Performance Limits of Multiple-Input Multiple-Output Wireless Communication Systems

1.1 Introduction

Demands for capacity in wireless communications, driven by cellular mobile, Internet and multimedia services have been rapidly increasing worldwide. On the other hand, the available radio spectrum is limited and the communication capacity needs cannot be met without a significant increase in communication spectral efficiency. Advances in coding, such as turbo [5] and low density parity check codes [6][7] made it feasible to approach the Shannon capacity limit [4] in systems with a single antenna link. Significant further advances in spectral efficiency are available through increasing the number of antennas at both the transmitter and the receiver [1][2].

In this chapter we derive and discuss fundamental capacity limits for transmission over multiple-input multiple-output (MIMO) channels. They are mainly based on the theoretical work developed by Telatar [2] and Foschini [1]. These capacity limits highlight the potential spectral efficiency of MIMO channels, which grows approximately linearly with the number of antennas, assuming ideal propagation. The capacity is expressed by the maximum achievable data rate for an arbitrarily low probability of error, providing that the signal may be encoded by an arbitrarily long space-time code. In later chapters we consider some practical coding techniques which potentially approach the derived capacity limits. It has been demonstrated that the Bell Laboratories Layered Space-Time (BLAST) coding technique [3] can attain the spectral efficiencies up to 42 bits/sec/Hz. This represents a spectacular increase compared to currently achievable spectral efficiencies of 2-3 bits/sec/Hz, in cellular mobile and wireless LAN systems.

A MIMO channel can be realized with multielement array antennas. Of particular interest are propagation scenarios in which individual channels between given pairs of transmit and receive antennas are modelled by an independent flat Rayleigh fading process. In this chapter, we limit the analysis to the case of narrowband channels, so that they can be
described by frequency flat models. The results are generalised in Chapter 8 to wide-band channels, simply by considering a wide-band channel as a set of orthogonal narrow-band channels. Rayleigh models are realistic for environments with a large number of scatterers. In channels with independent Rayleigh fading, a signal transmitted from every individual transmit antenna appears uncorrelated at each of the receive antennas. As a result, the signal corresponding to every transmit antenna has a distinct spatial signature at a receive antenna.

The independent Rayleigh fading model can be approximated in MIMO channels where antenna element spacing is considerably larger than the carrier wavelength or the incoming wave incidence angle spread is relatively large (larger than 30°). An example of such a channel is the down link in cellular radio. In base stations placed high above the ground, the antenna signals get correlated due to a small angular spread of incoming waves and much higher antenna separations are needed in order to obtain independent signals between adjacent antenna elements than if the incoming wave incidence angle spread is large.

There have been many measurements and experiment results indicating that if two receive antennas are used to provide diversity at the base station receiver, they must be on the order of ten wavelengths apart to provide sufficient decorrelation. Similarly, measurements show that to get the same diversity improvements at remote handsets, it is sufficient to separate the antennas by about three wavelengths.

### 1.2 MIMO System Model

Let us consider a single point-to-point MIMO system with arrays of $n_T$ transmit and $n_R$ receive antennas. We focus on a complex baseband linear system model described in discrete time. The system block diagram is shown in Fig. 1.1. The transmitted signals in each symbol period are represented by an $n_T \times 1$ column matrix $x$, where the $i$th component $x_i$, refers to the transmitted signal from antenna $i$. We consider a Gaussian channel, for which, according to information theory [4], the optimum distribution of transmitted signals is also Gaussian. Thus, the elements of $x$ are considered to be zero mean independent identically distributed (i.i.d.) Gaussian variables. The covariance matrix of the transmitted signal is given by

$$R_{xx} = E\{xx^H\}$$

(1.1)

where $E\{\cdot\}$ denotes the expectation and the operator $A^H$ denotes the Hermitian of matrix $A$, which means the transpose and component-wise complex conjugate of $A$. The total

![Figure 1.1 Block diagram of a MIMO system](image-url)
transmitted power is constrained to $P$, regardless of the number of transmit antennas $n_T$. It can be represented as

$$P = \text{tr}(R_{xx}) \quad (1.2)$$

where $\text{tr}(A)$ denotes the trace of matrix $A$, obtained as the sum of the diagonal elements of $A$. If the channel is unknown at the transmitter, we assume that the signals transmitted from individual antenna elements have equal powers of $P/n_T$. The covariance matrix of the transmitted signal is given by

$$R_{xx} = \frac{P}{n_T} I_{n_T} \quad (1.3)$$

where $I_{n_T}$ is the $n_T \times n_T$ identity matrix. The transmitted signal bandwidth is narrow enough, so its frequency response can be considered as flat. In other words, we assume that the channel is memoryless.

The channel is described by an $n_R \times n_T$ complex matrix, denoted by $H$. The $ij$-th component of the matrix $H$, denoted by $h_{ij}$, represents the channel fading coefficient from the $j$th transmit to the $i$th receive antenna. For normalization purposes we assume that the received power for each of $n_R$ receive branches is equal to the total transmitted power. Physically, it means that we ignore signal attenuations and amplifications in the propagation process, including shadowing, antenna gains etc. Thus we obtain the normalization constraint for the elements of $H$, on a channel with fixed coefficients, as

$$\sum_{j=1}^{n_T} |h_{ij}|^2 = n_T, \quad i = 1, 2, \ldots, n_R \quad (1.4)$$

When the channel matrix elements are random variables, the normalization will apply to the expected value of the above expression.

We assume that the channel matrix is known to the receiver, but not always at the transmitter. The channel matrix can be estimated at the receiver by transmitting a training sequence. The estimated channel state information (CSI) can be communicated to the transmitter via a reliable feedback channel.

The elements of the channel matrix $H$ can be either deterministic or random. We will focus on examples relevant to wireless communications, which involve the Rayleigh and Rician distributions of the channel matrix elements. In most situations we consider the Rayleigh distribution, as it is most representative for non-line-of-sight (NLOS) radio propagation.

The noise at the receiver is described by an $n_R \times 1$ column matrix, denoted by $n$. Its components are statistically independent complex zero-mean Gaussian variables, with independent and equal variance real and imaginary parts. The covariance matrix of the receiver noise is given by

$$R_{nn} = E\{nn^H\} \quad (1.5)$$

If there is no correlation between components of $n$, the covariance matrix is obtained as

$$R_{nn} = \sigma^2 I_{n_R} \quad (1.6)$$

Each of $n_R$ receive branches has identical noise power of $\sigma^2$.

The receiver is based on a maximum likelihood principle operating jointly over $n_R$ receive antennas. The received signals are represented by an $n_R \times 1$ column matrix, denoted by $r$, where each complex component refers to a receive antenna. We denote the average power
at the output of each receive antenna by $P_r$. The average signal-to-noise ratio (SNR) at each receive antenna is defined as

$$\gamma = \frac{P_r}{\sigma^2}$$  \hspace{1cm} (1.7)

As we assumed that the total received power per antenna is equal to the total transmitted power, the SNR is equal to the ratio of the total transmitted power and the noise power per receive antenna and it is independent of $n_T$. Thus it can be written as

$$\gamma = \frac{P}{\sigma^2}$$  \hspace{1cm} (1.8)

By using the linear model the received vector can be represented as

$$\mathbf{r} = \mathbf{Hx} + \mathbf{n}$$  \hspace{1cm} (1.9)

The received signal covariance matrix, defined as $E\{\mathbf{rr}^H\}$, by using (1.9), is given by

$$\mathbf{R}_{rr} = \mathbf{HR}_{xx}\mathbf{H}^H,$$  \hspace{1cm} (1.10)

while the total received signal power can be expressed as $\text{tr}(\mathbf{R}_{rr})$.

### 1.3 MIMO System Capacity Derivation

The system capacity is defined as the maximum possible transmission rate such that the probability of error is arbitrarily small.

Initially, we assume that the channel matrix is not known at the transmitter, while it is perfectly known at the receiver.

By the singular value decomposition (SVD) theorem [11] any $n_R \times n_T$ matrix $\mathbf{H}$ can be written as

$$\mathbf{H} = \mathbf{UDV}^H$$  \hspace{1cm} (1.11)

where $\mathbf{D}$ is an $n_R \times n_T$ non-negative and diagonal matrix, $\mathbf{U}$ and $\mathbf{V}$ are $n_R \times n_R$ and $n_T \times n_T$ unitary matrices, respectively. That is, $\mathbf{UU}^H = \mathbf{I}_{n_R}$ and $\mathbf{VV}^H = \mathbf{I}_{n_T}$, where $\mathbf{I}_{n_R}$ and $\mathbf{I}_{n_T}$ are $n_R \times n_R$ and $n_T \times n_T$ identity matrices, respectively. The diagonal entries of $\mathbf{D}$ are the non-negative square roots of the eigenvalues of matrix $\mathbf{HH}^H$. The eigenvalues of $\mathbf{HH}^H$, denoted by $\lambda$, are defined as

$$\mathbf{HH}^H \mathbf{y} = \lambda \mathbf{y}, \quad \mathbf{y} \neq 0$$  \hspace{1cm} (1.12)

where $\mathbf{y}$ is an $n_R \times 1$ vector associated with $\lambda$, called an eigenvector.

