CHAPTER 1

PROPOSITIONAL LOGIC

1.1 SYMBOLIC LOGIC

Let us define mathematics as the study of number and space. Although representations can be found in the physical world, the subject of mathematics is not physical. Instead, mathematical objects are abstract, such as equations in algebra or points and lines in geometry. They are found only as ideas in minds. These ideas sometimes lead to the discovery of other ideas that do not manifest themselves in the physical world as when studying various magnitudes of infinity, while others lead to the creation of tangible objects, such as bridges or computers.

Let us define logic as the study of arguments. In other words, logic attempts to codify what counts as legitimate means by which to draw conclusions from given information. There are many variations of logic, but they all can be classified into one of two types. There is inductive logic in which if the argument is good, the conclusion will probably follow from the hypotheses. This is because inductive logic rests on evidence and observation, so there can never be complete certainty whether the conclusions reached do indeed describe the universe. An example of an inductive argument is:

A red sky in the morning means that a storm is coming.
We see a red sky this morning.
Therefore, there will be a storm today.

Whether this is a trust-worthy argument or not rests on the strength of the predictive abilities of a red sky, and we know about that by past observations. Thus, the argument is inductive. The other type is deductive logic. Here the methods yield conclusions with complete certainty, provided, of course, that no errors in reasoning were made. An example of a deductive argument is:

All geometers are mathematicians.
Euclid is a geometer.
Therefore, Euclid is a mathematician.

Whether Euclid refers to the author of the *Elements* or is Mr. Euclid from down the street is irrelevant. The argument works because the third sentence must follow from the first two.

As anyone who has solved an equation or written a proof can attest, deductive logic is the realm of the mathematician. This is not to say that there are not other aspects to the discovery of mathematical results, such as drawing conclusions from diagrams or patterns, using computational software, or simply making a lucky guess, but it is to say that to accept a mathematical statement requires the production of a deductive proof of that statement. For example, in elementary algebra, we know that given

\[ 2x - 5 = 11, \]

we can conclude

\[ 2x = 6 \]

and then

\[ x = 3. \]

As each of the steps is legal, it is certain that the conclusion of \( x = 3 \) follows. In geometry, we can write a two-column proof that shows that

\[ \angle B \cong \angle D \]

is guaranteed to follow from

*ABCD* is a parallelogram.

The study of these types of arguments, those that are deductive and mathematical in content, is called mathematical logic.

**Propositions**

To study arguments, one must first study sentences because they are the main parts of arguments. However, not just any type of sentence will do. Consider

all squares are rectangles.
The purpose of this sentence is to affirm that things called squares also belong to the category of things called rectangles. In this case, the assertion made by the sentence is correct. Also, consider,

\textit{circles are not round.}

This sentence denies that things called circles have the property of being round. This denial is incorrect. If a sentence asserts or denies accurately, the sentence is \textbf{true}, but if it asserts or denies inaccurately, the sentence is \textbf{false}. These are the only \textbf{truth values} that a sentence can have, and if a sentence has one, it does not have the other. As arguments intend to draw true conclusions from presumably true given sentences, we limit the sentences that we study to only those with a truth value. This leads us to our first definition.

\begin{definition}
A sentence that is either true or false is called a \textbf{proposition}.
\end{definition}

Not all sentences are propositions, however. Questions, exclamations, commands, or self-contradictory sentences like the following examples can neither be asserted nor be denied.

- \textit{Is mathematics logic?}
- \textit{Hey there!}
- \textit{Do not panic.}
- \textit{This sentence is false.}

Sometimes it is unclear whether a sentence identifies a proposition. This can be due to factors such as imprecision or poor sentence structure. Another example is the sentence

\textit{it is a triangle.}

Is this true or false? It is impossible to know because, unlike the other words of the sentence, the meaning of the word \textit{it} is not determined. In this sentence, the word \textit{it} is acting like a variable as in $x + 2 = 5$. As the value of $x$ is undetermined, the sentence $x + 2 = 5$ is neither true nor false. However, if $x$ represents a particular value, we could make a determination. For example, if $x = 3$, the sentence is true, and if $x = 10$, the sentence is false. Similarly, if \textit{it} refers to a particular object, then \textit{it is a triangle} would identify a proposition.

There are two types of propositions. An \textbf{atom} is a proposition that is not comprised of other propositions. Examples include

\textit{the angle sum of a triangle equals two right angles}

and

\textit{some quadratic equations have real solutions.}
A proposition that is not an atom but is constructed using other propositions is called a **compound proposition**. There are five types.

- A **negation** of a given proposition is a proposition that denies the truth of the given proposition. For example, the negation of \(3 + 8 = 5\) is \(3 + 8 \neq 5\). In this case, we say that \(3 + 8 = 5\) has been **negated**. Negating the proposition *the sine function is periodic* yields *the sine function is not periodic*.

- A **conjunction** is a proposition formed by combining two propositions (called **conjuncts**) with the word *and*. For example,

  \[
  \text{the base angles of an isosceles triangle are congruent,}
  \text{and a square has no right angles}
  \]

is a conjunction with *the base angles of an isosceles triangle are congruent* and *a square has no right angles* as conjuncts.

- A **disjunction** is a proposition formed by combining two propositions (called **disjuncts**) with the word *or*. The sentence

  \[
  \text{the base angles of an isosceles triangle are congruent,}
  \text{or a square has no right angles}
  \]

is a disjunction.

- An **implication** is a proposition that claims a given proposition (called the **antecedent**) entails another proposition (called the **consequent**). Implications are also known as **conditional propositions**. For example,

  \[
  \text{if rectangles have four sides, then squares have four sides} \quad (1.1)
  \]

is a conditional proposition. Its antecedent is *rectangles have four sides*, and its consequent is *squares have four sides*. This implication can also be written as

  \[
  \text{rectangles have four sides implies that squares have four sides,}
  \]

  \[
  \text{squares have four sides if rectangles have four sides,}
  \]

  \[
  \text{rectangles have four sides only if squares have four sides,}
  \]

and

  \[
  \text{if rectangles have four sides, squares have four sides.}
  \]

A conditional proposition can also be written using the words *sufficient* and *necessary*. The word *sufficient* means “adequate” or “enough,” and *necessary* means “needed” or “required.” Thus, the sentence
rectangles having four sides is sufficient for squares to have four sides

translates (1.1). In other words, the fact that rectangles have four sides is enough for us to know that squares have four sides. Likewise,

squares having four sides is necessary for rectangles to have four sides

is another translation of the implication because it means that squares must have four sides because rectangle have four sides. Summing up, the antecedent is sufficient for the consequent, and the consequent is necessary for the antecedent.

- A biconditional proposition is the conjunction of two implications formed by exchanging their antecedents and consequents. For example,

  if rectangles have four sides, then squares have four sides,
  and if squares have four sides, then rectangles have four sides.

To remove the redundancy in this sentence, notice that the first conditional can be written as

rectangles have four sides only if squares have four sides

and the second conditional can be written as

rectangles have four sides if squares have four sides,

resulting in the biconditional being written as

rectangles have four sides if and only if squares have four sides

or the equivalent

rectangles having four sides is necessary and sufficient
for squares to have four sides.

Propositional Forms

As a typical human language has many ways to express the same thought, it is beneficial to study propositions by translating them into a notation that has a very limited collection of symbols yet is still able to express the basic logic of the propositions. Once this is done, rules that determine the truth values of propositions using the new notation can be developed. Any such system designed to concisely study human reasoning is called a symbolic logic. Mathematical logic is an example of symbolic logic.

Let \( p \) be a finite sequence of characters from a given collection of symbols. Call the collection an alphabet. Call \( p \) a string over the alphabet. The alphabet chosen so that \( p \) can represent a mathematical proposition is called the proposition alphabet and consists of the following symbols.
• **Propositional variables**: Uppercase English letters, \( P, Q, R, \ldots \), or uppercase English letters with subscripts, \( P_n, Q_n, R_n, \ldots \), where \( n = 0, 1, 2, \ldots \)

• **Connectives**: \( \neg, \land, \lor, \rightarrow, \leftrightarrow \)

• **Grouping symbols**: (, ), [, ].

The sequences \( P \lor Q \) and \( P_1 Q_1 \land \leftrightarrow (((\text{and the empty string, a string with no characters, are examples of strings over this alphabet, but only certain strings will be chosen for our study. A string is selected because it is able to represent a proposition. These strings will be determined by a method called a grammar. The grammar chosen for our present purposes is given in the next definition. It is given recursively. That is, the definition is first given for at least one special case, and then the definition is given for other cases in terms of itself.}

**DEFINITION 1.1.2**

A **propositional form** is a nonempty string over the proposition alphabet such that

- every propositional variable is a propositional form.
- \( \neg p \) is a propositional form if \( p \) is a propositional form.
- \( (p \land q), (p \lor q), (p \rightarrow q) \), and \( (p \leftrightarrow q) \) are propositional forms if \( p \) and \( q \) are propositional forms.

We follow the convention that parentheses can be replaced with brackets and outermost parenthesis or brackets can be omitted. As with propositions, a propositional form that consists only of a propositional variable is an **atom**. Otherwise, it is **compound**.

The strings \( P, Q_1, \neg P, (P_1 \lor P_2) \land P_3 \), and \( (P \rightarrow Q) \land (R \leftrightarrow \neg P) \) are examples of propositional forms. To prove that the last string is a propositional form, proceed using Definition 1.1.2 by noting that \( (P \rightarrow Q) \land (R \leftrightarrow \neg P) \) is the result of combining \( P \rightarrow Q \) and \( R \leftrightarrow \neg P \) with \( \land \). The propositional form \( P \rightarrow Q \) is from \( P \) and \( Q \) combined with \( \rightarrow \), and \( R \leftrightarrow \neg P \) is from \( R \) and \( \neg P \) combined with \( \leftrightarrow \). These and \( \neg P \) are propositional forms because \( P, Q \), and \( R \) are propositional variables. This derivation yields the following **parsing tree**:

```
  (P \rightarrow Q) \land (R \leftrightarrow \neg P)
     /     \
   P \rightarrow Q  R \leftrightarrow \neg P
     /     \
P     Q  R     \neg P
     /  \\  /  \
P P  \
```
The parsing tree yields the formation sequence of the propositional form:

\[ P, Q, R, \neg P, P \rightarrow Q, R \leftrightarrow \neg P, (P \rightarrow Q) \land (R \leftrightarrow \neg P). \]

The sequence is formed by listing each distinct term of the tree starting at the bottom row and moving upwards.

**EXAMPLE 1.1.3**

Make the following assignments:

\[
\begin{align*}
p &:= R \leftrightarrow (P \land Q), \\
q &:= (R \leftrightarrow P) \land Q.
\end{align*}
\]

The symbol := indicates that an assignment has been made. It means that the propositional form on the right has been assigned to the lowercase letter on the left. Using these designations, we can write new propositional forms using \(p\) and \(q\). The propositional form \(p \land q\) is

\[
(R \leftrightarrow (P \land Q)) \land ((R \leftrightarrow P) \land Q)
\]

with the formation sequence,

\[
P, Q, R, P \land Q, R \leftrightarrow P, \\
R \leftrightarrow (P \land Q), (R \leftrightarrow P) \land Q, [R \leftrightarrow (P \land Q)] \land [(R \leftrightarrow P) \land Q],
\]

and \(\neg q \rightarrow p\) is

\[
\neg[(R \leftrightarrow P) \land Q] \rightarrow [R \leftrightarrow (P \land Q)]
\]

with the formation sequence

\[
P, Q, R, R \leftrightarrow P, P \land Q, (R \leftrightarrow P) \land Q, R \leftrightarrow (P \land Q), \\
\neg[(R \leftrightarrow P) \land Q], \neg[(R \leftrightarrow P) \land Q] \rightarrow [R \leftrightarrow (P \land Q)].
\]

**Interpreting Propositional Forms**

Notice that determining whether a string is a propositional form is independent of the meaning that we give the symbols. However, as we do want these symbols to convey meaning, we assume that the propositional variables represent atoms and set this interpretation on the connectives:

\[
\begin{align*}
\neg & \quad \text{not} \\
\land & \quad \text{and} \\
\lor & \quad \text{or} \\
\rightarrow & \quad \text{implies} \\
\leftrightarrow & \quad \text{if and only if}
\end{align*}
\]

Because of this interpretation, name the compound propositional forms as follows:
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¬p   negation
p ∧ q  conjunction
p ∨ q  disjunction
p → q  implication
p ↔ q  biconditional

EXAMPLE 1.1.4

To see how this works, assign some propositions to some propositional variables:

\[ P := \text{The sine function is not one-to-one.} \]
\[ Q := \text{The square root function is one-to-one.} \]
\[ R := \text{The absolute value function is not onto.} \]

The following symbols represent the indicated propositions:

- \( \neg R \)
  - The absolute value function is onto.
- \( \neg P \lor \neg Q \)
  - The sine function is one-to-one, or the square root function is not one-to-one.
- \( Q \rightarrow R \)
  - If the square root function is one-to-one, the absolute function is not onto.
- \( R \leftrightarrow P \)
  - The absolute value function is not onto if and only if the sine function is not one-to-one.
- \( P \land Q \)
  - The sine function is not one-to-one, and the square root function is one-to-one.
- \( \neg P \land Q \)
  - The sine function is one-to-one, and the square root function is one-to-one.
- \( \neg(P \land Q) \)
  - It is not the case that the sine function is not one-to-one and the square root function is one-to-one.

The proposition

- the absolute value function is not onto if and only if
- both the sine function is not one-to-one and the square root function is one-to-one

is translated as \( R \leftrightarrow (P \land Q) \) and
the absolute value function is not onto if and only if the sine function is not one-to-one, and the square root function is one-to-one

is translated as \( (R \leftrightarrow P) \land Q \). If the parenthesis are removed, the resulting string is \( R \leftrightarrow P \land Q \). It is simpler, but it is not clear how it should be interpreted. To eliminate its ambiguity, we introduce an order of connectives as in algebra. In this way, certain strings without parentheses can be read as propositional forms.

**DEFINITION 1.1.5 [Order of Connectives]**

To interpret a propositional form, read from left to right and use the following precedence:

- propositional forms within parentheses or brackets (innermost first),
- negations,
- conjunctions,
- disjunctions,
- conditionals,
- biconditionals.

**EXAMPLE 1.1.6**

To write the propositional form \( \neg P \lor Q \land R \) with parentheses, we begin by interpreting \( \neg P \). According to the order of operations, the conjunction is next, so we evaluate \( Q \land R \). This is followed by the disjunction, and we have the propositional form \( \neg P \lor (Q \land R) \).

**EXAMPLE 1.1.7**

To interpret \( P \land Q \lor R \) correctly, use the order of operations. We discover that it has the same meaning as \( (P \land Q) \lor R \), but how is this distinguished from \( P \land (Q \lor R) \) in English? Parentheses are not appropriate because they are not used as grouping symbols in sentences. Instead, use *either...or*. Then, using the assignments from Example 1.1.4, \( (P \land Q) \lor R \) can be translated as

> either the sine function is not one-to-one
> and the square root function is one-to-one,
> or the absolute value function is not onto.

Notice that *either...or* works as a set of parentheses. We can use this to translate \( P \land (Q \lor R) \):

> the sine function is not one-to-one,
> and either the square root function is one-to-one
> or the absolute value function is not onto.
Be careful to note that the either-or phrasing is logically inclusive. For instance, some colleges require their students to take either logic or mathematics. This choice is meant to be exclusive in the sense that only one is needed for graduation. However, it is not logically exclusive. A student can take logic to satisfy the requirement yet still take a math class.

