Fundamentals of Electrodynamics

Most of electrical engineering rests upon classical electrodynamics. Classical electrodynamics summarizes the classical field theory which models the interaction between electric charges and the electromagnetic field. The related field theoretical concepts are necessary to develop transmission-line models which describe in a general way the interaction between electric charges on transmission lines and the electromagnetic field.

Within this first chapter the fundamental concepts and results of classical electrodynamics which form the physical and mathematical basis of this book will be collected and developed. To begin with, an axiomatic approach is presented which derives the Maxwell equations, that is the equations of motion of electrodynamics, from conservation laws. This approach is followed by an alternative gauge field approach which also yields the Maxwell equations and, additionally, conveys profound insights into the concept of gauge symmetry. Both approaches will be compared and interrelated. Then the dynamical and nondynamical properties of the electromagnetic field are investigated in order to set up electromagnetic boundary value problems in terms of appropriate differential equations and integral equations.

1.1 Maxwell Equations Derived from Conservation Laws – an Axiomatic Approach

An axiomatic approach to classical electrodynamics is based on electric charge conservation, the Lorentz force, magnetic flux conservation and the existence of local and linear constitutive relations [1, 2]. The inhomogeneous Maxwell equations, expressed in terms of \( D^i \) and \( H^i \), turn out to be a consequence of electric charge conservation, whereas the homogeneous Maxwell equations, expressed in terms of \( E_i \) and \( B^i \), are derived from magnetic flux conservation. The excitations \( D^i \) and \( H^i \), by means of constitutive relations, are linked to the field strengths \( E_i \) and \( B^i \). Whereas the axiomatic approach has been presented in a relativistic framework [1, 2], basically no relativistic notions are used in this book. This is quite remarkable and requires, in particular for the derivation of the homogeneous Maxwell equations from magnetic flux conservation, some steps that are not necessary if the complete framework of relativity were available.
The axiomatic approach is not only characterized by simplicity and beauty, but is also of appreciable pedagogical value. The more clearly a structure is presented, the easier it is to memorize. Moreover, an understanding of how the fundamental electromagnetic quantities $D_i$, $H_i$, $E_i$ and $B_i$ are related to each other facilitates the formulation and solution of actual electromagnetic problems.

As is appropriate for an axiomatic approach, as few prerequisites as possible are assumed. What will be needed is some elementary mathematical background that comprises differentiation and integration in the framework of tensor analysis in three-dimensional space. In particular, the concept of integration is necessary for introducing electromagnetic objects as integrands in a natural way. To this end, a tensor notation is used in which the components of mathematical quantities are explicitly indicated by means of upper (contravariant) or lower (covariant) indices [3]. The advantage of this notation is that it allows geometric properties to be represented clearly. In this way, the electromagnetic objects become more transparent and can be discussed more easily. For the formalism of differential forms, which provides similar conceptual advantages, the reader is referred to [2, 4].

Some mathematical material is compiled in Appendix A. This is helpful in order to become comfortable with the tensor notation. For a quick start the following conventions are introduced:

- Partial derivatives with respect to a spatial coordinate $x^i$ (with $i, j, \ldots = 1, 2, 3$) or with respect to time $t$ are abbreviated according to

$$\frac{\partial}{\partial x^i} \rightarrow \partial_i, \quad \frac{\partial}{\partial t} \rightarrow \partial_t. \quad (1.1)$$

- The ‘summation convention’ is used. It states that a summation sign can be omitted if the same index occurs both in a lower and an upper position. That is, there is for example the correspondence

$$\sum_{i=1}^{3} \alpha_i \beta^i \leftrightarrow \alpha_i \beta^i. \quad (1.2)$$

- The Levi-Civita symbols $\epsilon_{ijk}$ and $\epsilon^{ijk}$ are defined. They are antisymmetric with respect to all of their indices. Therefore, they vanish if two of their indices are equal. Their remaining components assume the values $+1$ or $-1$, depending on whether $ijk$ is an even or an odd permutation of 123:

$$\epsilon_{ijk} = \epsilon^{ijk} = \begin{cases} 1, & \text{for } ijk = 123, 312, 231, \\ -1, & \text{for } ijk = 213, 321, 132. \end{cases} \quad (1.3)$$

With these conventions for the gradient of a function $f$ the expression $\partial_i f$ is obtained. The curl of a (covariant) vector $v_i$ is written according to $\epsilon_{ijk} \partial_j v_k$ and the divergence of a (contravariant) vector (density) $w^i$ is given by $\partial_i w^i$.

In the next three subsections, classical electrodynamics will be established from electric charge conservation (axiom 1), the Lorentz force (axiom 2), magnetic flux conservation (axiom 3), and the existence of constitutive relations (axiom 4). This represents the core of classical
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electrodynamics: it results in the Maxwell equations together with the constitutive relations and the Lorentz force law.

In order to complete electrodynamics, one can require two more axioms, which are only mentioned briefly (see [2] for a detailed discussion). One can specify the energy–momentum distribution of the electromagnetic field (axiom 5) by means of its so-called energy–momentum tensor. This tensor yields the energy density \((D_i E_i + H_i B_i)/2\) and the energy flux density \(\epsilon_{ijk} E_j H_k\) (the Poynting vector). Moreover, if one treats electromagnetic problems of materials in macrophysics, one needs a further axiom by means of which the total electric charge (and the current) is split (axiom 6) in a bound or material charge (and current), which is also conserved, and in a free or external charge (and current).

1.1.1 Charge Conservation

In classical electrodynamics, the electric charge is characterized by its density \(\rho\). From a geometric point of view, the charge density \(\rho\) constitutes an integrand of a volume integral. This geometric identification is natural since, by definition, integration of \(\rho\) over a three-dimensional volume \(V\) yields the total charge \(Q\) enclosed in this volume

\[
Q := \int_V \rho \, dv .
\]

(1.4)

It is noted that, in the SI system, electric charge is measured in units of ‘ampere times second’ or coulomb, \([Q] = \text{As} = \text{C}\). Therefore, the SI unit of charge density \(\rho\) is \([\rho] = \text{As/m}^3 = \text{C/m}^3\).

It is instructive to invoke at this point the Poincaré lemma. There are different explicit versions of this lemma. Here the form (A.23) that is displayed in Appendix A is used. Then (if space fulfills suitable topological conditions) it is immediate to write the charge density \(\rho\) as the divergence of an integrand \(D_i\) of a surface integral. Thus,

\[
\partial_i D^i = \rho \quad \quad \quad (\nabla \cdot D = \rho) .
\]

(1.5)

This result already constitutes one inhomogeneous Maxwell equation, the Coulomb–Gauss law. The more conventional vector notation is displayed in parenthesis for comparison.

Electric charges often move. This motion is represented by a material velocity field \(u^i\), that is, locally a velocity is assigned to each portion of charge in space. The product of electric charge density \(\rho\) and material velocity \(u^i\) defines\(^1\) the electric current density \(J^i\),

\[
J^i = \rho u^i .
\]

(1.6)

Geometrically, the electric current density constitutes an integrand of surface integrals since integration of \(J^i\) over a two-dimensional surface \(S\) yields the total electric current \(I\) that crosses this surface,

\[
I = \int_S J^i \, da_i .
\]

(1.7)

In SI units, \([I] = \text{A}\) and \([J^i] = \text{A/m}^2\).

\(^1\)This definition is a microscopic one, since the movement of individual electric charges that constitute the electric charge density is considered. On a macroscopic ‘averaged’ level it is possible that the effective charge density vanishes while an electric current is present. An example is a configuration of an electric current that flows within a wire and exhibits no net charge density since negative charges of moving electrons are compensated by positive charges of atoms that constitute the wire.
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Now electric charge conservation comes into play, the first axiom of the axiomatic approach. To this end it is determined how individual packets of charge change in time as they move with velocity \( \mathbf{u} \) through space. A convenient way to describe this change is provided by the material derivative \( \frac{D}{Dt} \) which is also often called convective derivative [1, 5]. It allows the change of a physical quantity to be calculated as it appears to an observer or a probe that follows this quantity. Then electric charge conservation can be expressed as

\[
\frac{D}{Dt} Q = 0,
\]

where the material derivative is taken with respect to the velocity field \( \mathbf{u} \). It can be rewritten by means of the Reynold’s transport theorem in the following way [5],

\[
\frac{D}{Dt} Q = \frac{D}{Dt} \int_{V(t)} \rho \, dv = \int_{V(t)} \frac{\partial \rho}{\partial t} \, dv + \oint_{\partial V(t)} \rho \mathbf{u} \cdot d\mathbf{a} = \int_{V(t)} \left( \frac{\partial \rho}{\partial t} + \partial_i (\rho \mathbf{u}^i) \right) \, dv.
\]

In the last line Stokes’ theorem in the form of (A.24) has been used. The volume \( V(t) \) that is integrated over depends in general on time since it moves together with the electric charge that it contains. By means of Equations (1.6), (1.8) and (1.9) the axiom of electric charge conservation in the local form as continuity equation is obtained,

\[
\partial_t \rho + \partial_i J^i = 0.
\]

This result can also be obtained from the direct application of the material derivative \( \frac{D}{Dt} \) to the charge density. Noting that the material derivative is the sum of the partial time derivative and the Lie derivative with respect to the velocity field \( \mathbf{u} \) [1],

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{l}_u,
\]

it is found with (A.39) for the material time derivative of the scalar density \( \rho \)

\[
\frac{D}{Dt} \rho = \frac{\partial \rho}{\partial t} + \mathbf{l}_u \rho = \frac{\partial \rho}{\partial t} + \partial_i (\rho \mathbf{u}^i) = \frac{\partial \rho}{\partial t} + \partial_i J^i.
\]

In this way, the continuity Equation (1.10) follows from \( \frac{D}{Dt} \rho / Dt = 0 \).