The non-negative square roots of the eigenvalues are also referred to as the singular values of $\mathbf{H}$. Furthermore, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{HH}^H$ and the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{H}^H\mathbf{H}$. By substituting (1.11) into (1.9) we can write for the received vector $\mathbf{r}$

$$\mathbf{r} = \mathbf{UDV}^H\mathbf{x} + \mathbf{n}$$  \hspace{1cm} (1.13)
Let us introduce the following transformations

\[ r' = U^H r \]
\[ x' = V^H x \]
\[ n' = U^H n \]  \hspace{1cm} (1.14)

as \( U \) and \( V \) are invertible. Clearly, multiplication of vectors \( r, x \) and \( n \) by the corresponding matrices as defined in (1.14) has only a scaling effect. Vector \( n' \) is a zero mean Gaussian random variable with i.i.d real and imaginary parts. Thus, the original channel is equivalent to the channel represented as

\[ r' = Dx' + n' \]  \hspace{1cm} (1.15)

The number of nonzero eigenvalues of matrix \( HH^H \) is equal to the rank of matrix \( H \), denoted by \( r \). For the \( n_R \times n_T \) matrix \( H \), the rank is at most \( m = \min(n_R, n_T) \), which means that at most \( m \) of its singular values are nonzero. Let us denote the singular values of \( H \) by \( \sqrt{\lambda_i}, i = 1, 2, \ldots, r \). By substituting the entries \( \sqrt{\lambda_i} \) in (1.15), we get for the received signal components

\[ r_i' = \sqrt{\lambda_i} x_i' + n_i', \quad i = 1, 2, \ldots, r \]
\[ r_i' = n_i', \quad i = r + 1, r + 2, \ldots, n_R \]  \hspace{1cm} (1.16)

As (1.16) indicates, received components, \( r_i', i = r + 1, r + 2, \ldots, n_R \), do not depend on the transmitted signal, i.e. the channel gain is zero. On the other hand, received components \( r_i' \), for \( i = 1, 2, \ldots, r \) depend only on the transmitted component \( x_i' \). Thus the equivalent MIMO channel from (1.15) can be considered as consisting of \( r \) uncoupled parallel sub-channels. Each sub-channel is assigned to a singular value of matrix \( H \), which corresponds to the amplitude channel gain. The channel power gain is thus equal to the eigenvalue of matrix \( HH^H \). For example, if \( n_T > n_R \), as the rank of \( H \) cannot be higher than \( n_R \), Eq. (1.16) shows that there will be at most \( n_R \) nonzero gain sub-channels in the equivalent MIMO channel, as shown in Fig. 1.2.

On the other hand if \( n_R > n_T \), there will be at most \( n_T \) nonzero gain sub-channels in the equivalent MIMO channel, as shown in Fig. 1.3. The eigenvalue spectrum is a MIMO channel representation, which is suitable for evaluation of the best transmission paths.

The covariance matrices and their traces for signals \( r', x' \) and \( n' \) can be derived from (1.14) as

\[ R_{r'r'} = U^H R_{rr} U \]
\[ R_{x'x'} = V^H R_{xx} V \]
\[ R_{n'n'} = U^H R_{nn} U \]
\[ \text{tr}(R_{r'r'}) = \text{tr}(R_{rr}) \]
\[ \text{tr}(R_{x'x'}) = \text{tr}(R_{xx}) \]
\[ \text{tr}(R_{n'n'}) = \text{tr}(R_{nn}) \]  \hspace{1cm} (1.17)
Figure 1.2  Block diagram of an equivalent MIMO channel if $n_T > n_R$

Figure 1.3  Block diagram of an equivalent MIMO channel if $n_R > n_T$
The above relationships show that the covariance matrices of $r'$, $x'$ and $n'$, have the same sum of the diagonal elements, and thus the same powers, as for the original signals, $r$, $x$ and $n$, respectively.

Note that in the equivalent MIMO channel model described by (1.16), the sub-channels are uncoupled and thus their capacities add up. Assuming that the transmit power from each antenna in the equivalent MIMO channel model is $P/n_T$, we can estimate the overall channel capacity, denoted by $C$, by using the Shannon capacity formula

$$C = W \sum_{i=1}^{r} \log_2 \left( 1 + \frac{P_{ri}}{\sigma^2} \right) \quad (1.19)$$

where $W$ is the bandwidth of each sub-channel and $P_{ri}$ is the received signal power in the $i$th sub-channel. It is given by

$$P_{ri} = \frac{\lambda_i P}{n_T} \quad (1.20)$$

where $\sqrt{\lambda_i}$ is the singular value of channel matrix $H$. Thus the channel capacity can be written as

$$C = W \sum_{i=1}^{r} \log_2 \left( 1 + \frac{\lambda_i P}{n_T \sigma^2} \right)$$

$$= W \log_2 \prod_{i=1}^{r} \left( 1 + \frac{\lambda_i P}{n_T \sigma^2} \right) \quad (1.21)$$

Now we will show how the channel capacity is related to the channel matrix $H$. Assuming that $m = \min(n_R, n_T)$, Eq. (1.12), defining the eigenvalue-eigenvector relationship, can be rewritten as

$$(\lambda I_m - Q)y = 0, \quad y \neq 0 \quad (1.22)$$

where $Q$ is the Wishart matrix defined as

$$Q = \begin{cases} HH^H, & n_R < n_T \\ H^H H, & n_R \geq n_T \end{cases} \quad (1.23)$$

That is, $\lambda$ is an eigenvalue of $Q$, if and only if $\lambda I_m - Q$ is a singular matrix. Thus the determinant of $\lambda I_m - Q$ must be zero

$$\det(\lambda I_m - Q) = 0 \quad (1.24)$$

The singular values $\lambda$ of the channel matrix can be calculated by finding the roots of Eq. (1.24).

We consider the characteristic polynomial $p(\lambda)$ from the left-hand side in Eq. (1.24)

$$p(\lambda) = \det(\lambda I_m - Q) \quad (1.25)$$

It has degree equal to $m$, as each row of $\lambda I_m - Q$ contributes one and only one power of $\lambda$ in the Laplace expansion of $\det(\lambda I_m - Q)$ by minors. As a polynomial of degree $m$
with complex coefficients has exactly \( m \) zeros, counting multiplicities, we can write for the characteristic polynomial

\[
p(\lambda) = \Pi_{i=1}^{m}(\lambda - \lambda_i)
\]

where \( \lambda_i \) are the roots of the characteristic polynomial \( p(\lambda) \), equal to the channel matrix singular values. We can now write Eq. (1.24) as

\[
\Pi_{i=1}^{m}(\lambda - \lambda_i) = 0
\]

Further we can equate the left-hand sides of (1.24) and (1.27)

\[
\Pi_{i=1}^{m}(\lambda - \lambda_i) = \det(\lambda I_m - Q)
\]

Substituting \( -\frac{nP_i}{\sigma^2} \) for \( \lambda \) in (1.28) we get

\[
\Pi_{i=1}^{m} \left( 1 + \frac{\lambda_i P}{nT\sigma^2} \right) = \det \left( I_m + \frac{P}{nT\sigma^2}Q \right)
\]

Now the capacity formula from (1.21) can be written as

\[
C = W \log_2 \det \left( I_m + \frac{P}{nT\sigma^2}Q \right)
\]

As the nonzero eigenvalues of \( HH^H \) and \( H^H H \) are the same, the capacities of the channels with matrices \( H \) and \( H^H \) are the same. Note that if the channel coefficients are random variables, formulas (1.21) and (1.30), represent instantaneous capacities or mutual information. The mean channel capacity can be obtained by averaging over all realizations of the channel coefficients.

### 1.4 MIMO Channel Capacity Derivation for Adaptive Transmit Power Allocation

\(^1\)When the channel parameters are known at the transmitter, the capacity given by (1.30) can be increased by assigning the transmitted power to various antennas according to the “water-filling” rule [2]. It allocates more power when the channel is in good condition and less when the channel state gets worse. The power allocated to channel \( i \) is given by (Appendix 1.1)

\[
P_i = \left( \mu - \frac{\sigma^2}{\lambda_i} \right)^+, \quad i = 1, 2, \ldots, r
\]

where \( a^+ \) denotes \( \max(a, 0) \) and \( \mu \) is determined so that

\[
\sum_{i=1}^{r} P_i = P
\]

\(^1\)In practice, transmit power is constrained by regulations and hardware costs.
We consider the singular value decomposition of channel matrix $H$, as in (1.11). Then, the received power at sub-channel $i$ in the equivalent MIMO channel model is given by

$$P_{ri} = (\lambda_i \mu - \sigma^2)^+$$  \hspace{1cm} (1.33)

The MIMO channel capacity is then

$$C = W \sum_{i=1}^{r} \log_2 \left( 1 + \frac{P_{ri}}{\sigma^2} \right)$$  \hspace{1cm} (1.34)

Substituting the received signal power from (1.33) into (1.34) we get

$$C = W \sum_{i=1}^{r} \log_2 \left[ 1 + \frac{1}{\sigma^2} (\lambda_i \mu - \sigma^2)^+ \right]$$  \hspace{1cm} (1.35)

The covariance matrix of the transmitted signal is given by

$$R_{xx} = V \text{diag}(P_1, P_2, \ldots, P_n) V^H$$  \hspace{1cm} (1.36)

### 1.5 MIMO Capacity Examples for Channels with Fixed Coefficients

In this section we examine the maximum possible transmission rates in a number of various channel settings. First we focus on examples of channels with constant matrix elements. In most examples the channel is known only at the receiver, but not at the transmitter. All other system and channel assumptions are as specified in Section 1.2.

**Example 1.1: Single Antenna Channel**

Let us consider a channel with $n_T = n_R = 1$ and $H = h = 1$. The Shannon formula gives the capacity of this channel

$$C = W \log_2 \left( 1 + \frac{P}{\sigma^2} \right)$$  \hspace{1cm} (1.37)

The same expression can be obtained by applying formula (1.30). Note that for high SNRs, the capacity grows logarithmically with the SNR. Also in this region, a 3 dB increase in SNR gives a normalized capacity $C/W$ increase of 1 bit/sec/Hz. Assuming that the channel coefficient is normalized so that $|h|^2 = 1$, and for the SNR ($P/\sigma^2$) of 20 dB, the capacity of a single antenna link is 6.658 bits/s/Hz.

