Two fruits upon one tree

1. The continuation of the Early Draft into philosophy of mathematics

The constructional history of the Investigations (see Volume I, Part II, ‘The history of the composition of the Philosophical Investigations’) is a matter of interest not only to chroniclers of the history of ideas. It bears directly on features of Wittgenstein’s thought. For it raises a general question about his later philosophy, namely: how can two such disparate fruits as his philosophy of logic and mathematics and examination of the nature of necessity, on the one hand, and his investigations into the possibility of a private language, knowledge of other minds, the inner and the outer, the nature of thinking, imagination, consciousness and the self, on the other, grow from the same trunk of Investigations §§1–189? For the Frühfassung (the Early Draft) of 1937–8 consisted of TS 220 (corresponding roughly to PI §§1–189(a)) and TS 221 (corresponding roughly to RFM I). It was only in 1944 that Wittgenstein decided to drop the logico-mathematical continuation from the book, and to allow the material on following rules to evolve not into a discussion of the nature of necessity (as in TS 221), but rather into the private language arguments and their sequel up to §421 (which was the terminus of the Zwischenfassung (the Intermediate Draft) of 1944). This incorporated a mere eight pages from TS 221, corresponding to PI §§189–97, which was then followed by new material composed in 1944. Furthermore, in the Spätfassung (the Late Draft) of 1946–7 Wittgenstein added yet further material (including §§422–693) incorporating inter alia discussions of topics in the philosophy of psychology (such as intentionality, the will, intention), which for the most part bear, directly or indirectly, on the general theme of the nature of linguistic representation.

What motivated this metamorphosis? What did he conceive to be the relationship between philosophy of mathematics and philosophy of psychology? What can we learn from this realignment about his tactics, strategy and grand-strategy? Clearly many issues can be probed. Our ambitions here are limited to a textual and a methodological question. The textual question concerns the integration of the remarks and the structure of the chain of reasoning in the published text in the light of the divergent earlier continuation. The methodological question concerns parallelisms between Wittgenstein’s philosophy of mathematics and his philosophy of psychology.
In comparison with standard works on philosophy of mathematics, Part I of the *Remarks on the Foundations of Mathematics* is unusual. Philosophers of mathematics might expect discussions of Russell’s paradox, of how to construct real numbers from the rationals, of the acceptability of indirect proofs, or of the cogency of mathematical induction. Here we find none of this. Instead Wittgenstein investigated the concepts of proof and inference, compared calculations with experiments, juxtaposed logical with legal compulsion, demythologized logical, mathematical and metaphysical necessity, and so on. His points are illustrated with very elementary examples, such as $25 \times 25 = 625$, or simple diagrammatic proofs of equations. Only the innumerate would have difficulty following these.

Apart from what the editors conjecture to be projected appendices on Cantor’s theory of infinity, Gödel’s incompleteness proof, Russell’s logic and logicist definitions of natural numbers, Wittgenstein avoided the subject called ‘the foundations of mathematics’ (an amalgam of formal logic, number theory and real analysis). He incorporated no mathematical or metamathematical results into his work and he criticized attempts to extract philosophical theses from such proofs. This has evoked hostility among philosophers of mathematics and logicians, who have often misinterpreted this methodological disagreement as a manifestation of ignorance of sophisticated mathematics or even of philistineism. He abstained from frontal attacks on standard positions in philosophy of mathematics, neither allying himself with logicists, intuitionists or formalists nor lining up against them under some other banner (strict finitism or constructivism). But in the course of his investigations into mathematical concepts he made devastating criticisms, almost *en passant*, of each of the familiar triad. His reluctance to locate himself in some available pigeonhole has not diminished others’ enthusiasm for doing so. Whatever positive conception he had, must, it has sometimes been thought, either be a synthesis of the three\(^1\) or be a purified version of one of them.\(^2\)

Disregarding the themes of the three so-called appendices, Wittgenstein explored five interrelated topics in the text of the original continuation of the Early Draft beyond what is now PI §189:

(i) **Inference** Inferring or drawing a conclusion is not a mental process or act, but a transformation of expressions according to paradigms. It is not answerable to something external but is a movement within grammar. Rules of inference do not flow from the meanings of the logical constants, but rather are constitutive of these meanings. To explain that $fa$ follows from $(x)fx$ is to give a partial explanation of what the universal quantifier means, and to explain

\(^1\) In the Manifesto of the Vienna Circle (1929), the signatories (Hahn, Neurath and Carnap) assert that ‘Some hold that the three views are not so far apart as it seems. They surmise that essential features of all three will come closer in the course of future development and probably, using the far-reaching ideas of Wittgenstein, will be united in the ultimate solution.’ (See *The Scientific Conception of the World: The Vienna Circle* (Dordrecht, Holland, 1973), p. 13.)

\(^2\) For example, intuitionism purged of psychologism.
the nature of an inference is to teach someone the technique of inferring (drawing conclusions, reasoning, thinking).

(ii) Proof and calculation

Proofs in mathematics are commonly conflated with proofs (inferences) outside mathematics (and logic). But the ‘diplomatic’ or ‘imperial’ role of a mathematical proof or proposition, the role that gives point to the body of mathematics, is to supply a paradigm for the transformation of empirical statements, i.e. to establish a pattern of inference. The nature of mathematics is obscured by the fact that we express mathematical results in the form of declarative sentences; but we might carry on mathematics without thinking that we were dealing with propositions at all (RFM 93, 117). We construe mathematical proofs as demonstrations of propositions from other propositions, but this too is inessential, since a proof may consist of a diagram or geometrical construction to which the concepts of premises, conclusions and inference are inapplicable (MS 161, 6, see Exg. §144). It is also misleading in suggesting that a mathematical proposition is fully intelligible independently of its proof; but a conclusion is best conceived as the end surface of a proof-body (AWL 10). A proof establishes internal relations; it connects concepts and thereby contributes to the determination of their identity. Proofs and calculations are thus radically unlike experiments (empirical verifications).

