2)[|t|^{2H} + |s|^{2H} - |t-s|^{2H}].
$$

(vi) $0 < H \leq 1$.

(vii) If $H = 1$, then $Z(t) \overset{\Delta}{=} tZ(1)$, $-\infty < t < \infty$.

Proof:

(i) Note that $Z(0) = Z(a.0) \overset{\Delta}{=} a^H Z(0)$ for any $a > 0$ by the self-similarity of the process $Z$. It is easy to see that this relation holds only if $Z(0) = 0$ a.s.

(ii) Suppose $H \neq 1$. Since $Z(2t) \overset{\Delta}{=} 2^H Z(t)$, it follows that

$$
2^H E(Z(t)) = E(Z(2t))
$$

$$
= E(Z(2t) - Z(t)) + E(Z(t))
$$

$$
= E(Z(t) - Z(0)) + E(Z(t))
$$

(by stationarity of the increments)

$$
= 2E(Z(t))
$$

(1.5)

for any $t \in R$. The last equality follows from the observation that $Z(0) = 0$ a.s. from (i). Hence $E(Z(t)) = 0$ since $H \neq 1$.

(iii) Observe that, for any $t \in R$,

$$
Z(-t) \overset{\Delta}{=} Z(-t) - Z(0)
$$

$$
\overset{\Delta}{=} Z(0) - Z(t) \text{ (by stationarity of the increments)}
$$

$$
\overset{\Delta}{=} -Z(t) \text{ (by Property (i))}.
$$

Therefore $Z(-t) \overset{\Delta}{=} -Z(t)$ for every $t \in R$.

(iv) It is easy to see that, for any $t \in R$,

$$
E(Z^2(t)) = E(Z^2(|t| \sgn t))
$$

$$
= |t|^{2H} E(Z^2(\sgn t)) \text{ (by self-similarity)}
$$

$$
= |t|^{2H} E(Z^2(1)) \text{ (by Property (iii))}
$$

$$
= \sigma^2 |t|^{2H} \text{ (by Property (ii))}.
$$

(1.7)
Here the function \( \text{sgn } t \) is equal to 1 if \( t \geq 0 \) and is equal to \(-1\) if \( t < 0 \). If \( \sigma^2 = 1 \), the process \( Z \) is called a standard \( H \)-ss process with stationary increments.

(v) Let \( R_H(t, s) \) be the covariance between \( Z(t) \) and \( Z(s) \) for any \(-\infty < t, s < \infty\). Then

\[
R_H(t, s) \equiv \text{Cov}(Z(t), Z(s)) = E[Z(t)Z(s)] \quad \text{(by Property (ii))}
\]

\[
= \frac{1}{2} \{ E[Z^2(t)] + E[Z^2(s)] - E[(Z(t) - Z(s))^2] \}
\]

(by stationarity of the increments)

\[
= \frac{1}{2} \{ E[Z^2(t)] + E[Z^2(s)] - E[(Z(t-s) - Z(0))^2] \}
\]

(by Property (ii))

\[
= \frac{\sigma^2}{2} \left| t \right|^{2H} + \left| s \right|^{2H} - \left| t - s \right|^{2H} \quad \text{(by Property (iv)).} \quad (1.8)
\]

In particular, it follows that the function \( R_H(t, s) \) is nonnegative definite as it is the covariance function of a stochastic process.

(vi) Note that

\[
E(\left| Z(2) \right|) = E(\left| Z(2) - Z(1) + Z(1) \right|)
\]

\[
\leq E(\left| Z(2) - Z(1) \right|) + E(\left| Z(1) \right|)
\]

\[
= E(\left| Z(1) - Z(0) \right|) + E(\left| Z(1) \right|)
\]

(by stationarity of the increments)

\[
= 2E(\left| Z(1) \right|) \quad \text{(by Property (i)).} \quad (1.9)
\]

Self-similarity of the process \( Z \) implies that

\[
E(\left| Z(2) \right|) = 2^H E(\left| Z(1) \right|). \quad (1.10)
\]

Combining relations (1.9) and (1.10), we get

\[
2^H E(\left| Z(1) \right|) \leq 2 E(\left| Z(1) \right|)
\]

which, in turn, implies that \( H \leq 1 \) since the process \( Z \) is a non degenerate process.
(vii) Suppose \( H = 1 \). Then \( E(Z(t)Z(1)) = tE(Z^2(1)) \) and \( E(Z^2(t)) = t^2E(Z^2(1)) \) by the self-similarity of the process \( Z \). Hence
\[
E(Z(t) - tZ(1))^2 = E(Z^2(t)) - 2t E(Z(t)Z(1)) + t^2E(Z^2(1)) \\
= (t^2 - 2t^2 + t^2)E(Z^2(1)) \\
= 0. 
\] (1.11)

This relation shows that \( Z(t) = tZ(1) \) a.s. for every \( t \in \mathbb{R} \).

**Remarks:** As was mentioned earlier, self-similar processes have been used for stochastic modeling in such diverse areas as hydrology (cf. Montanari (2003)), geophysics, medicine, genetics and financial economics (Willinger et al. (1999)) and more recently in modeling Internet traffic patterns (Leland et al. (1994)). Additional applications are given in Goldberger and West (1987), Stewart et al. (1993), Buldyrev et al. (1993), Ossandik et al. (1994), Percival and Guttrop (1994) and Peng et al. (1992, 1995a,b). It is important to estimate the Hurst index \( H \) for modeling purposes. This problem has been considered by Azais (1990), Geweke and Porter-Hudak (1983), Taylor and Taylor (1991), Beran and Terrin (1994), Constantine and Hall (1994), Feuerverger et al. (1994), Chen et al. (1995), Robinson (1995), Abry and Sellan (1996), Comte (1996), McCoy and Walden (1996), Hall et al. (1997), Kent and Wood (1997), and more recently in Jensen (1998), Poggi and Viano (1998), and Coeurjolly (2001).

It was observed that there are some phenomena which exhibit self-similar behavior locally but the nature of self-similarity changes as the phenomenon evolves. It was suggested that the parameter \( H \) must be allowed to vary as a function of time for modeling such data. Goncalves and Flandrin (1993) and Flandrin and Goncalves (1994) propose a class of processes which are called *locally self-similar* with dependent scaling exponents and discuss their applications. Wang et al. (2001) develop procedures using wavelets to construct local estimates for time-varying scaling exponent \( H(t) \) of a locally self-similar process.

A second-order stochastic process \( \{Z(t), t > 0\} \) is called **wide-sense \( H \)-self-similar** if it satisfies the following conditions for every \( a > 0 \):

(i) \( E(Z(at)) = a^H E(Z(t)), t > 0; \)

(ii) \( E(Z(at)Z(as)) = a^{2H} E(Z(t)Z(s)), t > 0, s > 0. \)

This definition can be compared with the definition of (strict) \( H \)-self-similarity which is that the processes \( \{Z(at)\} \) and \( \{a^H Z(t)\} \) have the same finite-dimensional distributions for every \( a > 0 \). The wide-sense definition is more general. However, it excludes self-similar processes with infinite second moments such as non-Gaussian stable processes. Given a wide-sense \( H \)-ss process \( Z \), it is possible to form a wide-sense stationary process \( Y \) via the Lamperti transformation
\[
Y(t) = e^{-Ht} Z(e^t). 
\]

1.3 Fractional Brownian motion

A Gaussian $H$-ss process $W^H = \{W^H(t), -\infty < t < \infty \}$ with stationary increments and with fractal index $0 < H < 1$ is termed fractional Brownian motion (fBm). Note that $E[W^H(t)] = 0, -\infty < t < \infty$. It is said to be standard if $Var(W^H(1)) = 1$.

For standard fractional Brownian motion,

$$Cov(W^H(t), W^H(s)) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

If $H = \frac{1}{2}$, then fBm reduces to the Brownian motion known as the Wiener process. It is easy to see that if $\{X(t), -\infty < t < \infty\}$ is a Gaussian process with stationary increments with mean zero, with $X(0) = 0$ and $E(X^2(t)) = \sigma^2 |t|^{2H}$ for some $0 < \sigma < \infty$ and $0 < H < 1$, then the process $\{X(t), -\infty < t < \infty\}$ is fBm. The following theorem gives some properties of standard fBm.

**Theorem 1.5**: Let $\{W^H(t), -\infty < t < \infty\}$ be standard fBm with Hurst index $H$ for some $0 < H < 1$. Then:

(i) There exists a version of the process $\{W^H(t), -\infty < t < \infty\}$ such that the sample paths of the process are continuous with probability one.

(ii) The sample paths of the process $\{W^H(t), -\infty < t < \infty\}$ are nowhere differentiable in the $L^2$-sense.

(iii) For any $0 < \lambda < H$, there exist constants $h > 0$ and $C > 0$ such that, with probability one,

$$|W^H(t) - W^H(s)| < C |t-s|^{\lambda}, \quad 0 \leq t, s \leq 1, \quad |t-s| \leq h.$$

(iv) Consider the standard fBm $W^H = \{W^H(t), 0 \leq t \leq T\}$ with Hurst index $H$. Then

$$\lim_{n \to \infty} \sum_{j=0}^{2^n-1} \left| W^H \left( \frac{j+1}{2^n} T \right) - W^H \left( \frac{j}{2^n} T \right) \right|^p = 0 \text{ a.s. if } pH > 1$$

$$= \infty \text{ a.s. if } pH < 1$$

$$= T \text{ a.s. if } pH = 1. \quad (1.12)$$
Property (i) stated above follows from Kolmogorov’s sufficient condition for a.s. continuity of the sample paths of a stochastic process and the fact that

\[ E[|W^H(t) - W^H(s)|^\alpha] = E[|W^H(1)|^\alpha]|t - s|^\alpha H, \ -\infty < t, s < \infty \]

for any \( \alpha > 0 \). The equation given above follows from the observation that fBm is an \( H \)-ss process with stationary increments. The constant \( \alpha > 0 \) can be chosen so that \( \alpha H > 1 \) satisfies Kolmogorov’s continuity condition.

Property (ii) is a consequence of the relation

\[ E\left[ \left| \frac{W^H(t) - W^H(s)}{t - s} \right|^2 \right] = E[|W^H(1)|^2]|t - s|^{2H - 2} \]

and the last term tends to infinity as \( t \to s \) since \( H < 1 \). Hence the paths of fBm are not \( L^2 \)-differentiable.

For a discussion and proofs of Properties (iii) and (iv), see Doukhan et al. (2003) and Decreusefond and Ustunel (1999). If the limit

\[ \lim_{n \to \infty} \sum_{j=0}^{2^n-1} \left| W^H\left( \frac{j + 1}{2^n} T \right) - W^H\left( \frac{j}{2^n} T \right) \right|^p \]

exists a.s., then the limit is called the \( p \)-th variation of the process \( W^H \) over the interval \([0, T]\). If \( p = 2 \), then it is called the quadratic variation over the interval \([0, T]\). If \( H = \frac{1}{2} \) and \( p = 2 \), in (iv), then the process \( W^H \) reduces to the standard Brownian motion \( W \) and we have the well-known result

\[ \lim_{n \to \infty} \sum_{j=0}^{2^n-1} \left| W\left( \frac{j + 1}{2^n} T \right) - W\left( \frac{j}{2^n} T \right) \right|^2 = T \quad \text{a.s.} \]

for the quadratic variation of the standard Brownian motion on the interval \([0, T]\).

