1.1 INTRODUCTION

In this chapter we introduce the main, and first, concepts that one has to grasp in order to build, evaluate, purchase and sell financial structured products. Structured finance denotes the art (and science) of designing financial products to satisfy the different needs of investors and borrowers as closely as possible. In this sense, it represents a specific technique and operation of the financial intermediation business. In fact, the traditional banking activity, i.e. designing loans to provide firms with funds and deposits to attract funds from retail investors, along with managing the risk of a gap in their payoffs, was nothing but the most primitive example of a structuring process. Nowadays, the structured finance term has been provided with a more specialized meaning, i.e. that of a set of products involving the presence of derivatives, but most of the basic concepts of the old-fashioned intermediation business carry over to this new paradigm. Building on this basic picture, we will make it more and more involved, in this chapter and throughout the book, adding to these basic demands and needs the questions that professionals in the modern structured finance business address to make the products more and more attractive to investors and borrowers.

The very reason of existence of the structured finance market, as it is conceived today, rests on the same arguments as the old-fashioned banking business. That was motivated as the only way for investors to provide funds to borrowers, just in the same way as any sophisticated structured finance product is nowadays constructed to enable someone to do something that could not be done in any other way (or in a cheaper way) under the regulation. In this sense, massive use of derivatives and financial engineering appears as the most natural development of the old intermediation business.

To explain, take the simplest financial product you may imagine, a zero coupon bond, i.e. a product paying interest and principal in a single shot at the end of the investment. The investor’s question is obviously whether it is worth giving up some consumption today for some more at the end of the investment, given the risk that may be involved. The borrower’s question is whether it is worth using this instrument as an effective funding solution for his projects. What if the return is too low for the investors or so high that the borrower cannot afford it? That leads straight to the questions typically addressed by the structurer: what’s wrong with that structure? Maybe the maturity is too long, so what about designing a different coupon structure? Or maybe investors would prefer a higher expected return, even at the cost of higher risk, so why not make the investment contingent on some risky asset, perhaps the payoff of the project itself? If the borrower finds the promised return too high, what about making the project less risky by asking investors to provide some protection? All of these questions would lead to the definition of a “structure” for the bond as close as possible to those needs, and this structure will probably be much more sophisticated than any traditional banking product.
The production process of a structured finance tool involves individuation of a business idea and the design of the product, the determination and analysis of pricing, and the definition of risk measurement and management procedures. Going back again to the commercial banking example, the basic principles were already there: design of attractive investment and funding products, determination of interest rates consistent with the market, management of the misalignment between asset and liabilities (or asset liability management, ALM). Mostly the same principles apply to modern structured finance products: how should we assemble derivatives and standard products together, how should we price them and manage risk?

The hard part of the job would then be to explain the structure, as effectively as possible, to the investors and borrowers involved, and convince them that it is made up to satisfy their own needs. The difficulty of this task is something we are going to share in this book. What are you actually selling or buying? What are the risks? Could you do any better? We will see that asking the right questions will lead to an answer that will be found to be straightforward, almost self-evident: why did not I get it before? It is the replicating portfolio. The bad and good news is that many structured products have their own replicating portfolios, peculiar to them and different from those of any other. Bad news because this makes the design of a taxonomy of these products an impossible task; good news because the analysis of any new product is as surprising and thrilling as a police story.

### 1.2 Arbitrage-Free Valuation and Replicating Portfolios

All of the actors involved in the production process described above, i.e. the structurer, the pricer and the risk manager, share the same working tools: arbitrage-free valuation and the identification of replicating strategies for every product. Each and every product has to be associated to a replicating portfolio, or a dynamic strategy, well suited to deliver the same payoff at some future date, and its value has to be equal to that of its replicating portfolio. The argument goes that, if it were not so, unbounded arbitrage profits could be earned by going long in the cheaper portfolio and going short in the dearer one. This concept is the common fabric of work for structurers, pricers and risk managers. The structurer assembles securities in a replicating portfolio to design the product, the pricer evaluates the products as the sum of the prices of the securities in the replicating portfolio, and the risk manager uses the replicating portfolio to identify the risk factors involved and make the appropriate hedging decisions. Here we will elaborate on this subject to provide a bird’s-eye review of the most basic concepts in finance, developed along the replicating portfolio idea. This would require the reader to be well acquainted with them. For intermediate readers, mandatory references for a broad introduction to finance are reported at the end of the chapter.

Under a standard finance textbook model the production process of a structured product would be actually deterministic. In fact, the basic assumption is that each product is endowed with an “exact” replicating strategy (the payoff of each product is “attainable”): this is what we call the “market completeness” hypothesis. Everybody knows that this assumption is miles away from reality. Markets are inherently “incomplete”, meaning that no “exact” replicating portfolio exists for many products, and it is particularly so for the complex products in the structured finance business. Actually, market incompleteness makes life particularly difficult in structured finance. In fact, the natural effect is that the production process of these securities involves a set of decisions over stochastic outcomes. The structurer would
compare the product being constructed against the cheapest alternative directly available to the customers on the market. The pricer has to select the “closest” replicating portfolio to come up with a reasonable price from both the buyer’s and the seller’s point of view. Finally, the risk manager has to face the problem of the “hedging error” he would bear under alternative hedging strategies.

