Thermal Radiation

General objective

Gain knowledge on energy-related quantities and the laws of thermal radiation.

Specific objectives

On completing this chapter, the reader should be able to:

– define thermal radiation;
– know the origin of thermal radiation;
– know the relations between energy, photon wavelength and frequency;
– distinguish between black body and gray body;
– define the energy-related quantities (flux, exitance, radiance and intensity);
– apply the law of conservation of radiant flux;
– state Lambert’s, Kirchhoff’s, Stefan-Boltzmann and Wien’s laws;
– provide an interpretation of Planck’s law;
– apply the laws of thermal radiation;
– find the useful spectrum from a given isotherm of the black body.

Prerequisites

– structure of matter;

For color versions of the figures in this book, see www.iste.co.uk/sakho/quantum1.zip.
– modes of energy transfer;
– range of electromagnetic waves.

1.1. Radiation

1.1.1. Definition

An object at temperature $T$ can emit or absorb light waves of several frequencies [PÉR 86, PER 11, SAK 12]. The distribution of the energy exchanged by the object with its external environment depends on the temperature $T$. When two objects at different temperatures are in contact, thermal energy (heat) is transferred from the hot object to the cold object. By contrast, radiation is energy that is carried by an electromagnetic wave. In this case, energy is transferred by emission and absorption of light waves. Hence, by definition, thermal radiation is the electromagnetic radiation emitted by any object at non-zero temperature $T$.

1.1.2. Origin of radiation

In 1900, Max Planck laid the foundations of quantum physics by studying the black body emission spectrum within the theory of quanta [BRO 25, PAI 82, PLO 16]. He formulated the fundamental hypothesis according to which the energy generated by a periodic movement of frequency $\nu$ (rotation or vibration) has, similar to matter, a discontinuous structure. Consequently, radiant energy can only exist as bundles or quanta of energy $h \nu$. The number $h$ is a universal constant known as the Planck constant. In 1905, Albert Einstein stated that light is made of particles subsequently called photons, each of which has an energy $h \nu$. Radiation results from electronic transitions between discrete levels of atomic or molecular systems. The energy exchanged during these transitions corresponds to photon absorption and emission processes. The energy $E$, angular frequency $\omega$, frequency $\nu$ and wavelength $\lambda$ of the photon are related by the following relations:

$$\begin{cases} E = h \nu = h \omega \\ E = \frac{hc}{\lambda} \end{cases} \implies \begin{cases} \nu = \frac{\omega}{2\pi} = \frac{c}{\lambda} \\ h = \frac{\hbar}{2\pi} \end{cases} \quad [1.1]$$

In relations [1.1], $E$ is expressed in joules (J), $\nu$ in hertz (Hz), $\omega$ in radian per second (rad $\cdot$ s$^{-1}$) and $\lambda$ in meters (m). The quantity $c$ designates the speed of light in a vacuum and $\hbar$ is the $h$-bar (or reduced) Planck constant.
Numerical expression:

\[ h = 6.62606896 \times 10^{-34} \text{ J} \cdot \text{s}; \quad h = 1.054571628 \times 10^{-34} \text{ J} \cdot \text{s}. \]

\[ c = 299792458 \text{ m} \cdot \text{s}^{-1} \text{ (exact value)}. \]

The figures designate the absolute errors \( \Delta X \) (uncertainties) of the given values of the measured \( X \) quantity. For example, \( h = (6.626 \, 068 \, 96 \pm 0.000 \, 000 \, 33) \times 10^{-34} \text{ J} \cdot \text{s}. \)

This means an absolute error \( \Delta h = (0.000 \, 000 \, 33) \times 10^{-34} \text{ J} \cdot \text{s}. \)

At the microscopic scale, it is convenient to use the electronvolt (eV) as a unit of energy:

\[ 1 \text{ eV} = 1.602179487 \times 10^{-19} \text{ J}. \]

Photon absorption and emission processes are illustrated in Figure 1.1.

![Figure 1.1. Electronic transition between two discrete levels](image)

**APPLICATION 1.1.–**

A He-Ne laser in a laboratory emits radiation whose wavelength is 633 nm. Calculate the energy \( E \), frequency \( \nu \) and angular frequency \( \omega \) of a photon of this radiation. Express \( E \) in eV.

*Given data.* \( h = 6.63 \times 10^{-34} \text{ J} \cdot \text{s}; \ c = 3.0 \times 10^{8} \text{ m} \cdot \text{s}^{-1}; \ 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}. \)

*Solution.* \( E = 1.96 \text{ eV}; \ \nu = 4.74 \times 10^{14} \text{ Hz}; \ \omega = 2.98 \times 10^{15} \text{ rad} \cdot \text{s}^{-1}. \)

Max Planck, in full Max Karl Ernst Ludwig Planck, was a German physicist. He founded quantum physics in 1900 with his fundamental hypothesis on the theory of quanta. He was awarded the Nobel Prize for physics in 1918 for his essential contribution to the
theory of quanta. Planck is also one of the founding fathers of quantum mechanics. He is also well known for his law giving monochromatic radiant exitance, which makes it possible to interpret the experimental observations related to black body isotherms.

Box 1.1. Planck (1854–1947)

1.1.3. Classification of objects

Objects susceptible to exchange energy are classified into three categories:

– transparent objects that allow radiation to pass through without attenuation. This is the case with glass, transparent plastic material, etc.;

– opaque objects that absorb radiation and get heated. This is the case with solid bodies (metals, rocks, etc.), cardboard and some viscous liquids, such as paint;

– translucent objects that absorb a part of the radiation and allow the rest to pass through. For these objects, radiation propagation is accompanied by absorption that increases the energy of the medium. A familiar example is that of oil.

1.2. Radiant flux

1.2.1. Definition of radiant flux, coefficient of absorption

By definition, radiant flux denoted by $\Phi$ is the power emitted by a source throughout the space in which it can radiate. Radiant flux is expressed in Watts (W).

Let us consider an object receiving an incident energy flux $\Phi_i$. The surface of the object is chosen to allow radiation reflection, absorption and transmission (Figure 1.2).

