INTEREST RATE

Interest rate is the basis for financial transactions. Not only is there one rate at a given time, but the rate depends on the duration or time to maturity. This has interesting consequences as we shall see in Section 1.2. We shall in Section 1.1 start with the simple case when the interest rate is *flat*, which means that it is the same regardless of its duration.

One essential thing in this chapter is to give an understanding of (a) the connection between present value and arbitrage and (b) how to construct arbitrage.

1.1 FLAT RATE

1.1.1 Compound Interest

If the interest is compounded *n* times a year each time with the rate \( r/n \), the value of one EUR after *t* years equals

\[
F = P \left(1 + \frac{r}{n}\right)^{nt}
\]
\[
R = \left(1 + \frac{r}{n}\right)^n \quad \text{for } n = 1, 2, 3, \ldots \quad \text{and} \quad R = \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^{rt}
\]
when the interest is compounded continuously.

For \( r = 5\% \) and \( t = 1 \) these values are given by Table 1.1. These values thus depend

<table>
<thead>
<tr>
<th>( n )</th>
<th>Compounding every</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>year</td>
<td>1.05</td>
</tr>
<tr>
<td>2</td>
<td>half year</td>
<td>( \left(1 + \frac{0.05}{2}\right)^2 )</td>
</tr>
<tr>
<td>4</td>
<td>quarter</td>
<td>( \left(1 + \frac{0.05}{4}\right)^4 )</td>
</tr>
<tr>
<td>12</td>
<td>month</td>
<td>( \left(1 + \frac{0.05}{12}\right)^{12} )</td>
</tr>
<tr>
<td>52</td>
<td>week</td>
<td>( \left(1 + \frac{0.05}{52}\right)^{52} )</td>
</tr>
<tr>
<td>365</td>
<td>day</td>
<td>( \left(1 + \frac{0.05}{365}\right)^{365} )</td>
</tr>
</tbody>
</table>

not only on \( r \) but also on \( n \). The essential thing here is the growth factor, \( R \). This can also be expressed by the interest rate, \( r \), but then we must specify which interest rate we have in view. The most common is perhaps to define the rate as the return

\( r_1 = R - 1. \)

**EXERCISE 1.1.** Show that if the return equals \( r' \) for one part of a period and \( r'' \) for the remainder, then the return for the entire period equals \( r' + r'' + r'r'' \).

The return is thus not additive, but this holds, however, for the continuous rate:

\( r_\infty = \ln R. \)

**EXERCISE 1.2.** Show that if the continuous rate equals \( r' \) for one part of a period and \( r'' \) for the remainder, then the continuous rate for the entire period equals \( r' + r'' \).

We have \( R = e^{rt} \). The continuous rate can therefore also be defined as the instantaneous return per time unit:

\[
\lim_{t \to 0} \frac{e^{rt} - 1}{t} = r_{\infty}.
\]
1.1.2 Present Value

$X_0$ EUR today is worth $X_T$ EUR in $T$ years. Here

$$X_T = P_Y X_0$$

the future value of $X_0$ and

$$X_0 = d_T X_T$$

is the present value of $X_T$. Here $P_Y$ is the growth factor during $T$ years and

$$d_T = P_Y^{-1}$$

is the discount factor. We shall also write

$$X_0 = PV(X_T).$$

In order to value future payments, one compares their present values.

EXERCISE 1.3. Compare the value of 417 EUR in one year and 430 EUR in two years with 395 EUR today if the yearly return equals 5%.

EXERCISE 1.4. Express the time it takes to double a capital as a function of the continuous rate. In particular: How long will it take when the interest rate equals 5%? We shall sometimes also write $d(0, T)$ instead of $d_T$, and, in general,

$$d(t, T) = d_T / d_t$$

for the discount factor from time $t$ to time $T$.

1.1.3 Cash Streams

A cash stream is a sequence of real numbers, $x = (x_0, x_1, \ldots, x_n)$, together with a sequence of points of time $0 = t_0 < t_1 < \cdots < t_n$. The holder of the cash stream receives $x_t$ EUR at time $t$. (This means that the holder pays $|x_t|$ EUR if $x_t < 0$.) The opposite party, the writer of the cash stream, holds the cash stream $- x$.

The payments are thus divided into $n$ periods: $(t_{i-1}, t_i)$, $i = 1, \ldots, n$.

Here are three examples of cash streams:

Loan. You borrow today $S$ EUR and pay back $K$ EUR at the end of every period. This corresponds to the cash stream
\( (S, -K, \ldots, -K) \).

**Saving.** You put \( K \) EUR in the bank at the beginning of every period, and you take out the entire amount at the end of the last period. This gives the cash stream

\( (-K, \ldots, -K, S) \).

**Annuity.** You put in \( S \) EUR today and receive \( K \) EUR at the end of each period. This gives the cash stream

\( (-S, K, \ldots, K) \).

This is also the cash stream of the lender when you take a loan.

Unless otherwise stated, we shall assume that the periods have the same lengths: \( t_0 = 0, t_1 = 1, \ldots, t_n = n \) in some unit; day, month, or year for example. It is always possible to achieve this by letting \( x_k = 0 \) for certain \( k \). The discount factor per period is in this case denoted by \( d \). The present value of the cash stream is therefore given by

\[ PV(x) = x_0 + dx_1 + \cdots + d^n x_n. \]

**EXERCISE 1.5.** You receive 2000 EUR every year for 10 years with the first payment in one year. Calculate the present value of this cash stream if the rate of return is 5% per year.