**EXAMPLE 1.1.8**

Let us interpret \( \neg(P \land Q) \). We can try translating this as *not* \( P \) and \( Q \), but this represents \( \neg P \land Q \) according to the order of operations. To handle a propositional form such as \( \neg(P \land Q) \), use a phrase like *it is not the case* or *it is false* and the word *both*. Therefore, \( \neg(P \land Q) \) becomes

\[
\text{it is not the case that both } P \text{ and } Q
\]

or

\[
\text{it is false that both } P \text{ and } Q.
\]

For instance, make the assignments.

\[ P := \text{quadratic equations have at most two real solutions}, \]
\[ Q := \text{the discriminant can be negative}. \]

Then,

\[
\text{quadratic equations do not have at most two real solutions, and the discriminant can be negative}
\]

is a translation of \( \neg P \land Q \). On the other hand, \( \neg(P \land Q) \) can be

\[
\text{it is not the case that both quadratic equations have at most two real solutions and the discriminant can be negative.}
\]

To interpret \( \neg P \land \neg Q \), use *neither-nor*:

\[
\text{neither do quadratic equations have at most two real solutions, nor can the discriminant be negative.}
\]

**Valuations and Truth Tables**

Propositions have truth values, but propositional forms do not. This is because every propositional form represents any one of infinitely many propositions. However, once a propositional form is identified with a proposition, there should be a process by which the truth value of the proposition is associated with the propositional form. This is done with a rule \( v \) called a *valuation*. The input of \( v \) is a propositional form, and its output is \( T \) or \( F \). Suppose that \( P \) is a propositional variable. If \( P \) has been assigned a proposition,

\[
v(P) = \begin{cases} T & \text{if } P \text{ is true,} \\ F & \text{if } P \text{ is false.} \end{cases}
\]
For example, if \( P := 2 + 3 = 5 \), then \( v(P) = T \), and if \( P := 2 + 3 = 7 \), then \( v(P) = F \). If \( P \) has not been assigned a proposition, then \( v(P) \) can be defined arbitrarily as either \( T \) or \( F \).

The valuation of a compound propositional form is defined using truth tables. Let \( p \) and \( q \) be given propositional forms. Along the top row write \( p \) and, if needed, \( q \). Draw a vertical line. To its right identify the desired propositional form consisting of \( p \), possibly \( q \), and a single connective. In the body of the table, on the left of the vertical line are all combinations of \( T \) and \( F \) for \( p \) and possibly \( q \). On the right are the results of applying the connective. Each connective will have its own truth table, and we want to define these tables so that they match our understanding of the meaning of each connective.

Since the truth value of the negation of a given proposition is the opposite of that proposition’s truth value,

\[
\begin{array}{c|c}
    p & \neg p \\
    \hline
    T & F \\
    F & T \\
\end{array}
\]

This means that \( v(\neg p) = F \) if \( v(p) = T \) and \( v(\neg p) = T \) if \( v(p) = F \).

The conjunction,

\[ 3 + 6 = 9, \text{ and all even integers are divisible by two}, \]

is true, but

\[ \text{all integers are rational, and 4 is odd} \]

is false because the second conjunct is false. The disjunction

\[ 3 + 7 = 9, \text{ or all even integers are divisible by three}, \]

is false since both disjuncts are false. On the other hand,

\[ 3 + 7 = 9, \text{ or circles are round} \]

is true. This illustrates that

\begin{itemize}
  \item a conjunction is true when both of its conjuncts are true, and false otherwise, and
  \item a disjunction is true when at least one disjunct is true, and false otherwise.
\end{itemize}

We use these principles to define the truth tables for \( p \land q \) and \( p \lor q \):

\[
\begin{array}{c|c|c}
    p & q & p \land q \\
    \hline
    T & T & T \\
    T & F & F \\
    F & T & F \\
    F & F & F \\
\end{array}
\quad
\begin{array}{c|c|c}
    p & q & p \lor q \\
    \hline
    T & T & T \\
    T & F & T \\
    F & T & T \\
    F & F & F \\
\end{array}
\]

We must remember that only one disjunct needs to be true for the entire disjunction to be true. For this reason, the logical disjunction is sometimes called an **inclusive or**. The propositional form for the **exclusive or** is

\[(p \lor q) \land \neg(p \land q).\]
There are many ways to understand an implication. Sometimes it represents causation as in

*if I score at least 70 on the exam, I will earn a passing grade.*

Other times it indicates what would have been the case if some past event had gone differently as in

*if I had not slept late, I would not have missed the meeting.*

Study of such conditional propositions is a very involved subject, one that need not concern us here because in mathematics a simpler understanding of the implication is enough. Suppose that \( P \) and \( Q \) are assigned propositions so that \( P \rightarrow Q \) is a true implication. In mathematics, this means that it is not the case that \( P \) is true but \( Q \) is false. This understanding of the conditional is known as **material implication**. For example,

*if rectangles have four sides, then squares have four sides,*

*if rectangles have three sides, then squares have four sides,*

and

*if rectangles have three sides, then squares have three sides*

are all true, but

*if rectangles have four sides, then squares have three sides*

is false. Generalizing, in mathematics, \( p \rightarrow q \) means \( \neg(p \land \neg q) \), which has the following truth table:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg q )</th>
<th>( p \land \neg q )</th>
<th>( \neg(p \land \neg q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The truth table for \( p \rightarrow q \) is then defined as follows:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The truth table for \( p \leftrightarrow q \) is simpler because we understand \( p \leftrightarrow q \) to mean

\[(p \rightarrow q) \land (q \rightarrow p).\]

The truth table for this propositional form requires five columns:
Therefore, define the truth table of $p \leftrightarrow q$ as:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \leftrightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

This understanding of the biconditional is known as **material equivalence**.

Using these truth tables, the valuation of an arbitrary propositional form can be defined.

**DEFINITION 1.1.9**

Let $p$ and $q$ be propositional forms.

- $v(\neg p) = \begin{cases} T & \text{if } v(p) = F, \\ F & \text{if } v(p) = T. \end{cases}$

- $v(p \land q) = \begin{cases} T & \text{if } v(p) = T \text{ and } v(q) = T, \\ F & \text{otherwise}. \end{cases}$

- $v(p \lor q) = \begin{cases} F & \text{if } v(p) = F \text{ and } v(q) = F, \\ T & \text{otherwise}. \end{cases}$

- $v(p \rightarrow q) = \begin{cases} F & \text{if } v(p) = T \text{ and } v(q) = F, \\ T & \text{otherwise}. \end{cases}$

- $v(p \leftrightarrow q) = \begin{cases} T & \text{if } v(p) = v(q), \\ F & \text{otherwise}. \end{cases}$

**EXAMPLE 1.1.10**

Consider the propositional form $(P \leftrightarrow Q) \lor (R \rightarrow P)$ where

$v(P) = F$, $v(Q) = T$, and $v(R) = F$.

Then, $v(P \leftrightarrow Q) = F$ because $v(P) \neq v(Q)$, and $v(R \rightarrow P) = T$ because $v(R) = F$. Therefore, because $v(R \rightarrow P) = T$,

$v([P \leftrightarrow Q] \lor [R \rightarrow P]) = T,$
We now generalize the definition of a truth table to create truth tables for more complicated propositional forms and then use the tables to find the valuation of a propositional form given the valuations of its proposition variables.

**EXAMPLE 1.1.11**

To write the truth table of \( P \rightarrow Q \land \neg P \), identify the column headings by drawing the parsing tree for this form:

```
P \rightarrow Q \land \neg P
    |      |
    P    Q \land \neg P
        |      |
        Q    \neg P
            |
            \neg P
```

Reading from the bottom, we see that a formation sequence for the propositional form is

\[ P, Q, \neg P, Q \land \neg P, P \rightarrow Q \land \neg P. \]

Hence, the truth table for this form is

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>\neg P</th>
<th>Q \land \neg P</th>
<th>P \rightarrow Q \land \neg P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
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<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

So, if \( v(P) = T \) and \( v(Q) = F \),

\[ v(P \rightarrow Q \land \neg P) = F. \]

That is, any proposition represented by \( P \rightarrow Q \land \neg P \) is false when the proposition assigned to \( P \) is true and the proposition assigned to \( Q \) is false.

The propositional form in the next example has three propositional variables. To make clear the truth value pattern that is to the left of the vertical line, note that if there are \( n \) variables, the number of rows is twice the number of rows for \( n - 1 \) variables. To see this, start with one propositional variable. Such a truth table has only two rows. Add a variable, we obtain four rows. The pattern is obtained by writing the one variable case twice. For the first time, it has a \( T \) written in front of each row. The second copy has an \( F \) in front of each row. To obtain the pattern for three variables, copy the two-variable pattern twice as in Figure 1.1. To generalize, if there are \( n \) variables, there will be \( 2^n \) rows.
Example 1.1.12

Use a truth table to find the truth value of

\[ \text{if the derivative of the sine function is the cosine function} \]
\[ \text{and the second derivative of the sine function is the sine function,} \]
\[ \text{then the third derivative of the sine function is the cosine function.} \]

Define:

\[ P := \text{the derivative of the sine function is the cosine function}, \]
\[ Q := \text{the second derivative of the sine function is the sine function}, \]
\[ R := \text{the third derivative of the sine function is the cosine function}. \]

So \( P \) represents a true proposition, but \( Q \) and \( R \) represent false propositions. The proposition is represented by

\[ P \land Q \rightarrow R \]

with truth table:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( R )</th>
<th>( P \land Q )</th>
<th>( P \land Q \rightarrow R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Notice that we could have determined the truth value by simply writing one line from the truth table:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( R )</th>
<th>( P \land Q )</th>
<th>( P \land Q \rightarrow R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

We see that \( \nu(P \land Q \rightarrow R) = T \) when \( \nu(P) = T, \nu(Q) = F, \) and \( \nu(R) = F. \)
Therefore, the proposition is true.
EXAMPLE 1.1.13

Both \( P \lor \neg P \) and \( P \rightarrow P \) share an important property. Their columns in their truth tables are all T. For example, the truth table of \( P \lor \neg P \) is:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \neg P )</th>
<th>( P \lor \neg P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Therefore, \( v(P \lor \neg P) \) always equals T, no matter the choice of \( v \).

However, the columns for \( P \land \neg P \) and \( P \leftrightarrow \neg P \) are all F. To check the first one, examine its truth table:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \neg P )</th>
<th>( P \land \neg P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

This means that \( v(P \land \neg P) \) is always F for every valuation \( v \).

Based on the last example, we make the next definition.

DEFINITION 1.1.14

A propositional form \( p \) is a **tautology** if \( v(p) \) always equals T for every valuation \( v \), and \( p \) is a **contradiction** if \( v(p) \) always equals F for every \( v \). A propositional form that is neither a tautology nor a contradiction is called a **contingency**.

Exercises

1. Identify each sentence as either a proposition or not a proposition. Explain.
   (a) *Trisect the angle.*
   (b) *Some exponential functions are increasing.*
   (c) *All exponential functions are increasing.*
   (d) \( 3 + 8 = 18 \)
   (e) \( 3 + x = 18 \)
   (f) *Yea, logic!*
   (g) *A triangle is a three-sided polygon.*
   (h) *The function is differentiable.*
   (i) *This proposition is true.*
   (j) *This proposition is not true.*

2. Identify the antecedent and the consequent for the given implications.
   (a) *If the triangle has two congruent sides, it is isosceles.*
   (b) *The polynomial has at most two roots if it is a quadratic.*
   (c) *The data is widely spread only if the standard deviation is large.*
   (d) *The function being constant implies that its derivative is zero.*
   (e) *The system of equations is consistent is necessary for it to have a solution.*
(f) A function is even is sufficient for its square to be even.

3. Give the truth value of each proposition.
   (a) A system of equations always has a solution, or a quadratic equation always has a real solution.
   (b) It is false that every polynomial function in one variable is differentiable.
   (c) Vertical lines have no slope, and lines through the origin have a positive y-intercept.
   (d) Every integer is even, or every even natural number is an integer.
   (e) If every parabola intersects the x-axis, then an ellipse has only one vertex.
   (f) The sine function is periodic if and only if every exponential function is always nonnegative.
   (g) It is not the case that $2 + 4 \neq 6$.
   (h) The distance between two points is always positive if every line segment is horizontal.
   (i) The derivative of a constant function is zero is necessary for the product rule to be true.
   (j) The derivative of the sine function being cosine is sufficient for the derivative of the cosine function being sine.
   (k) Any real number is negative or positive, but not both.

4. For each sentence, fill in the blank using as many of the words and, or, if, and if and only if as possible to make the proposition true.
   (a) Triangles have three sides _______ $3 + 5 = 6$.
   (b) $3 + 5 = 6$ _______ triangles have three sides.
   (c) Ten is the largest integer _______ zero is the smallest integer.
   (d) The derivative of a constant function is zero _______ tangent lines for increasing functions have positive slope.

5. Extend Figure 1.1 by writing the typical pattern of Ts and Fs for the truth table of a propositional form with four propositional variables and then with five propositional variables.

6. Use a parsing tree to show that the given string is a propositional form.
   (a) $P \land Q \lor R$
   (b) $Q \leftrightarrow R \lor \neg Q$
   (c) $P \rightarrow Q \rightarrow R \rightarrow S$
   (d) $\neg P \land Q \lor (P \rightarrow Q) \land \neg S$
   (e) $(P \land Q \rightarrow Q) \land P \rightarrow Q$
   (f) $\neg \neg P \lor P \land S \rightarrow Q \lor [R \rightarrow \neg P \rightarrow \neg(Q \lor R)]$

7. Define:
   
   $P := \text{the angle sum of a triangle is 180},$
   $Q := 3 + 7 = 10,$
   $R := \text{the sine function is continuous}.$
Translate the given propositional forms into English.

(a) \( P \lor Q \)
(b) \( P \land Q \)
(c) \( P \land \neg Q \)
(d) \( Q \lor \neg R \)
(e) \( Q \leftrightarrow \neg R \)
(f) \( R \rightarrow Q \)
(g) \( P \lor R \rightarrow \neg Q \)
(h) \( Q \leftrightarrow R \land \neg Q \)
(i) \( \neg(P \land Q) \)
(j) \( \neg(P \lor Q) \)
(k) \( P \lor Q \land R \)
(l) \( (P \lor Q) \land R \)

8. Write the following sentences as propositional forms using the variables \( P, Q, \) and \( R \) as defined in Exercise 7.

(a) The sine function is continuous, and \( 3 + 7 = 10 \).
(b) The angle sum of a triangle is 180, or the angle sum of a triangle is 180.
(c) If \( 3 + 7 = 10 \), then the sine function is not continuous.
(d) The angle sum of a triangle is 180 if and only if the sine function is continuous.
(e) The sine function is continuous if and only if \( 3 + 7 = 10 \) implies that the angle sum of a triangle is not 180.
(f) It is not the case that \( 3 + 7 \neq 10 \).

9. Let \( \nu(P) = T, \nu(Q) = T, \nu(R) = F, \) and \( \nu(S) = F \). Find the given valuations. (See Exercise 6.)

(a) \( P \land Q \lor R \)
(b) \( Q \leftrightarrow R \lor \neg Q \)
(c) \( P \rightarrow Q \rightarrow R \rightarrow S \)
(d) \( \neg P \land Q \lor (P \rightarrow Q) \land \neg S \)
(e) \( (P \land Q \rightarrow Q) \land P \rightarrow Q \)
(f) \( \neg P \lor P \land S \lor Q \lor [R \rightarrow \neg P \rightarrow \neg(Q \lor R)] \)

10. Write the truth table for each of the given propositional forms.

(a) \( \neg P \rightarrow P \)
(b) \( P \rightarrow \neg Q \)
(c) \( (P \lor Q) \land \neg(P \land Q) \)
(d) \( (P \rightarrow Q) \lor (Q \leftrightarrow P) \)
(e) \( P \land (Q \lor R) \)
(f) \( P \lor Q \rightarrow R \)
(g) \( P \rightarrow Q \land \neg(R \lor P) \)
(h) \( P \rightarrow Q \leftrightarrow R \rightarrow S \)
(i) \( P \lor (\neg Q \leftrightarrow R) \land Q \)
11. Check the truth value of these propositions using truth tables as in Example 1.1.12.
   (a) If \(2 + 3 = 7\), then \(5 - 9 \neq 0\).
   (b) If a square is round implies that some functions have a derivative at \(x = 2\), then every function has a derivative at \(x = 2\).
   (c) Either four is odd or two is even implies that three is even.
   (d) Every even integer is divisible by 4 if and only if either 7 divides 21 or 9 divides 12.
   (e) The graph of the tangent function has asymptotes, and if sine is an increasing function, then cosine is a decreasing function.

12. If possible, find propositional forms \(p\) and \(q\) such that
   (a) \(p \land q\) is a tautology.
   (b) \(p \lor q\) is a contradiction.
   (c) \(\neg p\) is a tautology.
   (d) \(p \rightarrow q\) is a contradiction.
   (e) \(p \leftrightarrow q\) is a tautology.
   (f) \(p \leftrightarrow q\) is a contradiction.