According to the law of conservation of energy, we have:

$$\Phi_i = \Phi_r + \Phi_a + \Phi_t. \quad [1.2]$$

In this relation, $\Phi_r$ is the reflected radiant flux, $\Phi_a$ designates the absorbed radiant flux and $\Phi_t$ represents the transmitted radiant flux. Let us consider $\rho$, $\alpha$ and $\tau$ as the coefficients of reflection, absorption and transmission, respectively, of the radiant flux. Their expressions are given as:

$$\rho = \frac{\Phi_r}{\Phi_i} ; \quad \alpha = \frac{\Phi_a}{\Phi_i} ; \quad \tau = \frac{\Phi_t}{\Phi_i} \quad [1.3]$$
Implementing relations [1.3] in [1.2], the conservation of energy can be written as:

\[ \rho + \alpha + \tau = 1. \]  

[1.4]

Coefficients \( \rho \), \( \alpha \) and \( \tau \) characterize the behavior of an object that is subjected to radiation. It is worth noting that the absorption coefficient \( \alpha \) is the most important parameter. This coefficient measures the proportion in which incident electromagnetic radiation is converted into thermal energy.

**APPLICATION 1.2.–**

Let us consider an arbitrary process of reflection, absorption and transmission of an incident radiant flux. Calculate the absorbed flux.

*Given data.* \( \rho = 30 \%; \tau = 20 \%; \) transmitted flux: 200 W.

*Solution.* \( \Phi_a = 500 \text{ W} \).

**1.2.2. Black body and gray body**

There are two types of bodies:

– *gray bodies* for which \( \alpha < 1 \);

– *black bodies* for which \( \alpha = 1 \).
By definition, a black body is an ideal (therefore fictitious) object that has the specific property of perfectly absorbing the radiations of the visible spectrum irrespective of their frequency. The adjective “black” highlights only the fact that the object absorbs all the radiations of the visible spectrum so that it appears to be black. A black body can be actually realized by piercing a small orifice in the wall of a temperature-controlled cavity (whose walls are brought to a given temperature $T$). No radiation entering this cavity can escape. Hence the orifice behaves as a black body. It is nevertheless worth noting that an insignificant amount of thermal radiation leaves the cavity, but it is not sufficient to perturb the thermal equilibrium established in the cavity (but it is sufficient to be studied experimentally). Black velvet and black ink are simple examples of black bodies. Let us finally note that a gray body is not necessarily gray. This term designates any object whose absorption coefficient is $\alpha < 1$.

1.3. Black body emission spectrum

1.3.1. Isotherms of a black body: experimental facts

By definition, the electromagnetic energy density denoted by $du$ in the band of angular frequency between $\omega$ and $\omega + d\omega$ (or of wavelength between $\lambda$ and $\lambda + d\lambda$) is given by the expression:

$$du = u(\omega) \, d\omega = u(\lambda) \, d\lambda$$  \[1.5\]

In relations [1.5], the physical quantity $u(\omega)$ or $u(\lambda)$ is called the spectral density of electromagnetic energy. $u(\omega)$ is expressed in $\text{J} \cdot \text{rad}^{-1} \cdot \text{s}$ and $u(\lambda)$ in $\text{J} \cdot \text{m}^{-1}$.

Let us study the variation of the spectral density of electromagnetic energy depending on wavelength $\lambda$ for each temperature $T$ of the black body. Experience shows that these are asymmetrical curves known as black body isotherms. For each temperature value $T$, there is a corresponding curve that reaches a maximum for a specific wavelength value denoted as $\lambda_{\text{max}}$ (Figure 1.3). It should be kept in mind that $\lambda_{\text{max}}$ does not correspond to the maximal value of the wavelength of an isotherm taking place at temperature $T$ of the black body. It is rather the wavelength corresponding to the peak of each isotherm. For example, for the isotherm at 5,500 K, $\lambda_{\text{max}} \approx 520$ nm.
Solid angle

An angle $\theta$ (in radian) is defined as the length $l$ of the arc cut off from a circle, centered at the vertex of the angle, divided by the radius $R$ of this circle, which is $\theta = \frac{l}{R}$. By analogy, the solid angle denoted as $\Omega$ (expressed in steradian) of a cone is defined as the area $S$ cut off by this cone on a sphere centered at its vertex divided by the squared radius of the sphere (Figure 1.4), which is:

$$\Omega = \frac{S}{R^2}$$ \[1.6\]

- For the whole space, $S = 4\pi R^2 \Rightarrow \Omega = 4\pi$ steradian.
- For a half-space, $S = 2\pi R^2 \Rightarrow \Omega = 2\pi$ steradian.
The solid angle under which an elementary area $dS$, whose dimensions are small compared to its distance $r$ to point $O$, and whose normal makes an angle $\theta$ with the direction of the unit vector $\vec{e}_r$ (Figure 1.5), can be seen from a point $O$ is very often interesting to consider. This elementary solid angle is present, for example, in the definition of the flux of a field of vectors through an elementary area $dS$. By definition, the elementary solid angle is given by the following relation:

$$d\Omega = \frac{dS \cdot \vec{e}_r}{r^2} = \frac{dS \cos \theta}{r^2} \quad [1.7]$$

Let us finally express the solid angle of a cone of revolution of vertex $O$ and half-angle at vertex $\theta$. Let us consider for this purpose two cones of the same vertex $O$ and the same axis, and half-angles $\alpha$ and $\alpha + d\alpha$ (Figure 1.6).
The (hatched) surface cut off from the sphere of radius $R$ by the interval between the two cones is given by the relation:

$$dS = 2\pi R \sin \alpha R \, d\alpha = 2\pi R^2 \sin \alpha \, d\alpha$$

The corresponding elementary solid angle $d\Omega$ is then:

$$d\Omega = \frac{dS}{R^2} = \frac{2\pi \sin \alpha \, d\alpha}{R^2} = \frac{2\pi}{R^2} \sin \alpha \, d\alpha$$ \hspace{1cm} [1.8]

The solid angle $\Omega$ of a cone of revolution of vertex $O$ and half-angle at the vertex $\theta$ is obtained by integration of equation [1.8] between the limits $0$ and $\theta$. This leads to:

$$\Omega = 2\pi \int_0^\theta \sin \alpha \, d\alpha = 2\pi (1 - \cos \theta)$$ \hspace{1cm} [1.9]

– For the entire space, $\theta = \pi \Rightarrow \Omega = 4\pi$ steradian.
– For a half-space, $\theta = \pi/2 \Rightarrow \Omega = 2\pi$ steradian.