(iii) Essence and convention

Precisely because a mathematical proof establishes internal relations, it also creates essence (RFM 50). For in fixing novel internal relations, a proof extends the grammar of number or space. By extending grammar, essences are modified and created – normative (conceptual) connections are determined. The mathematician is an inventor, not a discoverer (RFM 99), for he determines new relations among concepts. To accept a proof is to accept a new rule, a new convention – which the proof has woven into the tapestry of mathematics. Indeed, talk about essences is no more than noting conventions (RFM 65). For ‘internal properties’ are marks of concepts (RFM 64). They seem adamantine, unassailable, independent of the vagaries of fortune – but their unassailability is that of shadows (RFM 74). Statements of essences seem to describe the language-independent, de re necessities of things, but they are actually determinations of forms of description and of forms of transformations of descriptions of things.

(iv) Logical compulsion

Logical inference seems inexorable. Anybody who believes the premises of a valid argument seems to be logically compelled to believe the conclusion. Misleading pictures surround the ‘hardness of the logical “must”’, for example, that the conclusion is somehow already contained in its premises, that ‘the mathematical proof drives us along willy-nilly’ until we arrive at the conclusion, or that if we wish to ‘remain faithful’ to the concepts we are employing, then logic compels us to accept the theorem as proven – we have no more choice in the matter than does the appropriately programmed computer. But, Wittgenstein argues, these pictures are no more than mythological representations of necessity. They are to be countered by clarifying the
concept of inference. We are instructed in the techniques of inferring and compelled by our teachers and peers to adhere to them. A specification of what conclusion to draw from given premises is an intrinsic feature of the technique of inference. Hence we are not forced by logic to draw a conclusion – no matter what conclusion we draw, logic will not seize us by the throat! That such-and-such a conclusion follows (or even: ‘follows necessarily’) is not a form of compulsion. Rather, we are constrained in our judgements about what is to be called ‘a correct inference’. Wittgenstein’s account of inference does not derogate from the inexorability of logic. It merely eliminates misconceptions of it.

(v) The natural history of mankind – regularity and the role of agreement The practices of inference, proof, calculation and reasoning presuppose a ramifying network of regularities in nature and human behaviour. We typically respond similarly to patterns of instruction in arithmetical techniques. We normally agree in our judgements when applying such techniques. Mathematicians rarely quarrel over whether something is a proof. Certain patterns or resemblances are memorable for us, whereas we have ‘blind spots’ for other possibilities or similarities. Such regularities of agreement and response are not parts of (do not define) our concept of proof or inference. Rather, these regularities are part of the framework within which we exercise these concepts. Without them our language-games would lose their point. Hence, in clarification of our concepts it is useful to note our established patterns of action and speech, our form of life or culture, and also other contrasting ones, whether real or imaginary.

Even this brief survey discloses affinities between the early mathematical continuations of the Early Draft and the published text. Both give central positions to clarifying the concepts of a rule, of correctness according to a rule, and of following a rule. Wittgenstein argued that arithmetical equations and geometrical theorems should be viewed not as descriptions of numbers and shapes, but as rules the general point and purpose of which lies in the transformation of empirical propositions concerning, for example, magnitudes, quantities, distances, velocities and spatial relationships of things. The immediate rationale of the initial continuation of the Early Draft is the elucidation of the concepts of mathematical proposition, mathematical necessity and the nature of proof for the purpose of removing prevalent philosophical confusions. This task is interwoven with the project of illuminating central normative3 concepts characteristic not only of mathematics but of language use in general. Only the latter material has direct parallels in the published text.

At the cost of a certain amount of rearrangement and perhaps some supplementation of the early text, it seems that Wittgenstein could have separated out the remarks on rules, accord with a rule and following rules, and then treated these as a preface to his discussion of logical inference and mathematical proof. The result might have been something like the published text of

3 By ‘normative’ we mean merely ‘pertaining to a norm (rule)’.
§§189–242, followed by material on philosophy of mathematics rather than on the ‘private language’ and its sequel. Apparently something roughly like this idea occurred to him, probably in 1943 or 1944. Under the heading ‘Plan’ he outlined this programme:

How can the rule determine what I have to do?
To follow a rule presupposes agreement.
It is essential to the phenomenon of language that we do not dispute about certain things.
How can agreement be a condition of language? . . . Were agreement lacking, i.e. were we to be unable to bring our expressions into agreement, then the phenomena of communication and language would disappear too.
In what does the inexorability of mathematics consist?
The way goes from what is not inexorable to inexorability. The word ‘oben’ has four sounds.
Is a mathematical proof an experiment? (MS 165, 30ff.)

Here the envisaged argument has the same general contour as §§189–242, but it then diverges into a discussion of mathematics in order to remove the objection that describing agreement in judgements as a presupposition of language is inconsistent with the inexorability of mathematical propositions and proofs. Apparently Wittgenstein contemplated deploying remarks on inference, proof and logical compulsion in order to show in detail that the need for agreement in judgement does not abolish logic though it seems to do so, i.e. that acknowledging this framework condition for language is not incompatible with recognizing the hardness of the logical ‘must’ (in so far as this is intelligible; cf. ‘Agreement in definitions, judgements and forms of life’ and ‘Grammar and necessity’.) Consequently we might view the continuation of the Early Draft as an important complement to the final version. In elaborating the implications of agreement, it removes potential misunderstandings.