1.2 INFERENCE

Now that we have a collection of propositional forms and a means by which to interpret them as either true or false, we want to define a system that expands these ideas to include methods by which we can prove certain propositional forms from given propositional forms. What we will define is familiar because it is similar to what Euclid did with his geometry. Take, for example, the familiar result,

\[\text{opposite angles in a parallelogram are congruent.}\]

In other words,

\[\text{if } ABCD \text{ is a parallelogram, then } \angle B \cong \angle D,\]

which translates to:

\[
\begin{align*}
\text{Given: } & \quad ABCD \text{ is a parallelogram,} \\
\text{Prove: } & \quad \angle B \cong \angle D.
\end{align*}
\]

To demonstrate this, we draw a diagram,
Euclid’s geometry consists of geometric propositions that are established by proofs like the above. These proofs rely on rules of logic, previously proved propositions (lemmas, theorems, and corollaries), and propositions that are assumed to be true (the postulates). Using this system of thought, we can show which geometric propositions follow from the postulates and conclude which propositions are true, whatever it means for a geometric proposition to be true. Euclidean geometry serves as a model for the following modern definition.

**DEFINITION 1.2.1**

A logical system consists of the following:

- An alphabet
- A grammar
- Propositional forms that require no proof
- Rules that determine truth
- Rules that are used to write proofs.

Although Euclid did not provide an alphabet or a grammar specifically for his geometry, his system did include the last three aspects of a logical system. In this chapter we develop the logical system known as propositional logic. Its alphabet, grammar, and rules that determine truth were defined in Section 1.1. The remainder of this chapter is spent establishing the other two components.

Consider the following collection of propositions:

If squares are rectangles, then squares are quadrilaterals.  
Squares are rectangles.  
Therefore, squares are quadrilaterals.

This is an example of a deduction, a collection of propositions of which one is supposed to follow necessarily from the others. In this particular case,

if squares are rectangles, then squares are quadrilaterals

and
squares are rectangles
are the premises, and

squares are quadrilaterals

is the conclusion. We recognize that in this case, the conclusion does follow from the premises because whenever the premises are true, the conclusion must also be true. When this is the case, the deduction is semantically valid, else it is semantically invalid.

Notice that not only do we see that the deduction works because of the meaning of the propositions, but we also see that it is valid based on the forms of the sentences. In other words, we also recognize this deduction as valid:

If Hausdorff spaces are preregular, their points can be separated. 
Hausdorff spaces are preregular. 
Therefore, their points can be separated.

Although we might not know the terms Hausdorff space, preregular, and separated, we recognize the deduction as valid because it is of the same pattern as the first deduction:

\[
p \rightarrow q \\
p \\
\therefore q
\]

When the deduction is found to work based on its form, the deduction is syntactically valid, else it is syntactically invalid.

We study both types of validity by examining general patterns of deductions and choosing rules that determine which forms correspond to deductions that are valid semantically and which forms correspond to deductions that are valid syntactically.

Semantics

The study of meaning is called semantics. We began this study when we wrote truth tables. These are characterized as semantic because the truth value of a proposition is based on its meaning. Our goal is to use truth tables to determine when an argument form, an example being (1.2), corresponds to a deduction that is semantically valid. We begin with a definition.

DEFINITION 1.2.2

Let \( p_0, p_1, \ldots, p_{n-1} \) and \( q \) be propositional forms.

- If \( q \) is a tautology, write \( \models q \).
- Define \( p_0, p_1, \ldots, p_{n-1} \) to logically imply \( q \) if

\[
\models p_0 \land p_1 \land \cdots \land p_{n-1} \rightarrow q.
\]
When \( p_0, p_1, \ldots, p_{n-1} \) logically imply \( q \), write

\[
p_0, p_1, \ldots, p_{n-1} \models q.
\]

and say that \( q \) is a consequence of \( p_0, p_1, \ldots, p_{n-1} \). Call the propositional forms \( p_0, p_1, \ldots, p_{n-1} \) the premises of the implication and \( q \) the conclusion.

Notice that if \( p_0, p_1, \ldots, p_{n-1} \models q \), then for any valuation \( v \), whenever \( v(p_i) = T \) for all \( i = 0, 1, \ldots, n - 1 \), it must be the case that \( v(q) = T \). Moreover, any deduction with premises represented by \( p_0, p_1, \ldots, p_{n-1} \) and conclusion by \( q \) is semantically valid if \( p_0, p_1, \ldots, p_{n-1} \models q \).

**EXAMPLE 1.2.3**

Because of Example 1.1.13, both \( \models P \rightarrow P \) and \( \models P \lor \neg P \).

**EXAMPLE 1.2.4**

Prove: \( P \rightarrow Q, P \models Q \)

To accomplish this, show that the propositional form

\[
(P \rightarrow Q) \land P \rightarrow Q,
\]

with antecedent equal to the conjunction of the premises and consequent consisting of the conclusion is a tautology.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \rightarrow Q )</th>
<th>( (P \rightarrow Q) \land P \rightarrow Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Therefore, \( \models (P \rightarrow Q) \land P \rightarrow Q \), so

\[
P \rightarrow Q, P \models Q,
\]

and any deduction based on this form is semantically valid.

**EXAMPLE 1.2.5**

Prove: \( P \lor Q \rightarrow Q, P \models Q \)

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \lor Q )</th>
<th>( P \lor Q \rightarrow Q )</th>
<th>( (P \lor Q \rightarrow Q) \land P \rightarrow Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
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<td>T</td>
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<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

If it is possible for \( v(p_i) = T \) for \( i = 0, 1, \ldots, n - 1 \) yet \( v(q) = F \), the propositional form

\[
p_0 \land p_1 \land \cdots \land p_{n-1} \rightarrow q
\]
is not a tautology, so $q$ is not a consequence of $p_0, p_1, \ldots, p_{n-1}$. If this is the case, write

\[ p_0, p_1, \ldots, p_{n-1} \not\models q. \]

**EXAMPLE 1.2.6**

Prove: $P \land Q \rightarrow Q, P \not\models Q$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \land Q$</th>
<th>$(P \land Q) \land P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Notice that $F$ appears for $(P \land Q) \land P \rightarrow Q$ on a line when $v(P \land Q) = v(P) = T$ yet $v(Q) = F$. Because of this, we can shorten the procedure for showing that a propositional form is not a consequence of other propositional forms.

**EXAMPLE 1.2.7**

Prove: $P \land Q \rightarrow R \not\models P \rightarrow R$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$P \land Q \rightarrow R$</th>
<th>$P \rightarrow R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

Observe that this shows that the valuation of $P \land Q \rightarrow R$ can be $T$ at the same time that the valuation of $P \rightarrow R$ is $F$.

**Syntactics**

Although we will return to semantics, it is important to note that using truth tables to check for logical implication has its limitations. If the argument form involves many propositional forms or if the propositional forms are complicated, the truth table used to show or disprove the logical implication can become unwieldy. Another issue is that in practice, truth tables are not the method of choice when determining whether a conclusion follows from the premises. What is typically done is to follow Euclid’s example (page 20), basing the conclusions on the syntax of the argument form, namely, based only on its pattern and structure.

Let us return to the deduction on page 20 and work with it differently. Start with the two propositions,

\[ \text{if squares are rectangles, then squares are quadrilaterals} \]

and

\[ \text{squares are rectangles}. \]

Because of the combined structure of the two sentences, we know that we can write
squares are quadrilaterals.

This act of writing (on paper or a blackboard or in the mind) means that we have the proposition and that it follows from the first two. Similarly, if we start with

squares are triangles, or squares are rectangles

and

squares are not triangles,

we can write

squares are rectangles.

Determining a method that will model this reasoning requires us to find rules by which propositional forms can be written from other propositional forms. Since every logical system requires a starting point, the first step in this process is to choose which propositional forms can be written without any prior justification. Each such propositional form is called an axiom. Playing the same role as that of a postulate in Euclidean geometry, an axiom can be considered as a rule of the game. Certain propositional forms lend themselves as good candidates for axioms because they are regarded as obvious. That is, they are self-evident. Other propositional forms are good candidates to be axioms, not because they are necessarily self-evident, but because they are helpful. In either case, the number of axioms should be as few as possible so as to minimize the number of assumptions. For propositional logic, we choose only three. They were first found in work of Gottlob Frege (1879) and later in that of Jan Łukasiewicz (1930).

**AXIOMS 1.2.8** [Frege–Łukasiewicz]

Let \( p, q, \) and \( r \) be propositional forms.

- [FL1] \( p \to (q \to p) \)
- [FL2] \( p \to (q \to r) \to (p \to q \to [p \to r]) \)
- [FL3] \( \neg p \to \neg q \to (q \to p) \).

The next step in defining propositional logic is to state when it is legal to write a propositional form from given propositional forms.

**DEFINITION 1.2.9**

The propositional forms \( p_0, p_1, \ldots, p_{n-1} \) infer \( q \) if \( q \) can be written whenever \( p_0, p_1, \ldots, p_{n-1} \) are written. Denote this by

\[ p_0, p_1, \ldots, p_{n-1} \Rightarrow q. \]

This is known as an inference.

To make rigorous which propositional forms can be inferred from given forms, we establish some rules. These are chosen because they model basic reasoning. They are also not proved, so they serve as postulates for our logic.
INFERENCE RULES 1.2.10

Let \( p, q, r, \) and \( s \) be propositional forms.

- **Modus Ponens** [MP]
  \( p \rightarrow q, p \Rightarrow q \)

- **Modus Tolens** [MT]
  \( p \rightarrow q, \neg q \Rightarrow \neg p \)

- **Constructive Dilemma** [CD]
  \( (p \rightarrow q) \land (r \rightarrow s), p \lor r \Rightarrow q \lor s \)

- **Destructive Dilemma** [DD]
  \( (p \rightarrow q) \land (r \rightarrow s), \neg q \lor \neg s \Rightarrow \neg p \lor \neg r \)

- **Disjunctive Syllogism** [DS]
  \( p \lor q, \neg p \Rightarrow q \)

- **Hypothetical Syllogism** [HS]
  \( p \rightarrow q, q \rightarrow r \Rightarrow p \rightarrow r \)

- **Conjunction** [Conj]
  \( p, q \Rightarrow p \land q \)

- **Simplification** [Simp]
  \( p \land q \Rightarrow p \)

- **Addition** [Add]
  \( p \Rightarrow p \lor q. \)

To use Inference Rules 1.2.10, match the form exactly. For example, even though \( P \rightarrow R \) appears to follow from \((P \land Q) \rightarrow R\) as an application of simplification, it does not. The problem is that simplification can only be applied to propositional forms with the \( p \land q \) pattern, but \((P \land Q) \rightarrow R\) is of the form \( p \rightarrow q \). With this detail in mind, we make some inferences.

EXAMPLE 1.2.11

Each inference is justified by the indicated rule.

- **Modus ponens**
  \( P \land Q \rightarrow \neg R, P \land Q \Rightarrow \neg R \)

- **Addition**
  \( P \Rightarrow P \lor Q \land R \)

- **Modus tolens**
  \( \neg \neg P, Q \lor R \rightarrow \neg P \Rightarrow \neg(Q \lor R) \).
EXAMPLE 1.2.12

Since it is possible that some propositional forms are not needed for the inference, we also have the following:

- **Modus ponens**
  \[ P \land Q \rightarrow \neg R, \, P \lor Q, \, P \land Q, \, Q \leftrightarrow S \Rightarrow \neg R \]

- **Addition**
  \[ P, \, R, \, S \rightarrow T \Rightarrow P \lor Q \land R \]

- **Modus tolens**
  \[ \neg S, \, \neg\neg P, \, P \land T, \, Q \lor R \rightarrow \neg P \Rightarrow \neg(Q \lor R). \]

Inference is a powerful tool, but it can only be used to check simple deductions. Sometimes multiple inferences are needed to move from a collection of premises to a conclusion. For example, if we write

\[ p \lor q, \neg p, \, q \rightarrow r, \]

based on the first two propositional forms, we can write

\[ q \]

by DS, and then based on this propositional form and the third of the given propositional forms, we can write

\[ r \]

by MP. This is a simple example of the next definition.

DEFINITION 1.2.13

- A **formal proof** of the propositional form \( q \) (the **conclusion**) from the propositional forms \( p_0, p_1, \ldots, p_{n-1} \) (the **premises**) is a sequence of propositional forms,

\[ p_0, p_1, \ldots, p_{n-1}, q_0, q_1, \ldots, q_{m-1}, \]

such that \( q_{m-1} = q \), and for all \( i = 0, 1, \ldots, m - 1 \), either \( q_i \) is an axiom,

- if \( i = 0 \), then \( p_0, p_1, \ldots, p_{n-1} \Rightarrow q_i \), or

- if \( i > 0 \), then \( p_0, p_1, \ldots, p_{n-1}, q_0, q_1, \ldots, q_{i-1} \Rightarrow q_i \).

If there exists a formal proof of \( q \) from \( p_0, p_1, \ldots, p_{n-1} \), then \( q \) is **proved** or **deduced** from \( p_0, p_1, \ldots, p_{n-1} \) and we write

\[ p_0, p_1, \ldots, p_{n-1} \vdash q. \]

- If there are no premises, a **formal proof** of \( q \) is a sequence,

\[ q_0, q_1, \ldots, q_{m-1}, \]
such that \( q_0 \) is an axiom, \( q_{m-1} = q \), and for all \( i > 0 \), either \( q_i \) is an axiom or

\[
q_0, q_1, \ldots, q_{i-1} \Rightarrow q_i.
\]

In this case, write \( \vdash q \) and call \( q \) a theorem.

Observe that any deduction with premises represented by \( p_0, p_1, \ldots, p_{n-1} \) and conclusion by \( q \) is syntactically valid if \( p_0, p_1, \ldots, p_{n-1} \vdash q \).

We should note that although \( \Rightarrow \) and \( \vdash \) have different meanings as syntactic symbols, they are equivalent. If \( p \Rightarrow q \), then \( p \vdash q \) using the proof \( p, q \). Conversely, suppose \( p \vdash q \). This means that there exists a proof

\[
p, q_0, q_1, \ldots, q_{n-1}, q,
\]

so every time we write down \( p \), we can also write down \( q \). That is, \( p \Rightarrow q \). We summarize this as follows.

**THEOREM 1.2.14**

For all propositional forms \( p \) and \( q \), \( p \Rightarrow q \) if and only if \( p \vdash q \).

We use a particular style to write formal proofs. They will be in two-column format with each line being numbered. In the first column will be the sequence of propositional forms that make up the proof. In the second column will be the reasons that allowed us to include each form. The only reasons that we will use are

- *Given* (for premises),
- FL1, FL2, or FL3 (for an axiom),
- An inference rule.

An inference rule is cited by giving the line numbers used as the premises followed by the abbreviation for the rule. Thus, the following proves \( P \vee Q \rightarrow Q \wedge R, P \vdash Q \):

\[
\begin{array}{ll}
1. & P \vee Q \rightarrow Q \wedge R \quad \text{Given} \\
2. & P \quad \text{Given} \\
3. & P \vee Q \quad 2 \text{ Add} \\
4. & Q \wedge R \quad 1, 3 \text{ MP} \\
5. & Q \quad 4 \text{ Simp}
\end{array}
\]

Despite the style, we should remember that a proof is a sequence of propositional forms that satisfy Definition 1.2.13. In this case, the sequence is

\[
P \vee Q \rightarrow Q \wedge R, P, P \vee Q, Q \wedge R, Q.
\]

The first two examples involve proofs that use the axioms.
EXAMPLE 1.2.15

Prove: $\vdash P \rightarrow Q \rightarrow (P \rightarrow P)$

1. $P \rightarrow (Q \rightarrow P)$ FL1
2. $P \rightarrow (Q \rightarrow P) \rightarrow (P \rightarrow Q \rightarrow [P \rightarrow P])$ FL2
3. $P \rightarrow Q \rightarrow (P \rightarrow P)$ MP

This proves that $P \rightarrow Q \rightarrow (P \rightarrow P)$ is a theorem. Also, by adding $P \rightarrow Q$ as a given and an application of MP at the end, we can prove

$$P \rightarrow Q \vdash P \rightarrow P.$$ 

This result should not be surprising since $P \rightarrow P$ is a tautology. We would expect any premise to be able to prove it.

EXAMPLE 1.2.16

Prove: $\neg(Q \rightarrow P), \neg P \vdash \neg Q$

1. $\neg(Q \rightarrow P)$ Given
2. $\neg P$ Given
3. $\neg P \rightarrow \neg Q \rightarrow (Q \rightarrow P)$ FL3
4. $\neg P \rightarrow \neg Q$ 1, 3 MT
5. $\neg Q$ 2, 4 MP

The next three examples do not use an axiom in their proofs.