**Example 1.2: A MIMO Channel with Unity Channel Matrix Entries**

For this channel the matrix elements $h_{ij}$ are

$$h_{ij} = 1, \quad i = 1, 2, \ldots, n_R, \quad j = 1, 2, \ldots, n_T$$  \hspace{1cm} (1.38)
Coherent Combining

In this channel, with the channel matrix given by (1.38), the same signal is transmitted simultaneously from $n_T$ antennas. The received signal at antenna $i$ is given by

$$r_i = n_T x$$

and the received signal power at antenna $i$ is given by

$$P_{ri} = n_T^2 \frac{P}{n_T} = n_T P$$

where $P/n_T$ is the power transmitted from one antenna. Note that though the power per transmit antenna is $P/n_T$, the total received power per receive antenna is $n_T P$. The power gain of $n_T$ in the total received power comes due to coherent combining of the transmitted signals.

The rank of channel matrix $H$ is 1, so there is only one received signal in the equivalent channel model with the power

$$P_r = n_R n_T P$$

Thus applying formula (1.19) we get for the channel capacity

$$C = W \log_2 \left( 1 + n_R n_T \frac{P}{\sigma^2} \right)$$

In this example, the multiple antenna system reduces to a single effective channel that only benefits from higher power achieved by transmit and receive diversity. This system achieves a diversity gain of $n_R n_T$ relative to a single antenna link. The cost of this gain is the system complexity required to implement coordinated transmissions and coherent maximum ratio combining. However, the capacity grows logarithmically with the total number of antennas $n_T n_R$. For example, if $n_T = n_R = 8$ and $10 \log_{10} \frac{P}{\sigma^2} = 20$ dB, the normalized capacity $C/W$ is 12.65 bits/sec/Hz.

Noncoherent Combining

If the signals transmitted from various antennas are different and all channel entries are equal to 1, there is only one received signal in the equivalent channel model with the power of $n_R P$. Thus the capacity is given by

$$C = W \log_2 \left( 1 + n_R \frac{P}{\sigma^2} \right)$$

For an SNR of 20 dB and $n_R = n_T = 8$, the capacity is 9.646 bits/sec/Hz.

Example 1.3: A MIMO Channel with Orthogonal Transmissions

In this example we consider a channel with the same number of transmit and receive antennas, $n_T = n_R = n$, and that they are connected by orthogonal parallel sub-channels, so there is no interference between individual sub-channels. This could be achieved for example,
by linking each transmitter with the corresponding receiver by a separate waveguide, or by spreading transmitted signals from various antennas by orthogonal spreading sequences. The channel matrix is given by

$$H = \sqrt{n} I_n$$

The scaling by $\sqrt{n}$ is introduced to satisfy the power constraint in (1.4).

Since

$$HH^H = nI_n$$

by applying formula (1.30) we get for the channel capacity

$$C = W \log_2 \left( \det \left( I_n + \frac{nP}{n\sigma^2} I_n \right) \right)$$

$$= W \log_2 \left[ \det \left( 1 + \frac{P}{\sigma^2} \right) \right]$$

$$= W \log_2 \left( 1 + \frac{P}{\sigma^2} \right)^n$$

$$= nW \log_2 \left( 1 + \frac{P}{\sigma^2} \right)$$

For the same numerical values $n_T = n_R = n = 8$ and SNR of 20 dB, as in Example 1.2, the normalized capacity $C/W$ is 53.264 bits/sec/Hz. Clearly, the capacity is much higher than in Example 1.2, as the sub-channels are uncoupled giving a multiplexing gain of $n$.

**Example 1.4: Receive Diversity**

Let us assume that there is only one transmit and $n_R$ receive antennas. The channel matrix can be represented by the vector

$$H = (h_1, h_2, \ldots, h_{n_R})^T$$

where the operator $(\cdot)^T$ denotes the matrix transpose. As $n_R > n_T$, formula (1.30) should be written as

$$C = W \log_2 \left[ \det \left( I_{n_T} + \frac{P}{n_T\sigma^2} H^HH \right) \right]$$

(1.44)

As $H^HH = \sum_{i=1}^{n_R} |h_i|^2$, by applying formula (1.30) we get for the capacity

$$C = W \log_2 \left( 1 + \sum_{i=1}^{n_R} |h_i|^2 \frac{P}{\sigma^2} \right)$$

(1.45)

This capacity corresponds to linear maximum combining at the receiver. In the case when the channel matrix elements are equal and normalized as follows

$$|h_1|^2 = |h_2|^2 = \cdots |h_{n_R}|^2 = 1$$
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the capacity in (1.45) becomes

$$C = W \log_2 \left( 1 + n_R \frac{P}{\sigma^2} \right)$$  \hspace{1cm} (1.46)

This system achieves the diversity gain of $n_R$ relative to a single antenna channel. For $n_R = 8$ and SNR of 20 dB, the receive diversity capacity is 9.646 bits/s/Hz.

Selection diversity is obtained if the best of the $n_R$ channels is chosen. The capacity of this system is given by

$$C = \max_i \left\{ W \log_2 \left( 1 + \frac{P}{\sigma^2} |h_i|^2 \right) \right\}$$

$$= W \log_2 \left( 1 + \frac{P}{\sigma^2} \max_i |h_i|^2 \right)$$  \hspace{1cm} (1.47)

where the maximization is performed over $i, i = 1, 2, \ldots, n_R$.

Example 1.5: Transmit Diversity

In this system there are $n_T$ transmit and only one receive antenna. The channel is represented by the vector

$$\mathbf{H} = (h_1, h_2, \ldots, h_{n_T})$$

As $\mathbf{H}^H \mathbf{H} = \sum_{j=1}^{n_T} |h_j|^2$, by applying formula (1.30) we get for the capacity

$$C = W \log_2 \left( 1 + \sum_{j=1}^{n_T} |h_j|^2 \frac{P}{n_T \sigma^2} \right)$$  \hspace{1cm} (1.48)

If the channel coefficients are equal and normalized as in (1.4), the transmit diversity capacity becomes

$$C = W \log_2 \left( 1 + \frac{P}{\sigma^2} \right)$$  \hspace{1cm} (1.49)

The capacity does not increase with the number of transmit antennas. This expression applies to the case when the transmitter does not know the channel. For coordinated transmissions, when the transmitter knows the channel, we can apply the capacity formula from (1.35). As the rank of the channel matrix is one, there is only one term in the sum in (1.35) and only one nonzero eigenvalue given by

$$\lambda = \sum_{j=1}^{n_T} |h_j|^2$$

The value for $\mu$ from the normalization condition is given by

$$\mu = P + \frac{\sigma^2}{\lambda}$$
So we get for the capacity

$$C = W \log_2 \left( 1 + \sum_{j=1}^{n_T} |h_j|^2 \frac{P}{\sigma^2} \right) \quad (1.50)$$

If the channel coefficients are equal and normalized as in (1.4), the capacity becomes

$$C = W \log_2 \left( 1 + n_T \frac{P}{\sigma^2} \right) \quad (1.51)$$

For $n_T = 8$ and SNR of 20 dB, the transmit diversity with the channel knowledge at the transmitter is 9.646 bits/s/Hz.

1.6 Capacity of MIMO Systems with Random Channel Coefficients

Now we turn to a more realistic case when the channel matrix entries are random variables. Initially, we assume that the channel coefficients are perfectly estimated at the receiver but unknown at the transmitter. Furthermore, we assume that the entries of the channel matrix are zero mean Gaussian complex random variables. Its real and imaginary parts are independent zero mean Gaussian i.i.d. random variables, each with variance of $1/2$. Each entry of the channel matrix thus has a Rayleigh distributed magnitude, uniform phase and expected magnitude square equal to unity, $E[|h_{ij}|^2] = 1$.

The probability density function (pdf) for a Rayleigh distributed random variable $z = \sqrt{z_1^2 + z_2^2}$, where $z_1$ and $z_2$ are zero mean statistically independent orthogonal Gaussian random variables each having a variance $\sigma_r^2$, is given by

$$p(z) = \frac{z}{\sigma_r^2} e^{-\frac{z^2}{2\sigma_r^2}} \quad z \geq 0 \quad (1.52)$$

In this analysis $\sigma_r^2$ is normalized to $1/2$. The antenna spacing is large enough to ensure uncorrelated channel matrix entries. According to frequency of channel coefficient changes, we will distinguish three scenarios.

1. Matrix $H$ is random. Its entries change randomly at the beginning of each symbol interval $T$ and are constant during one symbol interval. This channel model is referred to as *fast fading* channel.
2. Matrix $H$ is random. Its entries are random and are constant during a fixed number of symbol intervals, which is much shorter than the total transmission duration. We refer to this channel model as *block fading*.
3. Matrix $H$ is random but is selected at the start of transmission and kept constant all the time. This channel model is referred to as *slow* or *quasi-static fading* model.

In this section we will estimate the maximum transmission rate in various propagation scenarios and give relevant examples.
1.6.1 Capacity of MIMO Fast and Block Rayleigh Fading Channels

In the derivation of the expression for the MIMO channel capacity on fast Rayleigh fading channels, we will start from the simple single antenna link. The coefficient \(|h|^2\) in the capacity expression for a single antenna link (1.37), is a chi-squared distributed random variable, with two degrees of freedom, denoted by \(\chi^2\). This random variable can be expressed as

\[ y = \chi^2 = z_1^2 + z_2^2, \]

where \(z_1\) and \(z_2\) are zero mean statistically independent orthogonal Gaussian variables, each having a variance \(\sigma^2\), which is in this analysis normalized to 1/2. Its pdf is given by

\[ p(y) = \frac{1}{2\sigma_r^2} e^{-y/2\sigma_r^2}, \quad y \geq 0 \]

The capacity for a fast fading channel can then be obtained by estimating the mean value of the capacity given by formula (1.37)

\[ C = \mathbb{E} \left\{ W \log_2 \left( 1 + \chi^2 \frac{P}{\sigma^2} \right) \right\} \]

where \(\mathbb{E}[\cdot]\) denotes the expectation with respect to the random variable \(\chi^2\).