**EXERCISE 1.6.** A certain crop will give 1.05 EUR back for each invested EUR when harvested after one year. When the crop is harvested after two or three years, the corresponding figures are 1.11 and 1.14, respectively. Compare these cash streams, assuming that the entire income will be reinvested in planting.

### 1.1.4 Effective Rate

The effective rate is the rate for which the cash stream has the present value 0; it is therefore determined by the discount factor for which \( PV(x) = 0 \), provided that this is uniquely determined.

**EXERCISE 1.7.** Show that if \( x_0 > 0 \) and \( x_i < 0 \) for \( i = 1, \ldots, n \) (or if \( x_i < 0 \) for \( i = 0, \ldots, n-1 \) and \( x_n > 0 \)), then the discount factor is uniquely determined. Also show that the rate is positive \( (d < 1) \) in these cases if and only if

\[ |x_0| < \sum_{k=1}^{n} |x_k|, \text{ (or } |x_0| < \sum_{k=0}^{n-1} |x_k| \). \]
Let \( m \) denote the number of periods per year. The yearly discount factor then equals \( d^m \) and the continuous rate therefore equals

\[
\ln \frac{1}{d}
\]

per year, whereas the yearly return equals

\[
\frac{1}{d^m} - 1.
\]

**Exercise 1.8.** You borrow 1000 EUR and repay the loan by paying 507 EUR after one and two months. Determine the effective rate.

**Exercise 1.9.** Determine the effective rates for the cash streams in Exercise 1.6.

**Exercise 1.10.** Show that the effective rate for the loan and the saving above are given by discount factors that satisfies

\[
d^{\frac{1}{1-d}} - \frac{S}{K} \text{ respectively } d^{-n} \frac{1}{1-d} = \frac{S}{K}.
\]

In order to solve \( d \) from equations of this type, one can use Newton’s method to find zeros of a differentiable function, \( F(x) \): Start with a number \( x_0 \) which you think is close to the zero. Calculate then \( x_1, x_2, \ldots \) via the formula

\[
x_k = x_{k-1} - \frac{F(x_{k-1})}{F'(x_{k-1})},
\]

for \( k = 1, 2, \ldots \). This sequence converges toward a zero of \( F \). For each iteration the number of correct decimals will double.

**Exercise 1.11.** Show that \( x_2 \) is the point in which the tangent of \( F \) in the point \( x_1 \) cuts the \( x \)-axis, and use this to illustrate the construction of \( x_1, x_2, \ldots \) graphically.

**Exercise 1.12.** You borrow 1000 EUR and repay by paying 338 EUR per month in three months. What is the effective rate?

**Exercise 1.13.** You borrow 20000 EUR in a bank and pay 300 EUR at the end of every month. The effective rate is given by 0.7% return per month. What is the yearly return? Determine the time it takes to pay back the loan. How much do you have to pay per month in order to repay the entire loan in 5 years?

### 1.1.5 Bonds

A bond is a cash stream of the form
\[ P, e/m, \ldots, e/m, e/m + F. \]

Here the period length is 1/m years. The holder of the bond thus receives \(e/m\) EUR \(m\) times a year in \(T = n/m\) years. \(T\) is the time to maturity, \(e\) the coupon, \(F\) the face value, and \(P\) the price. A zero-coupon bond is a bond with \(e = 0\). A T-bill is a "zero-coupon bond" with maturity in at most a year.

The effective rate per year therefore is determined by the discount factor \(d^m\), where \(d\) satisfies

\[ P = \sum_{k=1}^{n} d^k F. \]

It is seen from this expression that the price is a decreasing function of the interest rate. Hence the prices of bonds decreases when the rate increases.

**EXERCISE 1.14.**

(a) Show that

\[ P = \frac{e}{m} \left( \frac{1 - d^m}{1 - d} \right) + d^m F. \]

(b) Define \(y\) by

\[ d = \frac{1}{1 - \frac{e}{m}}. \]

That is, \(y\) is the return during a period of length 1/m multiplied by \(m\). Show that

\[ P = \frac{e}{y} - d^m \left( F - \frac{e}{y} \right). \]

The number \(y\) is called the yield of the bond. The above expression becomes especially simple when the yield is par, \(e = yF\); \(P = F\).

**EXERCISE 1.15.** Let \(P_1\) and \(P_2\) denote the prices of two bonds where the second has longer time to maturity than the first, but otherwise the two bonds are equal (the same coupon, yield, face value, and period). Show that

\[ P_1 < P_2 \text{ for } y < e/F \quad \text{and} \quad P_1 > P_2 \text{ for } y > e/F. \]

**EXERCISE 1.16.** Calculate the effective rate for a five-year bond with face value 100 EUR, and yearly coupon payment 4 EUR. There are four coupon payments per year, and the price of the bond is 100 EUR.

One can form new cash streams by composing bonds into a portfolio.

**EXERCISE 1.17.** Consider two bonds having the same maturity date, period length and face value. One has the coupon \(e_1\) and the other \(e_2\), \(e_1 < e_2\). The prices are \(P_1\) respectively \(P_2\).
(a) Use these bonds to construct a bond that has the coupon \( r \), but the same face value. What is the price of this bond?