2. Hidden isomorphism

It is a moot question how a uniform foundation – a variant of Investigations §§1–189 – can underlie each of two such divergent extensions. How is it possible that a single chain of reasoning should lead smoothly into remarks on mathematics and logic, or alternatively into the private language arguments and discussions of psychological concepts? One response, natural in the light of the foregoing observations, would be that one must beware of exaggerating the divergence and of overlooking the shared features. As just noted, in both texts Wittgenstein discussed the internal relation of a rule to its applications, the autonomy of grammar, and the role of agreement as a framework-condition for following rules. Should one not view the final discussion of following rules as part of the shared nucleus of the two extensions? Then the
Two fruits upon one tree

early extension can be seen as exploring one main objection to the conceptual role assigned to agreement, viz. that logic and mathematics (or more generally, logical necessity) would be undermined. And the later extension can be viewed as examining a second fundamental objection, viz. that a language is conceivable (a ‘private language’) independently of even the possibility of agreement. Accordingly the two divergent continuations complement each other, and each graft fits perfectly on to the same trunk.

An alternative response, antithetical to the first, would beware of exaggerating the shared features. The pivot on which both continuations turn is a set of remarks about following a rule; but is the pivot identical? There seems to be a shift of emphasis in the discussion of following a rule between the Remarks on the Foundations of Mathematics, Part I and the Investigations. Wittgenstein’s focus of attention moved towards a sharper concentration upon the framework-conditions of rule-governed activities. Investigations §§185–242 stands at the culmination of this development, incorporating and moving onwards from the scrutiny of the internal relations between a rule and its application that began in The Big Typescript. One must not take for granted that there is no substantial evolution here, that the examination of following a rule in Part I of the Remarks has not been deepened and enriched in the Investigations. A fortiori one may not argue that the earlier extension of the Early Draft and Investigations §§243ff. are extensions of a homogeneous set of ideas dubbed ‘Wittgenstein’s rule-following considerations’.

Both responses are justified. We shall not attempt to adjudicate the question of what degree of continuity obtains and how extensive a change occurred. For our purposes, the important point to stress is that the text of the Early Draft was written after much reflection on philosophy of mathematics and on such topics in the philosophy of psychology that are pertinent to an overview of language and linguistic meaning. It was informed by a unified conception of philosophy and philosophical methods. It was expressly designed to highlight sources of philosophical confusion rampant both in philosophy of mathematics and in those parts of philosophy of psychology that are pertinent to the nature of thought, language and linguistic representation. Under the rubric of the Augustinian conception of language, the Early Draft drew together and surveyed a wide range of points that Wittgenstein had already made in earlier writings. The two divergent continuations fit on to this common foundation because it was crafted to support either, not as a result of personal idiosyncrasy, but for deeper philosophical reasons. We shall justify this claim by examining briefly two of the main ingredients of the Augustinian conception and by elucidating the manner in which it prevents philosophical understanding.

(i) Descriptions It is part of the Augustinian conception of language to take it for granted that the fundamental role of sentences, certainly of declarative sentences, is to describe something (see Volume 1, Part I: ‘The Augustinian conception of language’, §1 and §2(e)). Philosophers often begin their reflections from this presupposition. ‘Eight is greater than five’ is presumed to
be a description, and the serious philosophical question is what it describes—a relation between abstract objects, very general empirical facts or mental constructs? Similarly, ‘I have a toothache’ is assumed to be a description and the philosopher’s task is to determine whether it describes a private experience, a behaviour pattern or a brain state. This common presumption is a fundamental source of confusions about mathematics and the mind.

Mathematical propositions must be distinguished from descriptions. We should view them as instruments (AWL 157) and enquire into their roles, their use in practice (PR 134). We shall then see that their characteristic use is as rules (LFM 82, 246; AWL 154; PG 347) for transforming empirical propositions, or, more generally, as rules for framing descriptions (see ‘Grammar and necessity’, §§2, 4, 13). If one keeps the role of mathematical propositions in mind, then one will not mistake them for descriptions, just as ‘you can’t mistake a broom for part of the furnishing of a room as long as you use it to clean the furniture’ (PG 375). Similarly, geometrical theorems have the role of rules for describing, for example, shapes and sizes of objects and their spatial relations (LFM 44; PLP 47) and for making inferences about them. The contrast between descriptive propositions and mathematical propositions that play the part of rules of description is of the greatest importance (RFM 363). Failure to draw this distinction drags in its wake further muddles about the concepts of truth, falsehood, knowledge, belief and verification (see ‘Grammar and necessity’, §§4–8). It breeds philosophical mythologies such as Platonism, which notes correctly that mathematical propositions are not descriptions of signs and jumps to the conclusion that they must be descriptions of something else (AWL 151f.), namely abstract entities. The formalist reaction to Platonism is equally awry, equally ensnared in the web of the idea that mathematical propositions describe something, if not abstract entities, then signs and patterns of signs (LFM 112). Finally, it obliterates the distinction between applying mathematical techniques within as opposed to outside mathematics. Such a statement as ‘The real numbers cannot be put into a one–one correspondence with the natural numbers’ sounds like a discovery about mathematical objects, but it is part of the construction of a mathematical calculus—not a discovery of mathematical facts but the creation of new norms of description. Here we apply mathematical techniques to mathematics itself, as it were strengthening the internal structure of the mathematical scaffolding that we use to assist us in building the edifice of empirical knowledge of quantitative properties and relationships.

In philosophy of psychology it is vital to distinguish avowals or first-person utterances (Äusserungen) from descriptions. Utterances, such as groaning ‘I have a toothache’ or exclaiming ‘Now I understand’, are instruments, which have their uses. So too are descriptions (PI §290), such as ‘He has toothache’ and ‘Now he understands.’ But these uses are different. Utterances of this kind are not typically employed to convey information (cf. PI §363), but are cries for help, to solicit sympathy, or to signal an ability to do something. ‘I have a toothache’ sounds like ‘I have a matchbox’ but it is commonly used in an
utterly different way (LSD 44). A child is taught to use exclamations and sentences, such as ‘Ow!’, ‘Hurts!’, ‘It hurts!’, ‘I have a pain!’ to replace the moans which are the natural expressions of pain (PI §244). Here the verbal expression of pain replaces crying and describes neither crying out nor an ‘inner state’. Similarly the exclamation ‘Now I know how to go on’ is not a description, but corresponds to an instinctive sound, a glad start (PI §§180, 323). The contrast is crucial, even though it shades off in all directions (PPF §§83–5/p. 189).