If \( H < \frac{1}{2} \), then, for \( p = 2 \), we have \( pH < 1 \) and the process has infinite quadratic variation by Property (iv). If \( H > \frac{1}{2} \), then, for \( p = 2 \), we have \( pH > 1 \) and the process has zero quadratic variation by Property (iv). Such a process is called a Dirichlet process. Furthermore, the process \( W^H \) has finite \( p \)-th variation if \( p = 1/H \). In other words,

\[ \lim_{n \to \infty} \sum_{j=0}^{2^n-1} \left| W^H\left( \frac{j + 1}{2^n} T \right) - W^H\left( \frac{j}{2^n} T \right) \right|^{1/H} = T \quad \text{a.s.} \quad (1.13) \]

Let us again consider standard fBm \( W^H \) with Hurst index \( H > \frac{1}{2} \) over an interval \([0, T]\). Let \( 0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T \) be a subdivision of the interval \([0, T]\) such that

\[ \max_{0 \leq j \leq n-1} |t_{j+1}^{(n)} - t_j^{(n)}| \to 0 \]
as \( n \to \infty \). Note that
\[
E[(W^H(t) - W^H(s))^2] = |t - s|^{2H}, \quad 0 \leq t, s \leq T
\]
and hence
\[
\sum_{j=0}^{n-1} E[(W^H(t_{j+1}^{(n)}) - W^H(t_j^{(n)}))^2] = \sum_{j=0}^{n-1} (t_{j+1}^{(n)} - t_j^{(n)})^{2H} \leq \max_{0 \leq j \leq n-1} [t_{j+1}^{(n)} - t_j^{(n)}]^{2H-1} T
\]
\[
= \left( \max_{0 \leq j \leq n-1} [t_{j+1}^{(n)} - t_j^{(n)}] \right)^{2H-1} T. \quad (1.14)
\]
Therefore
\[
\lim_{n \to \infty} \sum_{j=0}^{n-1} E[(W^H(t_{j+1}^{(n)}) - W^H(t_j^{(n)}))^2] = 0
\]
or equivalently
\[
\lim_{n \to \infty} E \left[ \sum_{j=0}^{n-1} (W^H(t_{j+1}^{(n)}) - W^H(t_j^{(n)}))^2 \right] = 0.
\]
This statement in turn implies that
\[
\sum_{j=0}^{n-1} (W^H(t_{j+1}^{(n)}) - W^H(t_j^{(n)}))^2 \overset{p}{\to} 0 \text{ as } n \to \infty.
\]
As a consequence of this fact, it can be shown that the process \( W^H \) is not a semi-martingale from the results in Liptstter and Shirayev (1989). For the definition of a semi-martingale and its properties, see Prakasa Rao (1999b).

**Representation of fBm as a stochastic integral**

Suppose \( \{W(t), -\infty < t < \infty\} \) is standard Brownian motion and \( H \in \left( \frac{1}{2}, 1 \right) \). Define a process \( \{Z(t), -\infty < t < \infty\} \) with \( Z(0) = 0 \) by the relation
\[
Z(t) - Z(s) = \lim_{a \to -\infty} \left( c_H \int_a^t (t - \tau)^{H-\frac{1}{2}} dW_\tau - c_H \int_a^s (s - \tau)^{H-\frac{1}{2}} dW_\tau \right)
\]
\[
= c_H \int_s^t (t - \tau)^{H-\frac{1}{2}} dW_\tau + c_H \int_{-\infty}^s [(t - \tau)^{H-\frac{1}{2}} - (s - \tau)^{H-\frac{1}{2}}] dW_\tau.
\]
(1.15)
for $t > s$, where

$$c_H = \left(2H \Gamma\left(\frac{3}{2} - H\right) / \Gamma\left(\frac{1}{2} + H\right) \Gamma(2 - 2H)\right)^{1/2}$$

(1.16)

and $\Gamma(.)$ is the gamma function. Here the integrals are defined as Wiener integrals and the resulting process $Z$ is a mean zero Gaussian process. In order to show that the process $Z$ is in fact fBm, we have to prove that its covariance function is that of fBm. We will come back to this discussion later in this section.

**Integration with respect to fBm**

It is known that, in order to develop the theory of stochastic integration of a random process with respect to another stochastic process satisfying the usual properties of integrals such as linearity and dominated convergence theorem, it is necessary for the integrator to be a semimartingale. This can be seen from Theorem VIII.80 in Dellacherie and Meyer (1982). Semimartingales can also be characterized by this property. Since fBm is not a semimartingale, it is not possible to define stochastic integration of a random process with respect to fBm starting with the usual method of limiting arguments based on Riemann-type sums for simple functions as in the case of Ito integrals. However, the special case of a stochastic integration of a deterministic integrand with respect to fBm as the integrator can be developed using the theory of integration with respect to general Gaussian processes as given, say, in Huang and Cambanis (1978) and more recently in Alos et al. (2001). There are other methods of developing stochastic integration of a random process with respect to fBm using the notion of Wick product and applying the techniques of Malliavin calculus. We do not use these approaches throughout this book and unless specified otherwise, we consider fBm with Hurst index $H > \frac{1}{2}$ throughout this book. The reason for such a choice of $H$ for modeling purposes will become clear from our discussion later in this section.

Let $\{Z(t), -\infty < t < \infty\}$ be standard fBm with Hurst index $H > \frac{1}{2}$ and suppose $\{Y(t), -\infty < t < \infty\}$ is a simple process in the sense that $Y(t) = \sum_{j=1}^{k} X_j I(T_{j-1}, T_j)(t)$ where $-\infty < T_0 < T_1 < \cdots < T_k < \infty$. We define the stochastic integral of the process $Y$ with respect to $Z$ by the relation

$$\int_{-\infty}^{\infty} Y(t) dZ(t) = \sum_{j=1}^{k} X_j (Z(T_j) - Z(T_{j-1})).$$

(1.17)

If the process $Y$ is of locally bounded variation, then we can define the integral by using the integration by parts formula

$$\int_{a}^{b} Y(t) dZ(t) = Y(b)Z(b) - Y(a)Z(a) - \int_{a}^{b} Z(t) dY(t)$$

(1.18)
and the integral on the right of Equation (1.18) can be defined using the theory of Lebesgue–Stieltjes integration. Suppose the process $Y$ is non random, that is, deterministic. Under suitable conditions on the non random function $Y$, the integral on the left of (1.18) can be defined as an $L^2$-limit of Riemann sums of the type defined in (1.17) with nonrandom sequence $T_j$. Gripenberg and Norros (1996) give an example of a random process $Y$ illustrating the problem of non continuity in extending the above method of stochastic integration with respect to fBm for random integrands. We will consider integrals of non random functions only with integrator as fBm throughout this book unless otherwise stated.

An alternate way of defining the stochastic integral of a non random function $f$ with respect to fBm $Z$ is by the formula

$$\int_{-\infty}^{\infty} f(t) dZ(t) = c_H \left( H - \frac{1}{2} \right) \int_{-\infty}^{\infty} \left( \int_{\tau}^{\infty} (t - \tau)^{H - \frac{3}{2}} f(t) dt \right) dW_{\tau} \quad (1.19)$$

where $W$ is a standard Wiener process and the constant $c_H$ is as defined in (1.16). The integral defined on the right of (1.19) exists provided the function

$$\int_{\tau}^{\infty} (t - \tau)^{H - \frac{3}{2}} f(t) dt$$

as a function of $\tau$ is square integrable. A sufficient condition for this to hold is that $f \in L^2(R) \cap L^1(R)$. It is easy to see that the random variable

$$\int_{-\infty}^{\infty} f(t) dZ(t)$$

is Gaussian and

$$E \left( \int_{-\infty}^{\infty} f(t) dZ(t) \right) = 0$$

whenever $f \in L^2(R) \cap L^1(R)$. We now obtain the covariance formula for two such integrals.

**Theorem 1.6:** For functions $f, g \in L^2(R) \cap L^1(R)$,

$$E \left( \int_{-\infty}^{\infty} f(t) dZ(t) \int_{-\infty}^{\infty} g(t) dZ(t) \right) = H(2H - 1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) g(s) |t - s|^{2H - 2} dt ds. \quad (1.20)$$
Proof: Note that

\[
E \left( \int_{-\infty}^{\infty} f(t) dZ(t) \int_{-\infty}^{\infty} g(t) dZ(t) \right) \\
= c_H^2 \left( H - \frac{1}{2} \right)^2 E \left[ \left( \int_{-\infty}^{\infty} \left( \int_{\tau}^{\infty} (t-\tau)^{H-\frac{3}{2}} f(t) dt \right) dW_\tau \right) \right. \\
\left. \left( \int_{-\infty}^{\infty} \left( \int_{\tau}^{\infty} (t-\tau)^{H-\frac{3}{2}} g(t) dt \right) dW_\tau \right) \right] \\
= c_H^2 \left( H - \frac{1}{2} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) g(s) \left[ \int_{-\infty}^{\min(s,t)} (s-\tau)^{H-\frac{3}{2}} (t-\tau)^{H-\frac{3}{2}} d\tau \right] dt ds
\]

by the properties of Wiener integrals. From the results in Abramowitz and Stegun (1970), 6.2.1, 6.2.2, it follows that

\[
\int_{-\infty}^{\min(s,t)} (s-\tau)^{H-\frac{3}{2}} (t-\tau)^{H-\frac{3}{2}} d\tau = \int_{0}^{\infty} (|t-s| + \tau)^{H-\frac{3}{2}} \tau^{H-\frac{3}{2}} d\tau \\
= |t-s|^{2H-2} \int_{0}^{\infty} (1 + \tau)^{H-\frac{3}{2}} \tau^{H-\frac{3}{2}} d\tau \\
= |t-s|^{2H-2} \frac{\Gamma(H-\frac{1}{2}) \Gamma(2-2H)}{\Gamma(\frac{3}{2} - H)}. \quad (1.21)
\]

From the equations derived above, it follows that

\[
E \left( \int_{-\infty}^{\infty} f(t) dZ(t) \int_{-\infty}^{\infty} g(t) dZ(t) \right) \\
= H(2H-1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) g(s)|t-s|^{2H-2} dt ds. \quad (1.22)
\]

As a consequence of Theorem 1.6, we obtain that

\[
E \left[ \int_{-\infty}^{\infty} f(t) dZ(t) \right]^2 = H(2H-1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) f(s)|t-s|^{2H-2} dt ds. \quad (1.23)
\]

Our discussion of integration with respect to fBm given here is based on Gripenberg and Norros (1996). Another way of defining the integral of a non random function with respect to fBm is given later in this section. Zahle (1998) has defined path wise integration with respect to fBm when the integrand is possibly random using the methods of fractional calculus. We will briefly review this approach in the last section of this chapter.
Inequalities for moments of integrals with respect to fBm

We assume as before that $H > \frac{1}{2}$. Let $\Psi$ denote the integral operator

$$\Psi f(t) = H(2H - 1) \int_0^\infty f(s)|t - s|^{2H - 2}ds$$

(1.24)

and define the inner product

$$\langle f, g \rangle_{\Psi} = \langle f, \Psi g \rangle = H(2H - 1) \int_0^\infty \int_0^\infty f(t)g(s)|t - s|^{2H - 2}dtds$$

(1.25)

where $\langle ., . \rangle$ denote the usual inner product in $L^2([0, \infty))$. Let $L^2_\Psi$ be the space of functions $f$ such that $\langle f, f \rangle_{\Psi} < \infty$. Let $L^2_\Psi([0, T])$ be the space of functions $f$ such that $\langle f I_{[0, T]}, f I_{[0, T]} \rangle_{\Psi} < \infty$. Here $I_A$ denotes the indicator function of the set $A$. The mapping $Z(t) \to I_{[0,1]}$ can be extended to an isometry between the Gaussian subspace of the space generated by the random variables $Z(t), t \geq 0$, and the function space $L^2_\Psi$, as well as to an isometry between a subspace of the space generated by the random variables $Z(t), 0 \leq t \leq T$, and the function space $L^2_\Psi([0, T])$ (cf. Huang and Cambanis (1978)). For $f \in L^2_\Psi$, the integral

$$\int_0^\infty f(t)dZ(t)$$

is defined as the image of $f$ by this isometry. In particular, for $f, g \in L^2_\Psi([0, T])$,

$$E \left( \int_0^T f(t)dZ(t) \int_0^T g(t)dZ(t) \right) = H(2H - 1) \int_0^T \int_0^T f(t)g(s)|t - s|^{2H - 2}dtds$$

(1.26)

and

$$E \left( \int_u^v f(t)dZ(t) \right)^2 = H(2H - 1) \int_u^v \int_u^v f(t)f(s)|t - s|^{2H - 2}dt ds.$$ (1.27)

Let

$$|| f ||_{L^p((u,v))} = \left( \int_u^v |f(t)|^p \right)^{1/p}$$

(1.28)

for $p \geq 1$ and $0 \leq u < v \leq \infty$.