(b) Determine the weights of the bonds in the portfolio so that the result becomes a zero-coupon bond.

(c) For which values of \( r \) have both bonds positive weights in the portfolio?

### 1.1.6 The Effective Rate as a Measure of Valuation

The effective rate is a blunt tool to use to value cash streams in general. Consider the cash stream \( x = (ab, -a, -b, 1) \).

The present value of this is

\[
PV = ab + d(a - b) + d^2 = (d - a)(d - b).
\]

This present value equals zero for \( d = a \) and \( d = b \). The effective rate is therefore not uniquely determined when \( a \neq b \). Furthermore, the present value of the cash stream \(-x\) has the same zeros. It is therefore not clear how to use the effective rate to determine which of the two cash streams \( x \) and \(-x\) that is preferable (if any).

Assume that \( a = 1 \) and \( b = 3 \): \( x = (3, -4, 1) \). In this case the present value equals zero when \( d = 1 \) or \( d = 3 \). In the former case the effective rate equals zero, in the latter it is negative. The present value is positive for \( x \) and negative for \(-x\) when \( d < 1 \), which holds for normal interest rates. Therefore the cash stream \( x \) ought to be preferable to \(-x\).

Assume that the time between the payments is one year, that you can borrow money at the (return)rate 5% per year, and that you can lend at the rate 4%. The following procedure shows that it is advantageous to hold \( x \):  

At \( t = 0 \): Accept the cash stream \( x \). You receive 3 EUR which you lend for one year at the rate 4%.

At \( t = 1 \): The loan is repaid, and you get 3 \( \times 1.04 = 3.12 \) EUR. You borrow 0.88 EUR for one year and pay 4 EUR.

At \( t = 2 \): You receive 1 EUR and repay the loan with 0.88 \( \times 1.05 = 0.924 \) EUR. Remains 0.076 EUR.

In this way you get the cash stream \((0, 0, 0.076)\), and you have made a profit without taking any risk. That is, you have made arbitrage.

**EXERCISE 1.18.** In which way shall the lending rate be related to the interest on deposit in the above example in order to make it possible to make arbitrage in this way?
EXERCISE 1.19. You are offered two cash streams: \((1000, -3000, 2000)\) and \((-1000, 3000, -2000)\). The time between the payments is one year.

(a) Determine the effective rates of the cash streams.

(b) You can borrow at 5% per year and lend at 4%. Describe how you can make arbitrage.

1.2 DEPENDENCE ON THE MATURITY DATE

We shall now take into account that the interest rate varies with the maturity date, and we shall investigate the consequences of this fact.

1.2.1 Zero-Coupon Bonds

The holder of a \(k\)-year zero-coupon bond receives 1 EUR after \(k\) years: Let \(d_k\) denote the price of the \(k\)-year zero-coupon bond, \(k = 1, 2, \ldots, n\), and put \(d_0 = 1\). The price \(d_k\) defines the value today (the present value) of 1 EUR in \(k\) years. These prices also define the \(k\)-year spot rate, \(r_k\), by the relation
\[
d_k = e^{-kr_k},
\]
provided that we use the continuous rate. If instead \(r_k\) stands for the yearly returns, we have the relation
\[
d_k = (1 + r_k)^{-k}.
\]

Here we shall simplify the reasoning and assume that we can buy and sell these bonds in arbitrary quantities without transaction costs. Thus, for example, if we want to sell a fraction of a bond we don’t have, we can borrow this fraction without cost and sell it in order to buy it back later for a price that hopefully is lower (short-selling or shorting).

1.2.2 Arbitrage-Free Cash Streams

The present value of the cash stream \(x = (x_0, x_1, \ldots, x_n)\) is
\[
P_0(x) = \sum_{k=0}^{n} d_k x_k.
\]

Let \(x'\) denote the cash stream one gets by proceeding in the following way: Buy at \(t = 0\) \(x_k\) \(k\)-year bonds (this means that one sells \(-x_k\) if \(x_k < 0\), for \(k = 1, \ldots, n\).
Wait until \( t = n \). The cost at \( t = 0 \) is
\[
\sum_{k=1}^{n} x_k d_k = PV(x) - x_0
\]
and one receives \( x_k \) EUR at \( t = k \), \( k = 1, 2, \ldots, n \). Therefore
\[
x' = x - p, \text{ where } p = (PV(x), 0, \ldots, 0).
\]
Assume that \( PV(x) > 0 \). By accepting the cash streams \( x \) and \( -x' \), one gets the cash stream \( p = x - x' \) and hence a risk-free profit. If instead \( PV(x) < 0 \), one gets the cash stream \( -p \) by accepting \( -x \) and \( x' \) and hence a risk-free profit even in this case.

This is called arbitrage. If we assume that it is not possible to make arbitrage, we thus must have \( PV(x) = 0 \) for all cash streams. Observe that a geometrical way to express this is to say that \( x \) is orthogonal to the discount vector \( d := (d_0, d_1, \ldots, d_n) \):
\[
x \cdot d = 0.
\]

**Exercise 1.20.** You intend to borrow 1000 EUR and have to choose between the following two alternatives: \( x = (1000, -866, -181) \) and \( y = (1000, -426, -656) \). The first repayment is in one year, and the second one is in two years. The one-year zero-coupon bond costs 0.97 EUR, and the two-year bond costs 0.86 EUR.