Failure to distinguish expressions or manifestations of inner states from descriptions of people (including oneself) as being in certain mental states is the source of widespread confusion in reflections upon the mind. It drags in its wake misconceptions about self-knowledge and consciousness, doubt and certainty, belief and verification. It breeds mythologies of the mental such as the Cartesian and empiricist conception of the mind as an ‘inner world’ accessible only to its owner by introspection (see Volume 3, Part I, ‘The private language arguments’ and ‘Avowals and descriptions’). And the behaviourist reaction to this picture of the mental is no less ensnared in the web of misconceptions (see Volume 3, Part I, ‘Behaviour and behaviourism’). From these confusions grows the idea that the mental predicates one uses in the first-person present tense must be explained quite differently from the same predicates as applied to others. They seem to belong to a ‘private language’. The uncritical assumption that declarative sentences are uniformly descriptions plays as much havoc with our thought about the mind and the nature of self-consciousness or self-knowledge as it does with our reflections on mathematics.

(ii) Names

A second salient feature of the Augustinian conception of language is to treat every significant word as a name (see Volume 1, Part I, ‘The Augustinian conception of language’, §1 and §2(a)–(d)). This idea informs much reflection on philosophy of mathematics and of psychology alike. Mathematical expressions such as ‘0’, ‘−2’, ‘−\sqrt{1}’, ‘\aleph_0’, or even ‘+’, ‘x^4’, ‘e^x’ are taken to be names of entities, and the question ‘What do they mean?’ is thought to boil down to ‘What do they stand for?’ Similarly, it is assumed that ‘toothache’, ‘anger’, ‘understanding’, ‘thinking’, ‘knowing’, ‘believing’, ‘intending’, ‘meaning something’, etc. are names and hence that they stand for something, and the debate centres on the question of what they stand for. Do they stand for mental states, processes or events, or for qualia, or for neural states and processes, or for forms of behaviour and dispositions to behave, or for functional states.

It is commonplace in reflections on mathematics to contrast the claim that a symbol has content in virtue of standing for something with the claim that it is a meaningless mark and that the calculations in which it occurs are mere manipulations of empty symbols (see, e.g. Frege, The Basic Laws of Arithmetic, vol. ii, §§86–137, in his criticisms of the formalists). Mathematicians for long held that, by contrast with the signs for natural numbers, nothing corresponds to the use of negative number signs – that these do not stand for anything. Producing a rigorous explanation of what a sign stands for, e.g. identifying a
negative number with an equivalence-class of ordered pairs of positive integers, is taken to be a vindication of the claim that a sign does have a meaning. (It is striking that such explanations play no role in conveying to a neophyte how these symbols are used in the applications of mathematics, e.g. in employing negative integers in banking or in mechanics.) Wittgenstein held that the idea that all significant terms are names prevents recognition of logico-grammatical differences in use. It further promotes the misconception that differences in use flow from differences in the natures of the objects allegedly named.\footnote{Frege, for example, argued that the rules for the use of expressions are \emph{answerable} to the entities the expression stands for, that \textquote{the rules follow necessarily from the meanings (\emph{Bedeutungen}) of the signs} (BLA ii, §§91, 158).}

We should, Wittgenstein argued, look on words as tools and clarify their uses in our language-games (PI §§10f.). Number-words are instruments used in counting and measuring, and the foundation of elementary arithmetic, namely: mastery of the series of natural numbers, lies in training in counting. Philosophers stray from such platitudes in seeking deeper foundations for arithmetic. Frege exemplified this error. He began his investigation of numbers from an examination of extra-mathematical count-statements such as \textquote{Jupiter has four moons} (LFM 166, 262ff.). But he then threw away his insight through his conviction that numerals are names of Platonic objects. He succumbed to the mesmerizing power of the philosophical question \textquote{What are numbers?} and sought for a rigorous definition in response to it that will make clear what numerals and number-words stand for. Wittgenstein thought the question misleading\footnote{Cf. PI §1: \textquote{What is the meaning of the word \textquote{five}?} – \textquote{No such meaning was in question here, only how the word \textquote{five} is used.}} and the logicist answers useless: \textquote{What we are looking for is not a definition of the concept of number, but an exposition of the grammar of the word \textquote{number} and of the numerals} (PG 321). For the puzzlement about the question \textquote{What is a number?} is not resolved by a definition at all, but only by a clarification of the grammar of \textquote{number} and \textquote{numeral} (AWL 164). Assimilation of mathematical terms to names, and the idea that they are names of abstract objects, is a widespread confusion in mathematical reflection. But to say that an expression names an abstract object boils down to little more than that it looks like the name of a concrete object, but is not. Misconceptions about naming in the philosophy of mathematics and the mistaken idea that mathematical propositions are \textquote{about} the entities named (as typical chemical propositions are about the chemical substances named) is a target of Wittgenstein\textquote{s} criticism (LFM 33, 112–14; RFM 137, 262f.).