1.3 REPLICATING PORTFOLIOS FOR DERIVATIVES

Broadly speaking, designing a structured product means defining a set of payments and a set of rules determining each one of them. These rules define the derivative contracts embedded in the product, and the no-arbitrage argument requires that the overall value of the product has to be equal to the sum of the plain and the derivative part. But we may push our replicating portfolio argument even further. In principle, a derivative may be considered as a structure including a long or short position in a risk factor against debt or investment in the risk-free asset. This is the standard leverage feature that is the distinctive mark of a derivative contract.

1.3.1 Linear derivatives

As the simplest example, take a forward contract $\text{CF}(S, t; F(0), T)$, that is the value at time $t$ of a contract, stipulated at time 0, for delivery at time $T$ of one unit of the underlying $S$ at the price $F(0)$. The payoff to be settled at time $T$ is linear: $S(T) - F(0)$. By a straightforward no-arbitrage argument, it is easy to check that the same payoff can be attained by buying spot a unit of the underlying and issuing debt with maturity $T$ and nominal value $F(0)$. No-arbitrage requires that the value of the contract has to be equal to that of the replicating portfolio

$$\text{CF}(S, t; F(0), T) = S(t) - v(t, T)F(0) \quad (1.1)$$

where $v(t, T)$ is the discount factor function – that is, the value, at time $t$, of a unit of currency to be due at time $T$. By market convention, the delivery price is the forward price observed at time 0, when the contract is originated. The forward price is technically defined as $F(0) \equiv S(0)/v(0, T)$, so that $\text{CF}(S, 0; F(0), T) = 0$ and the value of the forward contract is zero at origin. Notice that the price of a linear contract does not depend on the distribution of the underlying asset. Furthermore, the replicating strategy does not call for a rebalancing of the portfolio as time elapses and the value of the underlying asset changes: it is a static replication strategy.

1.3.2 Nonlinear derivatives

Nonlinear products, i.e. options, can be provided with a replicating portfolio by the same line of reasoning. Take a European call contract, payoff $\text{max}(S(T) - K, 0)$, with a $K$ strike price for an exercise time $T$. By the same argument, we look for a replicating portfolio including a spot position in $\Delta_c$ units of the underlying and a debt position for a nominal value $W_c$. The price of the call option at time $t$ is

$$\text{CALL } (S, t; K, T) = \Delta_c S(t) - v(t, T) W_c \quad (1.2)$$
Notice that replicating portfolio can be equivalently represented in terms of two other elementary financial products. These two products are digital, meaning they yield a fixed payoff is the event of \( S(T) \geq K \), and 0 otherwise. The fixed payoff may be defined in terms of units of the asset or in units of currency. In the former case the digital option is called \textit{asset-or-nothing} (AoN), and in the latter case, \textit{cash-or-nothing} (CoN). It is easy to check that going long an AoN\((S, t; K, T)\) for one unit of the underlying and going short CoN\((S, t; K, T)\) options for \( K \) units of currency yields a payoff \( \max(S(T) - K, 0) \). We then have (see Figure 1.1)

\[
\text{CALL} (S, t; K, T) = \text{AoN} (S, t; K, T) \quad \text{and} \quad \text{CoN} (S, t; K, T)
\]

\[
\text{CALL} (S, t; K, T) = \text{AoN} (S, t; K, T) - K\text{CoN} (S, t; K, T) \quad (1.3)
\]

Nonlinearity of the payoff implies that the value of the product depends on the probability distribution of \( S(T) \). Without getting into the specification of such distribution, notice that for scenarios under which the event \( S(T) \geq K \) has measure 0 we have that both the AoN and the CoN products have zero value. For scenarios under which the event has measure 1, the AoN product will have a value of \( S(t) \) and the CoN option (with payoff of one unit of currency) will be worth \( v(t, T) \). This amounts to stating that \( 0 \leq \Delta_c \leq 1 \) and \( 0 \leq W_c \leq K \). Accordingly,

\[
0 \leq \text{CALL} (S, t; K, T) \leq \text{CF} (S, t; K, T) \quad (1.4)
\]

and the value of the call option has to be between zero and the value of a long position in a forward contract. This is the most elementary example of an incomplete market problem. Without further comment on the probability distribution of \( S(T) \), beyond the scenarios with probability 0 and 1, all we can state are the \textit{pricing bounds} of the product, and the corresponding replicating portfolios that are technically called its \textit{super-replicating portfolios}. The choice of a specific price then calls for the specification of a particular stochastic dynamic of the underlying asset and a corresponding \textit{dynamic replication} strategy.
Once a specific price is obtained for the call option, the replicating portfolio of the corresponding put option [payoff: \( \max(K - S(T), 0) \)] can be obtained from the well-known put–call parity relationship

\[
\text{CALL} (S, t; K, T) - \text{PUT} (S, t; K, T) = \text{CF} (S, t; K, T) \quad (1.5)
\]

which can be immediately obtained by looking at the payoffs. Notice that by using the replicating portfolios of the forward contract and the call option above, we have

\[
\text{CALL} (S, t; K, T) - \text{PUT} (S, t; K, T) = (\Delta_c - 1) S(t) + v(t, T) (K - W_c) \quad (1.6)
\]

Recalling the bounds for the delta and leverage of the call option, it is essential to check that a put option amounts to a short position in the underlying asset and a long position in the risk-free bond. The corresponding pricing bounds will then be zero and the value of a short position in a forward contract.