(a) Calculate the effective rates of the two loans.
(b) Calculate the present values of the two cash streams. Also calculate the one- and two-year spot rates.
(c) The loan \( y \) this is preferable despite the fact that it has higher effective rate than \( x \). Describe how to make arbitrage by accepting \( y \).
(d) Also describe how the lender can make arbitrage if you accept the loan \( x \).

### 1.2.3 The Arbitrage Theorem

Forget for a moment the concrete interpretation of \( d \) as prices of bonds. We shall here instead show that there is a unique discount vector in each market that satisfies certain conditions. Let \( x_1, x_2, \ldots, x_N \) be given cash streams in \( \mathbb{R}^{n+1} \), and let \( L \) denote the vector space generated by these. We shall say that there are arbitrage opportunities if there is a \( x \) in \( L \) such that \( x \neq 0 \) and \( x > 0 \) (the latter means that \( x_k > 0 \) for all \( k = 0, 1, \ldots, n \)). That \( L \) has no arbitrage opportunities can therefore be expressed in the following way:
\[
L \cap \mathbb{R}^{n+1}_+ = \{0\}.
\]
In this case we must have \( \dim(L) \leq n \). We shall also say that the market (i.e., \( L \)) is complete if \( \dim(L) = n \).
Theorem 1.1. The market $L$ is complete and has no arbitrage opportunities if and only if there is a $d$ in $\mathbb{R}^{n+1}$ with $d_0 = 1$ and $d_1 > 0, d_2 > 0, \ldots, d_n > 0$ such that $L = \{x; x \cdot d = 0\}$. The discount vector $d$ is uniquely determined by $L$.

Proof. The theorem is geometrically obvious when $n = 1$ and 2. Isn't it?

Assume that $L$ is complete and has no arbitrage opportunities. Let $c \neq 0$ be in the orthogonal complement of $L$; that is, $c \cdot x = 0$ for all $x$ in $L$. Then $c$ is uniquely determined up to a multiplicative constant and $L = \{x; c \cdot x = 0\}$ since $L$ is complete.

If $c_k = 0$, then $c_k \cdot c = 0$. Hence $c_k \in L$, a contradiction. Hence $c_k \neq 0$ for all $k$. Assume that not all $c_k$ have the same sign. Then there are indices $i$ and $j$ such that $c_i > 0$ and $c_j < 0$. Put $u = c_i e_j - c_j e_i$. Then $u \neq 0$, $u \cdot c = 0$; that is $u \in L$, a contradiction. It follows that all $c_k$ have the same sign.

The only it part now follows with $d_k = c_k/c_0$. Conversely, it is clear that if $x \cdot d = 0$, then it is impossible that $x \neq 0$ and $x \cdot d > 0$.

Assume that $L$ is complete and has no arbitrage opportunities and that a new cash stream of the form $(-p, a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are given numbers, is introduced on the market. This extended market is thus free of arbitrage if and only if $p = d_1 a_1 + \cdots + d_n a_n$.

The zero-coupon bonds $b_1 = (-d_1, 1, 0, 0, \ldots, 0, 0), b_2 = (-d_2, 0, 1, 0, \ldots, 0, 0), \ldots, b_n = (-d_n, 0, 0, 0, \ldots, 0, 1)$ is a basis for $L$. Aren't they?

EXERCISE 1.21. The market $L_1$ is generated by the two vectors $(5, -6, -6)$ and $(5, -5, -4)$. $L_2$ is generated by $(5, -6, -6)$ and $(-1, 0, 3)$, and $L_3$ is generated by $(5, -4, -2)$ and $(-8, 6, 3)$.

(a) Decide which of these markets that have no arbitrage opportunities.

(b) An additional cash stream of the form $(-p, 2, 3)$ is introduced on the market $L_2$. Determine its price (i.e., determine $p$) such that the extended market doesn't have any arbitrage opportunities.

EXERCISE 1.22. The market $L$ is generated by the cash stream $(2, -2, -1)$ and is therefore not complete. Another cash stream of the form $(-p, 1, 1)$ is going to be introduced. For which values of $p$ will the extended market be complete? Without arbitrage opportunities?

EXERCISE 1.23. Consider the following three bonds where the period length is one year: $A = (-P_A, 110, 0, 0)$, $B = (-P_B, 10, 110, 0)$, $C = (-P_C, 10, 10, 110)$.

(a) Determine the price, $P_C$, and the effective rate (yield) for the bond $C$ if the spot rates (measured by the yearly returns) are as follows: 1 year = 7%, 2 year = 9%, 3 year = 11%.

(b) Determine the prices and the spot rates if $A, B,$ and $C$ have the effective rates 5%, 9%, and 11%, respectively.
1.2.4 The Movements of the Interest Rate Curve

If we plot the yield of government bonds as a function of the maturity date, we will get a curve called the yield curve. The spot rate curve is the spot rate as a function of the maturity date. The two curves are similar but not identical because the latter is based only on zero-coupon bonds whereas the former is also based on coupon bonds. The spot rate curve is the relevant one for our purposes, but the yield curve is easier to determine. These curves are not constant but varies with time.