In philosophy of psychology there is a powerful temptation to construe psychological expressions as names of inner states, private experiences, or \textquote{qualia}, which the subject apprehends by introspection. Such names, it may seem, are given their meaning by a special form of ostensive definition. This is supposedly a mental analogue of public ostensive definition. One points
‘mentally’, as it were, at the inner entity, accessible to one through introspection, and names it. But, Wittgenstein argued, there is no such thing as a ‘private ostensive definition’. Moreover, we must emancipate ourselves from the preconception that to understand a term is to know what it stands for. For the role of psychological terms is not to stand for things on the model of a proper name’s standing for this = person, or even on the model of a colour word’s being the name of this = colour. We should conceive of words such as ‘pain’, ‘thinking’, ‘anger’ as tools and examine their functions in discourse. Scrutiny of the actual use of psychological concepts reveals complex and variegated patterns of use. We use many of them in (first-person) avowals and in (second- and third-person) descriptions. Some first-person forms are grafted on to natural, pre-linguistic behaviour (e.g. ‘pain’, ‘hurts’, ‘want’), others are grafted onto linguistic behaviour (‘believe’, ‘dreamt’), and all are embedded in highly diverse language-games. Neither the explanations nor the uses of such concepts have the formal simplicity and uniformity we naturally expect. Many are family-resemblance concepts (PG 74ff.), some are even more amorphous and heterogeneous, for example, thinking (Z §§110ff.) or understanding (PI §§532), and the use of many is related to complex behavioural criteria (cf. PI §§164, 182, 269, 580). In short, in every direction, psychological terms burst through the conceptual bounds typical of paradigmatic names.

(iii) The influence of the Augustinian conception The twin ideas that every meaningful word is a name and that every sentence is a description is a presupposition, often tacit, of much philosophy. It is, as it were, the magnetic pole of philosophical thought and hence something that must be identified by charting the movements of philosophical thought. Hence, to the extent that Wittgenstein’s earlier reflections on the philosophy of mathematics and of psychology can be seen to be exploring the ramifications of misconceptions about words and names, sentences and descriptions, each of these two groups of investigations would complement the central theme of the Early Draft. For each would clarify how this conception shapes and distorts a major branch of philosophy. The two continuations can be seen as parallel executions of a single strategy in respect of different domains of thought.

3. A common methodology

The foregoing discussion clarifies a crucial point of contact between Wittgenstein’s philosophy of mathematics and his philosophy of psychology. They involve parallel diagnostic investigations into ramifying aspects of a single syndrome. But there are many further parallelisms and uses of remarks in one domain to illuminate the other. For example, Wittgenstein took behaviourism as an object of comparison for standard manoeuvres in philosophy of mathematics: ‘Finitism and behaviourism are quite similar trends. Both say: but surely, all we have here is . . . Both deny the existence of something,
both with a view to escaping from a confusion’ (RFM 142). Similarly, he argued, ‘We might say that formalism in mathematics is behaviourism in mathematics. I could draw “2” and say “That is the number 2.” This is exactly the same as pinching and saying “This is pain.” – Mathematicians say “Surely it is not just the numeral, it is something more”’ (LSD 111). So too one says that pain is not just behaviour, but something more (LSD 114f.). In both cases this response is misleading. It leads the mathematician to the idea that arithmetical equations describe relations among abstract objects and that they are true in virtue of corresponding to a mathematical reality. This generates a debate between ‘formalism’ and ‘contentful mathematics’ (as Frege put it) in which both sides make unwarranted assertions at variance with their ordinary practice (PG 293).

In the psychological case it leads to the idea that each person attaches the term ‘toothache’ to a private experience ascertainable only by himself, and hence to the solipsistic claim ‘Only I have real toothache (or experiences).’ This generates a debate in which the antitheses ‘Only my experiences are real’ and ‘Everyone’s experiences are real’ are equally nonsensical (AWL 23). To escape from such absurdities one is tempted to deny that numbers are something more than signs or that sensations are something more than behaviour. But this too is misleading. Numbers and sensations are not (different kinds of) intangible objects. But neither are they just signs or just behaviour. The uses of ‘numeral’ and ‘number’, of ‘behaviour’ and ‘sensation’, are different (RFM 202). Formalism and behaviourism embody parallel grammatical insights into the uses of expressions, but both wrap them up in a form that generates further confusion.

Wittgenstein’s practice in philosophy of mathematics and philosophy of psychology indicates not only such parallelisms but also a shared methodology. Notes taken of his lectures in 1929–35 show him frequently juxtaposing remarks about psychological and mathematical concepts. His own notebooks reveal similar frequent transitions from one domain to the other. The parallelisms exemplifying the shared method fall into two main types (though there are transitional cases). The first are explicit interpolations of analogies to clarify or illustrate specific points, for example:

(i) ‘The relation of expectation and its fulfilment is precisely that of calculation and result’ (LWL 62). Expectation anticipates its fulfilment just as a calculation anticipates its conclusion. In both cases the relation is internal.

(ii) One is inclined to assert that we can never really know what another person feels. This is parallel to the claim that we can never draw an exact circle. Both look like statements about empirical possibilities, but in fact they are misconceived statements of grammar (LSD 133f.).

(iii) Viewing visual impressions as ‘inner pictures’ is misleading, for this concept is modelled on that of an ‘outer’ picture. But the uses of ‘visual impression’ and ‘picture’ are no more alike than the uses of ‘numeral’ and ‘number’ (PPF §133/p. 196). The category differences in grammar are crucial.

(iv) One no more has the concept of colour in virtue of seeing coloured objects than one has the concept of a negative number in virtue of running
up debts (Z §332). Concept-possession is not a matter of having had experiences, but of having mastered the uses of expressions.

(v) Calculating in one’s head is a kind of calculating. But only someone who has learnt to calculate on paper or out loud can learn how to calculate in his head, and the concept of calculating in one’s head can be acquired only by someone who has the concept of (perceptible) calculating. The concept of calculating in one’s head is confusing because it runs for a long stretch cheek by jowl with the concept of calculating aloud or on paper. These concepts are as closely related and also as different as the concepts of a cardinal number and a rational number (LW I, §857).

(vi) The contrast between mathematical proof and empirical verification parallels the contrast between giving the agent’s reason or motive for his action and specifying its cause. The connection between a reason and what it is a reason for, like that between calculation and result, is internal (AWL 4f).