### 1.4 NO-ARBITRAGE AND PRICING

Selecting a price within the pricing bound calls for the specification of the stochastic dynamics of the underlying asset. A world famous choice is that of a geometric Brownian motion.

\[
dS(t) = \mu S(t) \, dt + \sigma S(t) \, dz(t) \quad (1.7)
\]

where \( dz(t) \sim \Phi(0, dt) \) is defined a Wiener process and \( \mu \) and \( \sigma \) are constant parameters (drift and diffusion, respectively). Technically speaking, the stochastic process is defined with respect to a filtered probability space \( \{ \Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P} \} \). The filtration determines the dynamics of the information set in the economy, and the probability measure \( \mathcal{P} \) describes its stochastic dynamics. It is very easy to check that the transition probability of \( S \) at any time \( T > t \), conditional on the value \( S(t) \) observed at time \( t \), is log-normal. Assuming a constant volatility \( \sigma \) then amounts to assuming Gaussian log-returns.

#### 1.4.1 Univariate claims

To understand how the no-arbitrage argument enters into the picture just remember that the standard arbitrage pricing theory (APT) framework requires

\[
E \left( \frac{dS(t)}{S(t)} \right) = \mu \, dt = (r + \gamma \sigma) \, dt \quad (1.8)
\]

where \( r \) is the instantaneous interest rate intensity and \( \gamma \) is the market price of risk for the risk factor considered in the economy (the analysis can of course be easily extended to other risk factors). The key point is that the market price of risk (for any source of risk) must be the same across all the financial products. Financial products then differ from one another only in their sensitivity to the risk factors. Based on this basic concept, one can use the Girsanov theorem to derive

\[
dS(t) = (r + \gamma \sigma) S(t) \, dt + \sigma S(t) \, dz(t) = r S(t) \, dt + \sigma S(t) \, dz^\gamma(t) \quad (1.9)
\]
where $dz^* (t) \equiv dz (t) + \gamma dt$ is a Wiener process in the probability space $\{\Omega, \mathcal{F}, \mathbb{Q} \}$. The new $\mathbb{Q}$ measure is such that any financial product or contract yields an instantaneous interest rate intensity, without any risk premium. For this reason it is called the risk-adjusted measure. To illustrate, consider the call option written on $S$, described above. We have

$$d \text{CALL} (S, t; K, T) = \text{CALL} (S, t; K, T) (r dt + \sigma_{\text{Call}} dz^* (t)) \quad (1.10)$$

where $\sigma_{\text{Call}}$ is the instantaneous volatility that can be immediately obtained by Ito’s lemma. Notice that Ito’s lemma also yields

$$E_{\mathbb{Q}} (d \text{Call}) = \left( \frac{\partial \text{Call}}{\partial t} + \frac{\partial \text{Call}}{\partial S} rS (t) + \frac{1}{2} \frac{\partial^2 \text{Call}}{\partial S^2} \sigma^2 S (t)^2 \right) dt = r \text{Call} dt \quad (1.11)$$

from which it is immediate to recover the Black–Scholes fundamental PDE:

$$\frac{\partial \text{Call}}{\partial t} + \frac{\partial \text{Call}}{\partial S} rS (t) + \frac{1}{2} \frac{\partial^2 \text{Call}}{\partial S^2} \sigma^2 S (t)^2 - r \text{Call} = 0 \quad (1.12)$$

Derivative products must solve the fundamental PDE in order to rule out arbitrage opportunities. The price of specific derivative products (in our case a European call) requires specification of particular boundary solutions (in our case $\text{Call} (T) = \max (S (T) - K, 0)$).

Alternatively the solution may be recovered by computing an expected value under the measure $\mathbb{Q}$. Remember that under such a measure all the financial products yield a risk-free instantaneous rate of return. Assume that in the economy there exists a money market fund $B(t)$ yielding the instantaneous rate of return $r(t)$:

$$dB (t) = rB (t) \quad (1.13)$$

It is important to check that the special property of the measure $\mathbb{Q}$ can be represented as a martingale property for the prices of assets computed using the money market fund as the numeraire:

$$E_{\mathbb{Q}} \left[ \frac{S (T)}{B (T)} \right] = \frac{S (t)}{B (t)} \Rightarrow E_{\mathbb{Q}} \left[ \frac{\text{CALL} (T)}{B (T)} \right] = \frac{\text{CALL} (t)}{B (t)} \quad (1.14)$$

For this reason, the measure $\mathbb{Q}$ is also called an equivalent martingale measure (EMM), where the term equivalent refers to the technical requirement that the two measures must assign probability zero to the same events (complying with the super-replication bounds described above).