EXAMPLE 1.2.17

Prove: $P \rightarrow Q, Q \rightarrow R, S \lor \neg R, \neg S \vdash \neg P$

1. $P \rightarrow Q$ Given
2. $Q \rightarrow R$ Given
3. $S \lor \neg R$ Given
4. $\neg S$ Given
5. $P \rightarrow R$ 1, 2 HS
6. $\neg R$ 3, 4 DS
7. $\neg P$ 5, 6 MT

EXAMPLE 1.2.18

Prove: $P \rightarrow Q, P \rightarrow Q \rightarrow (T \rightarrow S), P \lor T, \neg Q \vdash S$

1. $P \rightarrow Q$ Given
2. $P \rightarrow Q \rightarrow (T \rightarrow S)$ Given
3. $P \lor T$ Given
4. \( \neg Q \) Given
5. \( T \rightarrow S \) 1, 2 MP
6. \( (P \rightarrow Q) \land (T \rightarrow S) \) 1, 5 Conj
7. \( Q \lor S \) 3, 6 CD
8. \( S \) 4, 7 DS

EXAMPLE 1.2.19

Prove: \( P \rightarrow Q, Q \rightarrow R, \neg R \vdash \neg Q \lor \neg P \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( P \rightarrow Q )</td>
<td>Given</td>
</tr>
<tr>
<td>2.</td>
<td>( Q \rightarrow R )</td>
<td>Given</td>
</tr>
<tr>
<td>3.</td>
<td>( \neg R )</td>
<td>Given</td>
</tr>
<tr>
<td>4.</td>
<td>( (Q \rightarrow R) \land (P \rightarrow Q) )</td>
<td>1, 2 Conj</td>
</tr>
<tr>
<td>5.</td>
<td>( \neg R \lor \neg Q )</td>
<td>3 Add</td>
</tr>
<tr>
<td>6.</td>
<td>( \neg Q \lor \neg P )</td>
<td>4, 5 DD</td>
</tr>
</tbody>
</table>

Exercises

1. Show using truth tables.
   (a) \( \neg P \lor Q, \neg Q \vdash \neg P \)
   (b) \( \neg (P \land Q), P \vdash \neg Q \)
   (c) \( P \rightarrow Q, P \vdash Q \lor R \)
   (d) \( P \rightarrow Q, Q \rightarrow R, P \vdash R \)
   (e) \( P \lor Q \lor R, \neg P \vdash R \)

2. Show the following using truth tables.
   (a) \( \neg (P \land Q) \not\equiv \neg P \)
   (b) \( P \rightarrow Q \lor R, P \not\equiv Q \)
   (c) \( P \land Q \rightarrow R \not\equiv Q \rightarrow R \)
   (d) \( (P \rightarrow Q) \lor (R \rightarrow S), P \lor R \not\equiv Q \lor S \)
   (e) \( \neg (P \land Q) \lor R, P \land Q \lor S \not\equiv R \land S \)
   (f) \( P \lor R, Q \lor S, R \leftrightarrow S \not\equiv R \land S \)

3. Identify the rule from Inference Rules 1.2.10.
   (a) \( P \rightarrow Q \rightarrow P, P \rightarrow Q \Rightarrow P \)
   (b) \( P, Q \lor R \Rightarrow P \land (Q \lor R) \)
   (c) \( P \Rightarrow P \lor (R \leftrightarrow \neg P \land \neg (Q \rightarrow S)) \)
   (d) \( P, P \rightarrow (Q \leftrightarrow S) \Rightarrow Q \leftrightarrow S \)
   (e) \( P \lor Q \lor Q, (P \lor Q \rightarrow Q) \land (Q \rightarrow S \land T) \Rightarrow Q \lor S \land T \)
   (f) \( P \lor (Q \land S), \neg P \Rightarrow Q \lor S \)
   (g) \( P \rightarrow \neg Q, \neg \neg Q \Rightarrow \neg P \)
   (h) \( (P \rightarrow Q) \land (Q \rightarrow R), \neg Q \lor \neg R \Rightarrow \neg P \lor \neg Q \)
   (i) \( (P \rightarrow Q) \land (Q \rightarrow R) \Rightarrow P \rightarrow Q \)
4. Arrange each collection of propositional forms into a proof for the given deductions and supply the appropriate reasons.

(a) \( P \to Q, R \to S, P \vdash Q \lor S \)
   - \( P \)
   - \( Q \lor S \)
   - \( R \to S \)
   - \( (P \to Q) \land (R \to S) \)
   - \( P \lor R \)
   - \( P \to Q \)

(b) \( P \to Q, Q \to R, P \vdash R \lor Q \)
   - \( P \)
   - \( P \to R \)
   - \( P \to Q \)
   - \( R \)
   - \( Q \to R \)
   - \( R \lor Q \)

(c) \((P \to Q) \lor (Q \to R), \neg(P \to Q), \neg R, Q \lor S \vdash S \)
   - \( \neg(P \to Q) \)
   - \( S \)
   - \( (P \to Q) \lor (Q \to R) \)
   - \( Q \to R \)
   - \( \neg R \)
   - \( \neg Q \)
   - \( Q \lor S \)

(d) \((P \lor Q) \land R, Q \lor S \to T, \neg P \vdash \neg P \land T \)
   - \( P \lor Q \)
   - \( Q \)
   - \( Q \lor S \)
   - \( T \)
   - \( (P \lor Q) \land R \)
   - \( Q \lor S \to T \)
   - \( \neg P \)
   - \( \neg P \land T \)

5. Prove using Axioms 1.2.8.

(a) \( \vdash P \to P \)

(b) \( \vdash \neg \neg P \to P \)

(c) \( \vdash P \to (P \to [Q \to P]) \)

(d) \( \neg P \vdash P \to Q \)

(e) \( P \to Q, Q \to R \vdash P \to R \) (Do not use HS.)

(f) \( P \to Q, \neg Q \vdash \neg P \) (Do not use MT.)
6. Prove. Axioms 1.2.8 are not required.

(a) \( P \rightarrow Q, P \lor (R \rightarrow S), \neg Q \vdash R \rightarrow S \)
(b) \( P \rightarrow Q, Q \rightarrow R, \neg R \vdash \neg P \)
(c) \( P \rightarrow Q, R \rightarrow S, \neg Q \lor \neg S \vdash \neg P \lor \neg R \)
(d) \( [P \rightarrow (Q \rightarrow R)] \land [Q \rightarrow (R \rightarrow P)], P \lor Q, \neg (Q \rightarrow R), \neg P \vdash \neg R \)
(e) \( P \rightarrow Q, Q \rightarrow R, \neg R \vdash \neg P \)
(f) \( P \rightarrow Q, R \rightarrow S, S \rightarrow T, P \rightarrow Q, R \vdash T \)
(g) \( P \rightarrow Q, Q \rightarrow R, (Q \land R), Q \land R \lor (\neg P \rightarrow S) \vdash S \)
(h) \( P \lor Q \rightarrow \neg R \land \neg S, Q \rightarrow R, P \vdash \neg Q \)
(i) \( N \rightarrow P, P \lor Q \lor R \rightarrow S \lor T, S \lor T \rightarrow T, N \vdash T \)
(j) \( P \rightarrow Q, Q \rightarrow R, R \rightarrow S, S \rightarrow T, P \lor R, \neg R \vdash T \)
(k) \( P \lor Q \rightarrow R \lor S, (R \rightarrow T) \land (S \rightarrow U), P, \neg T \vdash U \)
(l) \( P \rightarrow Q, Q \rightarrow R, R \rightarrow S, (P \lor Q) \land (R \lor S) \vdash Q \lor S \)
(m) \( P \lor \neg Q \lor R \rightarrow (S \rightarrow P), P \lor \neg Q \rightarrow (P \rightarrow R), P \vdash S \rightarrow R \)

1.3 REPLACEMENT

There are times when writing a formal proof that we want to substitute one propositional form for another. This happens when two propositional forms have the same valuations. It also happens when a particular sentence pattern should be able to replace another sentence pattern. We give rules in this section that codify both ideas.

Semantics

Consider the propositional form \( \neg (P \lor Q) \). Its valuation equals T when it is not the case that \( v(P) = T \) or \( v(Q) = T \) (Definition 1.1.9). This implies that \( v(P) = F \) and \( v(Q) = F \), so \( v(\neg P \land \neg Q) = T \). Conversely, the valuation of \( \neg P \land \neg Q \) is T implies that the valuation of \( \neg (P \lor Q) \) is T for similar reasons. Since no additional premises were assumed in this discussion, we conclude that

\[
v(\neg[P \lor Q]) = v(\neg P \land \neg Q),
\]

and this means that

\( \neg (P \lor Q) \leftrightarrow \neg P \land \neg Q \)

is a tautology. There is a name for this.

**DEFINITION 1.3.1**

Two propositional forms \( p \) and \( q \) are **logically equivalent** if \( \models p \leftrightarrow q \).

Observe that Definition 1.3.1 implies the following result.

**THEOREM 1.3.2**

All tautologies are logically equivalent, and all contradictions are logically equivalent.
Because of (1.3), we can use a truth table to prove logical equivalence.

**EXAMPLE 1.3.3**

Prove: \( \vdash \neg(P \lor Q) \iff \neg P \land \neg Q \).

```
<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \lor Q</th>
<th>\neg(P \lor Q)</th>
<th>\neg P</th>
<th>\neg Q</th>
<th>\neg P \land \neg Q</th>
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<tbody>
<tr>
<td>T</td>
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**EXAMPLE 1.3.4**

Because \( \vdash P \rightarrow Q \iff \neg P \lor Q \), we can replace \( P \rightarrow Q \) with \( \neg P \lor Q \), and vice versa, at any time. When this is done, the resulting propositional form is logically equivalent to the original. For example,

\( \vdash Q \land (P \rightarrow Q) \iff Q \land (\neg P \lor Q) \).

To see this, examine the truth table

```
<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \rightarrow Q</th>
<th>Q \land (P \rightarrow Q)</th>
<th>\neg P</th>
<th>\neg P \lor Q</th>
<th>Q \land (\neg P \lor Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>
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**EXAMPLE 1.3.5**

Consider \( R \land (P \rightarrow Q) \). Since \( \vdash P \rightarrow Q \iff \neg Q \rightarrow \neg P \) [Exercise 2(f)], we can replace \( P \rightarrow Q \) with \( \neg Q \rightarrow \neg P \) giving

\( \vdash R \land (P \rightarrow Q) \iff R \land (\neg Q \rightarrow \neg P) \).

When studying an implication, we sometimes need to investigate the different ways that its antecedent and consequent relate to each other.

**DEFINITION 1.3.6**

The **converse** of a given implication is the conditional proposition formed by exchanging the antecedent and consequent of the implication (Figure 1.2).

**DEFINITION 1.3.7**

The **contrapositive** of a given implication is the conditional proposition formed by exchanging the antecedent and consequent of the implication and then replacing them with their negations (Figure 1.3).
If the antecedent, then the consequent.

If the consequent, then the antecedent.

**Figure 1.2** Writing the converse.

If not the consequent, then not the antecedent.

**Figure 1.3** Writing the contrapositive.

For example, the converse of

*if rectangles have four sides, squares have for sides*

is

*if squares have four sides, rectangles have four sides,*

and its contrapositive is

*if squares do not have four sides, rectangles do not have four sides.*

Notice that a biconditional proposition is simply the conjunction of a conditional with its converse.

**EXAMPLE 1.3.8**

The propositional form \( P \rightarrow Q \) has \( Q \rightarrow P \) as its converse and \( \neg Q \rightarrow \neg P \) as its contrapositive. The first and fourth columns on the right of the next truth table show \( \models P \rightarrow Q \iff \neg Q \rightarrow \neg P \), while \( \not\models P \rightarrow Q \iff Q \rightarrow P \) is shown by the first and last columns.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>P</th>
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<th>( P \rightarrow Q )</th>
<th>( \neg Q )</th>
<th>( \neg P )</th>
<th>( \neg Q \rightarrow \neg P )</th>
<th>Q</th>
<th>P</th>
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</thead>
<tbody>
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Syntactics

If we limit our proofs to Inference Rules 1.2.10, we quickly realize that there will be little of interest that we can prove. We would have no reason on which to base such clear inferences as

\[ P \vdash Q \lor P \]

or

\[ P \lor Q, \neg Q \vdash P. \]

To fix this, we expand our collection of inference rules with a new type.

Suppose that we know that the form \( p \land q \) can replace \( q \land p \) at any time and vice versa. For example, in the propositional form

\[ P \land Q \rightarrow R, \quad (1.4) \]

\( P \land Q \) can be replaced with \( Q \land P \) so that we can write the new form

\[ Q \land P \rightarrow R. \quad (1.5) \]

This type of rule is called a replacement rule and is written using the \( \Leftrightarrow \) symbol. For example, the replacement rule that allowed us to write (1.5) from (1.4) is

\[ p \land q \Leftrightarrow q \land p. \]

Similarly, when the replacement rule

\[ \neg(p \land q) \Leftrightarrow \neg p \lor \neg q \]

is applied to \( P \lor (\neg Q \lor \neg R) \), the result is \( P \lor \neg(Q \land R) \). We state without proof the standard replacement rules.

**REPLACEMENT RULES 1.3.9**

Let \( p, q, \) and \( r \) be propositional forms.

- **Associative Laws [Assoc]**
  \[ p \land q \land r \Leftrightarrow p \land (q \land r) \]
  \[ p \lor q \lor r \Leftrightarrow p \lor (q \lor r) \]

- **Commutative Laws [Com]**
  \[ p \land q \Leftrightarrow q \land p \]
  \[ p \lor q \Leftrightarrow q \lor p \]

- **Distributive Laws [Distr]**
  \[ p \land (q \lor r) \Leftrightarrow p \land q \lor p \land r \]
  \[ p \lor q \land r \Leftrightarrow (p \lor q) \land (p \lor r) \]

- **Contrapositive Law [Contra]**
  \[ p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p \]
• Double Negation [DN]
  \[ p \iff \neg\neg p \]

• De Morgan’s Laws [DeM]
  \[ \neg(p \land q) \iff \neg p \lor \neg q \]
  \[ \neg(p \lor q) \iff \neg p \land \neg q \]

• Idempotency [Idem]
  \[ p \land p \iff p \]
  \[ p \lor p \iff p \]

• Material Equivalence [Equiv]
  \[ p \iff q \iff (p \rightarrow q) \land (q \rightarrow p) \]
  \[ p \iff q \iff p \land q \lor \neg p \land \neg q \]

• Material Implication [Impl]
  \[ p \rightarrow q \iff \neg p \lor q \]

• Exportation [Exp]
  \[ p \land q \rightarrow r \iff p \rightarrow (q \rightarrow r). \]

A replacement rule is used in a formal proof by appealing to the next inference rule.

**INSTRUCTION RULE 1.3.10**

For all propositional forms \( p \) and \( q \), if \( p \) if obtained from \( q \) using a replacement rule, \( p \Rightarrow q \) and \( q \Rightarrow p \).

As with Inference Rules 1.2.10, the replacement rule used when applying Inference Rule 1.3.10 must be used exactly as stated. This includes times when it seems unnecessary because it appears obvious. For example, \( P \lor Q \) does not follow directly from \( \neg P \rightarrow Q \) using Impl. Instead, include a use of DN to give the correct sequence,

\[ \neg P \rightarrow Q, \neg\neg P \lor Q, P \lor Q. \]

Similarly, the inference rule Add does not allow for \( Q \lor P \) to be derived from \( P \). Instead, we derive \( P \lor Q \). Follow this by Com to conclude \( Q \lor P \).

When writing formal proofs that appeal to Inference Rule 1.3.10, do not cite that particular rule but reference the replacement rule’s abbreviation using Replacement Rule 1.3.9 and the line which serves as a premise to the replacement. We use this practice in the following examples.