1.3.3. Lambert’s law, radiance

Let us consider an emissive surface $S$. The fraction of flux $d^2\Phi$ contained in the cone of solid angle $d\Omega$ in direction $Ox$ making an angle $\theta$ with the normal $N$ to the surface $dS$ (Figure 1.7) is given by the relation:

$$d^2\Phi = AdSd\Omega$$ \hspace{1cm} [1.10]
Surface $dS$ follows Lambert’s law if $A = L \cos \theta$. The fraction of flux [1.10] is then written as:

$$d^2 \Phi = L \cos \theta dS d\Omega$$ \[1.11\]

The energy-related quantity $L$ is known as the radiance of the emissive object. As relation [1.11] shows, radiance is by definition equal to the flux radiated by a solid angle unit and by a surface unit perpendicular to $Ox$. $L$ is expressed in $\text{W} \cdot \text{m}^{-2} \cdot \text{sr}^{-1}$. According to Lambert’s law, the radiance of an object is independent of the direction of axis $Ox$. Radiance depends only on temperature $T$ and on the nature of the object surface (color, roughness, etc.). Moreover, total radiant intensity $dI$ is equal to the fraction of the flux radiated in direction $Ox$ (Figure 1.7) per unit solid angle $d\Omega$, which is:

$$dI = AdS = L \cos \theta dS$$ \[1.12\]

Total radiant intensity is expressed in Watt per steradian ($\text{W} \cdot \text{sr}^{-1}$).

### 1.3.4. Kirchhoff’s laws

There are two Kirchhoff’s laws on thermal radiation. They explain the black body isotherms and the relation between radiance $L$ of the gray body and radiance $L_0$ of the black body. Kirchhoff’s laws can be stated as follows:

- **first law**: all black bodies at the same temperature have the same radiance;
- **second law**: among all the objects brought to the same temperature, the black body is the most luminous.

Considering a gray body of coefficient of absorption $\alpha$, the mathematical expression of Kirchhoff’s second law leads to the relation between radiance $L$ of the gray body and radiance $L_0$ of the black body, which is:

$$L = \alpha L_0$$ \[1.13\]
Johann Heinrich Lambert was a Swiss mathematician and astronomer. He is considered as one of the founders of photometry. He is well known in this field for the law that introduces the radian of an emissive object and for the Beer–Lambert law stating that the decrease in light intensity is proportional to the number of particles absorbing light.

Gustav Robert Kirchhoff was a German physicist. He is well known especially for his laws related to the conservation of currents and charges in electrical circuits. Kirchhoff is also known for his laws related to thermal radiation, which he formulated in 1859.

Box 1.2. Lambert (1728–1777); Kirchhoff (1824–1887)

1.3.5. Stefan–Boltzmann law, total energy exitance

By definition, total energy exitance $M$ is equal to the power radiated by the unit surface in all directions. Its relation with radiance is:

$$M = \int L \cos \theta d\Omega = 2\pi L \int_0^{\pi/2} \cos \theta \sin \theta d\theta.$$  

Considering $x = \cos \theta$, this leads to:

$$M = -2\pi L \int_1^0 \cos \theta d\cos \theta = -2\pi L \int_1^0 xdx$$

or:

$$M = \pi L.$$  \hspace{1cm} [1.14]

$M$ is expressed in $W \cdot m^{-2}$.

For the black body, total radiant exitance denoted by $M^0$ is, according to [1.14]:

$$M^0 = \pi L_0$$  \hspace{1cm} [1.15]

Given the Stefan–Boltzmann law, total radiant exitance $M^0$ of the black body is proportional to the fourth power of its temperature $T$ or (see demonstration in Appendix A.3):

$$M^0 = \sigma T^4$$  \hspace{1cm} [1.16]

In this relation, $\sigma$ is Stefan–Boltzmann constant.
Numerical expression:

\[ \sigma = 5.66897 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4} \]

Using relations [1.15] and [1.16], the radiance \( L_0 \) of the black body can be written as:

\[ L_0 = \frac{\sigma}{\pi} T^4 \]

[1.17]

There is no need to memorize expression [1.17] of the radiance \( L_0 \). It can be deduced, when needed. Only the laws and the definitions of the energy-related quantities studied in this section and in the following sections should be retained.

**APPLICATION 1.3.–**

Calculate the radiance and the radiant exitance of a gray body at temperature 3000 K whose coefficient of absorption is 85%.

**Given data.** Stefan–Boltzmann constant: \( \sigma = 5.67 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4} \).

**Solution.** \( L = 1.24 \text{ MW} \cdot \text{m}^{-2} \cdot \text{sr}^{-1} \); \( M = 3.9 \text{ MW} \cdot \text{m}^{-2} \).

**Joseph Stefan** was an Austrian physicist. He is especially renowned for his work published in 1879 on the radiation of the black body in which he stated the law that bears his name. Based on this law, Stefan determined the Sun’s surface temperature (5430°C). Then his student Boltzmann offered a theoretical justification for the Stefan law. This is why this law is commonly known as the Stefan–Boltzmann law (for Boltzmann, see Box A.3).

**Box 1.3. Stefan (1835–1893)**

**1.3.6. Wien’s laws, useful spectrum**

The two Wien’s laws give the abscissa \( \lambda_{\text{max}} \) of the wavelength and the ordinate \( M_{\text{max}}^0 \) of the maximum monochromatic exitance for each temperature \( T \) of the black body (Figure 1.8).

**Wien’s first law**

The wavelength \( \lambda_{\text{max}} \) at an isotherm peak shifts toward short wavelengths when temperature \( T \) increases according to the law:

\[ \lambda_{\text{max}} T = \sigma_w \]

[1.18]
In this relation, $\sigma_W$ designates the *Wien constant*: $\sigma_W = 2.898 \times 10^{-3} \text{ m \cdot K}$. 

Figure 1.8. *Isotherm curve of the black body at temperature $T$*

Wien’s law [1.19] shows that the value of $\lambda_{\text{max}}$ shifts toward short wavelengths when the temperature $T$ increases.