By using the singular value decomposition approach, the MIMO fast fading channel, with the channel matrix \(H\), can be represented by an equivalent channel consisting of \(r \leq \min(n_T, n_R)\) decoupled parallel sub-channels, where \(r\) is the rank of \(H\). Thus the capacities of these sub-channels add up, giving for the overall capacity

\[ C = \mathbb{E} \left\{ W \sum_{i=1}^{r} \log_2 \left( 1 + \lambda_i \frac{P}{n_T \sigma^2} \right) \right\} \]

where \(\sqrt{\lambda_i}\) are the singular values of the channel matrix. Alternatively, by using the same approach as in the capacity derivation in Section 1.3, we can write for the mean MIMO capacity on fast fading channels

\[ C = \mathbb{E} \left\{ W \log_2 \det \left( \mathbb{I}_r + \frac{P}{\sigma^2 n_T} Q \right) \right\} \]

where \(Q\) is defined as

\[ Q = \begin{cases} \mathbb{H} \mathbb{H}^H, & n_R < n_T \\ \mathbb{H}^H \mathbb{H}, & n_R \geq n_T \end{cases} \]

For block fading channels, as long as the expected value with respect to the channel matrix in formulas (1.55) and (1.56) can be observed, i.e. the channel is ergodic, we can calculate the channel capacity by using the same expressions as in (1.55) and (1.56).

While the capacity can be easily evaluated for \(n_T = n_R = 1\), the expectation in formulas (1.55) or (1.56) gets quite complex for larger values of \(n_T\) and \(n_R\). They can be evaluated with the aid of Laguerre polynomials [2][13] as follows

\[ C = W \int_0^\infty \log_2 \left( 1 + \frac{P}{n_T \sigma^2 \lambda} \right) \sum_{k=0}^{m-1} \frac{k!}{(k+n+m)!} [L^m_k(n^m)(\lambda)]^2 \lambda^{n-m} e^{-\lambda} d\lambda \]
where
\[ m = \min(n_T, n_R) \] (1.58)
\[ n = \max(n_T, n_R) \] (1.59)
and \( L_k^{n-m}(x) \) is the associate Laguerre polynomial of order \( k \), defined as [13]
\[
L_k^{n-m}(x) = \frac{1}{k!} e^x x^{m-n} \frac{d^k}{dx^k} (e^{-x} x^{n-k})
\] (1.60)

Let us define
\[ \tau = \frac{n}{m} \]

By increasing \( m \) and \( n \) and keeping their ratio \( \tau \) constant, the capacity, normalized by \( m \), approaches
\[
\lim_{n \to \infty} \frac{C}{m} = \frac{W}{2\pi} \int_{\nu_1}^{\nu_2} \log_2 \left( 1 + \frac{P n m}{\nu_1 \sigma^2} \right) \sqrt{\left( \frac{\nu_2}{\nu} - 1 \right) \left( 1 - \frac{\nu_1}{\nu} \right)} d\nu
\] (1.61)

where
\[ \nu_2 = (\sqrt{\tau} + 1)^2 \]
and
\[ \nu_1 = (\sqrt{\tau} - 1)^2 \]

**Example 1.6: A Fast and Block Fading Channel with Receive Diversity**

For a receive diversity system with one transmit and \( n_R \) receive antennas on a fast Rayleigh fading channel, specified by the channel matrix
\[ H = (h_1, h_2, \ldots, h_{n_R})^T \]

Formula (1.56) gives the capacity expression for maximum ratio combining at the receiver
\[
C = E \left[ W \log_2 \left( 1 + \frac{P}{\sigma^2} \chi_{2n_R}^2 \right) \right]
\] (1.62)

where
\[ \chi_{2n_R}^2 = \sum_{i=1}^{n_R} |h_i|^2 \]
is a chi-squared random variable with \( 2n_R \) degrees of freedom. It can be represented as
\[
y = \chi_{2n_R}^2 = \sum_{i=1}^{2n_R} z_i^2
\] (1.63)
where $z_i, i = 1, 2, \ldots, 2n_R$, are statistically independent, identically distributed zero mean Gaussian random variables, each having a variance $\sigma_r^2$, which is in this analysis normalized to $1/2$. Its pdf is given by

$$p(y) = \frac{1}{\sigma_r^{2n_R} 2^{n_R} \Gamma(n_R)} y^{n_R-1} e^{-\frac{y}{2\sigma_r^2}}, \quad y \geq 0 \quad (1.64)$$

where $\Gamma(p)$ is the gamma function, defined as

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt, \quad p > 0 \quad (1.65)$$

$$\Gamma(p) = (p-1)!, \quad p \text{ is an integer, } p > 0 \quad (1.66)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (1.67)$$

$$\Gamma\left(\frac{1}{3}\right) = \frac{\sqrt{\pi}}{2} \quad (1.68)$$

If a selection diversity receiver is used, the capacity is given by

$$C = E \left\{ W \log_2 \left[ 1 + \frac{P}{\sigma_2^2} \max_i (|h_i|^2) \right] \right\} \quad (1.69)$$

The channel capacity curves for receive diversity with maximum ratio combining are shown in Fig. 1.4 and with selection combining in Fig. 1.5.

**Example 1.7: A Fast and Block Fading Channel with Transmit Diversity**

For a transmit diversity system with $n_T$ transmit and one receive antenna on a fast Rayleigh fading channel, specified by the channel matrix

$$H = (h_1, h_2, \ldots, h_{n_T}),$$

formula (1.56) gives the capacity expression for uncoordinated transmission

$$C = E \left[ W \log_2 \left( 1 + \frac{P}{n_T \sigma_2^2} \chi_{2n_T}^2 \right) \right] \quad (1.70)$$

where

$$\chi_{2n_T}^2 = \sum_{j=1}^{n_T} |h_j|^2$$
is a chi-squared random variable with $2n_T$ degrees of freedom. As the number of transmit antennas increases, the capacity approaches the asymptotic value

$$\lim_{n_T \to \infty} C = W \log_2 \left( 1 + \frac{P}{\sigma^2} \right)$$  \hspace{1cm} (1.71)
That is, the system behaves as if the total power is transmitted over a single unfaded channel. In other words, the transmit diversity is able to remove the effect of fading for a large number of antennas.

The channel capacity curves for transmit diversity with uncoordinated transmissions are shown in Fig. 1.6. The capacity is plotted against the number of transmit antennas \( n_T \). The curves are shown for various values of the signal-to-noise ratio, in the range of 0 to 30 dB. The capacity of transmit diversity saturates for \( n_T \geq 2 \). That is, the capacity asymptotic value from (1.71) is achieved for the number of transmit antennas of 2 and there is no point in increasing it further.

In coordinated transmissions, when all transmitted signals are the same and synchronous, the capacity is given by

\[
C = E \left[ W \log_2 \left( 1 + \frac{P}{\sigma^2} \frac{\chi^2}{n_T} \right) \right]
\]  

Example 1.8: A MIMO Fast and Block Fading Channel with Transmit-Receive Diversity

We consider a MIMO system with \( n \) transmit and \( n \) receive antennas, over a fast Rayleigh fading channel, assuming that the channel parameters are known at the receiver but not at the transmitter. In this case

\[
m = n = n_R = n_T
\]

so that the asymptotic capacity, from (1.61), is given by

\[
\lim_{n \to \infty} \frac{C}{Wn} = \frac{1}{\pi} \int_0^4 \log_2 \left( 1 + \frac{P}{\sigma^2} \nu \right) \sqrt{\frac{1}{\nu} - \frac{1}{4}} \, d\nu
\]  

(1.73)
or in a closed form [2]

\[
\lim_{n \to \infty} \frac{C}{W_n} = \log_2 \frac{P}{\sigma^2} - 1 + \frac{1}{\sqrt{1 + \frac{4P}{\sigma^2}}} - 1 + 2 \tanh^{-1} \frac{1}{\sqrt{1 + \frac{4P}{\sigma^2}}}
\]  

(1.74)

Expression (1.73) can be bounded by observing that \(\log(1 + x) \geq \log x\), as

\[
\lim_{n \to \infty} \frac{C}{W_n} \geq \frac{1}{\pi} \int_0^4 \log_2 \left( \frac{P}{\sigma^2} \nu \right) \sqrt{1 - \frac{1}{4\nu}} d\nu
\]  

(1.75)

This bound can be expressed in a closed form as

\[
\lim_{n \to \infty} \frac{C}{W_n} \geq \log_2 \frac{P}{\sigma^2} - 1
\]  

(1.76)

The bound in (1.76) shows that the capacity increases linearly with the number of antennas and logarithmically with the SNR. In this example there is a multiplexing gain of \(n\), as there are \(n\) independent sub-channels which can be identified by their coefficients, perfectly estimated at the receiver.

The capacity curves obtained by using the bound in (1.76), are shown in Fig. 1.7, for the signal-to-noise ratio as a parameter, varying between 0 and 30 dB.

![Figure 1.7](image)

**Figure 1.7** Channel capacity curves obtained by using the bound in (1.76), for a MIMO system with transmit/receive diversity on a fast and block Rayleigh fading channel
The normalized capacity bound $C/n$ from (1.76), the asymptotic capacity from (1.74) and the simulated average capacity by using (1.56), versus the SNR and with the number of antennas as a parameter, are shown in Fig. 1.8. Note that in the figure the curves for $n = 2$, 8, and 16 antennas coincide. As this figure indicates, the simulation curves are very close to the bound. This confirms that the bound in (1.76) is tight and can be used for channel capacity estimation on fast fading channels with a large $n$.