Frye (1997) made a statistical study of the daily changes of the yield of ten American government securities during 1543 days from 1989 and 1995. The result was

\[ \partial r = \xi_1 a_1 + \cdots + \xi_{10} a_{10}. \]

Here \( \partial r = (\partial r_1, \ldots, \partial r_{10}) \) stands for the daily change of rate. The vectors \( a_1, \ldots, a_{10} \) are given in Table 1.2.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>3 mo</th>
<th>6 mo</th>
<th>1 yr</th>
<th>2 yr</th>
<th>3 yr</th>
<th>4 yr</th>
<th>5 yr</th>
<th>7 yr</th>
<th>10 yr</th>
<th>30 yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>0.21</td>
<td>0.25</td>
<td>0.32</td>
<td>0.35</td>
<td>0.36</td>
<td>0.36</td>
<td>0.36</td>
<td>0.34</td>
<td>0.31</td>
<td>0.25</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>-0.57</td>
<td>-0.49</td>
<td>-0.32</td>
<td>-0.10</td>
<td>0.02</td>
<td>0.14</td>
<td>0.17</td>
<td>0.27</td>
<td>0.30</td>
<td>0.33</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>0.50</td>
<td>0.23</td>
<td>-0.57</td>
<td>0.38</td>
<td>0.30</td>
<td>-0.12</td>
<td>-0.04</td>
<td>0.15</td>
<td>0.28</td>
<td>0.46</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>0.17</td>
<td>-0.37</td>
<td>-0.58</td>
<td>0.17</td>
<td>0.27</td>
<td>0.28</td>
<td>0.44</td>
<td>0.01</td>
<td>-0.10</td>
<td>-0.34</td>
</tr>
<tr>
<td>( a_5 )</td>
<td>-0.39</td>
<td>0.70</td>
<td>-0.52</td>
<td>0.04</td>
<td>0.07</td>
<td>0.16</td>
<td>0.08</td>
<td>0.00</td>
<td>-0.05</td>
<td>-0.18</td>
</tr>
<tr>
<td>( a_6 )</td>
<td>-0.02</td>
<td>0.01</td>
<td>-0.23</td>
<td>0.59</td>
<td>0.24</td>
<td>-0.08</td>
<td>-0.10</td>
<td>-0.12</td>
<td>0.01</td>
<td>0.33</td>
</tr>
<tr>
<td>( a_7 )</td>
<td>0.01</td>
<td>-0.04</td>
<td>-0.01</td>
<td>0.56</td>
<td>0.79</td>
<td>0.15</td>
<td>0.09</td>
<td>0.13</td>
<td>0.03</td>
<td>0.06</td>
</tr>
<tr>
<td>( a_8 )</td>
<td>0.00</td>
<td>-0.02</td>
<td>-0.05</td>
<td>0.12</td>
<td>0.00</td>
<td>0.55</td>
<td>0.26</td>
<td>-0.04</td>
<td>-0.23</td>
<td>0.52</td>
</tr>
<tr>
<td>( a_9 )</td>
<td>0.01</td>
<td>-0.00</td>
<td>0.00</td>
<td>-0.12</td>
<td>-0.09</td>
<td>-0.14</td>
<td>0.71</td>
<td>0.00</td>
<td>-0.08</td>
<td>0.26</td>
</tr>
<tr>
<td>( a_{10} )</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>-0.05</td>
<td>-0.06</td>
<td>-0.08</td>
<td>0.48</td>
<td>0.08</td>
<td>0.52</td>
<td>-0.18</td>
</tr>
</tbody>
</table>

These are pairwise orthogonal and have the length 1. The stochastic variables \( \xi_1, \ldots, \xi_{10} \) are uncorrelated and ordered after decreasing standard deviations, \( \sigma_1 > \sigma_2 > \cdots > \sigma_{10} \). These are given in Table 1.3.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_i )</td>
<td>1.74</td>
<td>0.35</td>
<td>3.10</td>
<td>2.17</td>
<td>1.97</td>
<td>1.60</td>
<td>1.27</td>
<td>1.24</td>
<td>0.80</td>
<td>0.79</td>
</tr>
</tbody>
</table>

The unit is basis points (bpi), that is, \( 1/100\% = 0.0001 \).

The expected value of \( \partial r \) is negligible compared to the fluctuations, and therefore the standard deviation is the relevant measure.
We therefore have

$$E[\theta r^2] \approx \sigma_1^2 + \cdots + \sigma_{10}^2 = 367.9.$$  

Movements of the yield along $\alpha_1$ explains $17.49^2 / 367.9 = 88\%$ of the total variance, and movements along $\alpha_2$ explains $10\%$, together $98\%$. By adding $\alpha_3$, one reaches $99\%$. In Fig. 1.1 is a plot of $\alpha_1$, $\alpha_2$, and $\alpha_3$. The first corresponds roughly to a parallel shift in the yield curve, the second to steepening; rates with time to maturity shorter than approximately 2 years moves in one direction, and the other rates in the other direction. The third factor corresponds to a bend; short and long yields moves in one direction, and the other rates in the other direction.

![Graph](image)

**Figure 1.1.** The three most important components that explains the movements of the yield curve.

We are not going to use the exact expressions for $\alpha_1$, $\alpha_2$, and $\alpha_3$, but we will approximate these with simple analytical expressions. Reasons for this is (except that is convenient):

1. The investigation comprises one market during one period, and it is not clear that one would get exactly the same result under other circumstances.
2. The investigation concern the yield curve and not the spot rate curve.