**Theorem 1.7:** Let $Z$ be standard fBm with Hurst index $H > \frac{1}{2}$. Then, for every $r > 0$, there exists a positive constant $c(H, r)$ such that for every $0 \leq u < v < \infty$,

$$E \left[ \int_u^v f(t)dZ(t) \right]^r \leq c(H, r) || f ||_{L^{1/H((u,v))}}^r$$

(1.29)
and
\[ E \left| \int_u^v f(t)dZ(t) \right|^r \leq c(H, r)||f||^r_{L^{1/H}(u, v)}||g||^r_{L^{1/H}(u, v)}, \]
\[ (1.30) \]

**Proof:** Since the random variable
\[ \int_0^T f(t)dZ(t) \]
is a mean zero Gaussian random variable, for every \( r > 0 \), there exists a constant \( k(r) > 0 \) such that
\[ E \left| \int_0^T f(t)dZ(t) \right|^r \leq k(r) \left( E \left| \int_0^T f(t)dZ(t) \right|^2 \right)^{r/2} \]
\[ = k(r) \left[ H(2H - 1) \int_0^T \int_0^T f(t)f(s)|t-s|^{2H-2}dtds \right]^{r/2}. \]
\[ (1.31) \]

Furthermore
\[ \int_0^T \int_0^T |f(t)f(s)||t-s|^{2H-2}dt ds \]
\[ = \int_0^T |f(t)| \left( \int_0^T |f(s)||t-s|^{2H-2}ds \right) dt \]
\[ \leq \left( \int_0^T |f(u)|^{1/H} du \right)^H \left( \int_0^T du \left( \int_0^T |f(v)||u-v|^{2H-2}dv \right)^{1/(1-H)} \right)^{1-H} \]
(by Holder’s inequality)
\[ \leq A \left( \frac{1}{H}, \frac{1}{1-H} \right) \left( \int_0^T |f(u)|^{1/H} du \right)^{2H} \]
\[ (1.32) \]
for some positive constant \( A(1/H, 1/1-H) \). The last inequality follows from the Hardy–Littlewood inequality (cf. Stein (1971), p. 119) stated below in Proposition 1.8. Combining the inequalities in (1.31) and (1.32), the inequality (1.29) stated in Theorem 1.7 is proved.

**Proposition 1.8:** Let \( 0 < \alpha < 1 \) and \( 1 < p < q < \infty, 1/q = (1/p) - \alpha \). Suppose \( f \in L^p((0, \infty)) \). Define
\[ I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty |x-y|^\alpha f(y)dy. \]
\[ (1.33) \]
Then there exists a positive constant $A(p, q)$ such that
\[
||I^\alpha f||_{L^q((0,\infty))} \leq A(p, q)||f||_{L^p((0,\infty))}.
\] (1.34)

We do not prove this inequality. For details, see Stein (1971), pp. 117–120.

The results stated above can be reformulated in the following manner. Let \{\(W_t, t \geq 0\)\} be standard fBm with Hurst index $H > \frac{1}{2}$. Suppose a function $f : [0, T] \to \mathbb{R}$ satisfies the condition
\[
||f||_{L^{1/H}([0,T])} = \left( \int_0^T |f(s)|^{1/H} ds \right)^H < \infty.
\] (1.35)

Then the Wiener integral
\[
Y^H_t = \int_0^t f(s) dW^H_s, \quad 0 \leq t \leq T
\]
exists and is a mean zero Gaussian process, and for every $r > 0$, there exists a positive constant $c(r, H)$ such that
\[
E|Y^H_{t_2} - Y^H_{t_1}|^r \leq c(r, H) \left( \int_{t_1}^{t_2} |f(s)|^{1/H} ds \right)^{rH}, \quad 0 \leq t_1 \leq t_2 \leq T.
\]

The inequalities proved in Theorem 1.7 are due to Memin et al. (2001). Extensions of these inequalities are discussed in Slominski and Ziemkiewicz (2005).

We now discuss some maximal inequalities for fBm due to Novikov and Valkeila (1999). Let \{\(Z(t), t \geq 0\)\} be standard fBm. Let \(\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}\) be the filtration generated by fBm $Z$ where $\mathcal{F}_t$ is the $\sigma$-algebra generated by the family \{\(Z(s), 0 \leq s \leq t\)\}. For any process $X$, define $X^*_t = \sup_{0 \leq s \leq t} |X_s|$. Since fBm is a self-similar process, it follows that $Z(at) \overset{\Delta}{=} a^H Z(t)$ for every $a > 0$. This in turn implies that $Z^*_t \overset{\Delta}{=} a^H Z^*(t)$ for every $a > 0$. In particular, we have the following important result.

**Proposition 1.9:** Let $T > 0$ and $Z$ be fBm with Hurst index $H$. Then, for every $p > 0$,
\[
E[(Z^*_T)^p] = K(p, H)T^{pH}
\] (1.36)
where $K(p, H) = E[(Z^*_1)^p]$.

**Proof:** The result follows from the observation $Z^*_T \overset{\Delta}{=} T Z^*_1$ by self-similarity.

Novikov and Valkeila (1999) proved the following result. Recall that a random variable $\tau$ is said to be a stopping time with respect to the filtration $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ if the event $[\tau \leq t] \in \mathcal{F}_t$ for every $t \geq 0$. 

Proposition 1.10: Let $\tau$ be any stopping time with respect to the filtration $\mathcal{F}$ defined above. Then, for any $p > 0$ and $H > \frac{1}{2}$, there exist positive constants $c_1(p, H)$ and $c_2(p, H)$ depending only on the parameters $p$ and $H$ such that

$$c_1(p, H)E(\tau^{pH}) \leq E[(Z^*_\tau)^p] \leq c_2(p, H)E(\tau^{pH}).$$

This result is the analogue of the classical Burkholder–Davis–Gundy inequality for martingales. However, recall that fBm is not a semimartingale. We point out that if $\{B_t, t \geq 0\}$ is standard Brownian motion, then, for any stopping time $\tau$ with respect to the filtration generated by the Brownian motion $B$ and for any $p > 0$, there exist positive constants $c_1(p)$ and $c_2(p)$ depending only on $p$ such that

$$c_1(p)E(\tau^{p/2}) \leq E[(B^*_\tau)^p] \leq c_2(p)E(\tau^{p/2}).$$

For a proof of Proposition 1.10, see Novikov and Valkeila (1999).

Deconvolution of fBm

Samorodnitsky and Taqqu (1994) proved that there is an integral transformation of standard Brownian motion $W$ which provides a representation for standard fBm $W^H$ with Hurst index $H \in (0, 1)$. It is the moving average representation of fBm and is given by

$$\{W^H(t), -\infty < t < \infty\} \overset{\Delta}{=} \left\{ \int_{-\infty}^{\infty} f_H(t, u)dW(u), -\infty < t < \infty \right\} \quad (1.37)$$

in the sense that the processes on both sides have the same finite-dimensional distributions where

$$f_H(t, u) = \frac{1}{C_1(H)}((t - u)^{H - \frac{1}{2}} - (-u)^{H - \frac{1}{2}}) \quad (1.38)$$

and

$$C_1(H) = \left[ \int_0^\infty ((1 + u)^{H - \frac{1}{2}} - u^{H - \frac{1}{2}})^2 du + \frac{1}{2H} \right]^{1/2} \quad (1.39)$$

where $a^\alpha_+ = a^\alpha$ if $a > 0$ and $a^\alpha_+ = 0$ if $a \leq 0$. Pipiras and Taqqu (2002) obtained a generalization of this result. They proved the following theorem.

Theorem 1.11: Let $W^{H_1}$ and $W^{H_2}$ be two standard fBms with the Hurst indices $H_i \in (0, 1), i = 1, 2$ respectively. Further suppose that $H_1 \neq H_2$. Then

$$\{W^{H_2}(t), -\infty < t < \infty\} \overset{\Delta}{=} \left\{ \int_{-\infty}^{\infty} f_{H_1, H_2}(t, u)dW^{H_1}(u), -\infty < t < \infty \right\} \quad (1.40)$$
where
\[ f_{H_1, H_2}(t, u) = \frac{C_1(H_1) \Gamma(H_2 + \frac{1}{2})}{C_1(H_2) \Gamma(H_1 + \frac{1}{2}) \Gamma(H_2 - H_1 + 1)} ((t - u)^{H_2 - H_1} - (-u)^{H_2 - H_1}) \] (1.41)

with \( C_1(\frac{1}{2}) = 1 \).

By taking \( H_2 = \frac{1}{2} \) in the above theorem, we get the following deconvolution formula or autoregressive representation for fBm proved in Pipiras and Taqqu (2002).

**Theorem 1.12:** Let \( W^H \) be standard fBm with index \( H \in (0, 1) \) with \( H \neq \frac{1}{2} \). Let \( W \) be standard Brownian motion. Then
\[ \{ W(t), -\infty < t < \infty \} \overset{\Delta}{=} \left\{ \int_{-\infty}^{\infty} f_{H, \frac{1}{2}}(t, u) dW^H(u), -\infty < t < \infty \right\} \] (1.42)

where
\[ f_{H, \frac{1}{2}}(t, u) = \frac{C_1(H)}{\Gamma(H + \frac{1}{2}) \Gamma(\frac{3}{2} - H)} ((t - u)^{\frac{1}{2} - H} - (-u)^{\frac{1}{2} - H}). \] (1.43)

Let \( \mathcal{F}_{H,t} \) denote the \( \sigma \)-algebra generated by the process \( \{ W^H(s), 0 \leq s \leq t \} \) and \( \mathcal{F}_{\frac{1}{2},t} \) denote the \( \sigma \)-algebra generated by the process \( \{ W(s), 0 \leq s \leq t \} \). Pipiras and Taqqu (2002) proved that the inversion formula
\[ W(t) = \int_{-\infty}^{\infty} f_{H, \frac{1}{2}}(t, u) dW^H(u) \] (1.44)
holds for each \( t \in \mathbb{R} \) almost everywhere and hence the \( \sigma \)-algebras \( \mathcal{F}_{H,t} \) and \( \mathcal{F}_{\frac{1}{2},t} \) are the same up to sets of measure zero for \( t > 0 \). The equality in (1.44) does not hold for \( t < 0 \).

**The fundamental martingale**

We noted earlier that fBm \( Z \) is not a semimartingale and its paths are continuous and locally of unbounded variation a.s. but with zero quadratic variation whenever \( H > \frac{1}{2} \). However, we will now show that such fBm can be transformed into a martingale by an integral transformation following the work of Norros et al. (1999) (cf. Molchan (1969)). We will first prove a lemma dealing with some equations for integrals of fractional powers.

**Lemma 1.13:** Let \( B(\alpha, \beta) \) denote the beta function with parameters \( \alpha \) and \( \beta \) given by
\[ B(\alpha, \beta) = \int_0^1 x^{\alpha - 1} (1 - x)^{\beta - 1} dx. \] (1.45)
The following identities hold:

(i) For $\mu > 0$, $\nu > 0$ and $c > 1$,
$$
\int_0^1 t^{\mu-1} (1-t)^{\nu-1} (c-t)^{-\mu-\nu} dt = c^{-\nu} (c-1)^{-\mu} B(\mu, \nu). \tag{1.46}
$$

(ii) For $\mu \in \mathbb{R}$, $\nu > -1$ and $c > 1$,
$$
\int_1^c t^{\mu} (t-1)^{\nu} dt = \int_0^{1-1/c} s^{\nu} (1-s)^{-\mu-\nu-2} ds. \tag{1.47}
$$

(iii) Suppose that $\mu > 0$, $\nu > 0$ and $c > 1$. Then
$$
\int_0^1 t^{\mu-1} (1-t)^{\nu-1} (c-t)^{-\mu-\nu+1} dt
= B(\mu, 1-\mu) - (\mu + \nu - 1)B(\mu, \nu)
\times \int_0^{1-1/c} s^{-\mu} (1-s)^{\mu+\nu-2} ds \quad (\text{if } \mu < 1)
= (\mu + \nu - 1)B(\mu, \nu) c^{-\nu+1} \int_0^1 s^{\nu-2} (c-s)^{-\mu} ds \quad (\text{if } \mu + \nu > 1). \tag{1.48}
$$

(iv) For $\mu \in (0, 1)$ and for $x \in (0, 1)$,
$$
\int_0^1 t^{-\mu} (1-t)^{-\mu} |x-t|^{2\mu-1} dt = B(\mu, 1-\mu). \tag{1.49}
$$

We give a proof of this lemma following Norros et al. (1999).