An alternative way of stating the martingale property is to say that under measure $\mathbb{Q}$ the expected value of each and every product at any future date $T$ has to be equal to its forward price for delivery at time $T$. So, for example, for our call option under examination we have

$$\text{Call} (t) = E_{\mathbb{Q}} \left[ \text{Call} (T) \frac{B (t)}{B (T)} \right] = E_{\mathbb{Q}} \left[ \exp \left( - \int_{t}^{T} r (u) du \right) \max (S (T) - K, 0) \right] \quad (1.15)$$

$$= v (t, T) E_{\mathbb{Q}} [\max (S (T) - K, 0)]$$
where we have assumed $r(t)$ to be non-stochastic or independent on the underlying asset $S(t)$. It is well known that the same result applies to cases in which this requirement is violated, apart for a further change of measure from the EMM measure $Q$ to the forward martingale measure (FMM) $Q(T)$: the latter is obtained by directly requiring the forward prices to be martingales, using the risk-free discount bond maturing at time $T (v(t, T))$ instead of the money market fund as the numeraire.

Under the log-normal distribution assumption in the Black–Scholes model we recover a specific solution for the call option price:

$$\text{CALL} (S, t; K, T) = \Delta cw (S(t) - v(t, T) W)$$

with

$$\Delta c = \Phi (d_1) \quad W_c = K \Phi (d_2)$$

$$d_1 = \frac{\ln (F(t)/K) + \sigma^2 / 2 (T - t)}{\sigma \sqrt{T - t}}$$

$$d_2 = d_1 - \sigma \sqrt{T - t}$$

$$F(t) = \frac{S(t)}{v(t, T)}$$

where $\Phi(.)$ denotes the standard normal cumulative distribution and $F(t)$ is the forward price of $S(t)$ for delivery at time $T$.

While the standard Black and Scholes approach is based on the assumption of constant volatility, there is vastly documented evidence that volatility, measured by whatever statistics, is far from constant. Non-constant volatility gives rise to different implied volatilities for different strikes (smile effect) and different exercise dates (term structure of volatility). Option traders “ride” the volatility surface betting on changes in skewness and kurtosis much in a same way as fixed income traders try to exploit changes in the interest rate term structure. Allowing for volatility risk paves the way to the need to design a reliable model for the stochastic dynamics of volatility. Unfortunately, no general consensus has as yet been reached on such a model. Alternatively, one could say that asset returns are not normally distributed, but the question of which other distribution could be a good candidate to replace the log-normal distribution of prices (and the corresponding geometric Brownian motion) has not yet found a definite satisfactory answer. This argument brings the concept of model risk as a paramount risk management issue for nonlinear derivative and structured products.

### 1.4.2 Multivariate claims

Evaluation problems are compounded in cases in which a derivative product is exposed to more than one risk factor. Take, for example, a derivative contract whose underlying asset is a function $f(S_1, S_2, \ldots, S_N)$. We may again assume a log-normal multivariate process for each risk factor $S_i$:

$$dS_i (t) = \mu_i S_i (t) dt + \sigma_i S_i (t) dz_i (t)$$

where we assume the shocks to be correlated $E(dz_i (t), dz_j (t)) = \rho_{ij} dt$. The correlation structure among the risk factors then enters into the picture.
Parallel to the Black–Scholes model in a univariate world, constant volatilities and correlations lead to the assumption of normality of returns in a multivariate setting. Extending the analysis beyond the Black–Scholes framework calls for a different multivariate probability distribution for the returns. The problem is even more compounded because the joint distribution must be such that the marginal distribution be consistent with the stochastic volatility behaviour analysed for every single risk factor. A particular tool, which will be used extensively throughout this book, enables us to break down the problem of identifying a joint distribution into that of identifying the marginals and the dependence structure independently. The methodology is known as the copula function approach. A copula function enables us to write

\[
\Pr (S_1 \leq K_1, S_2 \leq K_2, \ldots, S_n \leq K_n) = C(\Pr (S_1 \leq K_1), \Pr (S_2 \leq K_2), \ldots, \Pr (S_n \leq K_n))
\]

where \(C(u_1, u_2, \ldots, u_n)\) is a function satisfying particular requirements.

Alternatively – particularly for derivatives with a limited number of underlying assets – a possibility is to resort to the change of numeraire technique. This could apply to bivariate claims, such as, for example, the option to exchange (OEX), which gives the holder the right to exchange one unit of asset \(S_1\) against \(K\) units of asset \(S_2\) at time \(T\). The payoff is then \(\text{OEX}(T) = \max(S_1(T) - KS_2(T), 0)\). In this case, using \(S_2\) as the numeraire, we may use the Girsanov theorem to show that the prices of both \(S_1\) and \(S_2\), computed using \(S_2(T)\) as numeraire, are martingale. We then have

\[
\text{OEX}(t) = S_2(t) E_M \left[ \max \left( \frac{S_1(T)}{S_2(T)} - K, 0 \right) \right]
\]

with \(M\) a new martingale measure such that \(E_M(S_1(T)/S_2(T)) = S_1(t)/S_2(t)\). It is easy to check that if \(S_1\) and \(S_2\) are log-normal, it yields the famous Margrabe formula for exchange options.