**Wien’s second law**

The ordinate $M_{\lambda_{\text{max}}}^0$ of the maximum monochromatic exitance is proportional to the fifth power of temperature, which is:

$$M_{\lambda_{\text{max}}}^0 = BT^5 \quad \text{[1.19]}$$
In relation [1.19], $M_{\text{max}}^0$ is expressed in $W \cdot m^{-3}$ (if the wavelength $\lambda_{\text{max}}$ is expressed in m) or in $W \cdot m^{-2} \cdot \mu m^{-1}$ (if $\lambda_{\text{max}}$ is expressed in micrometers); the units of constant $B$ depend on the unit of wavelength $\lambda_{\text{max}}$.

\[
B = 1.28 \times 10^{-5} [W \cdot m^{-3} \cdot K^{-5}] \text{ if } \lambda_{\text{max}} \text{ is expressed in m.}
\]

\[
B = 1.28 \times 10^{-11} [W \cdot m^{-2} \cdot \mu m^{-1} \cdot K^{-5}] \text{ if } \lambda_{\text{max}} \text{ is expressed in } \mu \text{m.} \quad [1.20]
\]

Moreover, experience shows that when the wavelength of radiation is such that $\lambda < 0.5 \lambda_{\text{max}}$, there is practically no more radiated energy (approximately 1%) [PER 11].

Furthermore, there is practically no more radiated energy when $\lambda > 4.5 \lambda_{\text{max}}$. By definition, the wavelength range $0.5 \lambda_{\text{max}} < \lambda < 4.5 \lambda_{\text{max}}$ is called the *useful spectrum* of the considered isotherm (hatched part in Figure 1.9).

**APPLICATION 1.4.–**

Let us consider an isotherm at the surface temperature of the Sun, which is assimilated to a black body. Find the useful spectrum corresponding to $T = 6,000$ K.

*Given data.* $\sigma_W = 2.898 \times 10^{-3} m \cdot K$.

*Solution.* Useful spectrum: $0.5 \lambda_{\text{max}} < \lambda < 4.5 \lambda_{\text{max}} \Rightarrow (241.5 < \lambda < 2,173.5)$ nm.

*NOTE.*– The useful spectrum contains radiations from ultraviolet to infrared. Therefore it contains all the radiations of the visible spectrum.

**APPLICATION 1.5.–**

What is the maximal monochromatic exitance of the isotherm of the black body at wavelength $\lambda_{\text{max}} = 1.184 \mu m$ at $T = 2,500$ K?

*Given data.* $B = 1.28 \times 10^{-11} W \cdot m^{-2} \cdot \mu m^{-1} \cdot K^{-5}$.

*Solution.* $M_{\lambda_{\text{max}}}^0 = 1.25 MW \cdot m^{-2} \cdot \mu m^{-1}$.

*Wilhelm Wien* was a German physicist. He is well known for the laws published in 1896 that give the spectral distribution of the black body radiation for short wavelengths.

**Box 1.4. Wien (1864–1928)**
1.3.7. The Rayleigh–Jeans law, “ultraviolet catastrophe”

Let us consider a black body at thermodynamic equilibrium. The density of modes \( n(\omega) \) (or the number of types of oscillators) in the angular frequency range \([\omega, \omega + d\omega]\) is given by the following relation (see demonstration in Appendix A.1):

\[
n(\omega) = \frac{\omega^2}{\pi^2 c^3}
\]

[1.21]

In this relation, \( c \) designates the speed of light in a vacuum.

The spectral energy density is equal to the product of the density of modes \( n(\omega) \) and the average energy \( \langle E_\omega \rangle(T) \) of the field of a single mode:

\[
u(\omega) = n(\omega) \times \langle E_\omega \rangle(T)
\]

[1.22]

For a classical oscillator, the average energy is \( \langle E_\omega \rangle(T) = kT \). Hence using [1.21], the classical formula of the Rayleigh–Jeans can be written as:

\[
u(\omega) = \frac{\omega^2}{\pi^2 c^3} \times kT
\]

[1.23]

Expression [1.23] shows that spectral energy density \( \nu(\omega) \) is a parabolic arc. Let us draw the graphical representation of the variation of \( \nu(\omega) \) as a function of angular frequency \( \omega \) compared to experimental observations. The resulting curves are represented in Figure 1.10.

![Figure 1.10. Comparison of the classical Rayleigh–Jeans law with experimental observations](image-url)
Figure 1.10 shows that the classical Rayleigh–Jeans law is in agreement with the experimental observations for low frequencies $\nu (\nu = \omega/2\pi)$ or for long wavelengths $\lambda (\omega = 2\pi c/\lambda)$. On the other hand, it corresponds to an infinitely wide field of energy for high angular frequencies or for short wavelengths. This shift of the spectrum toward the ultraviolet region when angular frequency increases is known as “ultraviolet catastrophe”.

John William Strutt Rayleigh was a British physicist. In 1900, he applied the laws of classical statistical mechanics to the field of radiation to establish the law expressing the distribution of the energy radiated by the black body depending on frequency. A factor 8 error, due to erroneous counting in the phase space, was corrected in 1905 by Jeans [TAI 08].

James Hopwood Jeans was a British physicist, mathematician and astronomer. He had significant scientific contributions to several fields of physics, such as thermal radiation, in which he co-authored with Rayleigh the law bearing their names.

**Box 1.5. Rayleigh (1842–1919); Jeans (1877–1946)**

1.3.8. Planck’s law, monochromatic radiant exitance

In order to establish the quantum law of radiation by a generalization of the classical law [1.23] within the theory of quanta [BRO 25], Planck assimilated the black body cavity to a set of virtual harmonic oscillators. The problem posed is then to express the average energy of each of these oscillators. Considering that the average energy $\langle E_\omega \rangle (T)$ of a mode is determined by the quotient of the discrete sum energies of the set of elementary oscillators by the total number of oscillators, the result is (see Exercise 1.4.3 for the demonstration):

$$\langle E_\omega \rangle (T) = \frac{\hbar \omega}{(e^{\hbar \omega / kT} - 1)} \quad [1.24]$$

Using results [1.21] and [1.24], the quantum expression of the spectral density of electromagnetic energy known as Planck’s law is obtained:

$$u(\omega) = \frac{\hbar \omega^3}{\pi^2 c^3} \times \frac{1}{(e^{\hbar \omega / kT} - 1)} \quad [1.25]$$

NOTE.— Depending on frequency $\nu (\omega = 2\pi \nu)$, Planck’s law can be written as:

$$u(\nu) = \frac{8\pi \hbar \nu^3}{c^3} \times \frac{1}{(e^{\hbar \nu / kT} - 1)} \quad [1.26]$$
Figure 1.11 shows an illustration of classical [1.23] and quantum [1.26] predictions compared to experimental observations. It is worth noting that Planck’s quantum law is perfectly corroborated by experimental observations for all frequencies. Moreover, as indicated in Figure 1.11, the classical (Rayleigh–Jeans law) and quantum (Planck’s law) curves overlap for low values of angular frequency. This shows that Planck’s law is actually a generalization of the Rayleigh–Jeans’ law (see Application 1.6).