**Proof:** The identities in (i) and (ii) can be proved by using the transformations $t = cs/(c-1+s)$ and $t = 1/(1-s)$ respectively. We now prove (iii).

Suppose $\mu < 1$. Then
$$
\int_0^1 t^{\mu-1} (1-t)^{\nu-1} (c-t)^{-\mu-\nu+1} dt
= \int t^{\mu-1} (1-t)^{\nu-1} \left[ (1-t)^{-\mu-\nu+1} + (-\mu - \nu + 1) \int_1^c (v-t)^{-\mu-\nu} dv \right] dt
= B(\mu, 1-\mu) - (\mu + \nu - 1) \int_1^c \left[ \int_0^1 t^{\mu-1} (1-t)^{\nu-1} (v-t)^{-\mu-\nu} dt \right] dv
= B(\mu, 1-\mu) - (\mu + \nu - 1) B(\mu, \nu) \int_1^c v^{-\nu} (v-1)^{-\mu} dv \quad \text{(by (i))}
= B(\mu, 1-\mu) - (\mu + \nu - 1) B(\mu, \nu) \int_0^{1-1/c} s^{\mu} (1-s)^{\mu+\nu-2} ds \quad \text{(by (ii))}.
$$
\[ B(\mu, 1 - \mu) - (\mu + v - 1)B(\mu, v) \times \left[ B(1 - \mu, \mu + v - 1) - \int_0^{1/c} s^{\mu+v-2}(1-s)^{-\mu} ds \right] \]

\[ = (\mu + v - 1)B(\mu, v) \int_0^{1/c} s^{\mu+v-2}(1-s)^{-\mu} ds. \] (1.50)

The last equality follows from the identity

\[ (\mu + v - 1)B(\mu, v)B(1 - \mu, \mu + v - 1) = B(\mu, 1 - \mu). \] (1.51)

Since the first term and the last term in (1.50) are analytic in \( \mu \) for \( \mu > 0 \), the statement in (iii) holds for all \( \mu > 0 \). The last result given in (iv) follows from (iii) and (1.51) by elementary but tedious calculations. We omit the details.

Let \( \Psi \) denote the integral operator

\[ \Psi f(t) = H(2H - 1) \int_0^\infty f(s)|s - t|^{2H-2} ds. \]

**Lemma 1.14:** Suppose that \( H > \frac{1}{2} \). Let \( w(t, s) \) be the function

\[ w(t, s) = c_1 s^{\frac{1}{2} - H}(t - s)^{\frac{1}{2} - H} \quad \text{for} \quad 0 < s < t \]

\[ = 0 \quad \text{for} \quad s > t \] (1.52)

where

\[ c_1 = \left[ 2H \Gamma \left( \frac{3}{2} - H \right) \Gamma \left( H + \frac{1}{2} \right) \right]^{-1}. \] (1.53)

Then

\[ \Psi w(t, .)(s) = 1 \quad \text{for} \quad s \in [0, t) \]

\[ = \frac{(H - \frac{1}{2}) s^{H-\frac{1}{2}}}{(\frac{3}{2} - H)B(H + \frac{1}{2}, 2 - 2H)} \int_0^t u^{1-2H}(s - u)^{H-\frac{3}{2}} du \quad \text{for} \quad s > t. \] (1.54)

**Proof:** Recall that \( H > \frac{1}{2} \). For \( s \in [0, t] \), the result follows from (iv) of Lemma 1.13 by choosing \( \mu = H - \frac{1}{2} \). For \( s > t \), the result is obtained from (iii) of Lemma 1.13.

Let

\[ M_t = \int_0^t w(t, s) dZ_s, \quad t \geq 0. \] (1.55)
Theorem 1.15: The process \( \{M_t, t \geq 0\} \) is a mean zero Gaussian process with independent increments with \( E(M_t^2) = c_2 t^{2-2H} \) where

\[
c_2 = \frac{c_H}{2H(2 - 2H)^{1/2}}
\]

and

\[
c_H = \left[ \frac{2H \Gamma \left( \frac{3}{2} - H \right)}{\Gamma \left( H + \frac{1}{2} \right) \Gamma (2 - 2H)} \right]^{1/2}
\]

In particular, the process \( \{M_t, t \geq 0\} \) is a zero mean martingale.

Proof: From the properties of Wiener integrals, it follows that the process \( \{M_t, t \geq 0\} \) is a Gaussian process with mean zero. Suppose that \( s < t \). In view of Lemma 1.14, it follows that

\[
cov(M_s, M_t) = \langle w(s, .), \Psi w(t.) \rangle
\]

\[
= \langle w(s, .), I_{[0,t]} \rangle
\]

\[
= \int_0^s w(s, u) du
\]

\[
= c_1 B \left( \frac{3}{2} - H, \frac{3}{2} - H \right) s^{2-2H}
\]

\[
= c_2^2 s^{2-2H}. \quad (1.58)
\]

Here \( \langle ., . \rangle \) denotes the usual inner product in \( L^2((0, \infty)) \). Note that the last term is independent of \( t \) which shows that the process \( M \) has uncorrelated increments. Since it is a mean zero Gaussian process, the increments will be mean zero independent random variables. Let \( \mathcal{F}_t \) denote the \( \sigma \)-algebra generated by the random variables \( \{Z_s, 0 \leq s \leq t\} \). It is now easy to see that the process \( \{M_t, \mathcal{F}_t, t \geq 0\} \) is a zero mean Gaussian martingale.

The martingale \( M \) defined above is called the fundamental martingale associated with \( fBm \).

It is easy to check that the process

\[
W_t = \frac{2H}{c_H} \int_0^t s^{H-\frac{1}{2}} dM_s
\]

is standard Brownian motion. Stochastic integration with respect to a martingale is defined in the obvious way as in the case of Ito integrals. For details, see Prakasa Rao (1999b).
Furthermore, for $0 \leq s \leq t$,
\[
\text{cov}(Z_s, M_t) = \langle I_{[0,s]}, \Psi w(t, .) \rangle \\
= \langle I_{[0,s]}, I_{[0,t]} \rangle \\
= s. \quad (1.60)
\]

In particular, it follows that the increment $M_t - M_s$ is independent of $\mathcal{F}_s$ for $0 \leq s \leq t$. Let
\[
Y_t = \int_0^t s^{\frac{1}{2} - H} dZ_s.
\]

Observing that the process $\{Y_t, t \geq 0\}$ is a Gaussian process, it can be seen that
\[
Z_t = \int_0^t s^{H - \frac{1}{2}} dY_s.
\]

In fact, the process $Y$ generates the same filtration $\{\mathcal{F}_t, t \geq 0\}$ as the filtration generated by the process $Z$. It can be shown that
\[
E[M_t Y_T] = \frac{c^2}{2H} \int_0^t (T - s)^{H - \frac{1}{2}} s^{1 - 2H} ds \text{ if } t < T \\
= \frac{T^{\frac{3}{2} - H}}{\frac{3}{2} - H} \text{ if } t \geq T \quad (1.61)
\]

For proof, see Proposition 3.2 in Norros et al. (1999). We leave it to the reader to check that
\[
M_t = \frac{c}{2H} \int_0^t s^{\frac{1}{2} - H} dW_s. \quad (1.62)
\]

The martingale $M$ is the fundamental martingale associated with fBm $Z$ in the sense that the martingale $M$ generates, up to sets of measure zero, the same filtration as that generated by the process $Z$. Furthermore, the same holds for the related processes $W$ and $Y$ defined above. In fact, the process $Y$ has the representation
\[
Y_T = 2H \int_0^T (T - t)^{H - \frac{1}{2}} dM_t
\]
and the martingale $M_t$ can be represented in the form
\[
M_t = c_1 \int_0^t (t - s)^{\frac{1}{2} - H} dY_s.
\]

For detailed proofs of these results, see Norros et al. (1999). Let $\{\mathcal{F}_t^W\}$ denote the filtration generated by the process $W$ defined by (1.59). It is known that all
right-continuous square integrable \( \{ \mathcal{F}_t^W \} \)-martingales can be expressed as stochastic integrals with respect to the process \( W \). Since the filtrations \( \{ \mathcal{F}_t^W \} \) and \( \{ \mathcal{F}_t \} \) coincide up to sets of measure zero, it follows that all the right-continuous square integrable \( \{ \mathcal{F}_t \} \)-martingales can also be expressed as stochastic integrals with respect to \( W \).

**Baxter-type theorem for fBm**

In a fairly recent paper on the estimation of the Hurst index for fBm, Kurchenko (2003) derived a Baxter-type theorem for fBm.

Let \( f : (a, b) \to \mathbb{R} \) be a function and let \( k \) be a positive integer. Let \( \Delta_h^{(k)} f(t) \) denote the increment of \( k \)-th order of the function \( f \) in an interval \([t, t+h] \subset (a, b)\) as defined by

\[
\Delta_h^{(k)} f(t) = \sum_{i=0}^{k} (-1)^i k C_i f \left( t + \frac{i}{k} h \right).
\]

For any \( m \geq 0 \), positive integer \( k \geq 1 \) and \( 0 < H < 1 \), define

\[
V_k(m, H) = \frac{1}{2} \sum_{i,j=0}^{k} (-1)^{i+j+1} k C_i k C_j \left| m + \frac{i - j}{k} \right|^{2H}.
\]

It can be checked that \( V_1(0, H) = 1 \) and \( V_2(0, H) = 2^{2-2H} - 1 \). Note that

\[
\Delta_1^{(2)} f(t) = f(t) - 2 f \left( t + \frac{1}{2} \right) + f(t + 1).
\]

Kurchenko (2003) proved the following Baxter-type theorem for second-order increments for fBm among several other results.

**Theorem 1.16**: Let \( \{ W_H(t), t \geq 0 \} \) be standard fBm with Hurst index \( H \in (0, 1) \) as defined above. Then, with probability one,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} (\Delta_1^{(2)} W_H(m))^2 = V_2(0, H) \text{ a.s.}
\]

In other words,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} \left[ W_H(m) - 2 W_H \left( m + \frac{1}{2} \right) + W_H(m + 1) \right]^2 = V_2(0, H) \text{ a.s.}
\]

for any standard fBm with Hurst index \( H \in (0, 1) \).

For a proof of Theorem 1.16, see Kurchenko (2003).
Singularity of fBms for different Hurst indices

It is well known that if $P$ and $Q$ are probability measures generated by two Gaussian processes, then these measures are either equivalent or singular with respect to each other (cf. Feldman (1958), Hajek (1958)). For a proof, see Rao (2000), p. 226.

Let $\{W_{H_i}(t), t \geq 0\}, i = 1, 2,$ be two standard fBms with Hurst indices $H_1 \neq H_2$. From the result stated above, it follows that the probability measures generated by these processes are either equivalent or singular with respect to each other. We will now prove that they are singular with respect to each other (cf. Prakasa Rao (2008c)).

**Theorem 1.17:** Let $\{W_{H_i}(t), t \geq 0\}, i = 1, 2,$ be two standard fBms with Hurst indices $H_1 \neq H_2$. Let $P_i$ be the probability measure generated by the process $\{W_{H_i}(t, t \geq 0)\}$ for $i = 1, 2$. Then the probability measures $P_1$ and $P_2$ are singular with respect to each other.