Now the inhomogeneous Maxwell Equation (1.5) is used in order to replace within the continuity Equation (1.10) the charge density by the divergence of \( \mathbf{D} \). This yields

\[
\partial_i \left( \partial_i D^i + J^i \right) = 0.
\]
Again the Poincaré lemma is invoked, now in the form (A.22), and the sum \( \partial_t D^i + J^i \) is written as the curl of the integrand of a line integral which is denoted by \( H_i \). This yields

\[
\epsilon^{ijk} \partial_j H_k - \partial_t D^i = J^i \quad \left( \nabla \times H - \frac{\partial D}{\partial t} = J \right),
\]

Equation (1.16) constitutes the remaining inhomogeneous Maxwell equation, the Ampère–Maxwell law which, in this way, is derived from the axiom of electric charge conservation. The fields \( D^i \) and \( H_i \) are called electric excitation (historically, electric displacement) and magnetic excitation (historically, magnetic field), respectively. From Equations (1.5) and (1.16) it follows that their SI units are \( [D^i] = \text{As/m}^2 \) and \( [H_i] = \text{A/m} \).

Some remarks are appropriate now. The excitations \( D^i \) and \( H_i \) are obtained from the Poincaré lemma and charge conservation, respectively, without introducing the concept of force. This is in contrast to other approaches that rely on the Coulomb and the Lorentz force laws [6]. Furthermore, since electric charge conservation is valid not only on macroscopic scales but also in microphysics,\(^2\) the inhomogeneous Maxwell Equations (1.5) and (1.16) are microphysical equations as long as the source terms \( \rho \) and \( J^i \) are microscopically formulated as well. The same is valid for the excitations \( D^i \) and \( H_i \). They are microphysical quantities – in contrast to what is often stated in textbooks (see [8] for example). It is finally remarked that the inhomogeneous Maxwell Equations (1.5) and (1.16) can be straightforwardly put into a relativistically invariant form. This is not self-evident but suggested by electric charge conservation in the form of the continuity Equation (1.10) since this fundamental equation can also be shown to be relativistically invariant.

### 1.1.2 Lorentz Force and Magnetic Flux Conservation

During the discovery of the electromagnetic field, the concept of force has played a major role. Electric and magnetic forces are directly accessible to experimental observation. Experimental evidence shows that, in general, an electric charge is subject to a force if an electromagnetic field acts on it. For a point charge \( q \) at position \( x_q^i \) it follows \( \rho(x) = q \delta(x^i - x_q^i) \). If it has the velocity \( u^i \) the Lorentz force

\[
F_i = q(E_i + \epsilon_{ijk}u^j B^k)
\]

is postulated as the second axiom. It introduces the electric field strength \( E_i \) and the magnetic field strength \( B^i \). The Lorentz force already yields a prescription of how to measure \( E_i \) and \( B^i \) by means of the force that is experienced by an infinitesimally small test charge \( q \) which is either at rest or moving with velocity \( u^i \). Turning to the dimensions, voltage is introduced as ‘work per charge’. In SI, it is measured in volt (V). Then \( [F_i] = \text{VC/m} \) and, according to Equation (1.17), \( [E_i] = \text{V/m} \) and \( [B^i] = \text{Vs/m}^2 = \text{Wb/m}^2 = \text{T} \), with Wb an abbreviation for Weber and T for Tesla.

\(^2\) Microphysics is commonly understood to be physics on a small scale that describes the interaction between single, elementary particles. The concept of an elementary particle does not necessarily involve quantum effects and is also useful and important for classical electrodynamics [7]. On microphysical scales electric charges and their related currents are often represented by distributions that reflect the physical model of a point particle.
From the axiom of the Lorentz force (Equation (1.17)), it follows that the electric and the magnetic field strengths are not independent of each other. The corresponding argument is based on the special relativity principle: according to the special relativity principle, the laws of physics are independent of the choice of an inertial system [6]. Different inertial systems move with constant velocities $v^i$ relative to each other. The outcome of a physical experiment, as expressed by an empirical law, has to be independent of the inertial system where the experiment takes place.

It is assumed that a point charge $q$ with a certain mass moves with velocity $u^i$ in an electromagnetic field $E_i$ and $B_i$. The velocity and the electromagnetic field are measured in an inertial laboratory frame. The point charge can also be observed from its instantaneous inertial rest frame. If quantities that are measured with respect to this rest frame are denoted by a prime (i.e. by $u'^i$, $E'_i$ and $B'_i$) then it follows $u'^i = 0$. In the absence of an electric field in the rest frame (i.e. if additionally $E'_i = 0$) the charge experiences no Lorentz force and, therefore, no acceleration, so

$$F'_i = q(E'_i + \epsilon_{ijk}u'^jB'^k) = 0.$$  

(1.18)

The fact that the charge experiences no acceleration is also true in the laboratory frame. This is a consequence of the special relativity principle or, more precisely, of the fact that the square of the acceleration can be shown to form a relativistic invariant. Consequently,

$$F_i = q(E_i + \epsilon_{ijk}u^jB^k) = 0.$$  

(1.19)

Thus, in the laboratory frame, the electric and magnetic fields are related by

$$E_i = -\epsilon_{ijk}u^jB^k.$$  

(1.20)

This situation is depicted in Figure 1.1. Accordingly, it is found that the electric and magnetic field strengths cannot be viewed as independent quantities. They are connected to each other by transformations between different inertial systems.

![Figure 1.1](image)

(a) Charge observed from its rest frame  (b) Charge observed from inertial frame moving with respect to $q$

**Figure 1.1** (a) A charge which is, in some inertial frame, at rest and immersed in a purely magnetic field experiences no Lorentz force. The fact that there is no Lorentz force should be independent of the choice of the inertial system that is used to observe the charge. (b) A compensating electric field accompanies the magnetic field if viewed from an inertial laboratory system which is in motion relative to the charge.
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Figure 1.2 The tetrahedron of the electromagnetic field, according to Hehl [2]. The electric and magnetic excitations $D^i$ and $H_i$ and the electric and the magnetic field strengths $E_i$ and $B^i$ build up four-dimensional quantities in spacetime. These four fields describe the electromagnetic field completely. Of an electric nature are $D^i$ and $E_i$, of a magnetic nature $H_i$ and $B^i$.

So far, the four electromagnetic field quantities $D^i, H_i$ and $E_i, B^i$ have been introduced. These four quantities are interrelated by physical and mathematical properties. This is illustrated in Figure 1.2 by the ‘tetrahedron of the electromagnetic field’.

To introduce the notion of magnetic flux it is instructive to digress for a moment and to turn to hydrodynamics. Helmholtz was one of the first who studied rotational or ‘vortex’ motion in hydrodynamics (see [9]). He derived theorems for vortex lines. An important consequence of his work was the conclusion that vortex lines are conserved. They may move or change orientation but they are never spontaneously created nor annihilated. The vortex lines that pierce through a two-dimensional surface can be integrated to yield a scalar quantity that is called circulation. The circulation in a perfect fluid, which satisfies certain conditions, is constant provided the loop enclosing the surface moves with the fluid [9].

There are certainly fundamental differences between electromagnetism and hydrodynamics, but some suggestive analogies exist. A vortex line in hydrodynamics seems analogous to a magnetic flux line. The magnetic flux $\Phi$ is determined from magnetic flux lines, represented by the magnetic field strength $B^i$, that pierce through a two-dimensional surface $S$,

$$\Phi := \int_S B^i \, da_i \quad (1.21)$$

As the circulation in a perfect fluid is conserved, it is guessed that, in a similar way, the magnetic flux may be conserved. Of course, the consequences of such an axiom have to be borne out by experiment.

At first sight, to find the vortex lines of a fluid might be easier to visualize than magnetic flux lines. However, on a microscopic level, magnetic flux can occur in quanta.
The corresponding magnetic flux unit is called a flux quantum or fluxon and it carries
\[ \Phi_0 = \frac{\hbar}{2e} \approx 2.07 \times 10^{-15} \text{ Wb}, \]
with \( \hbar \) the Planck constant and \( e \) the elementary charge. Single quantized magnetic flux lines have been observed in the interior of type II superconductors if exposed to a sufficiently strong magnetic field (see [2, p. 131]). They can even be counted. The corresponding experiments provide good evidence that magnetic flux is a conserved quantity.