3. We shall consider the continuous rate rather than the return, and they are not exactly the same.

Assume that the rate changes from \( r = (r_1, \ldots, r_n) \) to \( r + \partial r \). The analytical expressions that approximate \( \partial r \) can be chosen in different ways. Here we shall first consider parallel shift,

\[
\partial r = 1 \partial p, \quad \text{where } 1 = (1, \ldots, 1)
\]

and then consider steepening,

\[
\partial r = \alpha \partial h.
\]

By also considering changes of the form

\[
\partial r = \rho \partial c, \quad \text{where } \rho^a = (\rho_1^a, \ldots, \rho_n^a),
\]

it is also possible to imitate a certain kind of bending.

By adding \( r^k = (r_1^k, \ldots, r_n^k) \) for \( k = 3, 4, \ldots \) one can increase the precision (but also the complexity) in the approximation of \( \partial r \). At \( k = n = 1 \) the fit will be perfect.

EXERCISE 1.24. Assume that all rates, \( r_1, \ldots, r_n \), are different. Show that \( r^k, k = 0, \ldots, n - 1 \) span \( \mathbb{R}^n \).

It is clear that \( \alpha_3 \) cannot be imitated by a parabola, but this fact isn’t crucial. We shall later use the results in this section to immunize bond portfolios. A possibility is to group the bonds by dividing the dates to maturity in a few intervals and treat each group separately.

In Fig. 1.2 a part of the yield curves for Swedish government securities are shown. The curves are from August 4 and September 4, 2000. We have adapted polynomial of degree 0, "o", degree 1, "+", and degree 2, "++" by least square to the lower curve.

The mean distance between the two yield curves,

\[
d = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (r_i^l - r_i)^2 / \delta_i}
\]

is 31 basis points, whereas the mean distances between the lower curve and the different approximations are 7.6, 4.3, and 3.8, respectively. Parallel shift explains thus the major part of change of rate in this case.

1.2.5 Sensitivity to Change of Rates

The price of a \( T \)-year zero-coupon bond with face value 1 is given by
Figure 1.2. The change of the yield curve, and approximation with polynomials of degree 0, "o", degree 1, "\( O \)”, and degree 2, "\( O^2 \).

\[ P = e^{-rT}, \]

where \( r \) is the T-year spot rate. The change of the price of this bond when \( r \rightarrow r + \delta r \) is given in the next exercise.

**Exercise 1.25.** Show that if one neglects terms of the order \((\delta r)^2\), then the following holds for the price of a zero-coupon bond with time to maturity \( T \):

\[ \frac{\partial P}{\partial r} = -T \delta r. \]

In this case the relative change of the price is proportional to the time to maturity. Zero-coupon bonds with long duration is thus especially sensitive to change of rates.

In order to study the effect of a change of rates \( r \rightarrow r + \delta r \), where \( r = (r_1, \ldots, r_n) \) and \( \delta r = (\delta r_1, \ldots, \delta r_n) \), on a general cash stream \( x = (-P, x_1, \ldots, x_k) \) we shall write...
\[ P(r) \sum_{k=1}^{n} d_k x_k \]

for the present value of the future payments as a function of \( r \). We have

\[ P(r + \partial r) - P(r) + \nabla P(r) \cdot \partial r + O(|\partial r|^2). \]

**EXERCISE 1.26.** Show that

\[ \frac{\partial P}{\partial r_k} = -kd_k x_k. \]

It follows that

\[ \frac{\partial P}{P} = -D \cdot \partial r + O(|\partial r|^2). \]

Here

\[ D = (1, 2, \ldots, n) \text{ and } \nu_k = \frac{d_k x_k}{P}. \]

In particular, if the spot rates change by a parallel shift, then

\[ \frac{\partial P}{P} = -D \partial p + O(|\partial p|^2), \]

where

\[ D = D \cdot 1 = 1, 2, \ldots, n. \]

is the duration of \( \nu \). Note that this is a weighted mean value of the payment time points \( 1, \ldots, n \), and where the weights are proportional to the present values of the payments. Also note that the duration of a zero-coupon bond with time to maturity \( T \) equals \( T \).

**EXERCISE 1.27.** Show that if two bond portfolios have the prices \( P_1 \) and \( P_2 \) and the durations \( D_1 \) and \( D_2 \), then the portfolio consisting of the two bonds has the duration

\[ \frac{P_1}{P_1 + P_2} D_1 + \frac{P_2}{P_1 + P_2} D_2. \]

It follows that a bond portfolio, formed by a number of bonds each having positive weight, will have a duration between the smallest and largest of the durations of the individual bonds.