Moreover, there is another formulation of Planck’s law expressing the monochromatic exitance $M_\lambda^0$ of the black body as a function of wavelength $\lambda$ and its absolute temperature $T$. This formulation can be written as:

$$M_\lambda^0 = \frac{C_1 \lambda^{-5}}{(e^{C_2/\lambda T} - 1)} \tag{1.27}$$

In [1.27], $C_1$ and $C_2$ are constants: $C_1 = 2\pi hc^2$ and $C_2 = hc/k$, where $h$ designates the Planck constant, $c$ designates the speed of light in a vacuum and $k$ denotes the Boltzmann constant. Planck’s law [1.25] makes it possible to find Wien’s first law [1.19] (see Application 1.6) and deduce from it the theoretical expression of the Wien constant.

**APPLICATION 1.6.–**

Use Planck’s law [1.25] to find the Rayleigh–Jeans law.

**Solution.** For low frequencies such that $\hbar \omega \ll kT$, Planck’s law [1.25] gives:
\[
\begin{align*}
u(\omega) &= \frac{\hbar \omega^3}{\pi^2 c^3} \left( \frac{1}{\hbar \omega/kT} - 1 \right) \\
&\Rightarrow u(\omega) = \frac{\omega^2}{\pi^2 c^3} kT
\end{align*}
\]

which actually corresponds to the classical Rayleigh–Jeans law [1.23].

1.4. Exercises

1.4.1. Exercise 1 – Calculation of the Stefan–Boltzmann constant

Let us consider a surface element \(dS\) of a black body. \(d^2\Phi\) is the power emitted by this element in the wavelength range \([\lambda, \lambda + d\lambda]\).

(1) Express \(d^2\Phi\) and the total exitance \(M^0\) of the black body.

(2) Deduce the expression of the Stefan–Boltzmann law.

(3) Calculate the Stefan–Boltzmann constant.

**Given data.**

- \(h = 6.62606896 \times 10^{-34} \text{ J} \cdot \text{s}; \ k = 1.3806504 \times 10^{-23} \text{ J} \cdot \text{K}^{-1}.
- \(c = 2.99792458 \times 10^8 \text{ m} \cdot \text{s}^{-1}.

- Spectral radiance of the black body:

\[
P^0_\lambda = \frac{2hc^2}{\lambda^5} \times \frac{1}{e^{hc/\lambda kT} - 1}
\]

For all practical purposes, the following integral is given:

\[
\int_0^{\infty} \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}
\]

1.4.2. Exercise 2 – Calculation of the Sun’s surface temperature

The Sun, of radius \(R\) and surface area \(S\), is assimilated to a black body at temperature \(T\). The part of solar radiation reaching the Earth situated at distance \(d\) from the Sun is considered. \(P_0\) designates the solar power received by the Earth disc of surface area \(S_0\) and radius \(R_0\). Throughout the exercise, the Earth is assimilated to a gray body.

(1) Express the power \(P\) radiated by the solar surface \(S\) as a function of \(R, T\) and \(\sigma\) (Stefan–Boltzmann constant).
(2) Draw a schematic representation of the relative positions of the Sun and Earth. Indicate on it the radii $R$ and $R_0$, surfaces $S$ and $S_0$ and the sphere of solar radiation of radius $d$ and surface $S'$. Express $P_0$ as a function of $R$, $R_0$, $T$, $d$ and $\sigma$.

(3) Find the expressions of power $P_a$ absorbed by the Earth and of power $P_r$ radiated through its entire spherical surface of radius $R_0$.

(4) Express the temperature $T$ of the Sun’s surface as a function of $T_0$, $d$ and $R$. Make the numerical application.

**Given data.**

– Ratio of the Sun’s radius $R$ to the Earth–Sun distance $d = R/d = 200$.
– Stefan–Boltzmann constant: $\sigma = 5.670 \times 10^{-8}$ W $\cdot$ m$^{-2} \cdot$ K$^{-4}$.

### 1.4.3. Exercise 3 – Average energy of a quantum oscillator, Planck's formula

According to Planck’s approach to black body radiation, the oscillations of an electromagnetic field can be assimilated to a set of elementary oscillators. Let us consider that $E_n$ is the energy of an elementary mode constituted of a number $n$ of photons.

(1) Establish the relation between $E_n$ and $\omega$ using Planck’s hypothesis.

(2) At thermodynamic equilibrium, the number $N_n$ of elementary oscillators obeys Boltzmann’s distribution law, which is:

$$N_n = N_0 e^{-\beta E_n}$$

In this expression, $\beta = 1/kT$, where $k$ is Boltzmann constant.

Find the physical significance of $N_0$.

(3) Knowing that the average energy $\langle E_\omega \rangle (T)$ of a mode can be calculated by the quotient of the discrete sum of energies of the set of elementary oscillators to the total number of oscillators, express $\langle E_\omega \rangle (T)$ as a function of $N_n$ and $E_n$.

(4) Considering $a = h\omega$ and $x = e^{-\beta a}$, show that $\langle E_\omega \rangle (T)$ can be written as:

$$\langle E_\omega \rangle (T) = \frac{a}{x^{-1} - 1}$$
(5) Use this result to deduce the expression of energy $\langle E_\omega \rangle(T)$ as a function of $h$, $\omega$, and $T$.

(6) Find Planck’s formula. The expression of mode density is:

$$n(\omega) = \frac{\omega^2}{\pi^2 c^3}$$

1.4.4. Exercise 4 – Deduction of Wien’s first law from Planck’s formula

Given Planck’s formula, the spectral exitance $M_\lambda$ (or spectral radiant exitance) can be written as follows (in vacuum):

$$M_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc\beta/\lambda} - 1}$$

(1) Express the spectral exitance in a medium of refractive index $n_\lambda$.