But how can magnetic flux conservation be formulated mathematically? In Section 1.1.1 the material derivative \( \frac{D_u}{Dt} \) was applied with respect to a velocity field \( u^i \) to the total electric charge \( Q \) and, equivalently, to the electric charge density \( \rho \). This yielded the continuity equation (1.10) which expresses electric charge conservation. The same pattern is followed to express magnetic flux conservation and to write down the conservation law

\[ \frac{D_u \Phi}{Dt} = 0. \]  

(1.22)

This expression has to be examined and this, in turn, requires the following two points to be clarified:

- What is the definition of a velocity field \( u^i \) with respect to a magnetic field \( B^i \)?
- What is a physically reasonable definition of the current of a magnetic flux?

To answer the first point it is necessary to know how to observe, in general, a magnetic field. The only means at our disposal is the Lorentz force law. With the Lorentz force law electric test charges can be used to measure the electric and magnetic field strength. It has already been noted in the last subsection that electric and magnetic field strengths are connected to each other by relativistic transformations. This makes it impossible to state in a relativistically invariant way which contribution to a Lorentz force is due to an electric field and which contribution is due to a magnetic field. Any observer who uses an electric test charge which is located in their rest frame might state that the Lorentz force on the test charge is ‘purely electric’ since for their test charge \( u^i = 0 \) in Equation (1.17) and, thus, \( F_i = qE_i \). They might furthermore draw the conclusion that in their rest frame the velocity of the magnetic field vanishes as well. This is, of course, a wrong conclusion since the velocity which appears in the Lorentz force law is the relative velocity between an observer and a test charge but it is not, a priori, the relative velocity between a test charge and the magnetic field.

Nevertheless to associate a velocity field to a magnetic field it is noted that the vanishing of the Lorentz force on a test charge is relativistically invariant. If the Lorentz force on a test charge vanishes in one inertial system it will vanish in all inertial systems. In this case there will be exactly one inertial system where this test charge is at rest. In this inertial system,

\[ F'_i = q(E'_i + \epsilon_{ijk} u^{ij} B^k) = 0 \]  

(1.23)

and it follows, since \( u^{ij} = 0 \), that \( E'_i = 0 \). Now this distinguished inertial system is defined to be, at the position considered, the rest frame of the magnetic field. This definition requires that an inertial system can always be found where \( F'_i = 0 \), that is that an inertial system can always be found where \( E'_i = 0 \). For an arbitrary electromagnetic field this will not be true, but this property is assigned to a purely magnetic field. A purely magnetic field is an electromagnetic field where, at any point in space and time, the electric field can be made to vanish in one
inertial system. In the definition of magnetic flux conservation only electromagnetic fields which are purely magnetic will be considered. Otherwise it is not possible to associate a velocity field to a magnetic field in a relativistically invariant way. Therefore, the answer to the first point above is that the velocity \( u^i \) associated to a (purely) magnetic field is the velocity of a specified inertial system which moves with respect to a laboratory system with velocity \( u^i \) and where the Lorentz force on a test charge vanishes.

An answer to the second point needs to provide a physically meaningful definition of magnetic flux current. To this end the notion of electric charge is reconsidered,

\[
Q = \int_V \rho \, dv ,
\]

(1.24)

together with the corresponding conservation law

\[
\partial_t Q + \int_{\partial V} J^i \, da_i = 0 .
\]

(1.25)

It is seen from this representation that the rate of change of the electric charge within a specified volume \( V \) is balanced by the out- or inflowing charge across the surface \( \partial V \). This charge transport is described by the electric charge current \( J^i \) that is integrated over the enveloping surface \( \partial V \). By means of the Stokes’ theorem in the form (A.24), Equation (1.25) yields the local continuity equation

\[
\partial_t \rho + \partial_i J^i = 0 .
\]

(1.26)

It is possible to follow the same pattern to define the current of a magnetic flux: starting with the definition (1.21) of the magnetic flux, the corresponding geometric conservation law, in analogy to (1.25), reads

\[
\partial_t \Phi + \int_{\partial S} J^i_\Phi \, dc^i = 0 ,
\]

(1.27)

where the magnetic flux current \( J^i_\Phi \) is introduced. This is a covariant vector that is integrated along a line \( \partial S \), that is along the curve bordering the two-dimensional surface \( S \). The conservation law (1.27) shows that the rate of change of the magnetic flux within a specified area \( S \) is balanced by the magnetic flux current \( J^i_\Phi \) that is integrated along the boundary \( \partial S \). Then the Stokes’ theorem in the form (A.25) yields the local continuity equation

\[
\partial_t B^i + \epsilon^{ijk} \partial_j J^j_\Phi = 0 .
\]

(1.28)

One interesting consequence is that the divergence of (1.28) reads

\[
\partial_i (\partial_t B^i) = 0 \quad \implies \quad \partial_t B^i = \rho_{\text{mag}} , \quad \partial_i \rho_{\text{mag}} = 0 .
\]

(1.29)

Thus, a time-independent term \( \rho_{\text{mag}} \) is found which tentatively acquires the meaning of a magnetic charge density. Let a specific reference system be chosen in which \( \rho_{\text{mag}} \) is constant in time, that is \( \partial_t \rho_{\text{mag}} = 0 \). Now an arbitrary reference system with time coordinate \( t' \) and spatial coordinates \( x'^i \) is considered. Clearly, in general \( \partial_t \rho_{\text{mag}} \neq 0 \). The only way to evade
a contradiction to (1.29) is to require $\rho_{\text{mag}} = 0$, that is the magnetic field strength $B^i$ has no sources, its divergence vanishes:

$$\partial_i B^i = 0 \quad (\nabla \cdot B = 0). \quad (1.30)$$

This is recognized as one of the homogeneous Maxwell equations.

To specify the magnetic flux current, magnetic flux conservation is explored, expressed by means of the material time derivative $D_u/ Dt$ with respect to a velocity field $u^i$ which is associated to a purely magnetic field. It follows

$$D_u \Phi = D_u \int_{S(t)} B^i \, da_i \quad (\nabla \cdot B = 0), \quad (1.31)$$

where the Helmholtz transport theorem [5, p. 456] has been applied. Alternatively, it is possible to work with the local expression

$$D_u B^i = \partial_t B^i + l_u B^i = \partial_t B^i + u^i \partial_j B^j - B^j \partial_j u^i + B^i \partial_j u^j, \quad (1.32)$$

where the formula (A.36) for the Lie derivative of a contravariant vector density has been used. The Lie derivative of $B^i$ can be rewritten according to

$$u^i \partial_j B^j - B^j \partial_j u^i + B^i \partial_j u^j = -\epsilon_{ijk} \partial_j \epsilon_{klm} u^l B^m + u^i \partial_j B^j. \quad (1.33)$$

Hence, it follows

$$D_u B^i = \partial_t B^i - \epsilon_{ijk} \partial_j \epsilon_{klm} u^l B^m + u^i \partial_j B^j \quad (1.34)$$

and it is recognized that Equation (1.34) is the local version of Equation (1.31).

According to Equation (1.30) the divergence of the magnetic field strength $B^i$ vanishes, $\partial_i B^i = 0$. Also, by virtue of Equation (1.20), the term $-\epsilon_{klm} u^l B^m$ can be locally identified with an electric field strength $E_k$. This is because it has been assumed that a purely magnetic field is presented, that is an electromagnetic field which does not exert a Lorentz force on an electric charge which moves with velocity $u^i$ in the laboratory frame. Then magnetic flux conservation, $D_u B^i / Dt = 0$, yields

$$\partial_t B^i + \epsilon_{ijk} \partial_j E_k = 0 \quad \left( \frac{\partial B}{\partial t} + \nabla \times E = 0 \right) \quad (1.35)$$

This equation reflects magnetic flux conservation. It constitutes the remaining homogeneous Maxwell equation, that is Faraday’s induction law. This result is compared to the continuity

\footnote{In a more conventional notation this identity reads:

$$(u \cdot \nabla) B - B(\nabla \cdot u) + (B \cdot \nabla) u = -\nabla \times (u \times B) + u(\nabla \cdot B).$$}
Equation (1.28) and it is deduced that the electric field that appears in the Faraday’s induction law has to be interpreted as a magnetic flux current.

1.1.3 Constitutive Relations and the Properties of Spacetime

So far, $4 \times 3 = 12$ unknown electromagnetic field components $D^i$, $H_i$, $E_i$ and $B^i$ have been introduced. These components have to fulfill the Maxwell Equations (1.5), (1.16), (1.30) and (1.35), which represent $1 + 3 + 1 + 3 = 8$ partial differential equations. In fact, among the Maxwell Equations, only Equations (1.16) and (1.35) contain time derivatives and are dynamical. The remaining Equations (1.5) and (1.30) are so-called ‘constraints’. They are, by virtue of the dynamical Maxwell equations, fulfilled at all times if they are fulfilled once. It follows that they do not contain information on the time evolution of the electromagnetic field. Therefore, there remain only six dynamical equations for 12 unknown field components. To make the Maxwell equations a determined set of partial differential equations it is also necessary to introduce the so-called ‘constitutive relations’ between the excitations $D^i$, $H_i$ and the field strengths $E_i$, $B^i$.