**EXAMPLE 1.3.11**

Although it is common practice to move parentheses freely when solving equations, inferences such as

\[ P \land Q \land (R \land S) \vdash P \land (Q \land R) \land S \]
must be carefully demonstrated. Fortunately, this example only requires two applications of the associative law. Using the boxes as a guide, notice that

\[ P \land Q \land (R \land S) \]

is of the same form as the right-hand side of the associative law. Hence, we can remove the parentheses to obtain

\[ P \land Q \land R \land S \]

Next, view the propositional form as

\[ P \land Q \land R \land S \]

One more application within the first box yields the result,

\[ P \land (Q \land R) \land S \]

We may, therefore, write a sequence of inferences by Inference Rule 1.3.10,

\[ P \land Q \land (R \land S) \Rightarrow (P \land Q \land R) \land S \Rightarrow [P \land (Q \land R)] \land S, \]

and we have a proof of \([P \land (Q \land R)] \land S\) from \(P \land Q \land (R \land S)\):

| 1. | \(P \land Q \land (R \land S)\) | Given |
| 2. | \(P \land Q \land R \land S\) | 1 Assoc |
| 3. | \(P \land (Q \land R) \land S\) | 2 Assoc |

**EXAMPLE 1.3.12**

Prove: \(R \land S \vdash \neg R \rightarrow P\)

| 1. | \(R \land S\) | Given |
| 2. | \(R\) | 1 Simp |
| 3. | \(R \lor P\) | 2 Add |
| 4. | \(\neg R \lor P\) | 3 DN |
| 5. | \(\neg R \rightarrow P\) | 4 Impl |

**EXAMPLE 1.3.13**

Prove: \(P \rightarrow Q, R \rightarrow Q \vdash P \lor R \rightarrow Q\)

| 1. | \(P \rightarrow Q\) | Given |
| 2. | \(R \rightarrow Q\) | Given |
| 3. | \((P \rightarrow Q) \land (R \rightarrow Q)\) | 1, 2 Conj |
| 4. | \((\neg P \lor Q) \land (\neg R \lor Q)\) | 3 Impl |
| 5. | \((Q \lor \neg P) \land (Q \lor \neg R)\) | 4 Com |
EXAMPLE 1.3.14

Prove: \( P \land Q \lor R \land S \vdash (P \lor S) \land (Q \lor R) \)

1. \( P \land Q \lor R \land S \) Given
2. \( (P \land Q) \land R \land S \) 1 Dist
3. \( (R \lor P) \land (P \lor Q) \land (S \lor P) \land (Q \lor S) \) 2 Com
4. \( (R \lor P) \land (P \lor Q) \land ((S \lor P) \land (Q \lor S)) \) 3 Dist
5. \( (R \lor Q) \land ((R \lor P) \land ((S \lor P) \land (Q \lor S))) \) 4 Com
6. \( (R \lor Q) \land ((R \lor P) \land ((S \lor P) \land (Q \lor S))) \) 5 Assoc
7. \( (P \lor S) \land (Q \lor R) \) 6 Simp
8. \( (S \lor P) \land (Q \lor R) \) 7, 10 Conj
9. \( (S \lor P) \land (Q \lor R) \) 8 Simp
10. \( S \lor P \) 9 Simp
11. \( (P \lor S) \land (Q \lor R) \) 10, 11 Com
12. \( (P \lor S) \land (Q \lor R) \) 11 Com

EXAMPLE 1.3.15

Both \( \vdash P \rightarrow P \) and \( \vdash P \lor \neg P \). Here is the proof for the second theorem:

1. \( P \rightarrow (P \rightarrow P) \) FL1
2. \( \neg P \lor (\neg P \lor P) \) 1 Impl
3. \( (\neg P \lor P) \lor P \) 2 Assoc
4. \( \neg P \lor P \) 3 Idem
5. \( P \lor \neg P \) 4 Com

This proof can be generalized to any propositional form \( p \), so that we also have \( \vdash p \rightarrow p \) and \( \vdash p \lor \neg p \).

The theorem \( p \lor \neg p \) is known as the law of the excluded middle, and the theorem \( \neg(p \land \neg p) \) is the law of noncontradiction. Notice that by De Morgan’s law with Com and DN, we have for all propositional forms \( p \)

\( \vdash (p \lor \neg p) \leftrightarrow \neg(p \land \neg p) \).

Exercises

1. The inverse of a given implication is the contrapositive of the implication’s converse. Write the converse, contrapositive, and inverse for each conditional proposition in Exercise 1.1.2.
2. Prove using truth tables.
   (a) \( \vdash P \lor \neg P \iff P \to P \)
   (b) \( \vdash P \lor \neg P \iff P \lor Q \lor \neg(P \land Q) \)
   (c) \( \vdash P \land Q \iff (P \iff Q) \land (P \lor Q) \)
   (d) \( \vdash R \land (P \to Q) \iff R \land (\neg Q \to \neg P) \)
   (e) \( \vdash P \land Q \to R \iff P \land \neg R \to \neg Q \)
   (f) \( \vdash P \to Q \iff \neg Q \to \neg P \)
   (g) \( \vdash P \to Q \iff (P \to Q) \land (P \to R) \)
   (h) \( \vdash P \to Q \iff (S \to R) \iff (P \to Q) \land S \to R \)

3. A propositional form is in **disjunctive normal form** if it is a disjunction of conjunctions. For example, the propositional form \( P \land (\neg Q \lor R) \) is logically equivalent to \( (P \land Q \land R) \lor (P \land \neg Q \land \neg R) \lor (\neg P \land Q \land \neg R) \), which is in disjunctive normal form. Find propositional forms in disjunctive normal form that are logically equivalent to each of the following.
   (a) \( P \lor Q \land (P \lor \neg R) \)
   (b) \( (P \lor Q) \land (\neg P \lor \neg Q) \)
   (c) \( (P \land \neg Q \lor R) \land (Q \land R \lor P \land \neg R) \)
   (d) \( P \lor (\neg Q \lor [P \land \neg R \lor P \land \neg Q]) \)

4. Identify the rule from Replacement Rule 1.3.9.
   (a) \( (P \to Q) \lor (Q \to R) \lor S \iff (P \to Q) \lor (Q \to R) \lor S \)
   (b) \( \neg\neg P \iff Q \lor R \iff P \iff Q \lor R \)
   (c) \( P \lor Q \lor R \iff Q \lor P \lor R \)
   (d) \( \neg P \lor (Q \lor R) \iff (P \lor Q) \lor (Q \lor R) \)
   (e) \( P \lor Q \iff Q \land (P \land Q) \)
   (f) \( P \lor Q \iff Q \land (P \land Q) \)
   (g) \( P \lor Q \land Q \iff Q \lor (P \lor Q) \land R \)
   (h) \( P \lor Q \land R \iff R \land (P \lor Q) \)
   (i) \( (P \lor Q) \land (Q \lor R) \iff (P \lor Q) \land (Q \lor R) \)
   (j) \( (P \lor [R \to Q]) \to S \iff P \to (R \to Q) \to S \)

5. For each given propositional form \( p \), find another propositional form \( q \) such that \( p \iff q \) using Replacement Rules 1.3.9.
   (a) \( \neg\neg P \)
   (b) \( P \lor Q \)
   (c) \( P \to Q \)
   (d) \( \neg (P \land Q) \)
   (e) \( (P \iff Q) \land (\neg P \iff Q) \)
   (f) \( (P \to Q) \lor (Q \to S) \)
   (g) \( (P \to Q) \lor P \)
   (h) \( \neg (P \to Q) \)
   (i) \( (P \to Q) \land (Q \iff R) \)
(j) \((P \lor \neg Q) \leftrightarrow T \land Q\)
(k) \(P \lor (\neg Q \leftrightarrow T) \land Q\)

6. Arrange each collection of propositional forms into a proof for the given deduction and supply the appropriate reasons.
(a) \(P \lor Q \rightarrow R \vdash (P \rightarrow R) \land (Q \rightarrow R)\)
- \(R \lor \neg P \land \neg Q\)
- \((\neg P \lor R) \land (\neg Q \lor R)\)
- \(\neg (P \lor Q) \lor R\)
- \((P \rightarrow R) \land (Q \rightarrow R)\)
- \((R \lor \neg P) \land (R \lor \neg Q)\)
- \(\neg P \land \neg Q \lor R\)
- \(P \lor Q \rightarrow R\)
(b) \(\neg (P \land Q) \rightarrow R \lor S, \neg P, \neg S \vdash R\)
- \(\neg (P \land Q) \rightarrow R \lor S\)
- \(\neg S\)
- \(S \lor R\)
- \(R\)
- \(\neg (P \land Q)\)
- \(\neg P\)
- \(R \lor S\)
- \(\neg P \lor \neg Q\)
(c) \(P \rightarrow (Q \rightarrow R), \neg P \rightarrow S, \neg Q \rightarrow T, R \rightarrow \neg R \vdash \neg T \rightarrow S\)
- \(S \lor T\)
- \(\neg R \lor \neg R\)
- \(R \rightarrow \neg R\)
- \(\neg R\)
- \(\neg (P \land Q)\)
- \(T \lor S\)
- \(\neg \neg T \lor S\)
- \(\neg P \lor \neg Q\)
- \(P \land Q \rightarrow R\)
- \(P \rightarrow (Q \rightarrow R)\)
- \(\neg P \rightarrow S\)
- \(\neg Q \rightarrow T\)
- \(\neg T \rightarrow S\)
- \((\neg P \rightarrow S) \land (\neg Q \rightarrow T)\)

7. Prove.
(a) \(\neg P \vdash P \rightarrow Q\)
(b) \(P \vdash \neg Q \rightarrow P\)
(c) \(\neg Q \lor (\neg R \lor \neg P) \vdash P \rightarrow \neg (Q \land R)\)
(d) \(P \rightarrow Q \vdash P \land R \rightarrow Q\)
(e) \( P \rightarrow Q \land R \vdash P \rightarrow Q \)
(f) \( P \lor Q \rightarrow R \vdash \neg R \rightarrow \neg Q \)
(g) \( P \rightarrow (Q \rightarrow R) \vdash Q \land \neg R \rightarrow \neg P \)
(h) \( P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R) \)
(i) \( P \land Q \lor R \land S \vdash \neg S \rightarrow P \land Q \)
(j) \( Q \rightarrow R \vdash P \rightarrow (Q \rightarrow R) \)
(k) \( P \rightarrow \neg (Q \rightarrow R) \vdash P \rightarrow \neg R \)
(l) \( P \lor (Q \lor R \lor S) \vdash (P \lor Q) \lor (R \lor S) \)
(m) \( Q \lor P \rightarrow R \land S \vdash Q \rightarrow R \)
(n) \( P \leftrightarrow Q \vdash Q \rightarrow P \)
(o) \( P \lor Q \rightarrow S \vdash Q \rightarrow P \)
(p) \( P \lor Q \lor P \lor R \rightarrow S \vdash Q \rightarrow Q \land R \)
(q) \( (P \lor Q) \land R \lor S \vdash P \lor R \land P \land S \lor (Q \lor R \lor Q \land S) \)
(r) \( P \land (Q \lor R) \rightarrow Q \land R \vdash P \rightarrow (Q \rightarrow R) \)
(s) \( P \leftrightarrow Q, \neg P \vdash \neg Q \)
(t) \( P \rightarrow Q \land R, \neg R \vdash Q \)
(u) \( P \rightarrow (Q \rightarrow R), R \rightarrow S \rightarrow T \vdash P \rightarrow (Q \rightarrow T) \)
(v) \( P \rightarrow (Q \rightarrow R), R \rightarrow S \rightarrow T \vdash (P \rightarrow S) \lor (Q \rightarrow T) \)
(w) \( P \rightarrow Q, P \rightarrow R \vdash P \rightarrow Q \land R \)
(x) \( P \lor Q \rightarrow R \land S, \neg P \rightarrow (T \rightarrow \neg T), \neg R \vdash \neg T \)
(y) \( (P \rightarrow Q) \land (R \rightarrow S), P \lor R, (P \rightarrow \neg S) \land (R \rightarrow \neg Q) \vdash Q \leftrightarrow \neg S \)
(z) \( P \land (Q \land R), P \land R \rightarrow S \lor (T \lor M), \neg S \land \neg T \vdash M \)

1.4 PROOF METHODS

The methods of Sections 1.2 and 1.3 provide a good start for writing formal proofs. However, in practice we rarely limit ourselves to these rules. We often use inference rules that give a straightforward way to prove conditional propositions and allow us to prove a proposition when it is easier to disprove its negation. In both the cases, the new inference rules will be justified using the rules we already know.

**Deduction Theorem**

Because of Axioms 1.2.8, not all of the inference rules are needed to write the proofs found in Sections 1.2 and 1.3. This motivates the next definition.

**DEFINITION 1.4.1**

Let \( p \) and \( q \) be propositional forms. The notation

\[ p \vdash_\ast q \]

means that there exists a formal proof of \( q \) from \( p \) using only Axioms 1.2.8, MP, and Inference Rule 1.3.10, and the notation

\[ \vdash_\ast q \]
means that there exists a formal proof of \( q \) from Axioms 1.2.8 using only MP and Inference Rule 1.3.10.

For example, \( P, Q, \neg P \lor (Q \rightarrow R) \vdash^* R \) because

\[
\begin{array}{ll}
1. & P \quad \text{Given} \\
2. & Q \quad \text{Given} \\
3. & \neg P \lor (Q \rightarrow R) \quad \text{Given} \\
4. & P \rightarrow (Q \rightarrow R) \quad 3 \text{ Impl} \\
5. & Q \rightarrow R \quad 1, 4 \text{ MP} \\
6. & R \quad 2, 5 \text{ MP} \\
\end{array}
\]

is a formal proof using MP and Impl as the only inference rules.

We now observe that the propositional forms that can be proved using the full collection of rules from Sections 1.2 and 1.3 are exactly the propositional forms that can be proved when all rules are deleted from Inference Rules 1.2.10 except for MP.

**THEOREM 1.4.2**

For all propositional forms \( p \) and \( q \), \( p \vdash^* q \) if and only if \( p \vdash q \).

**PROOF**

Trivially, \( p \vdash^* q \) implies \( p \vdash q \), so suppose that \( p \vdash q \). We show that the remaining parts of Inference Rules 1.2.10 are equivalent to using only Axioms 1.2.8, MP, and Replacement Rules 1.3.9. We show three examples and leave the proofs of the remaining inference rules to Exercise 6. The proof

\[
\begin{array}{ll}
1. & p \rightarrow q \quad \text{Given} \\
2. & \neg q \quad \text{Given} \\
3. & \neg \neg p \rightarrow \neg \neg q \quad \text{DN} \\
4. & \neg \neg p \rightarrow \neg \neg q \rightarrow (\neg q \rightarrow \neg p) \quad \text{FL3} \\
5. & \neg q \rightarrow \neg p \quad 3, 4 \text{ MP} \\
6. & \neg p \quad 2, 5 \text{ MP} \\
\end{array}
\]

shows that \( p \rightarrow q, \neg q \vdash^* \neg p \).

Thus, we do not need MT. The proof

\[
\begin{array}{ll}
1. & p \quad \text{Given} \\
2. & p \rightarrow (\neg q \rightarrow p) \quad \text{FL1} \\
3. & \neg q \rightarrow p \quad 1, 2 \text{ MP} \\
4. & \neg \neg q \lor p \quad 3 \text{ Impl} \\
5. & q \lor p \quad 4 \text{ DN} \\
6. & p \lor q \quad 5 \text{ Com} \\
\end{array}
\]

shows that \( p \vdash^* p \lor q \).
so we do not need Add. This implies that we can use Add to demonstrate that

\[ p \to q, \quad q \to r \models p \to r. \]

The proof is as follows:

1. \( p \to q \) \hspace{1cm} \text{Given}
2. \( q \to r \) \hspace{1cm} \text{Given}
3. \( \neg q \lor r \) \hspace{1cm} 2 \text{ Impl}
4. \( \neg q \lor r \lor \neg p \) \hspace{1cm} 3 \text{ Add}
5. \( \neg p \lor (\neg q \lor r) \) \hspace{1cm} 4 \text{ Com}
6. \( p \to (q \to r) \) \hspace{1cm} 5 \text{ Impl}
7. \( p \to (q \to r) \to (p \to q \to [p \to r]) \) \hspace{1cm} \text{FL2}
8. \( p \to q \to (p \to r) \) \hspace{1cm} 6, 7 \text{ MP}
9. \( p \to r \) \hspace{1cm} 1, 8 \text{ MP}

This implies that we do not need HS. □

At this point, there is an obvious question: If only MP is needed from Inference Rules 1.2.10, why were the other rules included? The answer is because the other inference rules are examples of common reasoning and excluding them would introduce unnecessary complications to the formal proofs. Try reproving some of the deductions of Section 1.2 with only MP and the axioms to confirm this.

When a formal proof in Section 1.3 involved proving an implication, the replacement rule Impl would often appear in the proof. However, as we know from geometry, this is not the typical strategy used to prove an implication. What is usually done is that the antecedent is assumed and then the consequent is shown to follow. That this procedure justifies the given conditional is the next theorem. Its proof requires a lemma.