Example 1.9: A MIMO Fast and Block Fading Channel with Transmit-Receive Diversity and Adaptive Transmit Power Allocation

The instantaneous MIMO channel capacity for adaptive transmit power allocation is given by formula (1.35). The average capacity for an ergodic channel can be obtained by averaging over all realizations of the channel coefficients. Figs. 1.9 and 1.10 show the capacities estimated by simulation of an adaptive and a nonadaptive system, for a number of receive antennas as a parameter and a variable number of transmit antennas over a Rayleigh MIMO channel, at an SNR of 25 dB. In the adaptive system the transmit powers were allocated according to the water-filling principle and in the nonadaptive system the transmit powers from all antennas were the same. As the figures shows, when the number of the transmit antennas is the same or lower than the number of receive antennas, there is almost no gain in adaptive power allocation. However, when the numbers of transmit antennas is larger than the number of receive antennas, there is a significant potential gain to be achieved by water-filling power distribution. For four transmit and two receive antennas, the gain is about 2 bits/s/Hz and for fourteen transmit and two receive antennas it is about 5.6 bits/s/Hz. The benefit obtained by adaptive power distribution is higher for a lower SNR and diminishes at high SNRs, as demonstrated in Fig. 1.11.
Figure 1.9 Achievable capacities for adaptive and nonadaptive transmit power allocations over a fast MIMO Rayleigh channel, for SNR of 25 dB, the number of receive antennas $n_R = 1$ and $n_R = 2$ and a variable number of transmit antennas.

Figure 1.10 Achievable capacities for adaptive and nonadaptive transmit power allocations over a fast MIMO Rayleigh channel, for SNR of 25 dB, the number of receive antennas $n_R = 4$ and $n_R = 8$ and a variable number of transmit antennas.
1.6.2 Capacity of MIMO Slow Rayleigh Fading Channels

Now we consider a MIMO channel for which $H$ is chosen randomly, according to a Rayleigh distribution, at the beginning of transmission and held constant for a transmission block. An example of such a system is wireless LANs with high data rates and low fade rates, so that a fade might last over more than a million symbols. As before, we consider that the channel is perfectly estimated at the receiver and unknown at the transmitter.

In this system, the capacity, estimated by (1.30), is a random variable. It may even be zero, as there is a nonzero probability that a particular realization of $H$ is incapable of supporting arbitrarily low error rates, no matter what codes we choose. In this case we estimate the capacity complementary cumulative distribution function (ccdf). The ccdf defines the probability that a specified capacity level is provided. We denote it by $P_c$. The outage capacity probability, denoted by $P_{out}$, specifies the probability of not achieving a certain level of capacity. It is equal to the capacity cumulative distribution function (cdf) or $1 - P_c$.

1.6.3 Capacity Examples for MIMO Slow Rayleigh Fading Channels

Example 1.10: Single Antenna Link

In this system $n_T = n_R = 1$. The capacity is given by

$$C = W \log_2 \left( 1 + \frac{P}{\sigma^2 \chi^2_2} \right)$$  \hspace{1cm} (1.77)

where $\chi^2_2$ is a chi-squared random variable with two degrees of freedom.
Example 1.11: Receive Diversity

In this system there is one transmit and \( n_R \) receive antennas. The capacity for receivers with maximum ratio combining is given by

\[
C = W \log_2 \left( 1 + \frac{P}{\sigma^2} \chi^2_{2n_R} \right)
\]  

(1.78)

where \( \chi^2_{2n_R} \) is a chi-squared random variable with \( 2n_R \) degrees of freedom.

Example 1.12: Transmit Diversity

In this system there is \( n_T \) transmit and one receive antenna. The capacity for receivers with uncoordinated transmissions is given by

\[
C = W \log_2 \left( 1 + \frac{P}{n_T \sigma^2} \chi^2_{2n_T} \right)
\]  

(1.79)

where \( \chi^2_{2n_T} \) is a chi-squared random variable with \( 2n_T \) degrees of freedom.

Example 1.13: Combined Transmit-Receive Diversity

In this system there is \( n_T \) transmit and \( n_R \) receive antennas. Assuming that \( n_T \geq n_R \) we can write for the lower bound of the capacity as

\[
C > W \sum_{i=n_T-(n_R-1)}^{n_T} \log_2 \left( 1 + \frac{P}{n_T \sigma^2} (\chi^2_i) \right)
\]  

(1.80)

where \( (\chi^2_i) \) is a chi-squared random variable with 2 degrees of freedom. The upper bound is

\[
C < W \sum_{i=1}^{n_T} \log_2 \left( 1 + \frac{P}{n_T \sigma^2} (\chi^2_{2n_R})_i \right)
\]  

(1.81)

where \( (\chi^2_{2n_R})_i \) is a chi-squared random variable with \( 2n_R \) degrees of freedom. This case corresponds to a system of uncoupled parallel transmissions, where each of \( n_T \) transmit antennas is received by a separate set of \( n_R \) receive antennas, so that there is no interference.

For \( n = n_R = n_T \) and \( n \) very large, the capacity is lower bounded as [1]

\[
\frac{C}{Wn} > \left(1 + \frac{\sigma^2}{P}\right) \log_2 \left(1 + \frac{P}{\sigma^2}\right) - \log_2 e + \varepsilon_n
\]  

(1.82)

where the random variable \( \varepsilon_n \) has a Gaussian distribution with mean

\[
E{\varepsilon_n} = \frac{1}{n} \log_2 \left(1 + \frac{P}{\sigma^2}\right)^{-1/2}
\]  

(1.83)

and variance

\[
\text{Var}{\varepsilon_n} = \left(\frac{1}{n \ln 2}\right)^2 \left[ \ln \left(1 + \frac{P}{\sigma^2}\right) - \frac{P/\sigma^2}{1 + P/\sigma^2} \right]
\]  

(1.84)
Figure 1.12 Capacity per antenna ccdf curves for a MIMO slow Rayleigh fading channel with constant SNR of 15 dB and a variable number of antennas.

Figure 1.13 Capacity per antenna ccdf curves for a MIMO slow Rayleigh fading channel with a constant number of antennas $n_T = n_R = 8$ and a variable SNR.

Figs. 1.12 and 1.13 show the ccdf capacity per antenna curves on a slow Rayleigh fading channel obtained by simulation from (1.19). Figure 1.12 is plotted for various numbers of antennas and a constant SNR. It demonstrates that the probability that the capacity achieves a given level improves markedly when the number of antennas increases. Figure 1.13 shows the ccdf curves for a constant number of antennas and variable SNRs. For large values of antennas, the ccdf curves, shown in
Fig. 1.14 exhibit asymptotic behavior which is more pronounced for lower SNRs. That is, the capacities per antenna remain approximately constant for increasing numbers of antennas. In other words, in MIMO channels with a large number of antennas, the capacity grows linearly with the number of antennas. That agrees with the analytical bound in (1.82). The analytical bounds on $C/Wn$ in (1.82) are plotted in Figs. 1.15 and 1.16 for similar system configurations as in Figs. 1.12 and 1.13, respectively. Comparing the simulated and analytical ccdf curves it is clear that the simulated capacities are slightly higher for a given probability level. Therefore, the bound in (1.82) can be used for reasonably accurate capacity estimations. Figure 1.17 depicts the capacity that can be achieved on slow Rayleigh fading channels 99% of the time, i.e. 1% outage. It shows that even with a relatively small number of antennas, large capacities are available. For example, at an SNR of 20 dB and with 8 antennas at each site, about 37 bits/sec/Hz could be achieved. This compares to the achievable spectral efficiencies between 1-2 bits/sec/Hz in the second generation cellular mobile systems.

1.7 Effect of System Parameters and Antenna Correlation on the Capacity of MIMO Channels

The capacity gain of MIMO channels, derived under the idealistic assumption that the channel matrix entries are independent complex Gaussian variables, might be reduced on real channels. The high spectral efficiency predicted by (1.30) is diminished if the signals arriving at the receivers are correlated. The effect of spatial fading correlation on the MIMO
channel capacity has been addressed in [8][16]. Correlation between antenna elements can be reduced by separating antennas spatially [8][16][17]. However, low correlation between antenna elements does not guarantee high spectral efficiency. In real channels, degenerate propagation conditions, known as “keyholes”, have been observed, leading to a considerable decrease in MIMO channel capacity. In indoor environments such propagation effect can occur in long hallways. A similar condition exists in tunnels or in systems with large separations between the transmit and the receive antennas in outdoor environments [10][15][16].
The “keyholes” reduce the rank of the channel matrix and thus lower the capacity.

In this section, we first define the correlation coefficients and introduce the correlation matrix models in MIMO systems. Then we proceed with the estimation of the correlation coefficients and the effect of correlation between antennas on the capacity in the MIMO channels with a line of sight (LOS) path and in the absence of scattering. It is followed by a correlation model for a Rician fading channel and a channel with no LOS propagation. Subsequently, we will demonstrate the “keyhole” effect and its influence on the channel matrix and capacity. Then we will derive a channel model for an outdoor channel with scattering, described by the system parameters such as the angular spread, transmit and receive scattering radii and the distance between the transmitter and receiver. By using this channel model we will discuss under what conditions degenerate propagation occurs and its effects on channel capacity. The discussion is supported by capacity curves for various system parameters.

A MIMO channel with $n_T$ transmit and $n_R$ receive antennas can be described by an $n_R \times n_T$ channel matrix $\mathbf{H}$. It can be represented in this form

$$\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_i, \ldots, \mathbf{h}_{n_R}]^T$$

where $\mathbf{h}_i$, $i = 1, 2, \ldots, n_R$, is given by

$$\mathbf{h}_i = [h_{i,1}, h_{i,2}, \ldots, h_{i,n_T}]$$
For the purpose of the calculation of the antenna correlation coefficients, we arrange vectors $\mathbf{h}_i$ in a vector $\mathbf{h}$ with $n_R n_T$ elements, as follows

$$
\mathbf{h} = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_i, \ldots, \mathbf{h}_{n_R}]
$$

We define an $n_R n_T \times n_R n_T$ correlation matrix $\Theta$ as follows

$$
\Theta = E[\mathbf{h}^H \mathbf{h}]
$$

where $\mathbf{h}^H$ denotes the Hermitian of $\mathbf{h}$.