**EXERCISE 1.28.** Consider the following three bonds where the period length is one year:

- \( A = (-100, 5, 5, 5, 5, 105) \)
- \( B = (-100, 4, 4, 4, 104, 0) \)
- \( C = (-100, 0, 100, 0, 0, 0) \)

Calculate the prices and durations in the case where the (continuous) spot rates are 1.0% \( \sqrt{1.01} \), 1.88% \( \sqrt{1.0192} \), 5.67% \( \sqrt{1.057} \), 5.14% \( \sqrt{1.0519} \), and 5.19%.
EXERCISE 1.29. Consider a bond with face value $F$ and coupon $c$ which is paid $m$ times a year in $T - n/m$ years. The period length thus equals $1/m$ years. Assume that the rate is flat: $d_k = d^k$, where

$$d = e^{-r/m} - \frac{1}{1 - \frac{r}{m}}.$$  

Show that

(a)  

$$D = \frac{1}{m} \left( e^{-c/m} \sum_{k=1}^{m} k d^k + m d^n T \right) / T^2$$

(b)  

$$\sum_{k=1}^{m} k d^{k-1} = \frac{1}{1 - d} \left( \frac{1 - d^{n+2}}{1 - d} - (n + 1) d^n \right)$$

(c)  

$$D = \left( \frac{c}{y} \left( \frac{1}{y} + \frac{1}{n} \right) (1 - d^c) + T \left( F - \frac{c}{y} \right) d^n \right) / F$$

(d)  

$$D = \left( \frac{1}{y} + \frac{1}{n} \right) (1 - d^n)$$

when $c = \mu F$.

If the spot rates change by steepening, $\delta r = r \delta h$, then

$$\frac{\partial P}{P} = D^{(1)} \delta h + O(\|\delta h\|^2),$$

where

$$D^{(1)} = D \cdot r = r_1 v_1 + 2r_2 v_2 + \cdots + n r_n v_n$$

is a weighted mean value of the rates until the different payments.

In a corresponding way the change of rates $\delta r = r^2 \delta r$ will give the change of the price

$$\frac{\partial P}{P} = D^{(2)} \delta h + O(\|\delta h\|^2),$$

where

$$D^{(2)} = D \cdot r^2 = r_1^2 v_1 + 2r_2^2 v_2 + \cdots + n r_n^2 v_n.$$  

Change of rates of the form $\delta r_k = \pi \delta p + \beta r_k \delta h + \gamma r_k^2 \delta r$ will thus give the change of price

$$\frac{\partial P}{P} \approx \pi D \delta p + \beta D^{(1)} \delta h + \gamma D^{(2)} \delta r.$$
EXERCISE 1.30. Calculate \( H^{(1)} \) and \( H^{(2)} \) for the bonds in Exercise 1.28.

Assume that Fig. 1.2 describes the spot rate (and not the yield). In Table 1.4 the relative change of price, \( \partial P / P \), of the three bonds \( A \), \( B \), and \( C \) are given. The values

\[-D \delta p = D \delta p = D^{(1)} \delta b \quad \text{and} \quad -D \delta p = D^{(2)} \delta c\]

are given in the three columns to the right. We have used the least-square estimates shown in the figure. The unit is basis points.

<table>
<thead>
<tr>
<th></th>
<th>( \partial P / P )</th>
<th>Parallel</th>
<th>+Steepening</th>
<th>+Bending</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>105</td>
<td>138</td>
<td>117</td>
<td>106</td>
</tr>
<tr>
<td>( B )</td>
<td>103</td>
<td>115</td>
<td>99</td>
<td>92</td>
</tr>
<tr>
<td>( C )</td>
<td>82</td>
<td>61</td>
<td>67</td>
<td>75</td>
</tr>
</tbody>
</table>

In this case the approximations are improved if one also immunizes against steepening, but only for \( A \) and \( C \) if one, in addition, immunizes against bending.

1.2.6 Immunization

Assume that we already today want to guarantee future payment obligations: \( x_1, x_2, \ldots, x_n \), where \( x_k \) shall be paid at time \( k \). This can also be formulated in the following way: We hold the cash stream

\( x_0 = (0, -x_1, \ldots, -x_n) \)

and want to replace this with a cash stream of the form

\( (-P, 0, \ldots, 0) \).

An imaginable possibility is that today buy \( x_k \) zero-coupon bonds with face values 1 and time to maturity \( k \), for \( k = 1, \ldots, n \), that is, to provide the cash stream

\( x = (-P, x_1, \ldots, x_n) \),

where

\( P = d_1 x_1 + \ldots + d_n x_n \)

is the price we have to pay for the bonds. Our payment obligations now have been reduced to

\( x_0 - x = (-P, 0, \ldots, 0) \).
and we have thus eliminated our future payment obligations, provided that the bonds are risk-free.

It may happen that there are no zero-coupon bonds with exactly these dates of maturity, or the payments may be so many and small that the above procedure is inconvenient.

An alternative in this case is to construct a bond portfolio, $y$, that consists of fewer bonds and that may have different dates of maturity, but they are chosen in such a way that $y$ has the same present value as $x$ and reacts in a similar way on changes of rates.

We have

$$ P_y(r + \partial r) \cdot P_x(r + \partial r) = P_y^2(r) - P_x^2(r) + (\nabla P_y(r) - \nabla P_x(r)) \cdot \partial r + O(\|\partial r\|^2) $$

and therefore

$$ P_y(r + \partial r) - P_y(r - \partial r) = O(\|\partial r\|^2) $$

if

$$ P_y(r) = P_x(r) \quad \text{and} \quad (\nabla P_y(r) - \nabla P_x(r)) \cdot \partial r = 0. $$

If we want to immunize $y - x$ against parallel shifts in the spot rate curve, then the second condition takes the form

$$ D_y = D_x $$

whereas the identity

$$ P^{(1)}_y = P^{(1)}_x $$

immunizes against steepening.