**Proof:** Applying Theorem 1.16, we obtain that

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} \left[ W_{H_i}(m) - 2W_{H_i} \left( m + \frac{1}{2} \right) + W_{H_i}(m + 1) \right]^2 = V_2(0, H_i) \text{ a.s.}\[P_i]\text{, } i = 1, 2.
$$

Since $V_2(0, H_1) \neq V_2(0, H_2)$ if $H_1 \neq H_2$, and since the convergence stated above is a.s. convergence under the corresponding probability measures, it follows that the measures $P_1$ and $P_2$ are singular with respect to each other.

Long-range dependence

Suppose $\{Z(t), -\infty < t < \infty\}$ is fBm with Hurst index $H$ for some $0 < H < 1$. Define $X_k = Z(k+1) - Z(k)$ for any integer $k \in R$. The process $\{X_k\}$ is called fractional Gaussian noise. Since the process $Z$ is $H$-ss with stationary increments, it is easy to see that the discrete time parameter process $\{X_k, -\infty < k < \infty\}$ is stationary with mean $E(X_k) = 0$, $E(X_k^2) = E(Z^2(1)) = \sigma^2$ (say) and the auto covariance

$$
\gamma(k) = E(X_k X_{k+i}) = \gamma(-k) = \frac{\sigma^2}{2} (|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}). \quad (1.63)
$$

Suppose $k \neq 0$. Then

$$
\gamma(k) = \begin{cases} 
0 & \text{if } H = \frac{1}{2} \\
< 0 & \text{if } 0 < H < \frac{1}{2} \\
> 0 & \text{if } \frac{1}{2} < H < 1.
\end{cases} \quad (1.64)
$$
This can be checked from the strict convexity of the function \( f(x) = x^{2H}, x > 0, \) for \( \frac{1}{2} < H < 1 \) and strict concavity of the function \( f(x) = x^{2H}, x > 0, \) for \( 0 < H < \frac{1}{2} \). Furthermore, if \( H \neq \frac{1}{2} \), then

\[
\gamma(k) \approx \sigma^2 H (2H - 1) |k|^{2H - 2}
\]
as \( |k| \to \infty \). In particular, \( \gamma(k) \to 0 \) as \( |k| \to \infty \) if \( 0 < H < 1 \). Observe that, if \( \frac{1}{2} < H < 1 \), then \( \sum_{k=-\infty}^{\infty} \gamma(k) = \infty \) and the process \( \{X_k, -\infty < k < \infty\} \) exhibits long-range dependence. In this case, the auto covariance tends to zero but so slowly that the sum of auto covariances diverges. If \( 0 < H < \frac{1}{2} \), then \( \sum_{k=-\infty}^{\infty} \gamma(k) < \infty \). This is the reason why the class of processes, driven by fBm with Hurst index \( H > \frac{1}{2} \), is used for modeling phenomena with long-range dependence.

### 1.4 Stochastic differential equations driven by fBm

#### Fundamental semimartingale

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t), P) \) be a stochastic basis satisfying the usual conditions. The natural filtration of a process is understood as the \( P \)-completion of the filtration generated by this process.

Let \( W^H = \{W^H_t, t \geq 0\} \) be standard fBm with Hurst parameter \( H \in (0, 1) \), that is, a Gaussian process with continuous sample paths such that \( W^H_0 = 0, E(W^H_t) = 0 \) and

\[
E(W^H_s W^H_t) = \frac{1}{2}[s^{2H} + t^{2H} - |s - t|^{2H}], t \geq 0, s \geq 0.
\]  
(1.65)

Let us consider a stochastic process \( Y = \{Y_t, t \geq 0\} \) defined by the stochastic integral equation

\[
Y_t = \int_0^t C(s) ds + \int_0^t B(s) dW^H_s, t \geq 0
\]  
(1.66)

where \( C = \{C(t), t \geq 0\} \) is an \( (\mathcal{F}_t) \)-adapted process and \( B(t) \) is a non vanishing, non random function. For convenience, we write the above integral equation in the form of a stochastic differential equation

\[
dY_t = C(t) dt + B(t) dW^H_t, t \geq 0
\]  
(1.67)

driven by fBm \( W^H \). Recall that the stochastic integral

\[
\int_0^t B(s) dW^H_s
\]  
(1.68)

is not a stochastic integral in the Ito sense, but one can define the integral of a deterministic function with respect to fBm as the integrator in a natural sense.
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(cf. Gripenberg and Norros (1996), Norros et al. (1999)) as we discussed earlier. Even though the process \( Y \) is not a semimartingale, one can associate a semimartingale \( Z = \{ Z_t, t \geq 0 \} \) which is called a fundamental semimartingale such that the natural filtration \( (\mathcal{F}_t) \) of the process \( Z \) coincides with the natural filtration \( (\mathcal{\mathcal{Y}}_t) \) of the process \( Y \) (Kleptsyna et al. (2000a)). Define, for \( 0 < s < t \),

\[
k_H = 2H \Gamma \left( \frac{3}{2} - H \right) \Gamma \left( H + \frac{1}{2} \right),
\]

\[
k_H(t, s) = k_H^{-1} s^{\frac{1}{2} - H} (t - s)^{\frac{1}{2} - H},
\]

\[
\lambda_H = \frac{2H \Gamma(3 - 2H) \Gamma(H + \frac{1}{2})}{\Gamma(\frac{3}{2} - H)}
\]

\[
w_H^t = \lambda_H^{-1} t^{2 - 2H},
\]

and

\[
M_H^t = \int_0^t k_H(t, s) dW_s^H, \quad t \geq 0.
\]

The process \( M_H \) is a Gaussian martingale, called the fundamental martingale (cf. Norros et al. (1999)), and its quadratic variation \( \langle M_H^t \rangle = w_H^t \). Furthermore, the natural filtration of the martingale \( M_H \) coincides with the natural filtration of fBm \( W^H \). In fact the stochastic integral

\[
\int_0^t B(s) dW_s^H
\]

can be represented in terms of the stochastic integral with respect to the martingale \( M_H \). For a measurable function \( f \) on \([0, T]\), let

\[
K_H^f(t, s) = -2H \frac{d}{ds} \int_s^t f(r) r^{H - \frac{1}{2}} (r - s)^{H - \frac{1}{2}} dr, \quad 0 \leq s \leq t
\]

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (see Samko et al. (1993) for sufficient conditions). The following result is due to Kleptsyna et al. (2000a).

**Theorem 1.18:** Let \( M_H \) be the fundamental martingale associated with fBm \( W^H \). Then

\[
\int_0^t f(s) dW_s^H = \int_0^t K_H^f(t, s) dM_s^H, \quad t \in [0, T]
\]

a.s. \([P]\) whenever both sides are well defined.
Suppose the sample paths of the process \( \{ C(t)/B(t), t \geq 0 \} \) are smooth enough (see Samko et al. (1993)) so that

\[
Q_H(t) = \frac{d}{dw_H} \int_0^t k_H(t, s) \frac{C(s)}{B(s)} ds, t \in [0, T]
\]

is well defined, where \( w_H \) and \( k_H(t, s) \) are as defined in (1.72) and (1.70) respectively and the derivative is understood in the sense of absolute continuity. The following theorem due to Kleptsyna et al. (2000a) associates a fundamental semimartingale \( Z \) associated with the process \( Y \) such that the natural filtration \( (\mathcal{Y}_t) \) coincides with the natural filtration \( (\mathcal{Z}_t) \) of \( Y \).

**Theorem 1.19:** Suppose the sample paths of the process \( Q_H \) defined by (1.77) belong \( P \)-a.s. to \( L^2([0, T], dw_H) \) where \( w_H \) is as defined by (1.72). Let the process \( Z = \{ Z_t, t \in [0, T] \} \) be defined by

\[
Z_t = \int_0^t k_H(t, s)[B(s)]^{-1} dY_s
\]

where the function \( k_H(t, s) \) is as defined in (1.70). Then the following results hold:

(i) The process \( Z \) is an \( (\mathcal{F}_t) \)-semimartingale with the decomposition

\[
Z_t = \int_0^t Q_H(s)dw_H + M_t^H
\]

where \( M^H \) is the fundamental martingale defined by (1.73).

(ii) The process \( Y \) admits the representation

\[
Y_t = \int_0^t K_B^H(t, s)dZ_s
\]

where the function \( K_B^H(., .) \) is as defined in (1.75).

(iii) The natural filtrations \( (\mathcal{Z}_t) \) and \( (\mathcal{Y}_t) \) coincide.

Kleptsyna et al. (2000a) derived the following Girsanov-type formula as a consequence of Theorem 1.19.

**Theorem 1.20:** Suppose the assumptions of Theorem 1.19 hold. Define

\[
\Lambda_H(T) = \exp \left( -\int_0^T Q_H(t) dM_t^H - \frac{1}{2} \int_0^T Q_H^2(t) dw_H^t \right).
\]

(1.81)
Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process $Y$ under $P^*$ is the same as that of the process $V$ defined by

$$V_t = \int_0^t B(s)dW^H_s, \quad 0 \leq t \leq T$$

(1.82)

under the $P$-measure.

**Stochastic differential equations**

It is possible to define the stochastic integral of a random process $\{\sigma(t, X_t), t \geq 0\}$ with respect to fBm $W^H$ as the integrator for some class of stochastic processes and to define a stochastic differential equation of the type

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW^H_t, \quad X_0, t > 0.$$ 

Sufficient conditions for the existence and uniqueness of solutions of such stochastic differential equations driven by fBm are discussed in Mishura (2008), p. 197. We do not go into the details here. The following result due to Nualart and Rascanu (2002) gives sufficient conditions for the existence and uniqueness of the solution.

For any $\lambda \in (0, 1]$, let $C^\lambda[0, T]$ be the space of continuous functions $f$ defined on the interval $[0, T]$ such that

$$\sup_{0 \leq x_1 \neq x_2 \leq T} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\lambda} < \infty.$$ 

Define the norm

$$||f||_{C^\lambda} = \max_{x \in [0, T]} |f(x)| + \sup_{0 \leq x_1 \neq x_2 \leq T} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\lambda} < \infty$$

on the space $C^\lambda[0, T]$. Let

$$C^{\mu}\tau = \cap_{\lambda < \mu} C^\lambda[0, T].$$

Define

$$C_0 = \frac{1}{2} \left[ \left( H - \frac{1}{2} \right) H(1 - H) B\left( \frac{3}{2} - H, \frac{3}{2} - H \right) B\left( H - \frac{1}{2}, \frac{3}{2} - H \right) \right]^{-1/2}$$

(1.83)

and

$$C_1 = C_0 B\left( \frac{3}{2} - H, \frac{3}{2} - H \right).$$

(1.84)
Let
\[ z(t, u) = C_0 u^{1-H} (t - u)^{\frac{1}{2}-H} \]  
and
\[ w(t, u) = C_0 u^{3-H} (t - u)^{\frac{1}{2}-H}. \]  

Norros et al. (1999) proved that
\[ M_t = \int_0^t z(t, u) dW^H_u \]  
is well defined as a pathwise integral and is a martingale with respect to the filtration \( \{ F_t, t \geq 0 \} \) generated by the process \( W^H \). The quadratic variation of the martingale \( M \) is
\[ \langle M \rangle_t = t^{2-2H}. \]
Furthermore,
\[ W_t = \int_0^t u^{H-\frac{1}{2}} dM_u \]
is a Wiener process \( W \) adapted to the same filtration.

**Theorem 1.21:** Let a function \( S : [0, T] \times R \to R \) be such that:

(i) for all \( N \geq 0 \), there exists \( L_N > 0 \) such that
\[ |S(t, x) - S(t, y)| \leq L_N |x - y|, |x| \leq N, |y| \leq N, 0 \leq t \leq T; \]  
(ii) and there exists \( M > 0 \) such that
\[ |S(t, x)| \leq M (1 + |x|), x \in R, 0 \leq t \leq T. \]

Then the stochastic integral equation
\[ X_t = x_0 + \int_0^t S(u, X_u) du + \epsilon W^H_t, 0 \leq t \leq T \]  
or equivalently the stochastic differential equation
\[ dX_t = S(t, X_t) dt + \epsilon dW^H_t, X_0 = x_0, 0 \leq t \leq T \]
has a unique solution \( \{ X_t, 0 \leq t \leq T \} \) and the sample paths of this process belong to \( C^{H-}[0, T] \) with probability one.