\section*{Lemma 1.4.3}

Let \( p \) and \( q \) be propositional forms. If \( \models q \), then \( \models p \to q \).

\textbf{Proof}

Let \( \vdash q \). By FL1, we have that \( \vdash q \to (p \to q) \), so \( \models p \to q \) follows by MP. □

\section*{Theorem 1.4.4 [Deduction]}

For all propositional forms \( p \) and \( q \), \( p \vdash q \) if and only if \( \vdash p \to q \).

\textbf{Proof}

Let \( p \) and \( q \) be propositional forms. Assume that \( \vdash p \to q \), so there exists propositional forms \( r_0, r_1, \ldots, r_{n-1} \) such that

\[ r_0, r_1, \ldots, r_{n-1}, p \to q \]

is a proof, where \( r_0 \) is an axiom, \( r_i \) is an axiom or \( r_0, r_1, \ldots, r_{i-1} \vdash r_i \) for \( i > 0 \), and \( p \to q \) follows from \( r_0, r_1, \ldots, r_{n-1} \). Then,
Section 1.4 PROOF METHODS

\[ p, r_0, r_1, \ldots, r_{n-1}, p \rightarrow q, q \]

is also a proof, where the last inference is due to MP. Therefore, \( p \vdash q \).

By Theorem 1.4.2, to prove the converse, we only need to prove that

if \( p \vdash q \), then \( \vdash p \rightarrow q \).

Assume \( p \vdash q \). First note that if \( q \) is an axiom, then \( \vdash q \), so \( \vdash p \rightarrow q \) by Lemma 1.4.3. Therefore, assume that \( q \) is not an axiom. We begin by checking four cases.

- Suppose that the proof has only one propositional form. In this case, we have that \( p = q \), so the inference is of the form \( p \vdash p \). By FL1,

\[ \vdash p \rightarrow (p \rightarrow p) \]

Because

\[ p \rightarrow (p \rightarrow p) \Rightarrow \neg p \lor (\neg p \lor p) \]
\[ \Rightarrow (\neg p \lor \neg p) \lor p \]
\[ \Rightarrow \neg p \lor p \]
\[ \Rightarrow p \rightarrow p, \]

we conclude that \( \vdash p \rightarrow p \).

- Next, suppose the proof has two propositional forms and cannot be reduced to the first case. This implies that \( p \vdash q \) by a single application of a replacement rule. Thus,

\[ p \rightarrow (\neg q \rightarrow p) \vdash p \rightarrow (\neg q \rightarrow q) \]

by a single application of the same replacement rule. Therefore,

\[ p \rightarrow (\neg q \rightarrow p) \Rightarrow p \rightarrow (\neg q \rightarrow q) \]
\[ \Rightarrow p \rightarrow (\neg q \lor q) \]
\[ \Rightarrow p \rightarrow (q \lor q) \]
\[ \Rightarrow p \rightarrow q. \]

This implies that \( \vdash p \rightarrow q \).

- We now consider the case when the proof of \( p \vdash q \) has three propositional forms and \( q \) follows by a rule of replacement. Let \( p, r, q \) be the proof. This implies that \( p \vdash r \), which implies

\[ \vdash p \rightarrow r \quad (1.6) \]

because either \( r \) is an axiom and Lemma 1.4.3 applies or the previous two cases apply. If \( q \) follows from \( p \) by a rule of replacement, then \( \vdash p \rightarrow q \) by
the previous case, so assume that \( q \) follows from \( r \) by a rule of replacement. Thus, \( \vdash \, r \rightarrow q \), which implies by Lemma 1.4.3 that

\[
\vdash \, p \rightarrow (r \rightarrow q). \tag{1.7}
\]

By FL2,

\[
\vdash \, p \rightarrow (r \rightarrow q) \rightarrow (p \rightarrow r \rightarrow [p \rightarrow q]). \tag{1.8}
\]

Therefore, by (1.7) and (1.8) with MP,

\[
\vdash \, p \rightarrow r \rightarrow (p \rightarrow q), \tag{1.9}
\]

and by (1.6) and (1.9) with MP,

\[
\vdash \, p \rightarrow q.
\]

• Again, let the proof have three propositional forms and write it as \( p, s, q \). Suppose that the inference that leads to \( q \) is MP. This means that \( r \) and \( r \rightarrow q \) are in the proof. Because either \( p = r \) and \( p \vdash \, r \rightarrow q \) or \( p = r \rightarrow q \) and \( p \vdash \, r \),

\[
\vdash \, p \rightarrow r
\]

and

\[
\vdash \, p \rightarrow (r \rightarrow q).
\]

Thus, as in the previous case, using (1.8), we obtain \( \vdash \, p \rightarrow q \).

These four cases exhaust the ways by which \( q \) can be proved from \( p \) with a proof with at most three propositional forms. Therefore, since these cases can be generalized to proofs of arbitrary length (Exercise 7), we conclude that \( p \vdash q \) implies \( \vdash \, p \rightarrow q \). \( \blacksquare \)

The deduction theorem (1.4.4) yields the next result. Its proof is left to Exercise 8.

\[ \text{COROLLARY 1.4.5} \]

For all propositional forms \( p_0, p_1, \ldots, p_{n-1}, q, r, \)

\[
p_0, p_1, \ldots, p_{n-1}, q \vdash r \quad \text{if and only if} \quad p_0, p_1, \ldots, p_{n-1} \vdash q \rightarrow r.
\]

**Direct Proof**

Most propositions that mathematicians prove are implications. For example,

*if a function is differentiable at a point, it is continuous at that same point.*

As we know, this means that whenever the function \( f \) is differentiable at \( x = a \), it must also be the case that \( f \) is continuous at \( x = a \). Proofs of conditionals like this
are typically very difficult if we are only allowed to use Inference Rules 1.2.10 and Replacement Rules 1.3.9. Fortunately, in practice another inference rule is used. To prove the differentiability result, what is usually done is that \( f \) is assumed to be differential at \( x = a \) and then a series of steps that lead to the conclusion that \( f \) is continuous at \( x = a \) are followed. We copy this strategy in our formal proofs using the next rule. Sometimes known as conditional proof, this inference rule follows by Corollary 1.4.5 and Theorem 1.2.14.

**Inference Rule 1.4.6 [Direct Proof (DP)]**

For propositional forms \( p_0, p_1, \ldots, p_{n-1}, q, r \),

if \( p_0, p_1, \ldots, p_{n-1}, q \vdash r \), then \( p_0, p_1, \ldots, p_{n-1} \Rightarrow q \rightarrow r \).

**Proof**

Suppose \( p_0, p_1, \ldots, p_{n-1}, q \vdash r \). Then, by Corollary 1.4.5,

\[ p_0, p_1, \ldots, p_{n-1} \vdash q \rightarrow r. \]

Therefore, by Simp and Com,

\[ p_0 \land p_1 \land \cdots \land p_{n-1} \vdash q \rightarrow r, \]

so by Theorem 1.2.14,

\[ p_0 \land p_1 \land \cdots \land p_{n-1} \Rightarrow q \rightarrow r. \]

Finally, we have by Conj and Theorem 1.2.14 that

\[ p_0, p_1, \ldots, p_{n-1} \Rightarrow q \rightarrow r. \]

To see how this works, let us use direct proof to prove

\[ P \lor Q \rightarrow (R \land S) \vdash P \rightarrow R. \]

To do this, we first prove

\[ P \lor Q \rightarrow (R \land S), P \vdash R. \]

Here is the proof:

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( P \lor Q \rightarrow (R \land S) )</td>
<td>Given</td>
</tr>
<tr>
<td>2.</td>
<td>( P )</td>
<td>Given</td>
</tr>
<tr>
<td>3.</td>
<td>( P \lor Q )</td>
<td>2 Add</td>
</tr>
<tr>
<td>4.</td>
<td>( R \land S )</td>
<td>1, 3 MP</td>
</tr>
<tr>
<td>5.</td>
<td>( R )</td>
<td>4 Simp</td>
</tr>
</tbody>
</table>
Therefore, by Inference Rule 1.4.6,

\[ P \lor Q \rightarrow (R \land S) \Rightarrow P \rightarrow R. \]

A proof of the original deduction can now be written as

| 1. \( P \lor Q \rightarrow (R \land S) \) | Given |
| 2. \( P \rightarrow R \) | 1 DP |

However, instead of writing the first proof off to the side, it is typically incorporated into the proof as follows:

| 1. \( P \lor Q \rightarrow (R \land S) \) | Given |
| 2. \( \neg P \) | Assumption |
| 3. \( P \lor Q \) | 2 Add |
| 4. \( R \land S \) | 1, 3 MP |
| 5. \( R \) | 4 Simp |
| 6. \( P \rightarrow R \) | 2–5 DP |

The proof that \( P \) infers \( R \) is a subproof of the main proof. To separate the propositional forms of the subproof from the rest of the proof, they are indented with a vertical line. The line begins with the assumption of \( P \) in line 2 as an additional premise. Hence, its reason is Assumption. This assumption can only be used in the subproof. Consider it a local hypothesis. It is only used to prove \( P \rightarrow R \). If we were allowed to use it in other places of the proof, we would be proving a theorem that had different premises than those that were given. Similarly, all lines within the subproof cannot be referenced from the outside. We use the indentation to isolate the assumption and the propositional forms that follow from it. When we arrive at \( R \), we know that we have proved \( P \rightarrow R \). The next line is this propositional form. It is entered into the proof with the reason DP. The lines that are referenced are the lines of the subproof.

**EXAMPLE 1.4.7**

Prove: \( P \rightarrow \neg Q, \neg R \lor S \vdash R \lor Q \rightarrow (P \rightarrow S) \)

| 1. \( P \rightarrow \neg Q \) | Given |
| 2. \( \neg R \lor S \) | Given |
| 3. \( \neg R \lor Q \) | Assumption |
| 4. \( \neg P \) | Assumption |
| 5. \( \neg Q \) | 1, 4 MP |
| 6. \( Q \lor R \) | 3 Com |
| 7. \( R \) | 5, 6 DS |
| 8. \( \neg \neg R \) | 7 DN |
| 9. \( S \) | 2, 8 DS |
| 10. \( P \rightarrow S \) | 4–9 DP |
| 11. \( (R \lor Q) \rightarrow (P \rightarrow S) \) | 3–10 DP |
EXAMPLE 1.4.8

Prove: \( P \land Q \to R \to S, \neg Q \lor R \vdash S \)

1. \( P \land Q \to R \to S \) Given
2. \( \neg Q \lor R \) Given
3. \( \to P \land Q \) Assumption
4. \( Q \land P \) 3 Com
5. \( Q \) 4 Simp
6. \( \neg \neg Q \) 5 DN
7. \( R \) 2, 6 DS
8. \( P \land Q \to R \) 3–7 DP
9. \( S \) 1, 8 MP

EXAMPLE 1.4.9

Prove: \( \vdash P \lor \neg P \)

1. \( \to P \) Assumption
2. \( P \to P \) 1 DP
3. \( \neg (p \land \neg p) \to \neg \neg q \) 2 Impl
4. \( P \lor \neg P \) 3 Com

Note that lines 1–2 prove \( \vdash P \to P \).

Indirect Proof

When direct proof is either too difficult or not appropriate, there is another common approach to writing formal proofs. Sometimes going by the name of proof by contradiction or reductio ad absurdum, this inference rule can also be used to prove propositional forms that are not implications.

INFERENCE RULE 1.4.10 [Indirect Proof (IP)]

For all propositional forms \( p \) and \( q \),

\[ \neg q \to (p \land \neg p) \Rightarrow q. \]

PROOF

Notice that instead of repeating the argument from Example 1.4.9 in this proof, the example is simply cited as the reason on line 2.

1. \( \neg q \to (p \land \neg p) \) Given
2. \( p \to p \) Example 1.4.9
3. \( \neg (p \land \neg p) \to \neg \neg q \) 1 Contra
4. \( \neg (p \land \neg p) \to q \) 3 DN
Chapter 1 PROPOSITIONAL LOGIC

5. \( \neg p \lor \neg \neg p \rightarrow q \) 4 DeM
6. \( \neg p \lor p \rightarrow q \) 5 DN
7. \( p \rightarrow p \rightarrow q \) 6 Impl
8. \( q \rightarrow q \) 2, 7 MP

The rule follows from Theorem 1.2.14.

To use indirect proof, assume each premise and assume the negation of the conclusion. Then, proceed with the proof until a contradiction is reached. (In Inference Rule 1.4.10, the contradiction is represented by \( p \land \neg p \).) At this point, deduce the original conclusion.

EXAMPLE 1.4.11

Prove: \( P \lor Q \rightarrow R \), \( R \lor S \rightarrow \neg P \land T \vdash \neg P \).

1. \( P \lor Q \rightarrow R \) Given
2. \( R \lor S \rightarrow \neg P \land T \) Given
3. \( \neg P \) Assumption
4. \( P \rightarrow \neg S \) 3 DN
5. \( P \lor Q \) 4 Add
6. \( R \rightarrow P \) 1, 5 MP
7. \( R \lor S \rightarrow \neg P \land T \) 6 Add
8. \( \neg P \land T \rightarrow P \rightarrow S \) 2, 7 MP
9. \( \neg P \rightarrow S \) 8 Simp
10. \( P \land \neg P \) 4, 9 Conj
11. \( \neg P \) 3–10 IP

Since IP involves proving an implication, the formal proof takes the same form as a proof involving DP.

Indirect proof can also be nested within another indirect subproof. As with direct proof, we cannot appeal to lines within a subproof from outside of it.

EXAMPLE 1.4.12

Prove: \( P \rightarrow Q \land R \), \( Q \rightarrow S \), \( \neg P \rightarrow S \vdash S \).

1. \( P \rightarrow Q \land R \) Given
2. \( Q \rightarrow S \) Given
3. \( \neg P \rightarrow S \) Given
4. \( \neg S \) Assumption
5. \( \neg Q \) 2, 4 MT
6. \( \neg P \) Assumption
7. \( Q \land R \) 1, 6 MP
8. \( Q \) 7 Simp
9. \( Q \land \neg Q \) 5, 8 Conj
Notice that line 11 was not the end of the proof since it was within the first sub-proof. It followed under the added hypothesis of \( \neg S \).

**EXAMPLE 1.4.13**

Prove: \( P \rightarrow R \vdash P \land Q \rightarrow R \lor S \).