If the entries of the channel matrix $\mathbf{H}$ are independent identically distributed (iid) variables, $\Theta$ is an identity matrix which produces a maximum capacity.

In order to simplify the analysis, we assume that the correlation between the receive antenna elements does not depend on the transmit antennas and vice versa. This assumption can be justified by the fact that only immediate antenna surroundings cause the correlation between array elements and have no impact on correlation observed between the elements of the array at the other end of the link [8][18]. In such a case we can define an $n_R \times n_R$ correlation coefficient matrix, denoted by $\Theta_R$, for the receive antennas and an $n_T \times n_T$ correlation matrix, denoted by $\Theta_T$, for the transmit antennas.

Assuming that we model the correlation of the receive and the transmit array elements independently, their respective correlation matrices can be represented as

$$
\Theta_R = \mathbf{K}_R \mathbf{K}_R^H
$$

and

$$
\Theta_T = \mathbf{K}_T \mathbf{K}_T^H
$$

where $\mathbf{K}_R$ is an $n_R \times n_R$ matrix and

where $\mathbf{K}_T$ is an $n_T \times n_T$ matrix. Matrices $\mathbf{K}_R$ and $\mathbf{K}_T$ are $n_R \times n_R$ and $n_T \times n_T$ lower triangular matrices, respectively, with positive diagonal elements. They can be obtained from their respective correlation matrices $\Theta_R$ and $\Theta_T$, by Cholesky decomposition [11] (Appendix 1.2). A correlated MIMO channel matrix, denoted by $\mathbf{H}_c$, can be represented as

$$
\mathbf{H}_c = \mathbf{K}_R \mathbf{H} \mathbf{K}_T
$$

where $\mathbf{H}$ is the channel matrix with uncorrelated complex Gaussian entries.

### 1.7.1 Correlation Model for LOS MIMO Channels

Let us consider a MIMO channel with a linear array of $n_T$ transmit and $n_R$ receive antennas, with the respective antenna element separation of $d_t$ and $d_r$, as shown in Fig. 1.18. We assume that the separation between the transmitter and the receiver, denoted by $R$, is much larger than $d_t$ or $d_r$.

We first consider a system with line of sight (LOS) propagation without scattering. The channel matrix entries, denoted by $h_{ki}$, are given by

$$
h_{ki} = e^{-j2\pi \frac{R_{ki}}{\lambda}} \quad k = 1, 2, \ldots, n_R, \quad i = 1, 2, \ldots, n_T,
$$

as their amplitudes are normalized and $R_{ki}$ is the distance between receive antenna $k$ and transmit antenna $i$. 

We assume that the receive and the transmit antenna element correlation coefficients are independent. That is, we calculate the receive antenna element correlation coefficients, for a fixed transmit antenna element, for example the first one, \( i = 1 \). Clearly this does not limit the generality of the analysis. In this case, the receive antenna correlation coefficients, for the LOS system model, shown in Fig. 1.18, are given by

\[
\theta_{mk} = E[h_{m1}^* h_{k1}] 
\]

\( m = 1, 2, \ldots, n_R, \quad k = 1, 2, \ldots, n_R \)  \hfill (1.92)

The correlation coefficients can be obtained by substituting the channel entries from (1.91) into the expression (1.92)

\[
\theta_{mk} = E \left[ e^{-j2\pi R_{k1} - R_{m1}} \right] 
\]

\( 1.93 \)

For the broadside array considered here with the angle of orientation of \( \pi/2 \) and large distances \( R_{k1} \) and \( R_{m1} \), the correlation coefficient in (1.93) can be approximated as

\[
\theta_{mk} = \begin{cases} 
 e^{-j2\pi d_{mk} \sin \alpha / \lambda} & m \neq k \\
 1 & m = k 
\end{cases} 
\]

\( 1.94 \)

where \( d_{mk} \) is the distance between the receive antenna elements \( m \) and \( k \), \( \alpha \) is the plane-wave direction of arrival (DOA).

As the expression (1.94) shows, the correlation coefficient is governed by the antenna element separation and will be largest between the adjacent antenna elements with the separation of \( d_r \). If the antenna separation is small compared to the wavelength \( \lambda \), all
correlation coefficients will be the same and equal to one. According to the expression in (1.94) the correlation coefficients are the same and equal to one, if the direction of arrival $\alpha$ is small, i.e., $\alpha \simeq 0$.

In both cases, for small antenna element separations and small directions of arrival, the channel matrix, denoted by $H_1$, is of rank one.

Assuming that $n_T = n_R = n$, matrix $H_1 H_1^H$ has only one eigenvalue equal to $n^2$ from (1.24). The capacity of this channel (1.21) is given by

$$C = W \log_2 \left( 1 + n \frac{P}{\sigma^2} \right)$$  (1.95)

In the case that the antenna elements are more widely separated, the channel matrix entries $h_{ij}$ will have different values. If they are chosen in such a way that the channel matrix, denoted by $H_n$, is of rank $n$ and that the matrix $H_n H_n^H$ is given by

$$H_n H_n^H = n I_n$$  (1.96)

the capacity can be expressed as

$$C = W \log_2 \left( 1 + \frac{P}{\sigma^2} \right)^n$$  (1.97)

This can be achieved, for example, if the values of the channel matrix entries are given by

$$h_{ik} = e^{j \gamma_{ik}}, \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, n$$  (1.98)

where

$$\gamma_{ik} = \frac{\pi}{n} \left( (i - i_0) - (k - k_0) \right)^2$$  (1.99)

where $i_0$ and $k_0$ are integers. If $n = 2$, then $i_0 = k_0 = 0$, giving for the channel matrix

$$H = \begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix}$$  (1.100)

This corresponds to two linear arrays broadside to each other.

### 1.7.2 Correlation Model for a Rayleigh MIMO Fading Channel

We consider a linear array of $n_R$ omnidirectional receive antennas, spaced at a distance $d_r$, surrounded by clutter, as shown in Fig. 1.19. There are $n_T$ transmit antenna radiating signals which are reflected by the scatterers surrounding the receiver. The plane-wave directions of arrival of signals coming from the scatterers towards the receive antennas is $\alpha$.

For a linear array with regularly spaced antennas and the orientation angle of $\pi/2$, the correlation coefficient of the signals received by antennas $m$ and $k$, separated by distance $d_{mk}$, can be obtained as

$$\theta_{mk} = E \left[ e^{-j 2\pi \frac{R_{ik} - R_{mk}}{\lambda}} \right]$$  (1.101)
where \((R_{k1} - R_{m1})\) is approximated as
\[
R_{k1} - R_{m1} \sim d_{mk} \sin \alpha
\] (1.102)

The correlation coefficient is given by
\[
\theta_{mk} = \begin{cases} 
\int_{\alpha_r/2}^{\alpha_r/2} e^{-j2\pi \frac{d_{mk}}{\lambda} \sin(\alpha)} p(\alpha) d\alpha, & m \neq k \\
1, & m = k 
\end{cases}
\] (1.103)

where \(p(\alpha)\) is the probability distribution of the direction of arrival or the angular spectrum, and \(\alpha_r\) is the receive antenna angular spread. For a uniformly distributed angular spectrum between \(-\pi\) and \(\pi\)

\[
p(\alpha) = \frac{1}{2\pi},
\] (1.104)

the correlation coefficient \(\theta_{mk}\) is given by [14]
\[
\theta_{mk} = J_0 \left(2\pi \frac{d_{mk}}{\lambda}\right), \quad m \neq k
\] (1.105)

where \(J_0(\cdot)\) is the zeroth order Bessel function. To achieve a zero correlation coefficient in this case, the antenna elements should be spaced by \(\lambda/2\), as \(J_0(\pi) \approx 0\). Base stations are typically positioned high above the ground and have a narrow angular spread, which makes the correlation coefficient high even for large antenna separations. It has been shown
that the angular spread at base stations with cell radii of 1km is about 2°. The correlation coefficients for a uniform direction of arrival distribution and various angle spreads are shown in Fig. 1.20. For small values of the angle spread very large antenna separations are needed to obtain low correlation. On the other hand, if the angle spread is reasonably large, for example 30°, low correlation (<0.2) can be obtained for antenna spacing not higher than two wavelengths. For low element separation (<λ/2), the correlation coefficient is high (>0.5) even for large angle spreads.

If the angular spectrum is Gaussian, the correlation coefficient goes monotonically down with antenna separation [8]. The correlation coefficient for a Gaussian distribution of the direction of arrival is shown in Fig. 1.21, for the same angle spreads as for the uniform distribution shown in Fig. 1.20. The pdf for the zero mean Gaussian distributed direction of arrival, denoted by \( p(\alpha) \), is given by

\[
p(\alpha) = \begin{cases} 
\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\alpha^2}{2\sigma^2}} & -\frac{\alpha_r}{2} \leq \alpha \leq \frac{\alpha_r}{2} \\
0 & |\alpha| > \frac{\alpha_r}{2}
\end{cases}
\] (1.106)

The standard deviation for the Gaussian distributed direction of arrival is calculated in such a way as to obtain the same rms values for the uniform and Gaussian distributions for a given angle spread \( \alpha_r \) and is given by

\[
\sigma = \alpha_r k
\] (1.107)

where \( k = 1/2\sqrt{3} \).
Assuming that \( n_R = n_T = n \), the capacity of a correlated MIMO fading channel can be expressed as

\[
C = \log_2 \left[ \det \left( I_n + \frac{P}{n\sigma^2} K_R H K_T^H H^H K_R^H \right) \right] \quad (1.108)
\]

By using the identity

\[
\det(I + AB) = \det(I + BA)
\]

we get for the capacity

\[
C = \log_2 \left[ \det \left( I_n + \frac{P}{n\sigma^2} \Theta_R^H H \Theta_T H^H \right) \right] \quad (1.110)
\]

The MIMO channel capacity for a system with \( n_R = n_T = 4 \) in a fast Rayleigh fading channel, with uniform distribution of the direction of arrival, obtained by averaging the expression in (1.110) and variable receive antenna angle spreads and antenna element separations, are shown in Fig. 1.22.