As a further special case, consider \(S_2(t) \equiv v(t, T)\), that is, the discount factor function. As we obviously have \(v(T, T) = 1\), we get

\[
\text{OEX}(t) = v(t, T) E_M \left[ \max (S_1(T) - K, 0) \right]
\]

and measure \(M\) is nothing but the forward martingale measure (FMM) \(Q(T)\) quoted above. Furthermore, if \(Q(T)\) is log-normal, we recover Black’s formula

\[
\text{CALL}(S, r; K, T) = v(t, T) [\Delta_c F(S, t; T) - W_c]
\]

where the delta \(\Delta_c\) and leverage \(W_c\) are defined as above.

### 1.5 THE STRUCTURING PROCESS

We are now in a position to provide a general view of the structuring process, with the main choices to be made in the design phase and the issues involved for the pricing and risk management functions. In a nutshell, the decision boils down to the selection of a set of maturities. For each maturity one has then to design the exposure to the risk factors. Choices
are to be made concerning both the nature of the risk factors to be selected (interest rate risk, equity, credit or others) and the specific kind of exposure (linear or nonlinear, long or short). In other words, designing a structure product amounts to assembling derivative contracts to design a specific payoff structure contingent on different realizations of selected risk factors.

1.5.1 The basic objects

Let us start with an abstract description of what structuring a financial product is all about. It seems that it all boils down to the design of three objects. The first is a set of maturity dates representing the due date of cash flow payments:

\[ \{t_1, t_2, \ldots, t_i, \ldots, t_n\} \]

The second is a set of cash flows representing the interest payments on the capital

\[ \{c_1, c_2, \ldots, c_i, \ldots, c_n\} \]

The third is the repayment plan of the capital

\[ \{k_1, k_2, \ldots, k_i, \ldots, k_n\} \]

or (the same concept stated in a different way) a residual debt plan

\[ \{w_1, w_2, \ldots, w_i, \ldots, w_n\} \]

Building up a structured finance product amounts to setting rules allowing univocal definition of each one of these objects. Note that all the objects may in principle be deterministic or stochastic. Repayment of capital may be decided deterministically at the beginning of the contract, according to standard amortizing schedules on a predefined set of maturities, and with a fixed coupon payment (as a percentage of residual debt); alternatively, a flat, and again deterministic, payment schedule can be designed to be split into interest and capital payments. Fixed rate bonds, such as the so-called bullet bonds, are the most standard and widespread examples of such structures. It is, however, in the design of the rules for the definition of stochastic payments that most of the creative nature of the structurer function comes into play. Coupon payments may be made contingent on different risk factors, ranging from interest rates to equity and credit indexes, and may be defined in different currencies. The repayment plan may instead feature rules to enable us to postpone (extendible bonds) or anticipate (retractable bonds) the repayment of capital, or to allow for the repayment to be made in terms of other assets, rather than cash (convertible bonds). These choices may be assigned to either the borrower or the lender, and may be made at one, or several dates: notice that this feature also contributes towards making the choice of the set of payment dates stochastic (early exercise feature). As one can glean directly from the jargon used, structuring a product means that we introduce derivative contracts in the definition of the coupon and the repayment plans.

1.5.2 Risk factors, moments and dimensions

The core of the structuring process consists of selecting the particular kind of risk exposure characterizing the financial product. With respect to such exposure, a structurer addresses
three basic questions. Which are the technical features of the product, or, in other words, which is the risk profile of the product? Is there some class of investors or borrowers that may be interested in such risk profile; that is, which is the demand side for this product? Finally, one should address the question whether investors and borrowers can achieve the same risk profile in an alternative, cheaper way – that is, which are the main competitors of the product?

In this book we are mainly concerned with the first question, i.e. that of the production technology of the structuring process, which is of course a mandatory prerequisite to addressing the other two, which instead are more related to the demand and supply schedules of the structured finance market.

In the definition of the risk profile of the product one has to address three main questions:

- Which kind of risk factors?
- Which moments of risk factors?
- Which dimension of risk factors?

Which kind of risk factors?

One has to decide the very nature of the risk exposure provided in the product. Standard examples are

- interest rates/term structure risk;
- equity risk;
- inflation risk/commodity risk;
- credit risk/country risk;
- foreign exchange risk.

Very often, or should we say always, a single product includes more than one risk factor. For example, interest rate risk is always present in the very nature of the product to provide exchange of funds at different times, and credit risk is almost always present as the issuer of the product often is a defaultable entity. Foreign exchange risk enters whenever the risk factor is referred to a different country with respect to that of the investor or borrower. Of course, these kinds of risk are, so to speak, “built-in to” the product, and are, loosely speaking, inherited from standard contractual specifications of the product such as the issuer, the currency in which payoffs and risk factors are denominated. Apart from this, of course, some risk factor characterizes the very nature, or the dominant risk exposure of the product, so that, for example, we denote one product equity linked and another one credit linked. More recent products, known as hybrids, include two sources of risk as the main feature of the product (such as forex and credit risk in the so-called “currency risk swap”).