**Theorem 1.22:** Suppose that the function \( S(t, x) \) satisfies the conditions stated in Theorem 1.21. Furthermore, suppose that the constant \( L_N \) in Equation (1.89)
does not depend on $N$, that is, $L_N = L$ for some $L$ for every $N \geq 1$. Let $\{X_t, 0 \leq t \leq T\}$ be the solution of Equation (1.92) and

$$x_t = x_0 + \int_0^t S(u, x_u)du, \quad 0 \leq t \leq T. \quad (1.93)$$

Then

$$\sup_{0 \leq t \leq T} |X_t - x_t| \leq \epsilon C \sup_{0 \leq t \leq T} |W_t^H| \quad (1.94)$$

where $C = e^{LT}$.

This inequality is a consequence of the Gronwall lemma (see Chapter 5).

**Absolute continuity of measures**

Consider the stochastic differential equations (SDEs)

$$dX_t = S_i(t, X_t)dt + \epsilon dW_t^H, \quad X_0 = x_0, \quad 0 \leq t \leq T, \quad i = 1, 2. \quad (1.95)$$

Suppose that sufficient conditions are satisfied by the functions $S_i$ so that there exist unique solutions for the SDEs defined above. Let $X^i$ be the solution of the equation for $i = 1, 2$. Let $P_T^i$ be the probability measure generated by the process $X^i$ on the space $C[0, T]$ associated with the Borel $\sigma$-algebra induced by the supremum norm on the space $C[0, T]$. The following theorem, due to Androshchuk (2005), gives sufficient conditions under which the probability measures $P_T^i, i = 1, 2$, are equivalent to each other and gives a formula for the Radon–Nikodym derivative. An alternate form for the Radon–Nikodym derivative via the fundamental semimartingale is discussed in Theorem 1.20.

**Theorem 1.23:** Suppose the functions $S_i(t, x), i = 1, 2,$ satisfy the following conditions: (i) $S_i(t, x) \in C^1([0, T] \times R)$; (ii) there exists a constant $M > 0$ such that $|S_i(t, x)| \leq M(1 + |x|), x \in R, 0 \leq t \leq T$. Then Equation (1.95) has unique solutions for $i = 1, 2$ and these solutions belong to $C^H- [0, T]$ a.s. In addition, the probability measures $P_T^i, i = 1, 2$, are absolutely continuous with respect to each other and

$$\frac{dP_T^2}{dP_T^1}(X^1) = \exp\left(\frac{1}{\epsilon}L_T - \frac{1}{2\epsilon^2}(L_T^2)\right) \quad (1.96)$$

where

$$L_T = \int_0^T \left\{ (2 - 2H)t^{1/2-H} \right. \times \left( C_1 \Delta S(0, x_0) + \int_0^t u^{2H-3} \left[ \int_0^u w(u, v) d(\Delta S(v, X^1_u)) \right] du \right)$$

$$+ \left. t^{H-\frac{3}{2}} \int_0^t w(t, u) d(\Delta S(u, X^1_u)) \right\} dW_t \quad (1.97)$$
with
\[ \Delta S(t, x) = S_2(t, x) - S_1(t, x), \] (1.98)

\[ C_1 \] as defined by (1.84), \( w(t, u) \) as given in (1.86) and \( W \) the Wiener process constructed from the process \( W^H \) using (1.88).

We omit the proof of this theorem. For details, see Androshchuk (2005).

### 1.5 Fractional Ornstein–Uhlenbeck-type process

We now study the fractional analogue of the Ornstein–Uhlenbeck process, that is, a process which is the solution of a one-dimensional homogeneous linear SDE driven by fBm \( W^H \) with Hurst index \( H \in \left( \frac{1}{2}, 1 \right) \).

Langevin (1908) suggested the following method to study the movement of a particle immersed in a liquid. He modeled the particle’s velocity \( v \) by the equation

\[ \frac{dv(t)}{dt} = -\frac{f}{m} v(t) + F(t) \] (1.99)

where \( m \) is the mass of the particle, \( f > 0 \) is a friction coefficient and \( F(t) \) is the fluctuating force resulting from the impact of the particles with the surrounding medium. Uhlenbeck and Ornstein (1930) studied a random version of the model by treating \( F(t), t \geq 0 \), as a random process and then derived that, for \( v(0) = x \), the random variable \( v(t) \) has a normal distribution with mean \( xe^{-\lambda t} \) and variance \( (\sigma^2/2\lambda)(1 - e^{-2\lambda t}) \) for \( \lambda = f/m \) and \( \sigma^2 = 2f k T / m^2 \) where \( k \) is the Boltzmann constant and \( T \) is the temperature. Doob (1942) observed that, if \( v(0) \) is a Gaussian random variable with mean zero and variance \( \sigma^2/2\lambda \) independent of the stochastic process \( \{F(t), t \geq 0\} \), then the solution \( \{v(t), t \geq 0\} \) of (1.99) is stationary and the process

\[ \left\{ I_{\{t > 0\}} t^{1/2}v \left( \frac{1}{2\lambda} \log t \right), t \geq 0 \right\} \]

is Brownian motion. Since the sample paths of Brownian motion are nowhere differentiable a.s., the differential equation (1.99) has to be interpreted as a stochastic integral equation formulation or as a stochastic differential equation of the form

\[ dX_t = -\lambda X_t dt + dW_t, \quad X_0 = x, \quad t \geq 0 \] (1.100)

where \( \{W_t, t \geq 0\} \) is Brownian motion. It can be shown that this equation has the unique solution

\[ X_t = e^{-\lambda t} \left( x + \int_0^t e^{\lambda s} dW_s \right), t \geq 0 \] (1.101)
by an application of Ito’s lemma (cf. Prakasa Rao (1999a)). Such a process is called the Ornstein–Uhlenbeck process. In analogy with this formulation, consider the SDE

$$dX_t = -\lambda X_t dt + \sigma dW^H_t, \quad X_0 = 0, \quad t \geq 0$$  \hspace{1cm} (1.102)

where $\lambda$ and $\sigma^2$ are constants. The existence and uniqueness of the solution of this SDE are discussed in Cheridito et al. (2003). This process is called a fractional Ornstein–Uhlenbeck-type process.

**Existence and uniqueness**

**Theorem 1.24:** Let $\{W^H_t, -\infty < t < \infty\}$ be fBm with index $H \in (0, 1]$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and $X(0, \omega) = \eta(\omega) \in R$. Let $-\infty \leq a < \infty$ and $\lambda, \sigma > 0$. Then, for almost every $\omega \in \Omega$, the following hold:

(a) for all $t > a$,

$$\int_a^t e^{\lambda u} dW^H_u(\omega)$$

exists as a Riemann–Stieltjes integral and is equal to

$$e^{\lambda t} W^H_t(\omega) - e^{\lambda a} W^H_a(\omega) - \lambda \int_a^t W^H_u(\omega) e^{\lambda u} du;$$

(b) the function

$$\int_a^t e^{\lambda u} dW^H_u(\omega), \quad t > a$$

is continuous in $t$; and

(c) the unique continuous function $x(t)$, that is, the solution of the integral equation

$$x(t) = \eta(\omega) - \lambda \int_0^t x(s) ds + \sigma W^H_t(\omega), \quad t \geq 0$$

or equivalently of the SDE

$$dX(t) = -\lambda X(t) dt + \sigma dW^H_t, \quad X(0) = \eta, \quad t \geq 0$$

is given by

$$x(t) = e^{-\lambda t} \left[ \eta(\omega) + \sigma \int_0^t e^{\lambda u} dW^H_u(\omega) \right], \quad t \geq 0.$$  

In particular, the unique continuous solution of the equation

$$x(t) = \sigma \int_{-\infty}^0 e^{\lambda u} dW^H_u(\omega) - \lambda \int_0^t x(s) ds + \sigma W^H_t(\omega), \quad t \geq 0$$
is given by

\[ x(t) = \sigma \int_{-\infty}^{t} e^{-\lambda(t-u)}dW_{u}^{H}(\omega), \ t \geq 0. \]

For a proof of this theorem, see Cheridito et al. (2003).

Let

\[ Y_{t}^{H,\eta} = e^{-\lambda t} \left( \eta + \sigma \int_{0}^{t} e^{\lambda u}dW_{u}^{H} \right) \]

where the stochastic integral is defined as the pathwise Riemann–Stieltjes integral. As a consequence of the above theorem, it follows that \( \{Y_{t}^{H,\eta}, \ t \geq 0\} \) is the unique a.s. continuous process that is a solution of the SDE

\[ dX(t) = -\lambda X(t)dt + \sigma dW_{t}^{H}, \ t \geq 0, \ X(0) = \eta. \] (1.103)

In particular, the process

\[ Y_{t}^{H} = \sigma \int_{-\infty}^{t} e^{-\lambda(t-u)}dW_{u}^{H}, \ 0 \leq t < \infty \]

is the a.s. continuous solution of (1.103) with the initial condition

\[ Y_{0}^{H} = \eta = \sigma \int_{-\infty}^{0} e^{\lambda u}dW_{u}^{H}. \]

Note that the process \( \{Y_{t}^{H}, -\infty < t < \infty\} \) is a Gaussian process and is a stationary process as the increments of fBm \( W^{H} \) are stationary. Furthermore, for every \( \eta \),

\[ Y_{t}^{H} - Y_{t}^{H,\eta} \overset{\Delta}{=} e^{-\lambda t}(Y_{0}^{H} - \eta) \]

and

\[ e^{-\lambda t}(Y_{0}^{H} - \eta) \to 0 \ \text{a.s. as} \ \ t \to \infty. \]

The process \( \{Y_{t}^{H,\eta}, \ t \geq 0\} \) is a fractional Ornstein–Uhlenbeck-type process with the initial condition \( \eta \) and \( \{Y_{t}^{H}, \ t \geq 0\} \) is a stationary fractional Ornstein–Uhlenbeck-type process. It can be checked that, for any fixed \( t, s \),

\[
\text{cov}(Y_{t}^{H}, Y_{t+s}^{H}) = \text{cov}(Y_{0}^{H}, Y_{s}^{H})
\]

\[ = E \left[ \sigma^{2} \int_{-\infty}^{0} e^{\lambda u}dW_{u}^{H} \int_{-\infty}^{s} e^{-\lambda(s-v)}dW_{v}^{H} \right] \]

\[ = \frac{1}{2} \sigma^{2} \sum_{n=1}^{N} \lambda^{-2n} (\Pi_{k=0}^{2n-1} (2H - k))s^{2H-2n} + O(s^{2H-2N-2}) \] (1.104)
and the last equality holds as $s \to \infty$ (for details, see Cheridito et al. (2003)). It can be shown that the process $\{Y_t^H, -\infty < t < \infty\}$ is ergodic and exhibits long-range dependence for $H \in (\frac{1}{2}, 1]$.

### 1.6 Mixed fBm

Cheridito (2001) introduced the class of mixed fBms. They are linear combinations of different fBms. We first consider a special case.

Let $a$ and $b$ be real numbers not both zero. **Mixed fractional Brownian motion (mfBm)** of parameters $a$, $b$ and $H$ is a process $M^H_t(a, b) = \{M^H_t(a, b), t > 0\}$ defined by

$$M^H_t(a, b) = aW_t + bW^H_t, \quad t > 0$$

where $\{W_t, t \geq 0\}$ is Brownian motion and the process $\{W^H_t, t \geq 0\}$ is independent fBm with Hurst index $H$. It is easy to check that the process $M^H(a, b)$ is a zero mean Gaussian process with $E[(M^H_t(a, b))^2] = a^2t + b^2t^{2H}$, and

$$\text{cov}(M^H_t(a, b), M^H_s(a, b)) = a^2 \min(t, s) + \frac{1}{2}b^2(|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

$t > 0, s > 0$.

Furthermore, the increments of the process $M^H(a, b)$ are stationary and it is mixed-self-similar in the sense that, for any $h > 0$,

$$\{M^H_t(a, b)\} \overset{\Delta}{=} \{M^H_t(ah^{1/2}, bh^H)\}.$$  

Recall that the notation $\{X_t\} \overset{\Delta}{=} \{Y_t\}$ means that the processes specified on both sides have the same finite-dimensional distributions.