The simplest case to begin with is to find constitutive relations for the case of electromagnetic fields in a vacuum. There are guiding principles that limit their structure. It is demanded that constitutive relations in a vacuum are invariant under translation and rotation, furthermore they should be local and linear, that is they should connect fields at the same position and at the same time. Finally, in a vacuum the constitutive relations should not mix electric and magnetic properties. These features characterize the vacuum and not the electromagnetic field itself. They cannot be proved but are postulated as the fourth axiom.

To relate the field strengths and the excitations it needs to be remembered that $E_i$, $H_i$ are natural integrands of line integrals and $D^i$, $B^i$ are natural integrands of surface integrals. Therefore, $E_i$, $H_i$ transform under a change of coordinates as covariant vectors while $D^i$, $B^i$ transform as contravariant vector densities. To compensate for these differences a symmetric metric field $g_{ij} = g_{ji}$ is introduced. The metric tensor determines spatial distances and introduces the notion of orthogonality. The determinant of the metric is denoted by $g$. It follows that $\sqrt{g} g^{ij}$ transforms like a density and maps a covariant vector into a contravariant vector density. Then the fourth axiom is given by the constitutive relations for vacuum,

$$D^i = \varepsilon_0 \sqrt{g} g^{ij} E_j, \quad (1.36)$$

$$H_i = (\mu_0 \sqrt{g})^{-1} g_{ij} B^j. \quad (1.37)$$

In flat spacetime and in Cartesian coordinates, $g = 1$, $g^{ii} = 1$, and $g^{ij} = 0$ for $i \neq j$, such that the familiar vacuum relations between field strengths and excitations are recognized. The electric constant $\varepsilon_0$ and the magnetic constant $\mu_0$ characterize the vacuum. They acquire the SI units $[\varepsilon_0] = \text{As/Vm}$ and $[\mu_0] = \text{Vs/Am}$.

What seems to be conceptually important about the constitutive Equations (1.36) and (1.37) is that they not only provide relations between the excitations $D^i$, $H_i$ and the field strengths $E_i$, $B^i$, but they also connect the electromagnetic field to the structure of spacetime, which here is represented by the metric tensor $g_{ij}$. The formulation of the first three axioms that were presented in the previous sections does not require information on this metric structure. The
connection between the electromagnetic field and spacetime, as expressed by the constitutive relations, indicates that physical fields and spacetime are not independent of each other. The constitutive relations might suggest the point of view that the structure of spacetime determines the structure of the electromagnetic field. However, the opposite conclusion could also be true: it can be shown that the propagation properties of the electromagnetic field determine the metric structure of spacetime [2, 10].

Constitutive relations in matter usually assume a more complicated form than Equations (1.36) and (1.37). In this case it would be appropriate to derive the constitutive relations, after an averaging procedure, from a microscopic model of matter. Such procedures are the subject of solid state or plasma physics, for example. A discussion of these subjects is outside the scope of this work but, without going into details, the constitutive relations of a general linear magnetoelectric medium are simply quoted:

\[
\begin{align*}
D^i &= \left( \epsilon_{ij} - \epsilon_{ijk} n_k \right) E_j + \left( \gamma^j + \tilde{\gamma}^j \right) B^i + (\alpha - s) B^i, \\
H_i &= \left( \mu_{ij}^{-1} - \epsilon_{ijk} m_k \right) B^j + \left( -\gamma^j i + \tilde{\gamma}^j \right) E_j - (\alpha + s) E_i.
\end{align*}
\] (1.38)

This formulation is due to Hehl and Obukhov [2, 11, 12], an equivalent formulation was given by Lindell and Olyslager [4, 13]. Both matrices \( \epsilon_{ij} \) and \( \mu_{ij}^{-1} \) are symmetric and possess six independent components each, \( \epsilon_{ij} \) is called the permittivity tensor and \( \mu_{ij}^{-1} \) the impermeability tensor (reciprocal permeability tensor). The magnetoelectric cross-term \( \gamma^j \), which is trace-free, \( \gamma^k k = 0 \), has eight independent components. It is related to the Fresnel–Fizeau effects. Accordingly, these pieces altogether, which are printed in Equations (1.38) and (1.39) in bold type for better visibility, add up to \( 6 + 6 + 8 + 1 = 20 + 1 = 21 \) independent components.

1.1.4 Remarks

An axiomatic approach to classical electrodynamics has been presented in which the Maxwell equations are derived from the conservation of electric charge and magnetic flux. In the context of the derivation of the inhomogeneous Maxwell equations, the electric and the magnetic excitation \( D^i \) and \( H_i \), respectively, are introduced. The explicit calculation is rather simple because the continuity equation for electric charge is already relativistically invariant such that for the derivation of the inhomogeneous Maxwell equations no additional ingredients from special relativity are necessary. The situation is more complicated for the derivation of the homogeneous Maxwell equations from magnetic flux conservation since it is not immediately clear how to formulate magnetic flux conservation in a relativistic invariant way. It should be mentioned that if the complete framework of relativity were available, the derivation of the axiomatic approach could be done with considerably more ease and elegance [1, 2].

At this point it is appropriate to comment on a question that sometimes leads to controversial discussions, as summarized in [5], for example. This is the question of how the quantities \( E_i \), \( D^i \), \( B^i \) and \( H_i \) should be grouped in pairs, that is the question of ‘which quantities belong together’? Some people like to form the pairs \((E_i, B^i)\) and \((D^i, H_i)\), while others prefer to build \((E_i, H_i)\) and \((D^i, B^i)\). Already from a dimensional point of view, the answer to this question is obvious. Both \( E_i \) and \( B^i \) are voltage-related quantities, that is, related to the notions of force and work: in SI, the units are \([E_i] = \text{V/m}\), \([B^i] = \text{T=Vs/m}^2\), or \([B^i] = [E_i]/\text{velocity}\). Consequently, they belong together. Analogously, \( D^i \) and \( H_i \) are current-related quantities: \([D^i] = \text{C/m}^2 = \text{As/m}^2\), \([H_i] = \text{A/m}\), or \([D^i] = [H_i]/\text{velocity}\).
These conclusions are made irrefutable by relativity theory. Classical electrodynamics is a relativistic invariant theory and the implications of relativity have been proven to be correct on macro- and microscopic scales over and over again. Also relativity tells us that the electromagnetic field strengths $E_i$, $B_i$ are inseparably intertwined by relativistic transformations, and the same is true for the electromagnetic excitations $D_i$, $H_i$. In the spacetime of relativity theory, the pair $(E_i, B_i)$ forms one single quantity, the tensor of electromagnetic field strength, while the pair $(D_i, H_i)$ forms another single quantity, the tensor of electromagnetic excitations. If compared to these facts, arguments in favor of the pairs $(E_i, H_i)$, namely that they are both covectors, and $(D_i, B_i)$ are both vector densities (see the tetrahedron in Figure 1.2), turn out to be of a secondary nature.

1.2 The Electromagnetic Field as a Gauge Field – a Gauge Field Approach

Modern descriptions of the fundamental interactions rely heavily on symmetry principles. In particular, this is true for the electromagnetic interaction which can be formulated as a gauge field theory that is based on a corresponding gauge symmetry. In recent articles this approach towards electromagnetism has been explained in an original and descriptive way [14, 15]. The gauge field approach is put next to the axiomatic approach since it furnishes further information that will complement the picture of classical electrodynamics. In particular, it allows the clarification of the concept of gauge invariance which often accompanies explicit calculations in the solution of electrodynamic boundary value problems. It also shows that the electromagnetic potentials, which are often viewed as mathematical auxiliary variables, are of major physical relevance. Furthermore, in the gauge field approach the inhomogeneous Maxwell equations turn out to be true equations of motion while the homogeneous Maxwell equations become a mere mathematical identity.