(2) What is spectral exitance at the peak of light emission of the black body and at temperature $T$?

(3) Use the previous results to deduce Wien’s first law. Consider $x = hc\beta/\lambda$.

(4) Estimate the Sun’s surface temperature corresponding to a maximum emission at a wavelength of 500 nm.

**Given data.**

– Unique positive solution of the equation: $e^{-x} + \frac{1}{5}x - 1 = 0 : x_0 = 4.965$.

– $k = 1.3806504 \times 10^{-23}$ J K$^{-1}$; $h = 6.62606896 \times 10^{-34}$ J s; $c = 2.99792458 \times 10^8$ m s$^{-1}$.

– Wien constant $\sigma_w = 2.898 \times 10^{-3}$ m K.

1.4.5. Exercise 5 – Total electromagnetic energy radiated by the black body

Let us consider a black cavity within which there is along all directions a light radiation emitted by the cavity walls at temperature $T$. Let us consider on the walls an emissive surface $dS$ that obeys Lambert’s law. Let us denote $d^2\Phi$ the flux contained in the cone of solid angle $d\Omega$ and direction $Ox$ that makes an angle $\theta$ with the normal $N$ to the surface $dS$. 
(1) Use Lambert’s law to find the expression of \( d^2 \Phi \). Deduce from it the electromagnetic energy \( d^3 E \) exiting a black surface orifice of cross-section \( dS \) in the solid angle \( d\Omega \) for a duration \( dt \) along the normal \( N \).

(2) The electromagnetic radiation is considered to be contained in a cylinder of height \( h = cd\) and base surface area \( dS \). Express the energy density per unit volume \( du \) as a function of \( d\Omega \), \( L_0 \) and \( c \) (speed of light in a vacuum).

(3) Show that the total energy per unit volume \( U(T) \) radiated by the black body can be written as \( U(T) = \sigma^* \times T^4 \), where \( \sigma^* \) is a constant. Calculate \( \sigma^* \).

(4) Calculate \( U(T) \) at \( T = 6,000 \) K.

Given data. \( \sigma = 5.670400 \times 10^{-8} \) W \( \cdot \) m\(^{-2} \) \( \cdot \) K\(^{-4} \). The values of other constants are among the given data in Exercise 1.5.3.

1.5. Solutions

1.5.1. Solution 1 – Calculation of the Stefan–Boltzmann constant

(1) Expression of flux and total exitance of the black body

An emissive surface following Lambert’s law radiates the light intensity:

\[
d^2 \Phi = \pi LdS \tag{1.28}
\]

The total exitance is \( M = d^2 \Phi/dS \). Hence for the black body:

\[
M^0 = \pi L_0 \tag{1.29}
\]

(2) Expression of Stefan–Boltzmann law

In the wavelength range \([\lambda, \lambda + d\lambda]\), radiance \( L_0 \) of the black body is given by the integral of spectral radiance \( L_0^0(T) \) on all the wavelengths, which is:

\[
L_0 = \int_0^\infty L_0^0(T)d\lambda \tag{1.30}
\]

Using [1.29], the total exitance \( M^0 \) of the black body is written as:

\[
M^0 = \pi \int_0^\infty L_0^0(T)d\lambda \tag{1.31}
\]
The spectral radiance is given by the following expression:

\[
L^0_\lambda(T) = \frac{2\hbar c^2}{\lambda^5} \frac{1}{e^{\hbar c / \lambda k T} - 1}
\]

Equation [1.31] becomes:

\[
M^0 = 2\pi \hbar c^2 \int_0^\infty \frac{1}{\lambda^5} \frac{1}{e^{\hbar c / \lambda k T} - 1} d\lambda \quad [1.32]
\]

Let us consider: \( x = \hbar c / \lambda k T \Rightarrow dx = - (\hbar c / \lambda^2 k T) d\lambda \). After simplification, [1.32] gives:

\[
M^0 = \frac{2\pi}{\hbar^3 c^2} k^4 T^4 \int_0^\infty \frac{x^3}{e^x - 1} dx \quad [1.33]
\]

Taking the following result into account:

\[
\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}
\]

leads to:

\[
M_0 = \frac{2\pi^5 k^4}{15\hbar^3 c^2} T^4 = \sigma T^4 \quad [1.34]
\]

(3) Calculation of the Stefan–Boltzmann constant

According to [1.34], the Stefan–Boltzmann constant can be written as:

\[
\sigma = \frac{2\pi^5 k^4}{15\hbar^3 c^2} \quad [1.35]
\]

N.A.– \( \sigma = 5.67040400 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4} \).

The accepted value of the Stefan–Boltzmann constant is actually found: \( \sigma = 5.670400 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4} \). Therefore, the result [1.34] corresponds to the Stefan–Boltzmann law.
1.5.2. Solution 2 – Calculation of the Sun’s surface temperature

(1) Expression of the power $P$ radiated by a solar surface

The power radiated by the solar surface $S = 4\pi R^2$ per unit time is $P = SM_0$, or according to the Stefan–Boltzmann law [1.17]:

$$P = 4\pi R^2 \sigma T^4$$  \[1.36\]

(2) Schematization and expression of the power $P_0$ received by the Earth disc

– Schematization: see Figure 1.12.

– Expression of power $P_0$.