**Proposition 1.25:** For all $0 < H < 1$, $H \neq \frac{1}{2}$, and $b \neq 0$, the process $M^H(a, b)$ is not a Markov process.

**Proof:** For notational convenience, we write $M^H$ for $M^H(a, b)$. If the process $M^H$ is Markov, then, for all $s < t < u$,

$$\text{cov}(M^H_s, M^H_u) \text{ var}(M^H_t) = \text{cov}(M^H_s, M^H_t) \text{ cov}(M^H_t, M^H_u)$$

from the results in Revuz and Yor (1991). It can be checked that the above identity does not hold, for instance, for $s = \frac{1}{2}, t = 1$ and $u = \frac{3}{2}$ whenever $0 < H < 1$ and $H \neq \frac{1}{2}$. For details, see Zili (2006).

Suppose $b \neq 0$. We leave it to the reader to check that the increments of the process $M^H$ are long-range dependent if $H > \frac{1}{2}$. 

Proposition 1.26: For all $T > 0$ and $0 < \gamma < \min(H, \frac{1}{2})$, the mixed fBm $M^H$ has a version of the process with sample paths which are Holder-continuous of order $\gamma$ in the interval $[0, T]$ with probability one.

Proof: Let $\alpha > 0$ and $0 \leq s \leq t \leq T$. By the stationarity of the increments and mixed-self-similarity of the process $M^H$, it follows that

$$E(|M^H_t - M^H_s|^{\alpha}) = E(|M^H_{t-s}|^{\alpha}) = E(|M^H_{t-s}|^{\alpha/2} b(t-s)^\gamma)^{\alpha/2}) \quad (1.105)$$

(i) Suppose $H \leq \frac{1}{2}$. Then, there exist positive constants $C_1$ and $C_2$ depending on $\alpha$ such that

$$E(|M^H_t - M^H_s|^{\alpha}) \leq (t-s)^{\alpha H} E(|M^H_{t-s}|^{\alpha/2}) b(t-s)^{\gamma} \leq (t-s)^{\alpha H} [C_1 |a|^{\alpha} (t-s)^{\alpha(\frac{1}{2} - H)} E|W_1|^{\alpha} + C_2 |b|^{\alpha} E(|W_1^H|^{\alpha})] \leq C' \gamma (t-s)^{\alpha H} \quad (1.106)$$

where

$$C' = C_1 |a|^{\alpha} (t-s)^{\alpha(\frac{1}{2} - H)} E|W_1|^{\alpha} + C_2 |b|^{\alpha} E(|W_1^H|^{\alpha})$$

(ii) Suppose $H > \frac{1}{2}$. Then, there exist positive constants $C_3$ and $C_4$ depending on $\alpha$ such that

$$E(|M^H_t - M^H_s|^{\alpha}) \leq (t-s)^{\alpha/2} E(|M^H_{t-s}|^{\alpha/2} b(t-s)^{H-\frac{1}{2}})^{\alpha/2}) \leq (t-s)^{\alpha/2} [C_3 |a|^{\alpha} E(|W_1|^{\alpha}) + C_4 |b|^{\alpha} (t-s)^{\alpha(H-\frac{1}{2})} E(|W_1^H|^{\alpha})] \leq C' \gamma (t-s)^{\alpha/2} \quad (1.107)$$

where

$$C' = C_3 |a|^{\alpha} E(|W_1|^{\alpha}) + C_4 |b|^{\alpha} (t-s)^{\alpha(H-\frac{1}{2})} E(|W_1^H|^{\alpha})$$

Hence, for every $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$E(|M^H_t - M^H_s|^{\alpha}) \leq C_\alpha |t-s|^{\alpha \min(\frac{1}{2}, H)}.$$
where \( \{W_t^{H_1}, t \geq 0\} \) and \( \{W_t^{H_2}, t \geq 0\} \) are independent fBms with Hurst indices \( H_1 \) and \( H_2 \) respectively, and studied the properties of such processes. For details, see Miao et al. (2008).

Suppose \( b \neq 0 \). Cheridito (2001) proved that the mixed fBm \( M_t^H(a, b) \) is not a semimartingale if \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \). Furthermore, it is equivalent to a multiple of the Wiener process if \( H = \frac{1}{2} \) and equivalent to a Wiener process if \( H \in (\frac{3}{4}, 1] \). For details, see Cheridito (2001).

A function \( f(t), t \geq 0 \), is said to belong to the lower class of a process \( X \) defined on a probability space \( (\Omega, \mathcal{F}, P) \) if, for almost all \( \omega \in \Omega \), there exists a function \( t_0 = t_0(\omega) \) such that \( X(t) \geq f(t) \) for every \( t > t_0 \). El-Nouty (2001, 2002) characterized such classes for fBm and extended the results to fractional mixed fBm in El-Nouty (2003a) and to integrated fBm in El-Nouty (2003b).

### 1.7 Donsker-type approximation for fBm with Hurst index \( H > \frac{1}{2} \)

Let \( Z = \{Z(t), t \geq 0\} \) be standard fBm with Hurst index \( H > \frac{1}{2} \). Norros et al. (1999) obtained the following kernel representation of the process \( Z \) with respect to standard Brownian motion \( W \):

\[
Z(t) = \int_0^t z(t, s)dW_s
\]  

where

\[
z(t, s) = c_H \left( H - \frac{1}{2} \right) s^{\frac{1}{2} - H} \int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{1}{2}}du, \quad s \leq t
\]  

with

\[
c_H = \left( \left[ 2H \Gamma \left( \frac{3}{2} - H \right) \right] / \left[ \Gamma \left( \frac{1}{2} + H \right) \Gamma(2 - 2H) \right] \right)^{1/2}.
\]

The function \( z(t, s) \) is defined to be zero if \( s \geq t \). We now briefly discuss an analogue of the Donsker-type approximation theorem for fBm as a limit of a random walk. This result is due to Sottinen (2001).

Let \( \psi_i^{(n)} \) be independent and identically distributed (i.i.d.) random variables with mean zero and variance one and define

\[
W_i^{(n)} = n^{-1/2} \sum_{i=1}^{[nt]} \psi_i^{(n)}
\]

where \([x]\) denotes the greatest integer not exceeding \( x \). Donsker’s theorem states that the process \( \{W_i^{(n)}, t \geq 0\} \) converges weakly to the standard Brownian
motion $W$ (cf. Billingsley (1968)). Let

$$Z^{(n)}_t = \int_0^t \left( \sum_{i=1}^{[nt]} \int_{(i-1)/n}^{i/n} z \left( \frac{[nt]}{n}, s \right) ds \right) (n^{-1/2} \psi_i^{(n)})$$

where $z(t, s)$ is the kernel that transforms standard Brownian into fBm. Note that the function $z^{(n)}(t, .)$ is an approximation to the function $z(t, .)$, namely,

$$z^{(n)}(t, s) = n \int_{s-1/n}^s z \left( \frac{[nt]}{n}, u \right) du.$$

Sottinen (2001) proved that the random walk $Z^{(n)}$ converges weakly to standard fBm with index $H$. For a detailed proof, see Sottinen (2001).

Weak convergence to fBm was also investigated by Beran (1994) and Taqqu (1975). The approximation schemes discussed by them involve Gaussian random variables. Dasgupta (1998) obtained approximations using binary random variables and the representation of fBm due to Mandelbrot and Van Ness (1968). Sottinen’s approximation scheme discussed above used i.i.d. random variables with finite variance.

### 1.8 Simulation of fBm

We mentioned earlier that the increments of fBm with Hurst index $H$ form a sequence called fractional Gaussian noise and this sequence exhibits long-range dependence whenever $\frac{1}{2} < H < 1$. Mandelbrot and Wallis (1969) provided a discrete approximation to fBm and Mandelbrot (1971) suggested a fast algorithm for simulating the fractional Gaussian noise. We now describe a few methods to simulate paths of fBm. Dieker (2004) has given an extensive discussion on this topic comparing different methods of simulation. Our remarks here are based on Dieker (2004).

#### Willinger, Taqqu, Sherman and Wilson method

The following method for the simulation of fBm paths is due to Willinger et al. (1997). Suppose there are $S$ i.i.d. sources transmitting packets of information. Each source $s$ has active and inactive periods modeled by a stationary time series $\{J^{(s)}(t), t \geq 0\}$ where $J^{(s)}(t) = 1$ if the source is sending a packet at time $t$ and $J^{(s)}(t) = 0$ if the source is not sending a packet at time $t$. Suppose the lengths of the active (‘ON’) periods are i.i.d. and those of the inactive (‘OFF’) periods are also i.i.d., and the lengths of ON and OFF periods are independent.
An OFF period follows an ON period and the ON and OFF period lengths may have different distributions. Rescaling time by a factor $T$, let

$$J_S(Tt) = \int_0^{Tt} \left[ \sum_{s=1}^{S} J^{(s)}(u) \right] du$$

be the aggregated cumulative packet counts in the interval $[0, t]$. Suppose the distributions of the ON and OFF periods are Pareto with parameter $1 < \alpha < 2$. Recall that a random variable $X$ has the Pareto distribution with parameter $\alpha > 0$ if $P(X > t) = t^{-\alpha}$ for $t \geq 1$. Note that the ON and OFF periods have infinite variance under this distribution when $1 < \alpha < 2$. Willinger et al. (1997) proved that

$$\lim_{T \to \infty} \lim_{S \to \infty} T^{-H} S^{-1/2} \left( J_S(Tt) - \frac{1}{2} TSt \right) = \sigma W_H(t)$$

for some $\sigma > 0$ where $H = (3 - \alpha/2)$ and $W_H(t)$ denotes fBm with Hurst index $H$. In other words, the random variable $J_S(Tt)$ closely resembles

$$\frac{1}{2} TSt + T^H \sqrt{S\sigma} W_H(t)$$

which is fractional Brownian traffic with mean $M = \frac{1}{2} T S$ and variance coefficient $a = 2\sigma^2 T^{2H-1}$. We say that $A(t) = Mt + \sqrt{aM} W_H(t)$ is fractional Brownian traffic with mean input rate $M > 0$ and variance coefficient $a$. The process $A(t)$ represents the number of bits (or data packets) that is transmitted in the time interval $[0, t]$.

The method given above can be used for simulation of fBm by aggregating a large number of sources with Pareto ON and OFF periods.

**Decreusfond and Lavaud method**

Decreusefond and Lavaud (1996) suggested the following method for the simulation of an fBm sample. Recall that fBm $\{W_H(t), t \geq 0\}$ can be represented in the form

$$W_H(t) = \int_0^t K_H(t, s)dW(s)$$

for a suitable function $K_H(t, s)$ where $\{W(t), t \geq 0\}$ is Brownian motion. Suppose that we need an fBm sample in the interval $[0, 1]$. Let $t_j = j/N, j = 0, 1, \ldots, N$. We estimate $W_H(t_j)$ at $t_j$ by the formula

$$W_H(t_j) = \sum_{i=0}^{j} \frac{1}{t_{i+1} - t_i} \left[ \int_{t_i}^{t_{i+1}} K_H(t_j, s)ds \right] (W(t_{i+1}) - W(t_i)). \quad (1.113)$$
Note that the integral
\[ \int_{t_i}^{t_{i+1}} K_H(t_j, s) ds \]
cannot be approximated by \( K_H(t_j, t_i) \) or \( K_H(t_j, t_{i+1}) \) since the function \( K_H(t_j, t) \) is not continuous with respect to \( t \) in \([0, t_j]\).