While it is rewarding to gain the additional insights that are provided by the gauge field approach it should be admitted that this approach, at first sight, extends on a rather abstract level. However, it only requires a small number of steps:

1. Accept the fact that physical matter fields (which represent electrons, for example) are described microscopically by complex wave functions.
2. Recognize that the absolute phase of these wave functions has no physical relevance. This arbitrariness of the absolute phase constitutes a one-dimensional rotational type symmetry $U(1)$ (the circle group). This is the gauge symmetry of electrodynamics.
3. To derive observable physical quantities from the wave functions needs the definition of derivatives of wave functions with respect to space and time. These derivatives need to be invariant under the gauge symmetry. The construction of such ‘gauge covariant’ derivatives makes it necessary to introduce gauge fields. One gauge field, the vector potential $A_i$, defines gauge covariant derivatives $D^A_i$ with respect to the three independent directions of space, while another gauge field, the scalar potential $\varphi$, defines a gauge covariant derivative $D^\varphi_t$ with respect to time.
4. Finally, the values of the gauge fields $\varphi$ and $A_i$ are obtained from equations of motion that turn out to be the inhomogeneous Maxwell equations. The gauge fields are related to the
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gauge invariant electric and magnetic field strengths via
\[ E_i = -\partial_i \phi - \partial_0 A_i, \]  
\[ B^i = \epsilon^{ijk} \partial_j A_k. \]  

1.2.1 Differences of Physical Fields that are Described by Reference Systems

To introduce the concept of gauge symmetry a physical field is formally denoted by \( F \), its components by \( F^i \), and a reference frame with respects to these components by \( e_i \),
\[ F = F^i e_i. \]  

The field \( F \) might change in space or time. Its components \( F^i \) depend on the reference frame \( e_i \) and usually do not have an absolute significance since usually there is a certain freedom to choose \( e_i \). Therefore, a change of \( F \) involves both its components and the corresponding reference frame. Then it follows
\[ \partial_i F = (\partial_i F^j) e_j + F^j (\partial_0 e_j), \]  
\[ \partial_0 F = (\partial_0 F^j) e_j + F^j (\partial_0 e_j). \]  

This simple looking ‘product rule for differentiation’ poses severe difficulties: It might be possible to determine the differences \( \partial_i F^j \) and \( \partial_0 F^j \), for example by measurements, but how is it possible to determine changes \( \partial_0 e_j \), \( \partial_i e_j \) of reference frames? What is an unchanged reference frame? How is a reference frame gauged? The fact needs to be faced that a priori changes of reference frames are not defined as long as they involve the change between different points in space or time.

At this point the concept of ‘interactions’ enters the stage. If a physical field changes its value intuition indicates that this is due to an interaction. The gauge principle states that the information on the change of reference frames is contained in interaction fields. Mathematically, this is formulated as follows. The changes \( \partial_0 e_j \), \( \partial_i e_j \) of a reference frame are determined by interaction fields \( A_{ij}^k \) and \( \varphi_{ij}^k \) according to
\[ \partial_i e_j : = A_{ij}^k e_k, \]  
\[ \partial_0 e_j : = \varphi_{ij}^k e_k. \]  

With these definitions the interaction fields provide a gauging of reference system at different positions in space and time. This is the reason why they are called gauge fields. A corresponding mathematical term is connection, because gauge fields connect reference systems at different positions in space and time.

With the definitions (1.45) and (1.46) the differences \( \partial_0 F, \partial_i F \) are written as
\[ \partial_0 F = (\partial_0 F^j) e_j + F^j A_{ij}^k e_k, \]  
\[ \partial_i F = (\partial_i F^j) e_j + F^j \varphi_{ij}^k e_k. \]  

So far it is not clear if the formal introduction of the gauge fields \( A_{ij}^k \) and \( \varphi_{ij}^k \) is physically meaningful. However, it turns out that these gauge fields describe the fundamental interactions
that are observed in nature correctly. In the following subsection this circumstance will be explained for the case of the electromagnetic interaction.

### 1.2.2 The Phase of Microscopic Matter Fields

As a first step, the physical field and its corresponding reference system which leads to the introduction of the electromagnetic field as a gauge field are identified. Then it is noted that microscopic matter fields, like electrons for example, are represented in the framework of quantum mechanics by wave functions \( \Psi(x^i, t) \) [16, 17]. A wave function assumes complex values and depends on space coordinates \( x^i \) and a time coordinate \( t \), \( \Psi(x^i, t) \in \mathbb{C} \). It follows that a microscopic particle with specified momentum \( p_i \) and specified energy \( E \) is represented by a wave function of the form

\[
\Psi(x^i, t) = \Psi_0 e^{-\frac{i}{\hbar}(p_i x^i - Et)},
\]

with \( \Psi_0 \in \mathbb{R} \) and \( \hbar \) a fundamental constant which carries the dimension of an action, \( \hbar = h/2\pi \approx 1.0546 \times 10^{-34} \text{ Js} \).

The plane wave in Equation (1.49) is a very special case of a wave function since the momentum and energy of a microscopic particle are usually not known exactly but are affected by an uncertainty. In this more general case microscopic particles are represented by wave packets that are obtained by the superposition of plane waves of the form (1.49) with different momenta and energies. The value \( \Psi(x^i, t) \) has no direct physical significance, but the square of its absolute value yields a probability density. This is the real value

\[
P(x^i, t) = |\Psi(x^i, t)|^2 \, dv \quad (1.50)
\]

which is the probability of finding a particle, which is described by \( \Psi(x^i, t) \), at time \( t \) within a volume \( dv \).

The wave function \( \Psi(x^i, t) \) of Equation (1.49) has the structure

\[
\Psi(x^i, t) = \Psi_0 e^{-\theta}
\]

with the phase

\[
\theta(x^i, t) = (p_i x^i - Et)/\hbar .
\]

It is seen that the characteristic quantities \( p_i \) and \( E \) of a free microscopic particle are included in its phase \( \theta(x^i, t) \). They can be obtained explicitly by differentiation of the phase of the wave function,

\[
p_i = \hbar \partial_i \theta ,
\]

\[
E = -\hbar \partial_t \theta ,
\]

and these relations reveal that the momentum and the energy of a microscopic particle is determined by the phase differences \( \partial_i \theta \) and \( \partial_t \theta \). These phase differences depend on reference frames that, a priori, can be chosen arbitrarily, as will be seen next.
1.2.3 The Reference Frame of a Phase

In this subsection a wave function $\Psi$ at one space point $x^i$ at a fixed time $t$ is considered. It is necessary to assign a specific phase $\theta$ to $\Psi$. The phase can be taken as a real number of the interval $[0, 2\pi]$, $0 \leq \theta < 2\pi$. In order to assign a fixed value to $\theta$ a reference system is necessary which determines a reference phase. This reference phase is denoted by $\beta_R$. The specification of $\theta$ is explained in Figure 1.3, where the wave function is displayed as a curly line. The arrow along this curly line indicates that the wave function is characterized by a certain phase. This phase can be thought of as a point on a circle and is given by an element of the interval $[0, 2\pi]$. The reference phase $\beta_R$ is drawn as another arrow which also indicates a certain direction, i.e., represents an element of the interval $[0, 2\pi]$. This value is taken as reference phase $\beta_R$. The choice of $\beta_R$ a priori is arbitrary. A convenient choice would be $\beta_R = 0$, for example.

To read off the value $\theta$ of the wave function the angle between the wave function and the reference system is determined. This angle is denoted by $\theta_R$ and refers to the reference system $\beta_R$. Therefore, $\theta_R$ can be understood as a component with respect to the basis $\beta_R$. The component $\theta_R$ and the basis $\beta_R$ yield the phase $\theta$ according to

$$\theta = \theta_R + \beta_R.$$  \hfill (1.55)

In terms of the wave function this is written as

$$\Psi = \Psi_0 e^{-j\theta} = \Psi_0 e^{-j(\theta_R + \beta_R)}$$  \hfill (1.56)

$$= \Psi_0 e^{-j\theta_R} e^{-j\beta_R}.$$  \hfill (1.57)

It follows that the wave function is of the form (1.42),

$$\Psi = \Psi e^R,$$  \hfill (1.57)

with the correspondences

$$\Psi e^R = \Psi_0 e^{-j\theta_R} \quad \text{(component)}$$  \hfill (1.58)

and

$$e^R = e^{-j\beta_R} \quad \text{(reference system).}$$  \hfill (1.59)

The choice of a reference system of a phase is not unique. This is clarified in Figure 1.4. There is the gauge freedom to choose between reference systems which differ from each other.
Figure 1.4  The phase $\theta^R$ of the wave function $\Psi$ depends on the choice of a reference system. A gauge transformation corresponds to the transition from one reference system to another equivalent reference system. This transition is accomplished by a rotation about an angle $(q/\hbar)\epsilon$.

The phase $\theta^R$ of the wave function $\Psi$ depends on the choice of a reference system. A gauge transformation corresponds to the transition from one reference system to another equivalent reference system. This transition is accomplished by a rotation about an angle $(q/\hbar)\epsilon$.

by a one-dimensional rotation. The choice of a fixed reference system is called the choice of a gauge. In this language the value $\theta^R$ is a gauge dependent quantity since it depends on the gauge, that is it depends on the choice of a reference system.

A gauge transformation is denoted by $\delta_\epsilon$. Its effect on the component $\theta^R$ of the phase is

$$\delta_\epsilon \theta^R := \theta^R - \theta^R = \frac{q}{\hbar} \epsilon .$$

(1.60)

Here, the difference $\theta^R - \theta^R$ is assigned to the value $q\epsilon/\hbar$. This notation seems a bit awkward but it is in accordance with conventions that have their origin in quantum mechanics. The phase difference carries no physical dimension and $q$ has the dimension of electric charge, $[q] = \text{As} = \text{C}$. Therefore, the parameter $\epsilon$ has the dimension of an action per charge, $[\epsilon] = 1/(\text{As}^2)$. The interpretation of a gauge transformation (1.60) might appear to be rather trivial: if there is a shift between two reference systems that differ by an angle $(q/\hbar)\epsilon$ then the component $\theta^R$ of the phase changes by an angle $(q/\hbar)\epsilon$.