(vii) We use arithmetical equations as norms of description, e.g. as criteria for saying that something has vanished during a count. In science, certain hypotheses have the same status as grammatical statements. Hertz used hypothetical ‘invisible masses’ to account for any deviation of observations from his laws. ‘Unconscious mental events’ play the same role in explanations of behaviour; they are introduced because we wish to say that there must be causes of human action. This is an arbitrary stipulation that makes determinism a property of the system of explanations of behaviour (AWL 15f).

(viii) We confuse categorial differences in grammar with differences between kinds. We say that transfinite numbers are another kind of number than rationals, that unconscious thoughts are a different kind of thought from conscious ones. But the differences are not analogous to that between different kinds of chair, but to that between a chair and permission to sit in a chair. For the words ‘thought’ and ‘number’ are differently used when prefixed with these adjectives (AWL 32; BB 64).

The second type of methodological parallel is a matter of isomorphism in argumentative manoeuvre between self-contained arguments in philosophy of mathematics and philosophy of psychology. Consider this pair of cases. In logic we introduce pupils to the concept of a set by using lists to define class membership. We then suggest that there is an alternative to such an extensional conception of a set. We may define a set by a general condition, satisfaction of which by an object is a necessary and sufficient condition for the object to belong to the set. And we then point out that this intensional conception is preferable because it does not rule out speaking of sets whose members are too numerous to list, especially sets whose members are infinite in number like the set of the natural numbers. Logicians then proceed to point out the remarkable properties of infinite sets. It is easy to prove that two finite sets have the same number of elements if and only if they can be put in one-to-one correspondence and that a finite set cannot be put into one-to-one correspondence with any of its proper subsets. But an infinite set can! This is
simple to show: the operation of doubling correlates the integers one-to-one with the even integers, but the even integers are a proper subset of the integers. With this paradoxical result the logician lures the pupil on into an exploration of the mysteries of sets, into the study of abstract set theory. And the expert prides himself in having surmounted the superstitions and prejudices that blinker the pupil and prevent him from seeing that only in the realm of finite sets does it hold that a proper part is never as great as the whole (cf. PG 465). Wittgenstein argued that this chain of reasoning rests on a series of conceptual confusions. It is a fallacy to suppose that there is a single concept of a set which subsumes both sets defined by lists of their elements and sets defined by satisfaction of a predicate (WWK 102f.). In fact, these are different grammatical structures (PG 464f.); for, in the first case, any true statement of set membership \((a \in A)\) will be a grammatical proposition, whereas in the second it will be equivalent to a proposition of the form \(\phi(a)\) which will typically express an empirical statement (cf. AWL 150). There is, of course, no such thing as an infinite list (AWL 206). Hence, if the concept of a set is introduced extensionally, there is no such thing as an infinite set. The ‘discovery’ of infinite sets is an alteration in the concept of a set. It is misleading to speak of finite sets and infinite sets as two kinds of sets (WWK 192; PG 463f.). Similarly, the concept of one-to-one correspondence is subtly stretched and altered (cf. LFM 161ff.; AWL 168f.). We pass from the claim that two sets have the same cardinal if their members are correlated one-to-one (as cups and saucers are by laying them out so that each cup stands on a single saucer) to the claim that the sets have the same cardinal if their members can be correlated one-to-one (cf. AWL 148ff., 161ff.). We then think that we can prove that two infinite sets have the same cardinal; for although in writing out the table

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 2 & 4 & 6 & 8 & 10
\end{array}
\]

one has arrived only at the sixth term of each series, one can correlate any specified even integer with an integer in the upper series, and vice versa (LFM 160). But in fact the concept of one-to-one correlation is now altered (LFM 161). There is no such thing as actually having listed against each integer the even integer which corresponds to it (PG 464), and equally no such thing as arriving at the end of the series of even integers to ascertain that none has been left out. The ‘discovery’ and corresponding astonishment that an infinite set may be one-to-one correlated with a proper subset is a muddled report of a redefinition of one-to-one correspondence (LFM 161; AWL 209).

Wittgenstein gave a parallel diagnosis of the conceptual confusions enveloping talk of the unconscious in psychology. We have our everyday concept of a desire or motive. An agent often avows his motives if challenged to explain his actions, and his avowals have a crucial role in determining what his motives are. A psychologist may call attention to the fact that an agent may act as if
he had a particular motive although he is unaware of it and would sincerely
disavow it. The psychologist suggests that we speak here of an ‘unconscious
motive’ (cf. RPP I §225), which he paraphrases by saying that the agent has
a motive but does not know it. This seems like a genuine discovery, opening
new domains to psychological investigations and new possibilities for pinpointing
the causes of behaviour (AWL 16). No longer is research limited to what is
conscious; one can study all motives, feelings, thoughts, whether conscious or
unconscious. The unconscious may turn out to obey remarkable laws and
to have extraordinary manifestations in behaviour (as claimed in Freudian
psychology). If anyone expresses scepticism or puzzlement as to how a desire
or motive can be unconscious, the psychologist will say that it is a proven fact
that there is such a thing, and ‘he will say it like a man who is destroying a
common prejudice’ (BB 23).

This drift of thought, Wittgenstein argued, is likewise confused. The phrase
‘unconscious thought (motive etc.)’ is misleading, for we suppose that uncon-
scious and conscious thoughts are two kinds of thoughts. But this is not implied
by the original explanation. By parity of reasoning someone might be said to
have ‘unconscious toothache’ when he has a rotten tooth but feels nothing.
In that case ‘conscious toothache’ must be understood to encompass everything
ordinarily called ‘toothache’ (BB 57f.) and ‘unconscious toothache’ has a totally
different use. It would be mistaken to object that there is no such thing as
unconscious toothache, for the phrase ‘unconscious toothache’ has been given
an intelligible explanation. Similarly, objectors to the unconscious do not appre-
ciate that they are objecting not to empirical discoveries but to a new form
of representation (cf. AWL 40). Psychoanalytic defenders of the unconscious
are equally confused. They misconstrue unconscious desires as a kind of desire
etc., and transfer parts of the grammar of ‘desire’ to ‘unconscious desire’. Misled
by their own novel convention of representation, they think they have, ‘in a
sense, discovered conscious thoughts which were unconscious’ (BB 57). They
think that an agent is cut off by a barrier from his own unconscious states,
that consciousness is a screen against the unconscious. Such ideas, with the
attendant aura of paradox and mystery, indicate that psychologists misunder-
stand their own discourse about the unconscious mind.