In order to consider the effect of the receive antenna elements correlation in a slow Rayleigh fading channel, a system with four receive and four transmit antennas is simulated. The receive antenna correlation coefficients are calculated by expression (1.105) assuming a uniform distribution of the direction of arrival. The remote antenna array elements are assumed to be uncorrelated, which is realistic for the case of an array immersed in clutter with a separation of a half wavelength between elements. Using (1.90) to impose correlation on a random uncorrelated matrix \( H \) we calculate the ccdf of the capacity for angle spreads.
Figure 1.22  Average capacity in a fast MIMO fading channel for variable antenna separations and receive antenna angle spread with constant SNR of 20 dB and $n_T = n_R = 4$ antennas.

Figure 1.23  Capacity ccdf curves for a correlated slow fading channel, receive antenna angle spread of 1° and variable antenna element separations of 1°, 5°, 40° and SNR = 20 dB and $n_R = n_T = 4$. The ccdf capacity curves are shown in Figs. 1.23–1.25 for variable antenna element separations. They show that for the angle spread of 1° the capacity of 3 bits/sec/Hz is exceeded with the probability of 90% for the antenna separation of 10$\lambda$, while for the angle spreads of 5° and 40° the same capacity is exceeded with the same probability for the antenna separations of 2$\lambda$ and $<\lambda/2$ wavelengths, respectively.
Figure 1.24  Capacity ccdf curves for a correlated slow fading channel, receive antenna angle spread of 5° and variable antenna element separations

Figure 1.25  Capacity ccdf curves for a correlated slow fading channel, receive antenna angle spread of 40° and variable antenna element separations

1.7.3 Correlation Model for a Rician MIMO Channel

A Rician model is obtained in a system with LOS propagation and scattering. The model is characterised by the Rician factor, denoted by $K$ and defined as the ratio of the line of sight and the scatter power components. The pdf for a Rician random variable $x$ is given by

$$p(x) = 2x(1 + K)e^{-K-(1+K)x^2} I_0 \left( 2x\sqrt{K(K+1)} \right) \quad x \geq 0$$

(1.111)
where

\[ K = \frac{D^2}{2\sigma_r^2} \]  \hspace{1cm} (1.112)

and \( D^2 \) and \( 2\sigma_r^2 \) are the powers of the LOS and scattered components, respectively. The powers are normalized such that

\[ D^2 + 2\sigma_r^2 = 1 \]  \hspace{1cm} (1.113)

The channel matrix for a Rician MIMO model can be decomposed as [1]

\[ H = DH_{LOS} + \sqrt{2}\sigma_r H_{Rayl} \]  \hspace{1cm} (1.114)

where \( H_{LOS} \) is the channel matrix for the LOS propagation with no scattering and \( H_{Rayl} \) is the channel matrix for the case with scattering only.

In one extreme case in LOS propagation, when the receive antenna elements are fully correlated, its LOS channel matrix, denoted by \( H_1 \), has all entries equal to one, as in (1.38), and its rank is one. The capacity curves for this case are shown in Fig. 1.26 for \( n_T = n_R = 3 \). For the Rician factor of zero (K in dB → −∞), which defines a Rayleigh channel, the capacity is equal to the capacity of the fully correlated Rayleigh fading channel. As the Rician factor increases (K → +∞), the capacity reaches the logarithmic expression in (1.95).

For the other LOS extreme case, when the receive antenna elements are uncorrelated, the LOS channel matrix, denoted by \( H_n \), is of rank \( n \), and with the entries given by (1.98). The capacity curves for this case with \( n_T = n_R = 3 \) are shown in Fig. 1.27. For the Rician factor of zero (K → −∞), the capacity is equal to the capacity of an uncorrelated Rayleigh fading channel. As the Rician factor increases (K → +∞), the capacity approaches the linear expression in (1.97).

### 1.7.4 Keyhole Effect

Let us consider a system with two transmit and two receive uncorrelated antennas surrounded by clutter. This system would under normal propagation conditions produce a matrix with independent complex Gaussian variables, giving a high capacity. However, if these two sets of antennas are separated by a screen with a small hole in it, as shown in Fig. 1.28, we get a propagation situation known as “keyhole”. The only way for the transmitted signals to propagate is to pass through the keyhole. If the transmitted signals are arranged in a vector

\[ x = (x_1, x_2)^T \]  \hspace{1cm} (1.115)
Figure 1.26  Ccdf capacity per antenna curves on a Rician channel with $n_R = n_T = 3$ and $SNR = 20$ dB, with a variable Rician factor and fully correlated receive antenna elements

Figure 1.27  Ccdf capacity per antenna curves on a Rician channel with $n_R = n_T = 3$ and $SNR = 20$ dB, with a variable Rician factor and independent receive antenna elements
$y = H_1 x$ \hfill (1.116)

$H_1 = (a_1, a_2)$ \hfill (1.117)

and $a_1$ and $a_2$ are the channel coefficients corresponding to transmitted signals $x_1$ and $x_2$, respectively. They can be described by independent complex Gaussian variables. The signal at the other side of the keyhole, denoted by $y_1$, is given by

$y_1 = g y$ \hfill (1.118)

where $g$ is the keyhole attenuation.

The signal vector at the receive antennas on the other side of the keyhole, denoted by $r$, is given by

$r = H_2 y_1$ \hfill (1.119)

where $H_2$ is the channel matrix describing the propagation on the right hand side of the keyhole. It can be represented as

$H_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ \hfill (1.120)
where \( b_1 \) and \( b_2 \) are the channel coefficients corresponding to the first and second receive antennas, respectively. Thus the received signal vector at the right hand side of the keyhole can be written as

\[
r = g H_2 H_1 x
\]  

(1.121)

In (1.121) we can identify the equivalent channel matrix, denoted by \( H \), as \( g H_2 H_1 \). It is given by

\[
H = g \begin{bmatrix} a_1 b_1 & a_2 b_1 \\ a_1 b_2 & a_2 b_2 \end{bmatrix}
\]  

(1.122)

The rank of this channel matrix is one and thus there is no multiplexing gain in this channel. The capacity is given by

\[
C = \log_2 \left( 1 + \lambda \frac{P}{2\sigma^2} \right)
\]  

(1.123)

where \( \lambda \) is the singular value of the channel matrix \( H \) and is given by

\[
\lambda = g^2 (a_1^2 + a_2^2)(b_1^2 + b_2^2)
\]  

(1.124)

### 1.7.5 MIMO Correlation Fading Channel Model with Transmit and Receive Scatterers

Now we focus on a MIMO fading channel model with no LOS path. The propagation model is illustrated in Fig. 1.29. We consider a linear array of \( n_R \) receive omnidirectional antennas and a linear array of \( n_T \) omnidirectional transmit antennas. Both the receive and transmit antennas are surrounded by clutter and large objects obstructing the LOS path. The scattering radius at the receiver side is denoted by \( D_r \) and at the transmitted side by \( D_t \). The distance between the receiver and the transmitter is \( R \). It is assumed to be much larger than the scattering radii \( D_r \) and \( D_t \). The receive and transmit scatterers are placed at the distance \( R_r \) and \( R_t \) from their respective antennas. These distances are assumed large enough from the antennas for the plane-wave assumption to hold. The angle spreads at the receiver, denoted by \( \alpha_r \), and at the transmitter, denoted by \( \alpha_t \), are given by

\[
\alpha_r = 2 \tan^{-1} \frac{D_r}{R_r}
\]  

(1.125)

\[
\alpha_t = 2 \tan^{-1} \frac{D_t}{R_t}
\]  

(1.126)

Let us assume that there are \( S \) scatterers surrounding both the transmitter and the receiver. The receive scatterers are subject to an angle spread of

\[
\alpha_S = 2 \tan^{-1} \frac{D_t}{R}
\]  

(1.127)

The elements of the correlation matrix of the received scatterers, denoted by \( \Theta_S \), depend on the value of the respective angle spread \( \alpha_S \).

The signals radiated from the transmit antennas are arranged into an \( n_T \) dimensional vector

\[
x = (x_1, x_2, \ldots, x_i, \ldots, x_{n_T})
\]  

(1.128)
The $S$ transmit scatterers capture and re-radiate the captured signal from the transmitted antennas. The $S$ receive scatterers capture the signals transmitted from the $S$ transmit scatterers.

We denote by $y_i$ an $S$-dimensional vector of signals originating from antenna $i$ and captured by the $S$ receive scatterers

$$y_i = (y_{1,i}, y_{2,i}, \ldots, y_{S,i})^T$$

It can be represented as

$$y_i = K_S g_i x_i$$

where the scatterer correlation matrix $\Theta_S$ is defined as

$$\Theta_S = K_S K_S^H$$

and $g_i$ is a vector column consisting of $S$ uncorrelated complex Gaussian components. It represents the channel coefficients from the transmit antenna to the $S$ transmit scatterers. All the signal vectors coming from $n_T$ antennas, $y_i$, for $i = 1, 2, \ldots, n_T$, captured and re-radiated by $S$ receive scatterers, can be collected into an $S \times n_T$ matrix, denoted by $Y$, given by

$$Y = K_S G_T X$$

where $G_T = [g_1, g_2, \ldots, g_{n_T}]$ is an $S \times n_T$ matrix with independent complex Gaussian random variable entries and $X$ is the matrix of transmitted signals arranged as the diagonal elements of an $n_T \times n_T$ matrix, with $x_{i,i} = x_i$, $i = 1, 2, \ldots, n_T$, while $x_{i,j} = 0$, for $i \neq j$. 