The reference system $\beta_R$ can also be used to gauge the reference system $\beta'_R$. In order to show this the value of $\beta'_R$ is set to

$$\beta'_R = \beta_R - \frac{q}{\hbar} \epsilon .$$

(1.61)

Then $\beta_R$ transforms under a gauge transformation according to

$$\delta_\epsilon \beta_R := \beta'_R - \beta_R = -\frac{q}{\hbar} \epsilon .$$

(1.62)

It follows that the phase $\theta$ is a gauge independent quantity,

$$\delta_\epsilon \theta = \delta_\epsilon \theta^R + \delta_\epsilon \beta_R = \frac{q}{\hbar} \epsilon - \frac{q}{\hbar} \epsilon = 0 .$$

(1.63)

However, in spite of this gauge independence the value of $\theta$ has no absolute significance since it depends, by virtue of $\theta = \theta^R + \beta_R$, on a choice of a reference frame $\beta_R$.

The situation is different if the difference between two phases $\theta_1$ and $\theta_2$ is considered. Since both phases are defined at the same point $x^i$ and to the same time $t$ they can be characterized
by a common reference system $\beta_R$,

$$\theta_1 = \theta_1^R + \beta_R$$  \hspace{1cm} (1.64)

$$\theta_2 = \theta_2^R + \beta_R.$$  \hspace{1cm} (1.65)

Taking the difference yields

$$\theta_1 - \theta_2 = \theta_1^R - \theta_2^R,$$  \hspace{1cm} (1.66)

that is, the difference $\theta_1 - \theta_2$ is both independent of the reference system $\beta_R$ and, due to

$$\delta_\epsilon(\theta_1 - \theta_2) = \delta_\epsilon \theta_1 - \delta_\epsilon \theta_2 = 0 - 0 = 0,$$  \hspace{1cm} (1.67)

a gauge independent quantity. Therefore, the phase difference $\theta_1 - \theta_2$ at one point $(x^i, t)$ in spacetime has an absolute significance.

### 1.2.4 The Gauge Fields of a Phase

So far, the phase $\theta$ at one point $(x^i, t)$ in spacetime has been considered. In a next step, the change of the phase between two different points in spacetime is investigated. To this end, consider two points $(x^i, t)$ and $(x^i + dx^i, t)$ that are separated by an infinitesimal spatial distance $dx^i$. Mathematically, from Equation (1.55) for the difference $\partial_i \theta$ between two points $x^i$ and $x^i + dx^i$ the expression

$$\partial_i \theta = \partial_i \theta^R + \partial_i \beta_R$$  \hspace{1cm} (1.68)

is found, that is the change of the phase $\theta$ is the sum of the change of its component and the change of its reference frame.

For a geometrical interpretation of Equation (1.68) it is useful to think of how to construct the difference $\partial_i \theta$. The construction is divided into several steps (see Figure 1.5):

1. According to the previous Section 1.2.3 first the phase $\theta(x^i, t)$ at the point $(x^i, t)$ is determined by a reference system $\beta_R$ according to $\theta(x^i, t) = \theta^R(x^i, t) + \beta_R(x^i, t)$.
2. At the point $(x^i + dx^i, t)$ an arbitrary reference frame $\beta^\prime_R$ is chosen. In Figure 1.5 this arbitrary reference frame is displayed by a dotted line. It can be used to read off the value

$$\theta^\prime_R(x^i + dx^i, t) = \theta^R(x^i, t) + \partial_i \theta^R dx^i.$$  \hspace{1cm} (1.69)

However, this value has no immediate physical relevance since its corresponding reference system has been arbitrarily chosen.
3. At the point $(x^i + dx^i, t)$ there exists a unique reference system $\beta_R$ which is defined to be unchanged if compared to the reference system $\beta_R$ at $(x^i, t)$. In mathematical terms, such an unchanged reference system is named a parallel reference system. It is obtained from the arbitrary reference system $\beta^\prime_R$ by a rotation about an angle $\partial_i \beta_R$,

$$\beta_R(x^i + dx^i, t) = \beta^\prime_R(x^i + dx^i, t) - \partial_i \beta_R dx^i$$  \hspace{1cm} (1.70)

and has the same phase value as the reference system at the point $(x^i, t)$,

$$\beta_R(x^i + dx^i, t) = \beta_R(x^i, t).$$  \hspace{1cm} (1.71)
Figure 1.5 Determination of parallel reference systems at two spatially separated points \((x^i, t)\) and \((x^i + dx^i, t)\). In the left part of the figure, at the point \((x^i, t)\), the phase is given by \(\theta = \theta^R + \beta_R\). In the right part of the figure, at the point \((x^i + dx^i, t)\), the phase is \(\theta = \theta^R + \partial_i \theta^R dx^i + \beta_R + \partial_i \beta_R dx^i\).

With respect to this parallel reference system the component of the phase is given by

\[
\theta^R(x^i + dx^i, t) = \theta^R(x^i, t) + \partial_i \theta^R dx^i + \partial_i \beta_R dx^i.
\]  
(1.72)

In summary, the relations are obtained:

\[
\theta(x^i, t) = \theta^R(x^i, t) + \beta_R(x^i, t),
\]
 (1.73)
\[
\theta(x^i + dx^i, t) = \theta^R(x^i + dx^i, t) + \beta_R(x^i + dx^i, t)
\]
\[= \theta^R(x^i, t) + \partial_i \theta^R dx^i + \beta_R(x^i, t) + \partial_i \beta_R dx^i,\]
(1.74)

which leads back to Equation (1.68),

\[
\partial_i \theta = \frac{\theta(x^i + dx^i, t) - \theta(x^i, t)}{dx^i}
\]
(1.75)
\[= \partial_i \theta^R + \partial_i \beta_R.
\]  
(1.76)

Equation (1.76) determines the difference \(\partial_i \theta\), but the contributions \(\partial_i \theta^R\) and \(\partial_i \beta_R\) need to be known explicitly and this leads to the conceptual problem that has been described at the end of Section 1.2.1: the change \(\partial_i \theta^R\) can be determined if the corresponding phases are properly read off but the difference \(\partial_i \beta_R\) a priori is not determined. It is simply not known which reference systems at different points are in parallel!

At this point the gauge fields come into play. In accordance with (1.45) and (1.46) the difference \(\partial_i \beta_R\) of the reference frame is determined from a gauge field \(A_i\) by

\[
\partial_i \beta_R := -\frac{q}{\hbar} A_i.
\]  
(1.77)

Similarly with (1.60) the factor \(q/\hbar\) has been chosen to arrive at results that comply with quantum mechanics. Therefore,

\[
\partial_i \theta = \partial_i \theta^R - \frac{q}{\hbar} A_i.
\]  
(1.78)

In view of Equation (1.53) it is realized that the momentum \(p_i\) of a microscopic particle can
only be defined by means of the gauge field $A_i$. It follows from Equation (1.78) that the SI unit of the gauge field $A_i$ is $[A_i] = \text{Vs/m}$.

So far, the difference $\partial_i \theta$ between two spatially separated points $(x^i, t)$ and $(x^i + dx^i, t)$ has been considered. In the same way the difference $\partial \theta$ between two temporally separated points $(x^i, t)$ and $(x^i, t + dt)$ can be considered. This leads to the relation

$$\partial \theta = \partial_i \theta^i + \partial_i \beta_i .$$

(1.79)

Since $\partial_i \beta_i$ is undetermined this requires the introduction of a gauge field $\varphi$,

$$\partial_i \beta_R := \frac{q}{\hbar} \varphi ,$$

(1.80)

and the result

$$\partial \theta = \partial_i \beta_R^i + \frac{q}{\hbar} \varphi$$

(1.81)

is obtained. Similarly it can be shown in view of Equation (1.54), that the energy $E$ of a microscopic particle can only be defined by means of the gauge field $\varphi$. Also it follows from Equation (1.81) that the SI unit of the gauge field $\varphi$ is $[\varphi] = \text{V}$.

A comparison between Equations (1.78) and (1.81) reveals that different signs in front of the fields $A_i$ and $\varphi$ have been chosen. This is done in order to be able to merge $A_i$ and $\varphi$ in accordance to common conventions into a single relativistically covariant quantity with four components.