Which moments of risk factors?

The second feature to address is the kind of sensitivity one wants to provide to the risk factors. The usual distinction in this respect is between linear and nonlinear products. Allowing for linear sensitivity to the risk factor enables us to limit the effect to the first moment. The inclusion of option-like features in the structure introduces a second dimension into the picture: dependence on volatility. In the post-Black and Scholes era, volatility is far from constant, and represents an important attribute of every risk factor. This means that
when evaluating a structured product that includes a nonlinear derivative, one should take into account the possibility that the value of the product could be affected by a change in volatility, even though the first moment of the risk factor stays unchanged.

Which dimension of risk factor?

The model risk problem is severely compounded in structured products in which the risk factor is made up of a “basket” of many individual risk factors. These products are the very frontier of structured finance and are widespread both in the equity and the credit-linked segments of the market. Using a basket rather than a single source of risk in a structured product is motivated on the obvious ground of providing diversification to the product, splitting the risk factor into systematic (or market) and idiosyncratic (or specific) parts. From standard finance textbooks we know that the amount of systematic risk in a product is determined by the covariance, or by the correlation between each individual risk factor and the market. But we should also note that, in that approach, volatilities and correlation of asset returns are assumed constant, and this is again clearly at odds with the evidence in financial market data. Correlation then is not constant, and the value of a financial product may be affected by a change in correlation even though neither the value of the risk factor nor its volatility has changed. Again, this paves the way to the need to devise a model for correlation dynamics, a question that has not yet found a unique satisfactory answer.

1.5.3 Risk management

The development of a structured finance market has posed a relevant challenge to the financial risk management practice and spurred the development of new risk measurement techniques. The increasing weight of structured financial products has brought into the balance sheet of the financial intermediaries – both those involved on the buy and the sell side – greater exposure to contingent claims and derivative contracts. Most of these exposures were new to the traditional financial intermediation business, not only for the nature of risk involved (well far beyond term structure risk) but also for the nonlinearity or exotic nature of the payoffs involved.

Optionality

The increased weight of nonlinear payoffs has raised the problem of accuracy of the parametric risk measurement techniques, in favour of simulation-based techniques. The development of exotic products, in particular, has given risk managers a two-fold problem: on the one hand, the need to analyse the pricing process in depth to unravel the risks nested in the product; on the other hand, the need to resort to acceptable pricing approximations in closed form, or at least light enough to be called in simulation routines as many times as necessary. Nonlinear payoffs have also raised the problem of evaluating the sensitivity of the market value of a position to changes in volatility and correlation, as well as the shape of the probability distribution representing the pricing kernel.

Measurement risk

Coping with a specification of volatility and correlation immediately leads to other risks that are brought into the picture. One kind of risk has to do with volatility and correlation
estimation. This *measurement risk* problem is common to every statistical application and has to do with how a particular sample may be considered representative of the universe of the events from a statistical inference point of view. Some technical methods can be used to reduce such estimation risk. In financial applications, however, this problem is compounded by the need to choose the proper information source – a choice that is more a matter of art than science and calls for good operating knowledge of the market. What is typical of financial applications is in fact the joint presence of “implied” and “historical” information and the need to choose between them. So, what is the true volatility figure? Is it the implied volatility backed out from a cross-section analysis of option prices, or is it to be estimated from the time series of prices of the underlying assets? Or do both cross-section prices and time series data include part of the information? And what about correlation?

**Model risk**

A different kind of risk has to do with the possible misspecification of the statistical model used. Apart from the information source used and the technique applied, the shape of the probability distribution we are using may not be the same as that generating the data. This *model risk* takes us back to the discussion above on possible statistical specifications for volatility and correlation dynamics in a post-Black and Scholes world. As we stated previously, no alternative model has been successful in replacing the Black–Scholes framework. Apart from choosing a specific model, however, one can cope with model risk by asking which is the sign of the position with respect to volatility and correlation and performing *stress testing analysis* using alternative scenarios.

**Long-term risk**

A particular feature of many structured finance products that compounds the problems of both *measurement* and *model* risk is that typically the contingent claims involved are referred to maturities that are very far in the future. It is not unusual to find embedded options to be exercised in five years or more. The question is then: Which is a reasonable volatility figure for the distribution of the underlying asset in five years? There is no easy way out from this *long-term risk* feature, other than sticking to the standard Black–Scholes constant volatility assumption, or sophisticated models to predict persistent changes in volatility. Again, a robust solution is to resort to extreme scenarios for volatility and correlation.

**Counterparty risk**

Last, but not least, structured finance has brought to the centre of the scene *counterparty risk*. Not only do these products expose the investor and/or the borrower to the possibility that the counterparty could not face its obligation, but very often these products are hedged, resorting to a *back-to-back* strategy on the over-the-counter (OTC) market. This is particularly so for products, including complex exotic derivatives, that may be particularly difficult to *delta–gamma hedge* on organized markets. So, to take the example of a very common product, if one is issuing an equity-linked note whose payoff is designed as a basket Asian option, he can consider hedging the embedded option position directly on the market, or can hedge it on the OTC market by buying an option with the same exact features from an investment bank. The cost of the former choice is the need to have sophisticated human resources, and some unavoidable degree of *basis risk* and/or *hedging risk*. The risk with the latter choice is
default of the counterparty selling the option, in which case one has to look for a different counterparty and to pay a new premium to keep the position hedged (substitution cost). Allowing for counterparty risk causes weird effects on the risk management of derivative products. Not only is it dangerous to overlook this source of risk per se, but it may also interfere with market risk inasmuch as counterparty risk is not taken into account in the pricing and hedging activity.