These two kinds of parallelisms between Wittgenstein’s philosophy of
mathematics and his philosophy of psychology are manifestations of his appli-
cation of a common methodology and an overarching conception of the nature
of philosophy, its problems and their resolution (cf. ‘Philosophy’, Volume 1,
Part I). We shall briefly catalogue and classify the methodological parallels:

(i) Pitting scrutiny of grammar against preconception Philosophical preconceptions
stand in the way of attaining an overview of how expressions are used. Our
craving for generality, simplicity and formal definitions is a source of confusion
in philosophy of mathematics and psychology alike, for we crave for a formal
definition of ‘number’ or ‘thought’. So we fail to recognize the existence and
character of family-resemblance concepts. Our natural disposition to construct
pictures’ of whole domains of thought leads us into mythologies of symbolism, on the one hand, and of psychology, on the other. The Platonist picture is profoundly appealing to mathematicians, who are prone to conceive of themselves as discoverers of the laws of non-empirical objects. The idioms of our psychological discourse foster the picture of the mind as a private realm, and psychologists are prone to think of themselves as investigating this hidden realm by indirect means, just as a physicist investigates unobservable particles by indirect means (PI §571): for example, they endeavour to discover the laws of motion of mental images in mental space. Such preconceptions and misconceptions are to be combated by scrutiny of the uses of expressions in the contexts and language-games in which they are at home.

(ii) Focusing on use rather than on grammatical form We are readily impressed by forms of expression, and take common form to be indicative of a shared meaning. But it is use, not form, that shows shared meaning. We talk of the certainty of empirical propositions, of knowing or believing them, of verifying or falsifying them. We speak similarly of mathematical propositions. Hence, we think that the certainty of mathematical propositions is akin to, but of a much higher degree than that of empirical ones. We suppose that we know mathematical truths just as we know empirical ones – the only difference lying in the object of knowledge; and so on (see ‘Grammar and necessity’, §8, i–ii.). Similarly in philosophy of mind: the pronoun ‘I’ seems to refer to a person in just the way ‘He’ does. ‘I have a toothache’ is deceptively like ‘I have a tooth.’ But the shared forms mask radical differences that come to light only when we examine the divergent uses of the similar forms in their different contexts.

(iii) Exposing surreptitious changes in concepts This important philosophical technique has already been illustrated with the parallel examples of the contrasts between conscious and unconscious thoughts or motives, and finite and infinite sets. In such cases we mistake the introduction of a new concept for the scientific discovery of hitherto unknown phenomena subsumable under the old concept.

(iv) Clarifying the completeness of language-games We are prone to view our language-games with all their complex articulations as essentially completions or culminations of the development of incomplete, gappy or essentially incorrect antecedent ones, or to view actual or imaginary language-games which lack such articulations as essentially incomplete. In arithmetic we think that the system of natural numbers contains gaps which are filled up by the negative integers (since we can frame within this system a problem which cannot be answered except by ‘extending the concept of number’ to include the signed integers). Similarly, in reflecting on perception, we are prone to think of colour-systems different from our own as incorrect or gappy. Wittgenstein castigated such tendencies. There are no gaps in a grammar. A question makes sense in a grammatical system only if it has an intelligible answer in that system. Adding the system of signed integers to the system of natural numbers adds
another member to the family of numbers. The new family has a greater multiplicity than the old one and the analogues of the members of the old system (positive integers) are different *kinds* of number than their cousins (natural numbers), admitting operations previously unintelligible, e.g. subtracting the greater from the lesser.

(v) **Crossing language-games** Consequently errors ensue from failure to realize that the transposition of a concept from one system or language-game into another involves a shift in meaning. Both in philosophy of mathematics and in philosophy of mind Wittgenstein laid bare such confusions. Philosophers of mathematics are prone to overlook the fact that, for example, the concepts of addition or subtraction are redefined as one moves from the system of natural numbers to that of the signed integers; or the fact that negative numbers are not an extension of natural numbers, but part of the system of signed integers in which positive numbers are *analogues* of natural numbers. Similarly, we use the expression ‘visual image’ in connection with both seeing and imagining (visualizing). So we assume, in philosophizing, a close similarity between these phenomena – that what it makes sense to say about seeing a tree (e.g. ‘I didn’t notice any birds’ nests, but there may have been some’) must also make sense about imagining a tree. In fact these language-games are radically different; the tie-ups are numerous, but there is no similarity ((RPP II §§70f.), see Volume 3, ‘Images and the imagination’).

(vi) **Transgressing the boundaries of language-games** Precisely because expressions have a meaning only within the language-game in which they are embedded, multiple confusions result from transgressing the boundaries of a language-game. For both in mathematics and in psychology we project one language-game not only into another but also into a grammatical void. It makes sense to ask whether there are four successive 7s in the first thousand (or two thousand) places of an expansion of $\pi$. So we misguidedly think that it makes sense to ask whether there are four successive 7s in the expansion of $\pi$, as if there were such a thing as *the* expansion of $\pi$. Similarly, we use ostensive definition to explain the meaning of predicates of perceptual qualities and assume misguidedly that it makes sense to explain (to oneself) the meaning of psychological predicates by a ‘private’ ostensive definition.