At the end of this subsection it is noted that the gauge fields $A_i$ and $\varphi$ are not gauge invariant. In fact, Equation (1.62) gives

$$\delta_\epsilon (\partial_i \beta_R) = - \frac{q}{\hbar} \partial_i \epsilon ,$$

(1.82)

$$\delta_\epsilon (\partial_i \beta_R) = - \frac{q}{\hbar} \partial_i \epsilon ,$$

(1.83)

and with the definitions (1.77) and (1.80) the behavior of $A_i$ and $\varphi$ under gauge transformations are obtained as

$$\delta_\epsilon A_i = \partial_i \epsilon ,$$

(1.84)

$$\delta_\epsilon \varphi = - \partial_i \epsilon .$$

(1.85)

If the combinations

$$E_i := - \partial_i \varphi - \partial_i A_i ,$$

(1.86)

$$B^i := e^{ijk} \partial_j A_k$$

(1.87)

are formed it can easily be verified that these are invariant under gauge transformations,

$$\delta_\epsilon E_i = - \partial_i \partial_i \epsilon + \partial_i \partial_i \epsilon = 0 .$$

(1.88)

$$\delta_\epsilon B^i = - e^{ijk} \partial_j \partial_i \epsilon = 0 .$$

(1.89)

It follows from (1.86) and (1.87) that the fields $E_i$ and $B^i$ have the same SI units as the electric and magnetic field strength, respectively, that have been introduced in Section 1.1.2 within the axiomatic approach, $[E_i] = \text{V/m}$, $[B^i] = \text{Vs/m}^2$. 


More important information is obtained if the integrability conditions
\[ \epsilon^{ijk} \partial_j E_k = - \epsilon^{ijk} \partial_j \varphi - \partial_t \epsilon^{ijk} \partial_j A_k \]
\[ = - \partial_t B^i \]
\[ \partial_t B^i = \epsilon^{ijk} \partial_i \partial_j A_k \]
\[ = 0 \] (1.90)
are considered. These conditions exactly resemble the homogeneous Maxwell Equations (1.35) and (1.30).

1.2.5 Dynamics of the Gauge Field

In the previous section the gauge fields \( A_i \) and \( \varphi \) were introduced in order to define parallel reference frames at different points in spacetime. The approach was general and the values of \( A_i \) and \( \varphi \) are still unknown. How are these values obtained?

If it is assumed that \( A_i \) and \( \varphi \) are physical fields it can further be assumed that they are determined by equations of motion which can be constructed according to the guidelines of classical field theory. These guidelines imply that equations of motion can often (but not always) be concisely characterized by a Lagrangian density
\[ \mathcal{L} = \mathcal{L}(\Psi, \partial_i \Psi, \partial_t \Psi) \] (1.92)
which, in the standard case, is a function of the fields \( \Psi \) of the theory and their first derivatives. Integration of the Lagrangian density \( \mathcal{L} \) over space yields the Lagrangian \( L \),
\[ L = \int \mathcal{L}(\Psi, \partial_i \Psi, \partial_t \Psi) \, dv, \] (1.93)
and further integration over time yields the action \( S \),
\[ S = \int L \, dt. \] (1.94)

There are guiding principles that show how to obtain an appropriate Lagrangian density for a given theory. Once an appropriate Lagrangian density is constructed, the properties of the fields \( \Psi \) can be conveniently derived. For example, the equations of motion which determine the dynamics of \( \Psi \) follow from extremization of the action \( S \) with respect to variations of \( \Psi \),
\[ \delta \Psi S = 0 \implies \text{equations of motion for } \Psi. \] (1.95)

More explicitly, the equations of motion are given by the well-known Euler–Lagrange equations
\[ \partial_j \left( \frac{\partial \mathcal{L}}{\partial \partial_j \Psi} \right) - \partial_i \left( \frac{\partial \mathcal{L}}{\partial \partial_i \Psi} \right) - \frac{\partial \mathcal{L}}{\partial \Psi} = 0. \] (1.96)

The Lagrangian density \( \mathcal{L}_{\text{gauge}} \) of the gauge fields \( A_i \) and \( \varphi \) has to fulfill a number of requirements:
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- It should have the geometric character of a scalar density in order to be a proper volume integrand (compare section A.3).
- It should be gauge invariant.
- It should be relativistically invariant.
- It should have the SI unit $[L] = \text{VAs/m}^3$.
- It should be no more than quadratic in the fields $A_i$ and $\varphi$.
- It should contain no higher-order derivatives than first-order derivatives to yield second-order equations of motion.

These requirements are quite stringent. In fact, with the fields $A_i$, $\varphi$ and the gauge invariant quantities $E_i$, $B_i$ alone it is not possible to construct a proper Lagrangian density. To construct a Lagrangian density it is necessary to introduce, as in Section 1.1.3, a metric structure $g_{ij} = g_{ji}$ that characterizes the geometry of spacetime. To get the dimensions right it is further necessary to introduce two constants $\varepsilon_0$ and $\mu_0$ with SI units $[\varepsilon_0] = \text{As/Vm}$ and $[\mu_0] = \text{Vs/Am}$, respectively. Then the only meaningful combination of $A_i$ and $\varphi$ can be written in terms of $E_i$ and $B_i$. It is given by the expression\(^4\)

$$L_{\text{gauge}} = \frac{1}{2} \left( \varepsilon_0 \sqrt{g} g^{ij} E_i E_j - (\mu_0 \sqrt{g})^{-1} g_{ij} B^i B^j \right).$$

(1.97)

So far, only the gauge fields have been taken into account. The Lagrangian density in Equation (1.97) corresponds to a free gauge field theory with no coupling to electrically charged matter fields. The inclusion of matter fields requires setting up a corresponding Lagrangian density $L_{\text{matter}}$. As indicated in Equation (1.92), this will involve derivatives $\partial_i \Psi$, $\partial_t \Psi$ of the fields $\Psi$. In order to be gauge invariant, that is to be independent of a specific choice of reference frames, these derivatives are expected from Equations (1.47) and (1.48) to involve the gauge fields. Indeed, if the relations in Equations (1.57), (1.59) and (1.77) are recalled it is possible to write

$$\partial_i \Psi = (\partial_i \Psi)^R e_R + \Psi^R (\partial_i e_R)$$

$$= (\partial_i \Psi^R) e_R - j \partial_i \beta_R \Psi^R e_R$$

$$= \left( \partial_i \Psi^R + j \frac{q}{\hbar} A_i \Psi^R \right) e_R$$

$$= D^A_i \Psi^R e_R$$

(1.98)

with the gauge covariant derivative

$$D^A_i := \partial_i + \frac{j}{\hbar} q A_i.$$

(1.99)

Analogously, it follows with Equation (1.80)

$$\partial_t \Psi = (\partial_t \Psi)^R e_R = (\partial_t \Psi^R) e_R + \Psi^R (\partial_t e_R)$$

$$= (\partial_t \Psi^R) e_R - j \partial_t \beta_R \Psi^R e_R$$

$$= \left( \partial_t \Psi^R - \frac{q}{\hbar} \varphi \Psi^R \right) e_R$$

$$= D^\varphi \Psi^R e_R.$$

(1.100)

\(^4\)The factor $1/2$ is introduced to yield the correct Hamilton function (energy function) which can be obtained from the Lagrangian density by means of a Legendre transformation.
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with the gauge covariant derivative

\[ D^\mu_i := \partial_i - j \frac{q}{\hbar} \phi. \]  

(1.101)

Therefore, in order to be gauge invariant, the matter Lagrangian density may only contain derivatives and gauge fields as combinations of gauge covariant derivatives. To explicitly obtain the matter Lagrangian density requires advanced knowledge of relativistic quantum mechanics. With reference to literature \[18, 19\] it is quoted as a result that the Lagrangian density of a certain class of electrically charged matter fields, like electrons, is given by\(^5\)

\[ \mathcal{L}_{\text{matter}} = - j \hbar c \overline{\Psi}^R \gamma^i \left( D^A_i - \frac{mc}{\hbar} \right) \Psi^R + j \hbar \overline{\Psi}^R \gamma^0 \left( D_i^\rho - \frac{me^2}{\hbar} \right) \Psi^R. \]  

(1.102)

The terms that couple the gauge fields to the matter fields can be written as

\[ \mathcal{L}_{\text{coupling}} = - A_i J^i - \phi \rho, \]  

(1.103)

with the definitions

\[ J^i := q c \overline{\Psi}^R \gamma^i \Psi^R, \]  

(1.104)

\[ \rho := q \overline{\Psi}^R \gamma^0 \Psi^R. \]  

(1.105)

Then the dynamics of the gauge fields \( A_i \) and \( \phi \) is determined from the combined Lagrangian density

\[ \mathcal{L}_{\text{em}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{coupling}}. \]  

(1.106)

The equations of motion that follow from this Lagrangian density are given by the Euler–Lagrange Equations (1.96). In the present case they acquire the form

\[ \partial_j \left( \frac{\partial \mathcal{L}_{\text{em}}}{\partial (D_j^A)} \right) - \partial_i \left( \frac{\partial \mathcal{L}_{\text{em}}}{\partial A_i} \right) = 0, \]  

(1.107)

\[ \partial_j \left( \frac{\partial \mathcal{L}_{\text{em}}}{\partial (\partial_j \phi)} \right) - \partial_i \left( \frac{\partial \mathcal{L}_{\text{em}}}{\partial \phi} \right) = 0. \]  

(1.108)

Inserting Equations (1.97) and (1.103) into Equation (1.106) yields

\[ \epsilon^{ijk} \partial_j H_k - \partial_i D^j = J^i, \]  

(1.109)

\[ \partial_i D^i = \rho, \]  

(1.110)

with the definitions

\[ H_k := (\mu_0 \sqrt{g})^{-1} g_{kl} B^l, \]  

(1.111)

\[ D^i := \varepsilon_0 \sqrt{g} g^{ij} E_j. \]  

(1.112)

Equations (1.109) and (1.110) are recognized as the inhomogeneous Maxwell Equations (1.16) and (1.5), respectively. It follows that the gauge fields \( A_i \) and \( \phi \), that have been introduced in the formal context of reference frames, do indeed constitute the electromagnetic potentials that are

\(^5\) In this expression \( \Psi^R \) denotes a four-component spinor, \( \overline{\Psi}^R \) is the adjoint spinor of \( \Psi^R \), and \( \gamma^0, \gamma^i \) are \( 4 \times 4 \) matrices [19].
familiar from classical electrodynamics. In this framework the homogeneous Maxwell equations turn out to be mathematical integrability conditions while the inhomogeneous Maxwell equations represent the equations of motion of the gauge fields $A_i$ and $\varphi$.