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( P \rightarrow R )</td>
<td>Given</td>
</tr>
<tr>
<td>2.</td>
<td>( P \land Q )</td>
<td>Assumption</td>
</tr>
<tr>
<td>3.</td>
<td>( \neg R )</td>
<td>Assumption</td>
</tr>
<tr>
<td>4.</td>
<td>( \neg P )</td>
<td>1, 3 MT</td>
</tr>
<tr>
<td>5.</td>
<td>( P )</td>
<td>2 Simp</td>
</tr>
<tr>
<td>6.</td>
<td>( P \land \neg P )</td>
<td>4, 5 Conj</td>
</tr>
<tr>
<td>7.</td>
<td>( R )</td>
<td>3–6 IP</td>
</tr>
<tr>
<td>8.</td>
<td>( R \lor S )</td>
<td>7 Add</td>
</tr>
<tr>
<td>9.</td>
<td>( P \land Q \rightarrow R \lor S )</td>
<td>2–8 DP</td>
</tr>
</tbody>
</table>

**Exercises**

1. Find all mistakes in the given proofs.
   
   (a) “\( P \land Q \rightarrow \neg R, R \rightarrow \neg Q \rightarrow S \lor Q \vdash S \)”
   
   **Attempted Proof**

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( P \land Q \rightarrow \neg R )</td>
<td>Given</td>
</tr>
<tr>
<td>2.</td>
<td>( R \rightarrow \neg Q \rightarrow S \lor Q )</td>
<td>Given</td>
</tr>
<tr>
<td>3.</td>
<td>( \neg R )</td>
<td>Assumption</td>
</tr>
<tr>
<td>4.</td>
<td>( \neg \neg R )</td>
<td>Assumption</td>
</tr>
<tr>
<td>5.</td>
<td>( \neg (P \lor Q) )</td>
<td>1, 4 MT</td>
</tr>
<tr>
<td>6.</td>
<td>( \neg P \land \neg Q )</td>
<td>5 DeM</td>
</tr>
<tr>
<td>7.</td>
<td>( \neg Q )</td>
<td>6 Simp</td>
</tr>
<tr>
<td>8.</td>
<td>( R \rightarrow \neg Q )</td>
<td>3–7 DP</td>
</tr>
<tr>
<td>9.</td>
<td>( S \lor Q )</td>
<td>2, 8 MP</td>
</tr>
<tr>
<td>10.</td>
<td>( Q \lor S )</td>
<td>9 Com</td>
</tr>
<tr>
<td>11.</td>
<td>( S )</td>
<td>7, 10 DS</td>
</tr>
</tbody>
</table>

   (b) “\( \neg P \lor Q \vdash P \rightarrow Q \rightarrow R \)”
   
   **Attempted Proof**

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \neg P \lor Q )</td>
<td>Given</td>
</tr>
<tr>
<td>2.</td>
<td>( \neg P )</td>
<td>Assumption</td>
</tr>
</tbody>
</table>
\(3. \quad \neg \neg P \quad 2 \text{ DN}
\)

\(4. \quad Q \quad 1, 3 \text{ DS}
\)

\(5. \quad P \rightarrow Q \quad 2–4 \text{ DP}
\)

\(6. \quad R \quad \text{MP}
\)

\(7. \quad P \rightarrow Q \quad 2–6 \text{ DP}
\)

(c) "\(\neg R \land S, \neg P \lor Q \rightarrow R \vdash \neg P \lor Q \rightarrow Q\)"

**Attempted Proof**

1. \(\neg R \land S\) Given
2. \(\neg P \lor Q \rightarrow R\) Given
3. \(\rightarrow \neg P\) Assumption
4. \(\rightarrow \neg P \lor Q\) Assumption
5. \(R\) 2, 3 MP
6. \(\neg R\) 1 Simp
7. \(R \land \neg R\) 5, 6 Conj
8. \(P\) 4–7 IP
9. \(\neg \neg P\) 8 DN
10. \(Q\) 4, 9 DS
11. \(\neg P \lor Q \rightarrow Q\) 3–10 DP

2. Prove using direct proof.
   (a) \(P \rightarrow Q \land R \vdash P \rightarrow Q\)
   (b) \(P \lor Q \rightarrow R \vdash P \rightarrow R\)
   (c) \(P \lor (Q \lor R) \rightarrow S \vdash Q \rightarrow S\)
   (d) \(P \rightarrow Q, R \rightarrow Q \vdash P \lor R \rightarrow Q\)
   (e) \(P \rightarrow Q, P \rightarrow R \vdash P \rightarrow Q \land R\)
   (f) \(P \rightarrow (Q \rightarrow R) \vdash Q \land \neg R \rightarrow \neg P\)
   (g) \(R \rightarrow \neg S \vdash P \lor Q \rightarrow (R \rightarrow \neg S)\)
   (h) \(P \rightarrow (Q \rightarrow R) \rightarrow Q \rightarrow (P \rightarrow R)\)
   (i) \(P \rightarrow (Q \rightarrow R), Q \rightarrow (R \rightarrow S) \vdash P \rightarrow (Q \rightarrow S)\)
   (j) \(P \rightarrow (Q \rightarrow R), R \rightarrow S \land T \vdash P \rightarrow (Q \rightarrow T)\)
   (k) \(P \leftrightarrow Q \land R \vdash P \rightarrow Q\)
   (l) \(P \land Q \lor R \rightarrow Q \land R \vdash P \rightarrow (Q \rightarrow R)\)

3. Prove using indirect proof.
   (a) \(P \rightarrow Q, Q \rightarrow R, \neg R \vdash \neg P\)
   (b) \(P \lor Q \land R, P \rightarrow S, Q \rightarrow S \vdash S\)
   (c) \(P \lor Q \land \neg R, P \rightarrow S, Q \rightarrow R \vdash S\)
   (d) \(P \leftrightarrow Q, \neg P \vdash \neg Q\)
   (e) \(P \land Q \lor R \rightarrow Q \land R \vdash \neg P \lor Q \land R\)
   (f) \(P \lor Q, Q \rightarrow R, S \rightarrow T, P \lor R, \neg R \vdash T\)
   (g) \(P \rightarrow \neg Q, R \rightarrow \neg S, T \rightarrow Q, U \rightarrow S, P \lor R \vdash \neg T \lor \neg U\)
Section 1.5 THE THREE PROPERTIES

We finish our introduction to propositional logic by showing that this logical system has three important properties. These are properties that are shared with Euclid’s geometry, but they are not common to all logical systems.

**Consistency**

Since we can need to consider infinitely many propositional forms, we now write our lists of propositional forms as

\[ p_0, p_1, p_2, \ldots, \]

allowing this sequence to be finite or infinite. Since proofs are finite, the notation

\[ p_0, p_1, p_2, \ldots \vdash q \]

4. Prove using both direct and indirect proof.
   (a) \( P \rightarrow Q, P \lor (R \rightarrow S), \neg Q \vdash R \rightarrow S \)
   (b) \( P \rightarrow \neg(Q \rightarrow \neg R) \vdash P \rightarrow R \)
   (c) \( P \rightarrow Q \vdash P \land R \rightarrow Q \)
   (d) \( P \leftrightarrow Q \lor R \vdash Q \rightarrow P \)

5. Prove by using direct proof to prove the contrapositive.
   (a) \( P \rightarrow Q, R \rightarrow S, S \rightarrow T, \neg Q \vdash \neg T \rightarrow \neg(P \lor R) \)
   (b) \( P \land Q \lor R \land S \vdash \neg S \rightarrow P \land Q \)
   (c) \( P \lor Q \rightarrow \neg R, S \rightarrow R \vdash P \lor \neg S \)
   (d) \( \neg P \rightarrow \neg Q, (\neg R \lor S) \land (R \lor Q) \vdash \neg S \rightarrow P \)

6. Prove to complete the proof of Theorem 1.4.2.
   (a) \( p \lor q, \neg p \vdash \ast q \)
   (b) \( p, q \vdash \ast p \land q \)
   (c) \( (p \rightarrow q) \land (r \rightarrow s), p \lor r \vdash \ast q \lor s \)
   (d) \( (p \rightarrow q) \land (r \rightarrow s), \neg q \lor \neg s \vdash \ast \neg p \lor \neg r \)
   (e) \( p \land q \vdash \ast p \)

7. Given there is a proof of \( q \) from \( p \) with four propositional forms, prove \( \vdash p \rightarrow q \). Generalize the proof for \( n \) propositional forms.

8. Prove Corollary 1.4.5.

9. Can MP be replaced with another inference rule in Definition 1.4.1 and still have Theorem 1.4.2 hold true? If so, find the inference rules.

10. Can any of the replacement rules be removed from Definition 1.4.1 and still have Theorem 1.4.2 hold true? If so, how many can be removed and which ones?
Chapter 1

PROPOSITIONAL LOGIC

means that there exists a subsequence \( i_0, i_1, \ldots, i_{n-1} \) of 0, 1, 2, \ldots such that

\[ p_{i_0}, p_{i_1}, \ldots, p_{i_{n-1}} \vdash q. \]

The notation

\[ p_0, p_1, p_2, \ldots \not\vdash q \]

means that no such subsequence exists.

\[ \text{DEFINITION 1.5.1} \]

- The propositional forms \( p_0, p_1, p_2, \ldots \) are consistent if for every propositional form \( q \),

\[ p_0, p_1, p_2, \ldots \not\vdash q \wedge \neg q, \]

and we write \( \text{Con}(p_0, p_1, p_2, \ldots) \). Otherwise, \( p_0, p_1, p_2, \ldots \) is inconsistent.

- A logical system is consistent if no contradiction is a theorem.

We have two goals. The first is to show that propositional logic is consistent. The second is to discover properties of sequences of consistent propositional forms that will aid in proving other properties of propositional logic. The next theorem is important to meet both of these goals. The equivalence of the first two parts is known as the compactness theorem

\[ \text{THEOREM 1.5.2} \]

If \( p_0, p_1, p_2, \ldots \) are propositional forms, the following are equivalent in propositional logic.

- \( \text{Con}(p_0, p_1, p_2, \ldots) \).

- Every finite subsequence of \( p_0, p_1, p_2, \ldots \) is consistent.

- There exists a propositional form \( p \) such that \( p_0, p_1, p_2, \ldots \not\vdash p \).

\[ \text{PROOF} \]

We have three implications to prove.

- Suppose there is a finite subsequence \( p_{i_0}, p_{i_1}, \ldots, p_{i_{n-1}} \) that proves \( q \wedge \neg q \) for some propositional form \( q \). This implies that there is a formal proof of \( q \wedge \neg q \) from \( p_0, p_1, p_2, \ldots \), therefore, not \( \text{Con}(p_0, p_1, p_2, \ldots) \).

- Assume that \( p_0, p_1, p_2, \ldots \) proves every propositional form. In particular, if we take a propositional form \( q \), we have that \( p_0, p_1, p_2, \ldots \vdash q \wedge \neg q \). This means that there is a finite subsequence \( p_{i_0}, p_{i_1}, \ldots, p_{i_{n-1}} \) that proves \( q \wedge \neg q \).

- Lastly, assume that there exists a propositional form \( q \) such that

\[ p_0, p_1, p_2, \ldots \not\vdash q \wedge \neg q. \]
This means that there exist subscripts $i_0, i_1, \ldots, i_{n-1}$ and propositional forms $r_0, r_1, \ldots, r_{m-1}$ such that

$$p_{i_0}, p_{i_1}, \ldots, p_{i_{n-1}}, r_0, r_1, \ldots, r_{m-1}, q \land \neg q$$

is a proof. Take any propositional form $p$. Then,

$$p_{i_0}, p_{i_1}, \ldots, p_{i_{n-1}}, r_0, r_1, \ldots, r_{m-1}, \neg p, q \land \neg q, p$$

is a proof of $p$ by IP. Therefore, $p_0, p_1, p_2, \ldots \vdash p$. ■

A sequence of propositional forms, such as $P \rightarrow Q, P, Q$, although consistent, has the property that there are propositional forms that can be added to the sequence so that the resulting list remains consistent. When the sequence can no longer take new forms and remain consistent, we have arrived at a sequence that satisfies the next definition.


\[\textbf{DEFINITION 1.5.3}\]

A sequence of propositional forms $p_0, p_1, p_2, \ldots$ is \textbf{maximally consistent} whenever $\text{Con}(p_0, p_1, p_2, \ldots)$ and for all propositional forms $p$, $\text{Con}(p, p_0, p_1, p_2, \ldots)$ implies that $p = p_i$ for some $i$.

It is a convenient result of propositional logic that every consistent sequence of propositional forms can be extended to a maximally consistent sequence. This is possible because all possible propositional forms can be put into a list. Following Definition 1.1.2, we first list the propositional variables:

$$A, \quad B, \quad C, \quad \ldots \quad X, \quad Y, \quad Z,$$

$$A_0, \quad A_1, \quad A_2, \quad \ldots$$

$$B_0, \quad B_1, \quad B_2, \quad \ldots$$

$$\vdots \quad \vdots \quad \vdots$$

$$Z_0, \quad Z_1, \quad Z_2, \quad \ldots$$

Then, we list all propositional forms with only one propositional variable:

$$\neg A, \quad \neg B, \quad \neg C, \quad \ldots \quad \neg X, \quad \neg Y, \quad \neg Z,$$

$$\neg A_0, \quad \neg A_1, \quad \neg A_2, \quad \ldots$$

$$\neg B_0, \quad \neg B_1, \quad \neg B_2, \quad \ldots$$

$$\vdots \quad \vdots \quad \vdots$$

$$\neg Z_0, \quad \neg Z_1, \quad \neg Z_2, \quad \ldots$$

Next, we list all propositional forms with exactly two propositional variables starting by writing $A$ on the right:

$$A \lor A, \quad B \lor A, \quad C \lor A, \quad \ldots \quad X \lor A, \quad Y \lor A, \quad Z \lor A,$$

$$A_0 \lor A, \quad A_1 \lor A, \quad A_2 \lor A, \quad \ldots$$

$$B_0 \lor A, \quad B_1 \lor A, \quad B_2 \lor A, \quad \ldots$$

$$\vdots \quad \vdots \quad \vdots$$

$$Z_0 \lor A, \quad Z_1 \lor A, \quad Z_2 \lor A, \quad \ldots$$
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\[ A \land A, \quad B \land A, \quad C \land A, \quad \ldots \quad X \land A, \quad Y \land A, \quad Z \land A, \]
\[ A_0 \land A, \quad A_1 \land A, \quad A_2 \land A, \quad \ldots \]
\[ B_0 \land A, \quad B_1 \land A, \quad B_2 \land A, \quad \ldots \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ Z_0 \land A, \quad Z_1 \land A, \quad Z_2 \land A, \quad \ldots \]

\[ A \rightarrow A, \quad B \rightarrow A, \quad C \rightarrow A, \quad \ldots \quad X \rightarrow A, \quad Y \rightarrow A, \quad Z \rightarrow A, \]
\[ A_0 \rightarrow A, \quad A_1 \rightarrow A, \quad A_2 \rightarrow A, \quad \ldots \]
\[ B_0 \rightarrow A, \quad B_1 \rightarrow A, \quad B_2 \rightarrow A, \quad \ldots \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ Z_0 \rightarrow A, \quad Z_1 \rightarrow A, \quad Z_2 \rightarrow A, \quad \ldots \]

\[ A \leftrightarrow A, \quad B \leftrightarrow A, \quad C \leftrightarrow A, \quad \ldots \quad X \leftrightarrow A, \quad Y \leftrightarrow A, \quad Z \leftrightarrow A, \]
\[ A_0 \leftrightarrow A, \quad A_1 \leftrightarrow A, \quad A_2 \leftrightarrow A, \quad \ldots \]
\[ B_0 \leftrightarrow A, \quad B_1 \leftrightarrow A, \quad B_2 \leftrightarrow A, \quad \ldots \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ Z_0 \leftrightarrow A, \quad Z_1 \leftrightarrow A, \quad Z_2 \leftrightarrow A, \quad \ldots \]

After following the same pattern by attaching \( A \) on the left, we continue by writing \( \neg A \) on the right, and then on the left, and then we adjoin \( B \) and \( \neg B \), etc., and then use 3 propositional variables, and then four, etc. Following a careful path through this infinite list, we arrive at a sequence

\[ q_0, q_1, q_2, \ldots \]

of all propositional forms. We are now ready for the theorem.

\[ \textbf{THEOREM 1.5.4} \]

A consistent sequence of propositional forms is a subsequence of a maximally consistent sequence of propositional forms.

\[ \textbf{PROOF} \]

Let \( p_0, p_1, p_2, \ldots \) be consistent and \( q_0, q_1, q_2, \ldots \) be a sequence of all propositional forms. Define the sequence \( r_i \) as follows:

- Let \( r_0 = p_0 \) and
  \[ r_1 = \begin{cases} 
  q_0 & \text{if } \text{Con}(q_0, p_0, p_1, p_2, \ldots), \\
  p_0 & \text{otherwise}, 
  \end{cases} \]
  so the sequence at this stage is \( p_0, q_0 \) or \( p_0, p_0 \). Both of these are consistent.

- Let \( r_2 = p_1 \) and
  \[ r_3 = \begin{cases} 
  q_1 & \text{if } \text{Con}(q_1, r_0, r_1, r_2, p_0, p_1, p_2, \ldots), \\
  p_1 & \text{otherwise}. 
  \end{cases} \]
At this stage, the sequence is still consistent and of the form \( r_0, r_1, p_1, q_1 \) or \( r_0, r_1, p_1, p_1 \). The first sequence is consistent by the definition of \( r_3 \), and the second sequence is consistent because \( \text{Con}(r_0, r_1, p_1) \).

- Generalizing, let \( r_{2k} = p_k \) and
  \[
  r_{2k+1} = \begin{cases} 
  q_k & \text{if } \text{Con}(q_k, r_0, r_1, \ldots, r_{2k}, p_0, p_1, p_2, \ldots), \\
  p_k & \text{otherwise,}
  \end{cases}
  \]
  resulting in a consistent sequence of the form
  \[
  r_0, r_1, r_2, r_3, \ldots, p_k, q_k
  \]
  or
  \[
  r_0, r_1, r_2, r_3, \ldots, p_k, p_k.
  \]
  Since \( p_0, p_1, p_3, \ldots \) is a subsequence of \( r_0, r_1, r_2, \ldots \), it only remains to show that the new sequence is maximally consistent.

- Let \( s \) be a propositional form such that \( r_0, r_1, r_2, \ldots \vdash s \land \neg s \). This implies that there exists a sequence \( i_j \) such that \( i_0 < i_1 < \cdots < i_k \) and
  \[
  r_{i_0}, r_{i_1}, \ldots, r_{i_k} \vdash s \land \neg s,
  \]
  but by Theorem 1.5.2, this is impossible because \( \text{Con}(r_0, r_1, \ldots, r_{i_k}) \).