(vii) **Conceptual topology** Wittgenstein criticized the idea that our concepts mirror the essential nature of things. His most straightforward method is to imagine that certain general features of the world, including culture and history (CV 37), were different, and to consider what concepts would then be natural to us. He employed this method in philosophy of mathematics in imagining very different ways of measuring, weighing, or buying and selling which would involve different grammars from our own. Even in the matter of counting, fundamental differences are conceivable, e.g. the number system ‘1, 2, 3, 4, 5, many’ (which would be a natural enough system for what might be called ‘visual numbers’ that can be taken in at a glance). Such a number system is not an incomplete part of our own, but an autonomous system. The
same method is conspicuous in his philosophy of psychology. Our concept of a person seems justified by the nature of persons. But our use of proper names, of ‘person’ and ‘same person’ is ‘based on the fact that many characteristics which we use as criteria for identity coincide in the vast majority of cases’ (BB 61). But if things were very different – for example, if all human beings looked alike and sets of character traits ‘circulated’ among these beings – our present concept would be useless and various different concepts, more or less remotely related to it, might replace it (BB 62).

In these ways (and doubtless others could be mentioned too) one can see Wittgenstein’s philosophy of mathematics and his philosophy of psychology as informed by a common conception of philosophy and a common array of methods.

4. The flatness of philosophical grammar

According to Wittgenstein philosophy is purely descriptive. It does not explain, in the sense in which scientific theory explains phenomena. It clarifies the grammar of our language, the rules for the construction of significant utterances, whose transgression yields nonsense. The purpose of such clarifications is to disentangle conceptual confusions and resolve conceptual puzzlement, and to enable us to handle philosophical questions without tying ourselves in knots. But philosophy does not explain the logico-grammatical articulations of our conceptual scheme. Such explanations would be possible only if it made sense to get behind the rules and supply a deeper foundation. But there is no behind, and rules are not answerable to reality in the currency of truth (see ‘Grammar and necessity’, §11). Any deeper explanation would simply be another rule of grammar standing in the same relation to the uses of expressions as the rules it allegedly explains. So philosophy must, in this sense, be ‘flat’.

The flatness of philosophy has two coordinate aspects. The first is methodological. A grammatical proposition such as ‘red is darker than pink’ has no significant negation. Its negation (‘red is not darker than pink’) is neither an empirical falsehood, nor an expression of a rule of grammar. The denial of a grammatical proposition is best considered to be a form of nonsense, since it employs expressions blatantly contrary to the rules for their use that are given by the grammatical proposition denied. Hence there is no such thing as demonstrating a grammatical proposition by proving the falsity of its negation. Rather, what Wittgenstein does is to show, by argument and example, the incoherent consequences of its transgression. His arguments commonly have the form of a literal reductio ad absurdum, rather than the reductio ad

---

6 The expression of a rule for the use of the constituent terms in the guise of a statement.
contradictionem that often goes by that name. The result of philosophizing is not philosophical knowledge of reality, but clarity about our modes of representing reality (cf. ‘Philosophy’, Volume 1, Part 1).

This point must shape our conception of what Wittgenstein’s arguments are meant to accomplish. We may be tempted to look at his discussion of avowals and ask where he proved that such an utterance is not a description of the speaker’s mental state, or that it is not something he can be said to know or not to know. Similarly, we may enquire where he proved that an arithmetical equation such as ‘25 × 25 = 625’ is a rule for the transformation of empirical propositions concerning magnitudes or quantities, or that a consistency proof is irrelevant to the usefulness of arithmetic in building bridges. This is misconceived. Wittgenstein’s grammatical remarks fall roughly into two categories. The first are evident grammatical truisms concerning our uses of expressions and undisputed rules for their use, e.g. that it makes sense to say ‘I know you have toothache’, that ‘to understand’ has no continuous present tense, or that ‘two’ may be correctly explained by ostension. The second are not grammatical truisms and they are often taken to express philosophical theses, e.g. that its method of verification is often a contribution to the grammar of a sentence, that the proper answer to ‘What are numbers?’ is a description of the grammar of ‘number’ and of ‘numeral’, or that inner states stand in need of outward criteria. But to concede that these are not indisputable truisms is not to cast them in the role of explanatory theses (for what Wittgenstein meant by ‘thesis’, see Exg. §128 and Volume 1, Part I, ‘Philosophy’, §6). They are rather intended to play the role of synoptic descriptions drawing together and interrelating a multitude of grammatical propositions that are truisms. Achievement here is to be measured by the usefulness of these propositions in dissolving philosophical problems and promoting philosophical understanding, which is exhibited in knowing one’s way about among our concepts. Whether these propositions do discharge this function may be disputed, but the dispute is not properly conducted by focusing on whether Wittgenstein has given rigorous enough proofs of his theses. What he offers us are not theses, but elucidations of problematic or confusing segments of our grammar, clarifications of where and why we so persistently misconstrue them in philosophical reflection, and reasoned argument to show the absurd consequences that ensue from our misconstruals.

This point about the absence of theoretical explanations is likewise apparent in the second aspect of the flatness of Wittgenstein’s ‘grammatical’ observations: there are no arguments from grammatical propositions in his work, i.e. no reasoning which takes the form ‘Arithmetical equations are grammatical rules, so . . .’ or ‘First-person psychological utterances are often manifestations, not descriptions of psychological states, so . . .’ Instead such synoptic grammatical propositions are introduced to promote insight and express overviews of complex networks of concepts, but they are not invoked as premises in arguments to prove a philosophical thesis.
Wittgenstein’s conception of philosophy informs his work in philosophy of mathematics and philosophy of psychology alike. For these are the two main branches that spring from his reflections on language, meaning and understanding in the Early Draft. Close scrutiny of his writings on philosophy of mathematics can contribute to clarifying the *Philosophical Investigations*, and the latter in turn can illuminate his philosophy of mathematics. In particular, both promote deeper understanding of his conception of grammar and of internal relations.