**Dzhaparidze and van Zanten method**

Dzhaparidze and van Zanten (2004) obtained a series expansion for fBm. This series involves the positive zeroes \( x_1 < x_2 < \ldots \) of the Bessel function \( J_{-H} \) of the first kind of order \(-H\) and the positive zeroes \( y_1 < y_2 < \ldots \) of the Bessel function \( J_{1-H} \). Then
\[
W_H(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1 - \cos(y_n t)}{y_n} Y_n. \tag{1.114}
\]

Equality holds in Equation (1.114) in the sense that random processes on both sides have the same finite-dimensional distributions. The random variables \( X_i, i \geq 1 \), and \( Y_i, i \geq 1 \), are independent Gaussian random variables with mean zero and with the variances given by
\[
\text{Var}(X_n) = 2C_H^2 x_n^{-2H} J_{1-H}^{-2}(x_n),
\]
\[
\text{Var}(Y_n) = 2C_H^2 y_n^{-2H} J_{-H}^{-2}(y_n),
\]
and
\[
C_H^2 = \frac{1}{\pi} \Gamma(1 + 2H) \sin(\pi H).
\]

Furthermore, the series on the right of (1.114) converge absolutely and uniformly in \( t \in [0, 1] \) a.s. The series expansion in (1.114) generalizes the result on Karhunen–Loeve-type expansion for Brownian motion. The representation (1.114) can be used for simulating an fBm sample from Gaussian samples. This method is useful as there are efficient algorithms to compute the zeroes of Bessel functions. Furthermore, the zeroes have to be computed only once regardless of the number of samples to be simulated. For computational purposes, the series on the right of (1.114) have to be truncated at some level \( N \). Dzhaparidze and van Zanten (2003) proved that
\[
\limsup_{N \to \infty} \frac{N^H}{\sqrt{\log N}} E \left[ \sup_{0 \leq t \leq 1} \left| \sum_{n > N} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n > N} \frac{1 - \cos(y_n t)}{y_n} Y_n \right| \right] < \infty.
\]

Kuhn and Linde (2002) proved the rate \( N^{-H} \sqrt{\log N} \) is the ‘best’ possible.
1.9 Remarks on application of modeling by fBm in mathematical finance

Geometric Brownian motion is used for modeling stock prices in the theory of mathematical finance but it was empirically observed that the model may not be suitable due to several reasons, including the fact that the log-share prices may follow long-range dependence. In recent years, fBm has been suggested as a replacement for Brownian motion as the driving force in modeling various real-world phenomena, including the modeling of stock prices. Absence of arbitrage, that is, the impossibility of receiving a risk-less gain by trading in a market, is a basic assumption or a condition that underlies all modeling in financial mathematics. For, if there is a strategy that is feasible for investors and promises a risk-less gain, then the investors would like to buy this strategy and will not sell. By the law of demand and supply, the price of this strategy would increase immediately, indicating that the market is not in equilibrium. Hence the absence of arbitrage is a basic requirement of any useful pricing model. See Rogers (1997) and Bender et al. (2006).

The first fundamental theorem of asset pricing (Delbaen and Schachermayer (1994)) links the no-arbitrage property to the martingale property of the discounted stock price process under a suitable pricing measure. Since fBm is not a semimartingale, except when $H = \frac{1}{2}$ (the Brownian motion case), the stock price process driven by fBm cannot be transformed into a martingale in general by an equivalent change of measure. Hence the fundamental theorem rules out these models as sensible pricing models.

Hu and Oksendal (2003) and Elliott and van der Hock (2003) suggested a fractional Black–Scholes model as an improvement over the classical Black–Scholes model using the notion of a Wick integral. Necula (2002) studied option pricing in a fractional Brownian environment using the Wick integral. Common to these fractional Black–Scholes models is that the driving force is fBm but the stochastic integral used is interpreted as the Wick integral. It was shown by these authors that the fractional Black–Scholes models are arbitrage free, contradicting earlier studies that the fractional Black–Scholes models do admit arbitrage. Bjork and Hult (2005) have, however, pointed out that the notion of self-financing trading strategies and the definition of value used by Hu and Oksendal (2003) and others, using the Wick integral, do not have a reasonable economic interpretation.

1.10 Pathwise integration with respect to fBm

Zahle (1998) developed the theory of pathwise integration with respect to fBm when the Hurst index $H > \frac{1}{2}$ using the methods of fractional calculus.
We will now briefly discuss these results. Zahle (1998) extends the classical Lebesgue–Stieltjes integral
\[ \int_a^b f(x) \, dg(x) \]
for real or complex-valued functions on a finite interval \((a, b)\) to a large class of integrands \(f\) and integrators \(g\) of unbounded variation. The techniques used are composition formulas and integration by parts rules for fractional integrals and fractional derivatives (cf. Samko et al. (1993)).

Note that if \(f\) or \(g\) is a smooth function on a finite interval \((a, b)\), the Lebesgue–Stieltjes integral can be written in the form
\[
\int_a^b f(x) \, dg(x) = \int_a^b f(x) g'(x) \, dx
\]
or
\[
\int_a^b f(x) \, dg(x) = -\int_a^b f'(x) g(x) \, dx + f(b-) g(b-) - f(a+) g(a+).
\]
Here \(f(a+) = \lim_{\delta \to 0} f(a + \delta)\) and \(g(b-) = \lim_{\delta \to 0} f(b - \delta)\) whenever the limits exist. The main idea of Zahle’s approach is to replace the ordinary derivatives by the fractional derivatives. Let
\[
f_{a+}(x) = (f(x) - f(a+)) 1_{(a,b)}(x)
\]
and
\[
g_{b-}(x) = (g(x) - g(b-)) 1_{(a,b)}(x)
\]
where \(1_{(a,b)}(x) = 1\) if \(x \in (a, b)\) and \(1_{(a,b)}(x) = 0\) otherwise. For a function \(f \in L_1(R)\) and \(\alpha > 0\), define
\[
I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha-1} f(y) \, dy
\]
and
\[
I_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y - x)^{\alpha-1} f(y) \, dy
\]
where \(\Gamma(.)\) is the gamma function. For \(p \geq 1\), let \(I_{a+}^\alpha(L_p)\) be the class of functions \(f\) which may be represented as \(I_{a+}^\alpha\)-integral for some function \(\phi\) in \(L_p(R)\). Similarly, let \(I_{b-}^\alpha(L_p)\) be the class of functions \(f\) which may be represented as \(I_{b-}^\alpha\)-integral for some function \(\phi\) in \(L_p(R)\). If \(p > 1\), then \(f \in I_{a+}^\alpha(L_p)\) if and only if \(f \in L_p(R)\) and the integrals
\[
\int_a^x \frac{f(x) - f(y)}{(x - y)^{\alpha+1}} \, dy
\]
converge in $L_p(R)$ as a function of $x$ as $\epsilon \downarrow 0$ defining $f(y) = 0$ if $x$ is not in $(a, b)$. Similarly, $f \in I^\alpha_b(L_p)$ if and only if $f \in L_p(R)$ and the integrals

$$
\int_{x+\epsilon}^{b} \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy
$$

converge in $L_p(R)$ as a function of $x$ as $\epsilon \downarrow 0$ defining $f(y) = 0$ if $x$ is not in $[a, b]$ (cf. Samko et al. (1993)).

Suppose $f_{a+} \in I^{\alpha}_{a+}(L_p)$ and $g_{b-} \in I^{\alpha}_{b-}(L_q)$ for some $1/p + 1/q \leq 1$ and $0 \leq \alpha \leq 1$. Define the integral

$$
\int_{a}^{b} f(x) \, dg(x) = (-1)^{\alpha} \int_{a}^{b} D^\alpha_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x) \, dx 
+f(a+)(g(b-) - g(a+)) \tag{1.115}
$$

for some $0 \leq \alpha \leq 1$ where

$$
D^\alpha_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left[ \frac{f(x)}{(x-a)^\alpha} + \alpha \int_{0}^{x} \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} \, dy \right] 1_{(a,b)}(x)
$$

and

$$
D^\alpha_{b-} f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left[ \frac{f(x)}{(b-x)^\alpha} + \alpha \int_{x}^{b} \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} \, dy \right] 1_{(a,b)}(x)
$$

and the convergence of the integrals above at the singularity $y = x$ holds pointwise for almost all $x$ if $p = 1$ and in the $L_p(R)$-sense if $p > 1$. It can be shown that the integral defined by Equation (1.115) is independent of the choice of $\alpha$ (cf. Zahle (1998)). Furthermore, for $\alpha p < 1$, the function $f_{a+} \in I^{\alpha}_{a+}(L_p)$ if and only if $f \in I^{\alpha}_{a+}(L_p)$ and $f(a+)$ exists.

It was pointed out earlier that fBm $W^H$ with Hurst index $H$ has a version with sample paths of Holder exponent $H$, that is, of Holder continuity of all orders $\lambda < H$, in any finite interval $[0, T]$ with probability one. Holder continuity implies existence pointwise of the integral (1.115) and hence of the integral

$$
\int_{0}^{t} f(s) \, dW^H_s, \ 0 \leq t \leq T
$$

with probability one for any function $f$ defined in the interval $[0, T]$ such that $f_{0+} \in I^{\alpha}_{0+}(L_1(0, T))$ with probability one for some $\alpha > 1-H$. Note that there is no condition where the function $f$ has to be adapted with respect to the filtration generated by the process $W^H$. Let $H^\lambda(0, T)$ be the family of functions which are Holder continuous of order $\lambda$ in the interval $[0, T]$. Suppose $\lambda > 1-H$. Then we can interpret the integral as a Riemann–Stieltjes integral and use the change-of-variable formula given below. In particular, we may define the stochastic integral with respect to $W^H$ for functions of the form $f(t) = \sigma(t, X(t))$ for some real-valued Lipschitz function $\sigma(., .)$ and any stochastic process with sample paths in
$H^\lambda(0, T)$ with probability one for some $\lambda > 1 - H$. Since $H > \frac{1}{2}$, it is possible to study SDEs of the type

$$dX(t) = aX(t)dt + bX(t)dW^H_t, t \geq 0$$

(1.116)

or equivalently

$$X(t) = X(0) + a \int_0^t X(s)ds + b \int_0^t X(s)dW^H_s, t \geq 0.$$  

(1.117)

It can be shown that the solution of the above SDE is

$$X(t) = X(0) \exp[at + bW^H_t], t \geq 0.$$  

(1.118)

**Change-of-variable formula**

It is known that the chain rule

$$dF(f(x)) = F'(f(x))df(x)$$

does not hold for functions $f$ of Holder exponent $\frac{1}{2}$ arising as sample paths of stochastic processes which are semimartingales. However, for functions of Holder exponent greater than $\frac{1}{2}$, the classical formula remains valid in the sense of Riemann–Stieltjes integration. The following change-of-variable formula can be proved (cf. Zahle (1998)).

**Theorem 1.27:** If $f \in H^\lambda(a, b)$ and $F \in C^1(R)$ is a real-valued function such that $F'(f(.)) \in H^\mu(a, b)$ for some $\lambda + \mu > 1$, then, for any $y \in (a, b)$,

$$F(f(y)) - F(f(a)) = \int_a^y F'(f(x))df(x).$$  

(1.119)

**Remarks:** The conditions in Theorem 1.27 will be satisfied if $f \in H^\lambda(a, b)$ for some $\lambda > \frac{1}{2}$ and $F \in C^1(R)$ with Lipschitzian derivative.

Theorem 1.27 can be extended to a more general version in the following way.

**Theorem 1.28:** Suppose $f \in H^\lambda(a, b)$, $F \in C^1(R \times (a, b))$ and $F'_1(f(.), .) \in H^\mu(a, b), \lambda + \mu > 1$. Then

$$F(f(y), y) - F(f(a), a) = \int_a^y F'_1(f(x), x)dx + \int_a^y F'_2(f(x), x)df(x)$$  

(1.120)

where $F'_1$ and $F'_2$ are the partial derivatives of $f$ with respect to the first and second variable respectively.
As an example, suppose that $f \in H^\lambda(a, b)$ for some $\lambda > \frac{1}{2}$ and $F(u) = u^2$. Then

$$\int_a^y f(x)df(x) = \frac{1}{2}(f^2(y) - f^2(a)).$$

As an application of the change-of-variable formula, it follows that

$$\int_x^y W_t^H dW_t^H = \frac{1}{2}[ (W_y^H)^2 - (W_x^H)^2 ], 0 \leq x \leq y < \infty$$

with probability one provided $H > \frac{1}{2}$. 