- Suppose that \( s \) is a propositional form so that \( \text{Con}(s, r_0, r_1, r_2, \ldots) \). Write \( s = q_i \) for some \( i \). Therefore,
  \[
  \text{Con}(q_i, r_0, r_1, \ldots, r_{2i}, p_0, p_1, p_2, \ldots),
  \]
  which means that \( s \) is a term of the sequence \( r_i \) because \( q_i \) was added at step \( 2i + 1 \).

**Soundness**

We have defined two separate tracks in propositional logic. One track is used to assign \( T \) or \( F \) to a propositional form, and thus it can be used to determine the truth value of a proposition. The other track focused on developing methods by which one propositional form can be proved from other propositional forms. These methods are used to write proofs in various fields of mathematics. The question arises whether these two tracks have been defined in such a way that they get along with each other. In other words, we want the propositional forms that we prove always to be assigned \( T \), and we want the propositional forms that we always assign \( T \) to be provable. This means that we want semantic methods to yield syntactic results and syntactic methods to yield semantic results.
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The statement form is a theorem.
The statement form is a tautology.

Sound

Complete

Sound and complete logics.

DEFINITION 1.5.5

- A logic is sound if every theorem is a tautology.
- A logic is complete if every tautology is a theorem.

There is no guarantee that the construction of the two tracks for a logic will have these two properties (Figure 1.4), but it does in the case of propositional logic.

To prove that propositional logic is sound, we need three lemmas. The proof of the first is left to Exercise 1.

LEMMA 1.5.6

The propositional forms of Axioms 1.2.8 are tautologies.

LEMMA 1.5.7

Let p, q, and r be propositional forms.

- If p \Rightarrow r, then p \rightarrow r is a tautology.
- If p, q \Rightarrow r, then p \land q \rightarrow r is a tautology.

PROOF

This is simply a matter of checking Inference Rules 1.2.10 and 1.3.10.

For example, to check the theorem for De Morgan’s law, we must show that

\( \neg(p \land q) \rightarrow \neg p \lor \neg q \)

and

\( \neg p \lor \neg q \rightarrow \neg(p \land q) \)

are tautologies. To do this, examine the truth table

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p \land q</th>
<th>\neg(p \land q)</th>
<th>\neg p</th>
<th>\neg q</th>
<th>\neg p \lor \neg q</th>
<th>\neg(p \land q) \rightarrow \neg p \lor \neg q</th>
<th>\neg p \lor \neg q \rightarrow \neg(p \land q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
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<td>F</td>
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<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
We also have to show that

\[ \neg(p \lor q) \rightarrow \neg p \land \neg q \]

and

\[ \neg p \land \neg q \rightarrow \neg(p \lor q) \]

are tautologies.

As another example, to check that the disjunctive syllogism leads to an implication that is a tautology, examining the truth table:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( p \lor q )</th>
<th>( \neg p )</th>
<th>( (p \lor q) \land \neg p )</th>
<th>( (p \lor q) \land \neg p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

The other rules are checked similarly (Exercise 2).

\[ \textbf{LEMMA 1.5.8} \]

If \( p \rightarrow q \) and \( p \) are tautologies, then \( q \) is a tautology.

\[ \textbf{PROOF} \]

This is done by examining the truth table of \( p \rightarrow q \) (page 12) where we see that \( \nu(q) \) is constant and equal to \( T \) because \( \nu(p) \) and \( \nu(p \rightarrow q) \) are constant and both equal to \( T \).

We now prove the first important property of propositional logic.

\[ \textbf{THEOREM 1.5.9 [Soundness]} \]

Every theorem of propositional logic is a tautology.

\[ \textbf{PROOF} \]

Let \( p \) be a theorem. Let \( p_0, p_1, \ldots, p_{n-1} \) be propositional forms such that

\[ p_0, p_1, \ldots, p_{n-1} \]

is a proof for \( p \) such that \( p_1 \) is an axiom, \( p_i \) with \( i > 0 \) is an axiom or follows by a rule of inference, and \( p_{n-1} = p \) (Definition 1.2.13). We now examine the propositional forms of the proof.

- By Lemma 1.5.6, \( p_0 \) is a tautology.

- The propositional form \( p_1 \) is a tautology for one of two reasons. If \( p_1 \) is an axiom, it is a tautology (Lemma 1.5.6). If it follows from \( p_0 \) because \( p_0 \Rightarrow p_1 \), then \( p_0 \rightarrow p_1 \) is a tautology (Lemma 1.5.7), so \( p_1 \) is a tautology by Lemma 1.5.8.
• If $p_2$ follows from $p_0$ or $p_1$, reason as in the previous case. Suppose $p_0, p_1 \Rightarrow p_2$. Then, by Lemma 1.5.7, $(p_0 \land p_1) \rightarrow p_2$ is a tautology. Because $p_0, p_1 \Rightarrow p_0 \land p_1$ by Conj, $p_0 \land p_1$ is a tautology. Thus, $p_2$ is a tautology by Lemma 1.5.8.

Since every $p_i$ with $i > 0$ is an axiom, follows from some $p_j$ with $j < i$, or follows from some $p_j, p_k$ with $j, k < i$, continuing in this manner, we find after finitely many steps that $p$ is a tautology.

\[\text{COROLLARY 1.5.10}\]

For all propositional forms $p_0, p_1, \ldots, p_{n-1}, q$,

\[
\text{if } p_0, p_1, \ldots, p_{n-1} \vdash q, \text{ then } p_0, p_1, \ldots, p_{n-1} \models q.
\]

The Law of Noncontradiction being a theorem of propositional logic (page 37) suggests that we have the following result, which is the second important property of propositional logic.

\[\text{COROLLARY 1.5.11}\]

Propositional logic is consistent.

\[\text{PROOF}\]

Let $p$ be a propositional form. Suppose that $p \land \neg p$ is a theorem. This implies that it is a tautology by the soundness theorem (1.5.9), but $p \land \neg p$ is a contradiction.

\[\text{Completeness}\]

We use the consistency of propositional logic to prove that propositional logic is complete. For this we need a few lemmas.

\[\text{LEMMA 1.5.12}\]

If not $\text{Con}(\neg q, p_0, p_1, p_2, \ldots)$, then $p_0, p_1, p_2, \ldots \vdash q$.

\[\text{PROOF}\]

If not $\text{Con}(p_0, p_1, p_2, \ldots)$, then $p_0, p_1, p_2, \ldots \vdash q$ by Theorem 1.5.2, so suppose $\text{Con}(p_0, p_1, p_2, \ldots)$. Assume that there exists a propositional form $r$ such that

\[
\neg q, p_0, p_1, p_2, \ldots \vdash r \land \neg r.
\]

This implies that there exists a formal proof

\[p_{i_0}, p_{i_1}, \ldots, p_{i_{n-1}}, \neg q, s_0, s_1, \ldots, s_{m-1}, r \land \neg r, \quad (1.10)\]

where $\neg q$ is in the proof because $\text{Con}(p_0, p_1, p_2, \ldots)$. Then,

\[p_{i_0}, p_{i_1}, \ldots, p_{i_{n-1}}, q\]
is a proof, where $q$ follows by IP with (1.10) as the subproof. Therefore,

$$p_0, p_1, p_2, \ldots \vdash q. \blacksquare$$

\[ \text{Lemma 1.5.13} \]

If $p_0, p_1, p_2, \ldots$ are maximally consistent, then for every propositional form $q$, either $q = p_i$ or $\neg q = p_i$ for some $i$.

**Proof**

Since $p_0, p_1, p_2, \ldots$ are consistent, both $q$ and $\neg q$ cannot be terms of the sequence. Suppose that it is $\neg q$ that is not in the list. By the definition of maximal consistency, we conclude that $\neg q, p_0, p_1, p_2, \ldots$ are not consistent. Therefore, by Lemma 1.5.12, we conclude that $p_0, p_1, p_2, \ldots \vdash q$, and since the sequence is maximally consistent, $q = p_i$ for some $i$. \(\blacksquare\)

To prove the next lemma, we use a technique called **induction on propositional forms**. It states that a property will hold true for all propositional forms if two conditions are met:

- The property holds for all propositional variables.
- If the property holds for $p$ and $q$, the property holds for $\neg p, p \land q, p \lor q, p \rightarrow q$, and $p \leftrightarrow q$.

In proving the second condition, we first assume that

$$\text{the property holds for the propositional forms } p \text{ and } q. \quad (1.11)$$

This assumption (1.11) is known as an **induction hypothesis**. Because

$$p \lor q \iff \neg p \rightarrow q,$$

$$p \land q \iff \neg(p \rightarrow \neg q),$$

and

$$p \leftrightarrow q \iff (p \rightarrow \neg q) \rightarrow \neg(p \rightarrow q),$$

we need only to show that the induction hypothesis implies that the property holds for $\neg p$ and $p \rightarrow q$.

\[ \text{Lemma 1.5.14} \]

If $\text{Con}(p_0, p_1, p_2, \ldots)$, there exists a valuation $\nu$ such that

$$\nu(p) = T \text{ if and only if } p = p_i$$

for some $i = 0, 1, 2, \ldots$. 
PROOF

Since \(\text{Con}(p_0, p_1, p_2, \ldots)\), we know that

\[ p_0, p_1, p_2, \ldots \]  
(1.12)

can be extended to a maximally consistent sequence of propositional forms by Theorem 1.5.4. If we find the desired valuation for the extended sequence, that valuation will also work for the original sequence, so assume that (1.12) is maximally consistent. Let \(X_0, X_1, X_2, \ldots\) represent all of the possible propositional variables (page 53). Define

\[ v(X_j) = \begin{cases} 
T & \text{if } X_j \text{ is a propositional variable of } p_i \text{ for some } i, \\
F & \text{otherwise.}
\end{cases} \]

We prove that this is the desired valuation by induction on propositional forms. We first claim that for all \(j\),

\[ v(X_j) = T \text{ if and only if } X_j = p_i \text{ for some } i. \]

To prove this, first note that if \(X_j\) is a term of (1.12), then \(v(X_j) = T\) by definition of \(v\). To show the converse, suppose that \(v(X_j) = T\) but \(X_j\) is not a term of (1.12). By Lemma 1.5.13, there exists \(i\) such that \(\neg X_j = p_i\). This implies that \(v(\neg X_j) = T\), and then \(v(X_j) = F\) (Definition 1.1.9), a contradiction.

Now assume that

\[ v(q) = T \text{ if and only if } q = p_i \text{ for some } i \]

and

\[ v(r) = T \text{ if and only if } r = p_i \text{ for some } i. \]

We first prove that

\[ v(\neg q) = T \text{ if and only if } \neg q = p_i \text{ for some } i. \]

- Suppose that \(v(\neg q) = T\). Then, \(v(q) = F\), and by induction, \(q\) is not in (1.12). Since \(p_0, p_1, p_2, \ldots\) is maximally consistent, \(\neg q\) is in the list (Lemma 1.5.13).

- Conversely, let \(\neg q = p_i\) for some \(i\). By consistency, \(q\) is not in the sequence. Therefore, by the induction hypothesis, \(v(q) = F\), which implies that \(v(\neg q) = T\).

We next prove that

\[ v(q \rightarrow r) = T \text{ if and only if } q \rightarrow r = p_i \text{ for some } i. \]

- Assume that \(v(q \rightarrow r) = T\). We have two cases to check. First, let \(v(q) = v(r) = T\), so both \(q\) and \(r\) are in (1.12) by the induction hypothesis. Suppose \(q \rightarrow r\) is not a term of the sequence. This implies that its
negation \( \neg(q \to r) \) is a term of the sequence by Lemma 1.5.13. Therefore, 
\( q \land \neg r \) is in the sequence by maximal consistency, so \( \neg r \) is also in the sequence, a contradiction. Second, let \( v(q) = F \). By induction, \( q \) is not a term of the sequence, which implies that \( \neg q \) is a term. Hence, \( \neg q \lor r \) is in the sequence, which implies that \( q \to r \) is also in the sequence.

- To prove the converse, suppose that \( q \to r = p_i \) for some \( i \). Assume 
\( v(q \to r) = F \). This means that \( v(q) = T \) and \( v(r) = F \). Therefore, by induction, \( q \) is a term of the sequence but \( r \) is not. This implies that \( \neg r \) is in the sequence, so \( \neg q \) is in the sequence by MT. This contradicts the consistency of (1.12).

\[ \square \]

**THEOREM 1.5.15 [Completeness]**

Every tautology of propositional logic is a theorem.

**PROOF**

Let \( p \) be a propositional form such that \( FL1, FL2, FL3 \not\vdash p \). By Lemma 1.5.12, 
\( \text{Con}(FL1, FL2, FL3, \neg p) \). Therefore, by Lemma 1.5.14, there exists a valuation 
such that \( v(p) = F \), which implies that \( \not\models p \).

\[ \square \]

**COROLLARY 1.5.16**

If \( p_0, p_1, p_2, \ldots \models q \), then \( p_0, p_1, p_2 \vdash q \) for all propositional forms \( p_0, p_1, p_2, \ldots \).

We conclude by Theorems 1.5.9 and 1.5.15 and their corollaries that the notions 
of semantically valid and syntactically valid coincide for deductions in propositional 
logic.

**Exercises**

1. Prove Lemma 1.5.6.
2. Provide the remaining parts of the proof of Lemma 1.5.7.
3. Let \( p, p_0, p_1, p_2, \ldots \) be propositional forms. Prove the following.
   - If \( p_0, p_1, p_2, \ldots \not\vdash p \), then \( \text{Con}(\neg p, p_0, p_1, p_2, \ldots) \).
   - If \( \text{Con}(p_0, p_1, p_2, \ldots) \) and \( p_0, p_1, p_2, \ldots \vdash p \), then \( \text{Con}(p, p_0, p_1, p_2, \ldots) \).
   - If \( \text{Con}(p_0, p_1, p_2, \ldots) \), then \( \text{Con}(p, p_0, p_1, p_2, \ldots) \) or \( \text{Con}(\neg p, p_0, p_1, p_2, \ldots) \).
4. Let \( p_0, p_1, p_2, \ldots \) be a maximally consistent sequence of propositional forms. Let \( p \) 
and \( q \) be propositional forms. Prove the following.
   - If \( p_0, p_1, p_2, \ldots \vdash p \), then \( p = p_k \) for some \( k \).
   - \( p \land q = p_k \) for some \( k \) if and only if \( p = p_i \) and \( q = p_j \) for some \( i \) and \( j \).
   - If \( (p \to q) = p_i \) and \( p = p_j \) for some \( i \) and \( j \), then \( q = p_k \) for some \( k \).
5. Use truth tables to prove the following. Explain why this is a legitimate technique.
   - \( \neg Q \lor (\neg R \lor \neg P) \vdash P \to \neg(Q \land R) \)
   - \( P \lor Q \to R \vdash \neg R \to \neg Q \)
(c) $P \rightarrow \neg(Q \rightarrow R) \vdash P \rightarrow \neg R$
(d) $P \leftrightarrow Q \lor R \vdash Q \rightarrow P$
(e) $P \lor Q \rightarrow R \land S, \neg P \rightarrow (T \rightarrow \neg T), \neg R \vdash \neg T$

6. Write a formal proof to show the following. Explain why this is a legitimate technique.
(a) $\neg P \lor Q, \neg Q \not\vdash \neg P$
(b) $\neg(P \land Q), P \not\vdash \neg Q$
(c) $P \rightarrow Q, P \not\vdash Q \lor R$
(d) $P \rightarrow Q, Q \rightarrow R, P \not\vdash R$
(e) $P \lor Q \land R, \neg P \not\vdash R$

7. Write a formal proof to show the following.
(a) $\neg(P \land Q) \not\equiv \neg P$
(b) $(P \rightarrow Q) \lor (R \rightarrow S), P \lor R \not\equiv Q \lor S$
(c) $P \lor R, Q \lor S, R \leftrightarrow S \not\equiv R \land S$

8. Write a formal proof to demonstrate the following.
(a) $p \lor \neg p$ is a tautology.
(b) $p \land \neg p$ is a contradiction.

9. Modify propositional logic by removing all replacement rules (1.3.9). Is the resulting logic consistent? sound? complete?

10. Modify propositional logic by removing all inference rules (1.2.10) except for Inference Rule 1.3.10. Is the resulting logic consistent? sound? complete?