The portfolio must be rebalanced at the first payment and possibly earlier if the change of rate is considerable.

Assume that the portfolio $y$ is composed by the bonds $b_1, \ldots, b_l$ and that $a_k$ is the number of the bond $b_k$, $k = 1, \ldots, l$, in the portfolio:

$$ y = a_1 b_1 + \cdots + a_l b_l. $$

Let $P_k, D_k$, etc., denote the prices, durations, etc., for $b_k$. The equations above then take the form

$$ a_1 P_1 + \cdots + a_l P_l = P_x $$

$$ a_1 D_1 + \cdots + a_l D_l = P_x D_x $$

$$ a_1 D^{(1)}_1 + \cdots + a_l D^{(1)}_l = P_x D^{(1)}_x $$
or

\[ v_1 + \cdots + v_t = 1 \]

\[ v_1 D_1 + \cdots + v_t D_t = D_x \]

\[ v_1 D_1^{(1)} + \cdots + v_t D_t^{(1)} = D_x^{(1)} \]

where

\[ v_k = \frac{a_k P_k}{P_x} \]

is the weight of the bond \( b_k \) in the portfolio.

**Example 1.2.** Assume that the spot rates are as in Exercise 1.28 and that we want to imitate the cash stream \( \omega = (\ldots, 100, 100, 100, 100, 100, 100, 100) \) by means of the bonds in that exercise. Calculations show that \( P = 430.47, D = 2.89 \), and \( D^{(1)} = 0.15 \).

Let us first treat the case when we use only two of the three bonds and want to immunize against parallel shifts. If we want to have positive weights, then we cannot choose \( A \) and \( B \) since this gives a portfolio with duration \( \geq 3.77 > D \). We therefore must choose \( C \) and one of the others — \( B \), for example.

We get the equations

\[ v_B + v_C = 1 \]

\[ v_B D_B + v_C D_C = D \]

that have the solution

\[ v_B = \frac{D - D_C}{D_B - D_C}, \quad v_C = \frac{D_B - D}{D_B - D_C} \]

The portfolio is therefore obtained by buying the bond \( B \) for \( v_B P = 217.30 \) EUR and \( C \) for \( v_C P = 212.98 \) EUR.

The number of the bonds \( B \) and \( C \) in the portfolio becomes \( v_B P/B_B = 2.28 \) and \( v_C P/B_C = 2.35 \), respectively. If we assume that the spot rate changes as in Fig. 1.2., then \( \partial P_x/P_x = 81 \) bp to be compared to \( \partial P_y/P_y = 93 \) bp.

**EXERCISE 1.31.** Make a portfolio by means of the bonds \( A \) and \( C \) which is immune relative to \( \omega \) against parallel shifts.

We have for this portfolio \( \partial P_y/P_y = 91 \) bp if the spot rate changes as in Fig. 1.2.

**Example 1.3.** Here we shall immunize also against steepening by using all three bonds: \( A, B, \) and \( C \). In this case we have the equations

\[ v_A + v_B + v_C = 1, \]

\[ v_A D_A + v_B D_B + v_C D_C = D, \]

\[ v_A D_A^{(1)} + v_B D_B^{(1)} + v_C D_C^{(1)} = D^{(1)} \]
which have the solution: −0.27, 0.89, 0.38. Here we thus need a short position in the bond A. We have for this portfolio $\partial P_y / P_y 94$ bp if the spot rate change as in Fig. 1.2. An impairment compared to the two portfolios in Example 1.2 and Exercise 1.31.

EXERCISE 1.32. Assume that there are zero-coupon bonds with optional time to maturity. Let $y$ consist of a number of one zero-coupon bond. Which condition must be satisfied to immunize against

(a) parallel shift.

(b) steepening.

If one immunizes the zero-coupon bond in (a) relative to $x$ against parallel shifts and the spot rate curve changes as in Fig. 1.2, then the relative change of price becomes 72 bp, to be compared to 81 bp for $x$.

EXERCISE 1.33. Assume that you wish to pay 1 EUR in $T$ years and want secure the payment by buying a zero-coupon bond with time to maturity $T$ years. This does not exist, but there are zero-coupon bonds with times to maturity $T_1$ and $T_2$ years, where $T_1 < T < T_2$.

(a) Form a portfolio consisting of zero-coupon bonds with times to maturity $T_1$ and $T_2$ years. Determine the weights such that the portfolio has the same present value and duration as a zero-coupon bond with duration $T$.

(b) Let $\Delta_v(r)$ denote the difference between the value at time 0 of the portfolio and the zero-coupon bond with duration $T$ as a function of the rate $r$.

Show that

$$\Delta_v(r + \delta r) = \frac{\delta r}{2} (T - T_1)(T_2 - T)(\delta p)^2 + O(|\delta p|^3).$$

The expression in (b) is thus positive for small $\delta p$. Assume that the interest rate at time $\partial t$ has changed to $r + \delta r$ and that $\partial t$ is small compared to $\delta p$; thus the expression in (b) is an approximation of the difference of the values of the two portfolios at time $\partial t$.

1.3 NOTES

My main source here has been Luenberger (1998).

I have benefited from discussions with Harald Lang concerning the arbitrage theorem in Section 1.2.

The data in Section 2.4 are taken from Frye (1997).