1.6 A TALE OF TWO BONDS

In the spirit of introducing the reader to the methodology of structured products, rather than to a classification, we now provide an example involving two of the easiest cases: an equity-linked and a reverse convertible bond. While these products are probably very well known even to non-professional readers, we think that following them in a sort of “parallel slalom”, rather than one by one, would help to summarize and highlight some of the basic methodological aspects discussed above, which are of general interest for the analysis of any other product.

Take a zero coupon bond by which investors provide funding to some borrower. The maturity of the zero coupon bond is $T$ and the nominal amount of principal is $L$. Define $S(T)$ the value of a risky asset at the date of maturity of the bond.

Consider the two following structures:

- **Equity-linked note**: At time $T$ the note will pay:
  
  (i) the principal $L$;
  
  (ii) a coupon equal to the greater between a guaranteed return $r_g$ (typically low and unattractive) and the rate of appreciation of the risky asset with respect to a given value $K$: $\max(r_g, S(T)/K - 1)$.

- **Reverse convertible note**: At time $T$ the note will pay:
  
  (i) the principal $L$ if the value of $S(T)$ is greater than some value $K$, and an amount of stocks equal to $n = L/K$ otherwise: $\min(L, n^{S(T)})$;
  
  (ii) a coupon equal to $r_g$ (typically pretty high and attractive).

We will now provide a comparative analysis of these two products, asking which are the similarities and the differences.

1.6.1 Contingent coupons and repayment plans

At a glance, it is immediately clear that both products include nonlinearities, and option-like derivatives. A first difference that emerges from mere description of the payoffs is that in the equity-linked note case the nonlinearity is introduced in the coupon payment, while in the reverse convertible case the derivative component is in the repayment plan.

We may be more precise and discover by straightforward manipulation that the coupon rate of the equity-linked note is given by

$$\text{Coupon} = r_g + \max [0, S(T) - (1 + r_g) K] / K \quad (1.22)$$
that is, a constant part plus the payoff of a call option. The repayment of the principal $L$ is guaranteed.

In the reverse convertible the coupon payment is guaranteed while it is the repayment of principal that is contingent on the risky asset $S(T)$. We have

$$\text{Repayment} = L - \max[0, L - nS(T)] = L - n \max[0, K - S(T)]$$

and the principal repaid will consist of the principal minus the payoff of a put option.

### 1.6.2 Exposure to the risky asset

The two products above include a part of the payoff contingent on the value of the risky asset $S$. The question that immediately follows is their sensitivity to changes in the value of that asset. Does the investor have a long or short position in the risky asset $S$? It may be surprising to discover that from this point of view the two products are similar.

Let us start with the equity note. We saw that this product includes a long position in a call option. It is well known that buying a call option is a way to take a long position on the underlying asset for a delta $(0 \leq \Delta_c \leq 1)$ quantity funded by leverage $(0 \leq W_c \leq \text{strike})$. The value of the equity-linked note (ELN) is then

$$\text{ELN} = v(t, T)(1 + r_g) + \frac{\text{CALL}(S, t; (1 + r_g)K, T)}{K}$$

$$= v(t, T)(1 + r_g) + \Delta_c \frac{S(t)}{K} - v(t, T) \frac{W_p}{K} \quad (1.24)$$

On the contrary, we know that a put option represents a short position for a delta $\Delta_p = \Delta_c - 1(-1 \leq \Delta_p \leq 1)$ and a long position in the risk-free asset, such that $W_p = K - W_c$. Notice, however, that the reverse convertible note (RCN) includes a short position in a put option. Assuming $L = 1$ we then have

$$\text{RCN} = v(t, T)(1 + r_g) - \frac{\text{PUT}(S, t; K, T)}{K}$$

$$= v(t, T)(1 + r_g) + (1 - \Delta_c) \frac{S(t)}{K} - v(t, T) \frac{K - W_c}{K} \quad (1.25)$$

We may then check that both the equity-linked and the reverse convertible products share the same feature of a long position in the risky asset funded by a leverage position.

### 1.6.3 Exposure to volatility

An increase in the value of the underlying asset would then have a positive effect on both of the structured products. What about a change in volatility? Standard option pricing theory suggests that response of the two products should now be opposite. The equity-linked note in fact embeds a long position in an option, and, unless the option itself is endowed with complex exotic features, that causes the value of the product to be positively affected by a volatility increase. An increase in volatility would also increase the value of the put option in the reverse convertible product, but as in this case the option is sold by the investor to the
issuer, that would subtract value from the product. So, recognizing a long or short position in volatility is another question that any investor has to address. Notice that in this case – where we have plain vanilla options – it coincides with being long or short in an option, but that is not a general result. If, for example, as often happens in real world cases, barrier option were used, the sign of exposure to volatility should be measured case by case.