The total power [1.36] is radiated in the sphere of radius $d$ and surface $S'$ (Figure 1.12). The power $P_0$ received by the Earth disc of surface $S_0$ is ($P \rightarrow S'$; $P_0 \rightarrow S_0$):

$$P_0 = P \times \frac{S_0}{S'}$$  \[1.37\]

![Figure 1.12. Solar radiation sphere of radius $d$ and surface $S'$](image)

Taking [1.36] into account and knowing that $S' = 4\pi d^2$, relation [1.37] can be written as:

$$P_0 = R^2 \sigma T^4 \times \frac{\pi R_0^2}{d^2}$$  \[1.38\]
(3) *Expressions of absorbed power $P_a$ and radiated power $P_r$*

– *Expression of absorbed power*

The Earth is assimilated to a gray body whose coefficient of absorption is $\alpha < 1$. Hence, it only absorbs a fraction of power $[1.38]$. Let us consider: $P_a = \alpha P_0$. Using $[1.38]$ leads to:

$$P_0' = \alpha \pi R^2 \sigma T^4 \frac{R_0^2}{d^2}$$  \[1.39\]

– *Expression of radiated power*

If the Earth is brought to constant temperature $T_0$, the power radiated throughout its spherical surface of radius $R_0$ is:

$$P_r = \alpha 4\pi R_0^2 \sigma T_0^4$$  \[1.40\]

(4) *Expression of the Sun’s surface temperature, application*

When in thermodynamic equilibrium, the Earth radiates as much as it absorbs. Hence $P_a = P_r$. Placing expressions $[1.39]$ and $[1.40]$ in this equality leads, after simplification, to:

$$T = \left(4 \frac{d^2}{R^2}\right)^{\frac{1}{4}} T_0 = \sqrt{2 \frac{d}{R} \times T_0}$$  \[1.41\]

N.A.– $T = 6,000$ K.

In fact, the temperature of the solar surface (photosphere) is slightly below 6,000 K. Its precise value is 5,800 K.

### 1.5.3. Solution 3 – Average energy of a quantum oscillator, Planck’s formula

(1) *Relation between $E_n$ and $\omega$*

According to Planck’s hypothesis, the energy of a quantum oscillator can only take discrete values:

$$E_n = n\hbar \omega$$  \[1.42\]
In this relation, \( n = 0, 1, 2, \ldots \) and \( h\omega \) is the energy of a quantum of energy.

(2) Physical significance \( N_0 \)

- **Expression of \( \langle E_\omega \rangle(T) \)**

At thermodynamic equilibrium, the number \( N_n \) of elementary oscillators of angular frequency \( \omega \) follows the Boltzmann distribution law or, using [1.42]:

\[
N_n = N_0 e^{-n\beta h\omega}
\]

\[\beta = \frac{1}{kT}\]

In relation [1.43], \( N_0 \) designates the number of elementary oscillators of angular frequency \( \omega \) at ground state of energy \( E_0 = 0 \) \((n = 0)\) of the black body cavity.

(3) **Expression of \( \langle E_\omega \rangle(T) \) as a function of \( N_n \) and \( E_n \)**

The average energy \( \langle E_\omega \rangle(T) \) of a mode can be calculated by the quotient of the discrete sum of energies of the set of elementary oscillators to the total number of oscillators:

- energy of the set of elementary oscillators:

\[
\sum_{n=0}^{\infty} N_n \times E_n
\]

In this expression, \( N_n \) is given by relation [1.43]:

- total number of oscillators:

\[
\sum_{n=0}^{\infty} N_n
\]

Hence, the average energy of the set of modes can be written as:

\[
\langle E_\omega \rangle(T) = \frac{\sum_{n=0}^{\infty} N_n \times E_n}{\sum_{n=0}^{\infty} N_n} \tag{1.44}
\]
(4) Demonstration

Using [1.43] and considering \( a = \hbar \omega \), expression [1.44] becomes:

\[
\langle E_\omega \rangle (T) = \frac{\sum_{n=0}^{\infty} n e^{-n \beta \omega}}{\sum_{n=0}^{\infty} e^{-n \hbar \omega \beta}} = a \frac{\sum_{n=0}^{\infty} n x^n}{\sum_{n=0}^{\infty} x^n} \quad \text{[1.45]}
\]

Since \( x < 1 \), we have at infinity:

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n + O(x^{n+1})
\]

Hence:

\[
\frac{d}{dx} \left( \frac{1}{1-x} \right) = 1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{1}{(1-x)^2}
\]

Taking these results into account, expression [1.45] is written as:

\[
\langle E_\omega \rangle (T) = a \times \frac{x + 2x^2 + 3x^3 + 4x^4 + \ldots}{1 + x + x^2 + x^3 + x^4 + \ldots} = a \times \frac{1 + 2x + 3x^2 + 4x^3 + \ldots}{1 + x + x^2 + x^3 + x^4 + \ldots} \quad \text{[1.46]}
\]

Using these last relations, [1.46] can be written as:

\[
\langle E_\omega \rangle (T) = a \times \frac{(1-x)}{(1-x)^2} = a \times \frac{x}{(1-x)}
\]

or by multiplying the numerator and denominator by \( x^{-1} \):

\[
\langle E_\omega \rangle (T) = \frac{a}{x^{-1} - 1} \quad \text{[1.47]}
\]
(5) Expression of \( \langle E_\omega \rangle(T) \) as a function of \( \omega, h \) and \( T \)

Let us consider: \( a = h \omega, \quad x = e^{-\beta a}, \quad \beta = 1/kT \). The average energy of the set of quantum oscillators is then written according to [1.47]:

\[
\langle E_\omega \rangle(T) = \frac{\hbar \omega}{\left( e^{\hbar \omega / kT} - 1 \right)}
\]  

[1.48]

Moreover, spectral energy density is given by the following relation:

\[
u(\omega) = n(\omega) \times \langle E_\omega \rangle(T)
\]  

[1.49]

(6) Planck’s law

The density of modes is given by the relation \( n(\omega) = \omega^2 / \pi^2 c^3 \). Planck’s law is written as:

\[
u(\omega) = \frac{\hbar \omega^3}{\pi^2 c^3} \times \frac{1}{\left( e^{\hbar \omega / kT} - 1 \right)}
\]  

[1.50]

1.5.4. Solution 4 – Deduction of Wien’s law from Planck’s law

Using Planck’s formula, spectral exitance \( M_\lambda \) (or spectral radiance exitance) can be written (for a vacuum) in the form [1.51]:

\[
M_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}
\]

(1) Expression of spectral exitance

In a medium of refractive index \( n_\lambda \), spectral exitance \( M_\lambda \) can be written as:

\[
M_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda c_\lambda kT} - 1}
\]  

[1.52]

with \( c_\lambda = c/n_\lambda \), where \( c \) is the speed of light in a vacuum, \( n_\lambda = 1 \). This leads to [1.51].
(2) **Spectral exitance at the peak of light emission of the black body**

At the peak of emission of the black body, spectral exitance is maximal. To put it differently, for $\lambda = \lambda_{\text{max}}$, $M_{\lambda} = \text{constant}$. This is mathematically expressed by the following relation:

$$\frac{dM_{\lambda}}{d\lambda} \bigg|_{\lambda = \lambda_{\text{max}}} = 0$$ \[1.53\]