**Figure 1.29** Propagation model for a MIMO correlated fading channel with receive and transmit scatterers.
Taking into account correlation between the transmit antenna elements, we get for the matrix $Y$

$$Y = K_S G_T K_T X$$ \hfill (1.133)$$

where the transmit correlation matrix $\Theta_T$ is defined as

$$\Theta_T = K_T K^H_T$$ \hfill (1.134)$$

The receive scatterers also re-radiate the captured signals. The vector of $n_R$ received signals, coming from antenna $i$, denoted by $r_i$, can be represented as

$$r_i = (r_{i,1}, r_{i,2}, \ldots, r_{i,n_R})^T \quad i = 1, 2, \ldots, n_T$$ \hfill (1.135)$$

It is given by

$$r_i = K_R G_R y_i$$ \hfill (1.136)$$

where $G_R$ is an $n_R \times S$ matrix with independent complex Gaussian random variables. The receive correlation matrix $\Theta_R$ is defined as

$$\Theta_R = K_R K^H_R$$ \hfill (1.137)$$

The received signal vectors $r_i$, $i = 1, 2, \ldots, n_T$, can be arranged into an $n_R \times n_T$ matrix $R = [r_1, r_2, \ldots, r_i, \ldots, r_{n_T}]$, given by

$$R = K_R G_R Y$$ \hfill (1.138)$$

Substituting $Y$ from (1.133) into (1.138) we get

$$R = \frac{1}{\sqrt{S}} K_R G_R K_S G_T K_T X$$ \hfill (1.139)$$

where the received signal vector is divided by a factor $\sqrt{S}$ for the normalization purposes. As the channel input-output relationship can in general be written as

$$R = H X$$ \hfill (1.140)$$

where $H$ is the channel matrix, by comparing the relationships in (1.139) and (1.140), we can identify the overall channel matrix as

$$H = \frac{1}{\sqrt{S}} K_R G_R K_S G_T K_T$$ \hfill (1.141)$$

A similar analysis can be performed when there are only transmit scatterers, or both transmit and receive scatterers.

### 1.7.6 The Effect of System Parameters on the Keyhole Propagation

As the expression for the channel matrix in (1.141) indicates, the behavior of the MIMO fading channel is controlled by the three matrices $K_R$, $K_S$ and $K_T$. Matrices $K_R$ and $K_T$
are directly related to the respective antenna correlation matrices and govern the receive and transmit antenna correlation properties.

The rank of the overall channel matrix depends on the ranks of all three matrices $K_R$, $K_S$ and $K_T$ and a low rank of any of them can cause a low channel matrix rank. The scatterer matrix $K_S$ will have a low rank if the receive scatterers angle spread is low, which will happen if the ratio $D_t/R$ is low. That is, if the distance between the transmitter and the receiver $R$ is high, the elements of $K_S$ are likely to be the same, so the rank of $K_S$ and thus the rank of $H$ will be low. In the extreme case when the rank is one, there is only one thin radio pipe between the transmitter and the receiver and this situation is equivalent to the keyhole effect. Note that if there is no scattering at the transmitter side, the parameter relevant for the low rank is the transmit antenna radius, instead of $D_t$.

The rank of the channel matrix can also be one when either the transmit or receive array antenna elements are fully correlated, which happens if either the corresponding antenna elements separations or angle spreads are low.

The fading statistics is determined by the distribution of the entries of the matrix obtained as the product of $G_R K_S G_T$ in (1.141). To determine the fading statistics of the correlated fading MIMO channel in (1.141) we consider the two extreme cases, when the channel matrix is of full rank and of rank one. In the first case, matrix $K_S$ becomes an identity matrix and the fading statistics is determined by the product of the two $n_R \times S$ and $S \times n_T$ complex Gaussian matrices $G_R$ and $G_T$. Each entry in the resulting matrix $H$, being a sum of $S$ independent random variables, according to the central limit theorem, is also a complex Gaussian matrix, if $S$ is large. Thus the signal amplitudes undergo a Rayleigh fading distribution.

In the other extreme case, when the matrix $K_S$ has a rank of one, the MIMO channel matrix entries are products of two independent complex Gaussian variables. Thus their amplitude distribution is the product of two independent Rayleigh distributions, each with the power of $2\sigma_r^2$, called the double Rayleigh distribution. The pdf for the double Rayleigh distribution is given by

$$ f(z) = \int_0^{\infty} \frac{z}{w\sigma_r^2} e^{-z^2/(2w^2\sigma_r^2)} \, dw, \quad z \geq 0, \quad (1.142) $$

For the channel matrix ranks between one and the full rank, the fading distribution will range smoothly between Rayleigh and double Rayleigh distributions.

The probability density functions for single and double Rayleigh distributions are shown in Fig. 1.30.

The channel matrices, given by (1.141), are simulated in slow fading channels for various system parameters and the capacity is estimated by using (1.30). It is assumed in all simulations that the scattering radii are the same and equal to the distances between the antenna and the scatterers on both sides in order to maintain high local angle spreads and thus low antenna element correlations. It is assumed that the number of scatterers is high (32 in simulations). The capacity increases as the number of scatterers increases, but above a certain value its influence on capacity is negligible. Now we focus on examining the effect of the scattering radii and the distance between antennas on the keyhole effect. The capacity
Figure 1.30  Probability density functions for normalized Rayleigh (right curve) and double Rayleigh distributions (left curve)

Figure 1.31  Capacity ccdf obtained for a MIMO slow fading channel with receive and transmit scatterers and $\text{SNR} = 20 \text{ dB}$ (a) $D_r = D_t = 50 \text{ m}, R = 1000 \text{ km}$, (b) $D_r = D_t = 50 \text{ m}, R = 50 \text{ km}$, (c) $D_r = D_t = 100 \text{ m}, R = 5 \text{ km}, \text{SNR} = 20 \text{ dB}$; (d) Capacity ccdf curve obtained from (1.30) (without correlation or keyholes considered)

curves for various combination of system parameters in a MIMO channel with $n_R = n_T = 4$ are shown in Fig. 1.31. The first left curve corresponds to a low rank matrix, obtained for a low ratio of $D_t/R$, while the rightmost curve corresponds to a high rank channel matrix, in a system with a high $D_t/R$ ratio.
The average capacity increase in a fast fading channel, as the scattering radius $D_t$ increases, while keeping the distance $R$ constant, is shown in Fig. 1.32. For a distance of 10 km, 80% of the capacity is attained if the scattering radius increases to 35m.

**Appendix 1.1 Water-filling Principle**

Let us consider a MIMO channel where the channel parameters are known at the transmitter. The allocation of power to various transmitter antennas can be obtained by a “water-filling” principle. The “water-filling principle” can be derived by maximizing the MIMO channel capacity under the power constraint [20]

$$\sum_{i=1}^{n_T} P_i = P \quad i = 1, 2, \ldots, n_T$$

(1.143)

where $P_i$ is the power allocated to antenna $i$ and $P$ is the total power, which is kept constant. The normalized capacity of the MIMO channel is determined as

$$C/W = \sum_{i=1}^{n_T} \log_2 \left[ 1 + \frac{P_i \lambda_i}{\sigma^2} \right]$$

(1.144)

Following the method of Lagrange multipliers, we introduce the function

$$Z = \sum_{i=1}^{n_T} \log_2 \left[ 1 + \frac{P_i \lambda_i}{\sigma^2} \right] + L \left( P - \sum_{i=1}^{n_T} P_i \right)$$

(1.145)

where $L$ is the Lagrange multiplier, $\lambda_i$ is the $i$th channel matrix singular value and $\sigma^2$ is the noise variance. The unknown transmit powers $P_i$ are determined by setting the partial
derivatives of $Z$ to zero

$$\frac{\delta Z}{\delta P_i} = 0$$  \hfill (1.146)

$$\frac{\delta Z}{\delta P_i} = \frac{1}{\ln 2} \frac{\lambda_i/\sigma^2}{1 + P_i \lambda_i/\sigma^2} - L = 0$$  \hfill (1.147)

Thus we obtain for $P_i$

$$P_i = \mu - \frac{\sigma^2}{\lambda_i}$$  \hfill (1.148)

where $\mu$ is a constant, given by $1/\ln 2$. It can be determined from the power constraint (1.143).

## Appendix 1.2: Cholesky Decomposition

A symmetric and positive definite matrix can be decomposed into a lower and upper triangular matrix $A = LL^T$, where $L$ (which can be seen as a square root of $A$) is a lower triangular matrix with positive diagonal elements. To solve $Ax = b$ one solves first $Ly = b$ and then $L^Tx = y$ for $x$.

$$A = LL^T$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} \\ l_{21} \\ \vdots \\ l_{n1} \end{bmatrix} = \begin{bmatrix} 0 \\ l_{22} \\ \vdots \\ l_{nn} \end{bmatrix}$$

where $a_{ij}$, and $l_{ij}$ are the entries of $A$ and $L$, respectively.

$$a_{11} = l_{11}^2 \rightarrow l_{11} = \sqrt{a_{11}}$$

$$a_{21} = l_{21}l_{11} \rightarrow l_{21} = a_{21}/l_{11}, \ldots l_{n1} = a_{n1}/l_{11}$$

$$a_{22} = l_{21}^2 + l_{22}^2 \rightarrow l_{22} = \sqrt{(a_{22} - l_{21}^2)}$$

$$a_{32} = l_{31}l_{21} + l_{32}l_{22} \rightarrow l_{32} = (a_{32} - l_{31}l_{21})/l_{22}$$

In general, for $i = 1, 2, \ldots n$, $j = i + 1, \ldots n$

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}$$

$$l_{ji} = \left( a_{ji} - \sum_{k=1}^{i-1} l_{jk} l_{ik} \right) / l_{ii}$$
Because $A$ is symmetric and positive, the expression under the square root is always positive.

**Bibliography**