1.6.4  Hedging

The difference between the two products also emerges, in quite a neat form, in a dynamic hedging perspective. Consider a hedging policy in discrete time for both of the products. In both cases, delta hedging would require us to take a short position in the underlying asset. From standard option pricing theory we know that delta hedging is effective against infinitely small changes of the underlying asset, but what about the impact of finite changes in the underlying? This question may be relevant if the hedging portfolio is not frequently rebalanced or the underlying moves a lot between the rebalance dates. It is easy to see that this second-order effect, called \textit{gamma} exposure, has different sign in the two cases. In the equity-linked note case changes of the underlying increase the value of the product, while they correspondingly decrease the value of the reverse convertible note. It should be remembered that as this is a second-order effect, the impact is due to the absolute change in the underlying, rather than its direction: so, a delta-hedged investment position in a reverse convertible note leaves the investor exposed to losses from finite changes of the underlying no matter what their directions, and a \textit{gamma-hedging} strategy would be strongly recommended.

1.7  STRUCTURED FINANCE AND OBJECT-ORIENTED PROGRAMMING

As we saw above, the job of every participant in the structured finance business is to assemble objects. Every product can be decomposed in a stream of cash flows and every cash flow in a set of long and short positions. The term “object” introduces another function that is particularly relevant in the structured finance team, and that is central in this book: IT design and software engineering.

\textit{Object-Oriented Programming} (OOP) denotes a particular programming technique that is based on the idea of partitioning the programming tasks in elementary units that are then linked together to perform the overall task. The main advantage in favour of OOP is in reusability of the code and updating. In case some adjustment is needed, one has to focus only on the interested part without rewriting the entire code from scratch. Furthermore, the objects are black boxes, including methods and attributes, that can be used without in-depth knowledge of their content. So, when one takes an object called “option”, for example, he would take something that would have some methods to compute prices, deltas, leverage, and the like, without any need to know anything about the model used to compute them.

The software engineer and the financial engineer look at the concept of “object” with different attitudes. For software engineers, an object is something in which to hide attributes and methods; it is something to \textit{forget about}. For financial engineers, an object is something to unbundle in order to understand more about its working; it is something to \textit{learn from}. But, curiously enough, in structured finance the objects are the same: they are the basic
Structured components of the replicating portfolio. For this reason, both the software and financial engineers very often find themselves designing a system of objects to represent and manage structured products. The result must be consistent with the aims of both of them:

- It must carry the information content with respect to the risk factors as required by the financial engineer.
- It must allow re-usability of code and code update as required by the software engineer.

Complying with the two targets is beneficial for the financial intermediary as a whole, and the benefits are particularly relevant for the risk management process. A well-built object oriented system

- would be able to speak out on the risks involved, the kind of risk, volatility and correlation;
- would allow a consistent update of prices and sensitivities of all the objects involved in the structured products: changes of models are consistently “inherited” by all the products in the portfolio;
- if the structure of the objects is finally shared with the counterparties, that could speed up the transmission of information and could enable automated execution of the deal. This source of execution risk is currently causing much concern to people in the market and to the regulators.

The aim of this book is to reach both the financial engineer and the software engineer, and to lay down a common set of tools for both of them. Our ambition is to make them meet and work together sharing language and concepts. For this reason, we have attempted to address every topic within the common language of the replicating portfolio, and the objects involved, spelled out in the jargon of both the software engineer (OOP) and the financial engineer (building block approach). Every topic would be discussed in an object-oriented framework, paying attention to: (i) the global structure of relationships among the objects; (ii) availability of data structures shared by people in the market in that specific instance of XML language called FpML.

Chapter 2 introduces the main concepts of object-oriented programming, and the layout of the basic language that the software engineer would share with the financial engineer. The latter would in turn look for analogies between this language and that of the replicating portfolio that is natural to him. Chapter 3 addresses the main concepts used by the financial engineer to analyse the joint distribution of the risk factors, namely volatility and correlation, both implied and historical.

Chapter 4 moves into the building of a structured financial product: here the software engineer would disclose the problems involved in the construction of a schedule of payments, and these arguments would be merged with the alternatives available to the structurer to design a stream of cash flows (a leg, to borrow the wording from swaps) to meet the need of a set of clients. Chapter 5 would address the use of derivative contracts to modify the repayment plan of the product: these are mainly convertible and reverse convertible bonds. Chapter 6 will investigate in detail the construction of coupon plans indexed to equity products, both univariate and multivariate. Chapter 7 will introduce credit-linked structured products, limiting the analysis to univariate risks. Multivariate credit-linked products, which represent the bulk of the structured finance market, will be addressed in Chapter 8.

Chapter 9 will finally address what is different about the structured finance business, as far as risk management is concerned. In particular, historical filtered simulation and scenario
analysis techniques will be addressed in detail, as well as counterparty risk in derivatives, which is one of the main reasons of concern in the finance world today.

REFERENCES AND FURTHER READING