(3) **Deduction of Wien’s law**

Let us consider $x = hc / \lambda$. In this case, Planck’s law [1.51] can be written as:

$$M_{\lambda} = \frac{2}{h^4 c^3 \beta^5} \times \frac{x^5}{e^x - 1}$$ \[1.54\]

Differentiating expression [1.54] and applying condition [1.53] lead to the required result. It is however simpler to differentiate the inverse of spectral exitance:

$$\frac{d}{d\lambda} \left( \frac{1}{M_{\lambda}} \right) = \frac{h^4 c^3 \beta^5}{2} \times \frac{d}{d\lambda} \left( \frac{e^x - 1}{x^5} \right) = \frac{h^4 c^3 \beta^5}{2} \times \left( \frac{x^5 e^x - 5x^4 (e^x - 1)}{x^{10}} \right)$$

If the result obtained is minimized with respect to $\lambda$, this leads to:

$$\frac{x^5 e^x - 5x^4 (e^x - 1)}{x^{10}} = 0 \Rightarrow xe^x - 5(e^x - 1) = 0$$

The division by $e^x / x^5$ leads to:

$$e^{-x} + \frac{1}{5} x - 1 = 0$$

This equation has a unique solution $x_0 = 4.9651$. Knowing that $x = hc / \lambda$, then:

$$x_0 = \frac{hc \beta}{\lambda_{\text{max}}} = \frac{hc}{kT \lambda_{\text{max}}}$$

or:

$$\lambda_{\text{max}} T = \frac{hc}{kx_0} = \text{Cst}$$ \[1.55\]

Let us find the value of constant Cst in [1.55].
Given \( k = 1.3806504 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} \), \( h = 6.62606896 \times 10^{-34} \text{ J} \cdot \text{s} \) and \( c = 2.99792458 \times 10^8 \text{ m} \cdot \text{s}^{-1} \), the result is: \( \text{Cst} = 2.89776802 \times 10^{-3} \text{ m} \cdot \text{K} \).

But the Wien constant is \( \sigma_W = 2.898 \times 10^{-3} \text{ m} \cdot \text{K} \). It can be noted that \( \text{Cst} = \sigma_W \).

Hence, relation [1.55] can be written as:

\[
\lambda_{\text{max}} T = \sigma_W.
\]

which actually corresponds to Wien’s first law [1.18].

(4) Estimation of Sun’s surface temperature

Using Wien’s first law [1.18] leads to:

\[
T = \frac{\sigma_W}{\lambda_{\text{max}}}.
\]  \[1.56\]

N.A.– \( \lambda_{\text{max}} = 5 \times 10^{-7} \text{ m} \); \( \sigma_W = 2.898 \times 10^{-3} \text{ m} \cdot \text{K} \). This leads to: \( T = 5,798 \text{ K} \).

1.5.5. Solution 5 – Total electromagnetic energy radiated by the black body

(1) Expression of the flux \( d^2\Phi \) and the electromagnetic energy \( d^3E \)

The flux contained in the solid angle \( d\Omega \) and of direction \( Ox \) making an angle \( \theta \) with the normal \( \vec{n} \) to the surface \( dS \) is given by the following relation:

\[
d^2\Phi = AdSd\Omega
\]  \[1.57\]

As the surface \( dS \) follows Lambert’s law, then \( A = L\cos\theta \). Hence [1.57] can be written as:

\[
d^2\Phi = L\cos\theta dSd\Omega
\]  \[1.58\]

In this relation, \( L \) designates the radiance of the emissive surface.

For an elementary duration \( dt \), the energy \( d^3E \) exiting an orifice of area section \( dS \) of black surface in the solid angle \( d\Omega \) is such that \( d^3E = d^2\Phi dt \). Or considering [1.58] this leads to:

\[
d^3E = L_0\cos\theta dSd\Omega dt
\]  \[1.59\]
Along the normal to the orifice ($\theta = 0$), we have:

$$d^3E = L_0dSd\Omega dt$$  \[1.60\]

(2) Expression of the energy density per unit volume

Electromagnetic radiation is constituted by photons moving at speed $c$. Light energy [1.60] is contained in an elementary cylinder of height $h = cdt$ and volume $d\tau = dS h = c dS dt$. Energy density per unit volume $du$ is then given by the following relation:

$$d^3E = du d\tau = ducdS dt$$  \[1.61\]

Equalization of [1.60] and [1.61] leads to:

$$du = \frac{d\Omega}{c} L_0$$  \[1.62\]

(3) Demonstration, calculation of $\sigma^*$

Over the whole space, $\int d\Omega = 4\pi$. Knowing that $L_0$ depends on temperature, the total energy per unit volume $U(T)$ radiated by the black body can be written according to [1.62]:

$$U(T) = \frac{4\pi}{c} L_0$$  \[1.63\]

According to the Stefan–Boltzmann law, the total energy exitance of the black body is $M = \pi L_0 = \sigma T^4$. Black body radiance is expressed as a function of temperature as follows:

$$L_0 = \frac{\sigma}{\pi} T^4$$  \[1.64\]

Putting [1.64] in [1.63], total energy per unit volume $U(T)$ radiated by the black body can be written as:

$$U(T) = \frac{4\pi}{c} \times \frac{\sigma}{\pi} T^4 \Rightarrow U(T) = \frac{4\sigma}{c} T^4 = \sigma^* T^4$$  \[1.65\]

with:

$$\sigma^* = \frac{4\sigma}{c}$$  \[1.66\]
– *Calculation of constant* $\sigma^*$

$$c = 2.99792458 \times 10^8 \text{ m} \cdot \text{s}^{-1}; \sigma = 5.670400 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4}$$

Hence according to [1.66]: $\sigma^* = 7.565767 \times 10^{-16} \text{ W} \cdot \text{s} \cdot \text{m}^{-3} \cdot \text{K}^{-4} = 7.565767 \times 10^{-16} \text{ J} \cdot \text{m}^{-3} \cdot \text{K}^{-4}$.

(4) *Calculation of the total energy per unit volume radiated by the black body*

Using [1.65] leads to the numerical result $T = 6000 \text{ K}$; $U = 0.98 \text{ J}$. 