### 1.3 The Relation Between the Axiomatic Approach and the Gauge Field Approach

In the following text some comments are made on the interrelation between the previously presented axiomatic approach and the gauge field approach. It is interesting to see how the axioms find their proper place within the gauge approach.

#### 1.3.1 Noether Theorem and Electric Charge Conservation

In field theory there is a famous result which connects symmetries of laws of nature to conserved quantities. This is the Noether theorem [20] which has been proven to be of great importance in both classical and quantum contexts. It is, in particular, discussed in books on classical electrodynamics, (see [7, 21] for example). The Noether theorem connects the symmetry of a Lagrangian density $\mathcal{L}(\Psi, \partial_\mu \Psi, \partial_\mu \bar{\Psi})$, compare Equation (1.93), to conserved quantities. Suppose, for example, that $\mathcal{L}$ is invariant under time translations $\delta_t$. From daily experience this assumption seems plausible since it is not expected that the laws of nature change in time. Then the Noether theorem implies a local conservation law which expresses the conservation of energy. Similarly, invariance under translations $\delta_{x_i}$ in space implies conservation of momentum, while invariance under rotations $\delta_{\omega_{ij}}$ yields the conservation of angular momentum,

$$\delta_t \mathcal{L} = 0 \implies \text{conservation of energy} \quad \text{(1.113)}$$

$$\delta_{x_i} \mathcal{L} = 0 \implies \text{conservation of momentum} \quad \text{(1.114)}$$

$$\delta_{\omega_{ij}} \mathcal{L} = 0 \implies \text{conservation of angular momentum} \quad \text{(1.115)}$$

These symmetries of spacetime are called external symmetries, but the Noether theorem also works for other types of symmetries that are called internal ones. Gauge symmetries often are internal symmetries. In this case, gauge invariance of the Lagrangian implies a conserved current with an associated charge. That is, if a gauge transformation is denoted by $\delta_\epsilon$ it can be concluded that

$$\delta_\epsilon \mathcal{L} = 0 \implies \text{charge conservation} \quad \text{(1.116)}$$

If this conclusion is applied to electrodynamics it is necessary to specify the Lagrangian density to be the one of matter fields that represent electrically charged particles. Then, invariance of this Lagrangian density under the gauge symmetry of electrodynamics yields the conservation of electric charge. Thus, if the validity of the Lagrangian formalism is accepted, electric charge conservation is obtained from gauge invariance via the Noether theorem.

#### 1.3.2 Minimal Coupling and the Lorentz Force

It has already been mentioned that, according to Equation (1.95), the equations of motion (1.96) of a physical theory can be derived from a Lagrangian density and its associated action.
This scheme can be used to derive the equations of motion of electrically charged particles. In this case, the corresponding Lagrangian density (that of the electrically charged particles) has to be gauge invariant.

For the electromagnetic case it has been demonstrated that the Lagrangian density will be gauge invariant if partial derivatives are converted to gauge covariant derivatives (1.99) and (1.101) according to

\[
\partial_i \rightarrow D_i^A := \partial_i + j A_i, \quad (1.117)
\]

\[
\partial_i \rightarrow D_i^\varphi := \partial_i - j \frac{\hbar}{\epsilon_0} \varphi, \quad (1.118)
\]

with \( q \) the electric charge of the particle under consideration. The substitutions of Equations (1.117) and (1.118) constitute the simplest way to ensure gauge invariance of the Lagrangian density of electrically charged particles. They constitute what is commonly called minimal coupling. Due to minimal coupling, electrically charged particles and the electromagnetic field are related in a natural way that is dictated by the requirement of gauge invariance.

If it is assumed, as in Section 1.2.2, that

\[
\Psi^R = \Psi_0 e^{-\frac{j}{\hbar} \int (p_i x_i - E t) dt}, \quad (1.119)
\]

it follows that

\[
j \hbar \partial_i \Psi^R = p_i \Psi^R, \quad (1.120)
\]

\[
- j \hbar \partial_t \Psi^R = E \Psi^R. \quad (1.121)
\]

These are relations that indicate how to pass from quantum physics to classical physics, that is how to pass from the action of the differential operators \( j \hbar \partial_i \) and \( - j \hbar \partial_t \) on a wave function to the momentum and energy of a classical particle. It follows that the classical analogues of Equations (1.117) and (1.118) are given by

\[
p_i \rightarrow p_i - q A_i, \quad (1.122)
\]

\[
E \rightarrow E - q \varphi. \quad (1.123)
\]

That is, if electrically charged particles are represented by classical particles, rather than by wave functions, it is necessary to replace within the corresponding classical Lagrangian function the energy \( E \) and the momentum \( p_i \) of each particle according to Equations (1.122) and (1.123).

As a general example a nonrelativistic classical particle with mass \( m \) and charge \( q \) can be considered. In the absence of an electromagnetic field\(^6\) the energy and momentum of the particle are related by

\[
E = \frac{p_i p^j}{2m}. \quad (1.124)
\]

In the presence of an electromagnetic field the replacements in Equations (1.122) and (1.123) lead to

\[
E = \frac{(p_i - q A_i)(p^j - q A^j)}{2m} + q \varphi. \quad (1.125)
\]

\(^6\) Also the absence of a gravitational field is assumed.
It follows from this expression for the energy \( E \) that the Lagrange function \( L \) of the particle is given by [22, p. 167]

\[
L = \frac{m}{2} (\partial_t x^i)(\partial_t x^i) + q A_i (\partial_t x^i) - q \varphi .
\]

(1.126)

The equation of motion is obtained from the Lagrange function via

\[
\frac{d}{dt} \frac{\partial L}{\partial (\partial_t x^i)} - \frac{\partial L}{\partial x^i} = 0 .
\]

(1.127)

This yields with Equation (1.126)

\[
m (\partial_t^2 x^i) + q (\partial_i A_i + \partial_i x^j (\partial_j A_i)) - q \partial_i x^j (\partial_t A_j) + q \partial_i \varphi = 0 .
\]

(1.128)

The first term represents the force \( F_i = m (\partial_t^2 x_i) \) that acts on the particle. Rearranging the other terms and noting, in particular, the identity

\[
\partial_i A_j - \partial_j A_i = \epsilon_{ijk} \epsilon^{klm} \partial_l A_m
\]

(1.129)
yields

\[
F_i = q (-\partial_i \varphi - \partial_i A_i) + q (\epsilon_{ijk} \partial_k x^j \epsilon^{klm} \partial_l A_m) .
\]

(1.130)

Finally, by means of Equations (1.86) and (1.87) the gauge fields \( A_i \) and \( \varphi \) are replaced by the field strengths \( E_i \) and \( B^i \). The result is the Lorentz force law (1.17),

\[
F_i = q (E_i + \epsilon_{ijk} \partial_k x^j B^k) .
\]

(1.131)

Therefore, in the gauge field approach the Lorentz force is a consequence of the minimal coupling procedure which couples electrically charged particles to the electromagnetic potentials.

1.3.3 Bianchi Identity and Magnetic Flux Conservation

The electromagnetic gauge fields \( A_i \) and \( \varphi \) are often introduced as mathematical tools to facilitate the integration of the Maxwell equations. Indeed, Equations (1.90) and (1.91) have revealed that the homogeneous Maxwell equations reduce to mere integrability conditions that are automatically fulfilled if the electromagnetic field strengths are expressed in terms of the gauge potentials. This is an interesting observation since within the gauge approach the gauge potentials are fundamental physical quantities and are not only the outcome of a mathematical trick. Thus it can be stated that the mathematical structure of the gauge potentials already implies the homogeneous Maxwell equations and, in turn, magnetic flux conservation. In this light, magnetic flux conservation, within the gauge approach, appears as the consequence of a geometric identity. This is in contrast to electric charge conservation that can be viewed as the consequence of gauge invariance, that is via the Noether theorem as the consequence of a physical symmetry.

The integrability conditions that are reflected in the homogeneous Maxwell equations are special cases of Bianchi identities. Bianchi identities are the result of differentiating a potential twice. For example, in electrostatics the electric field strength \( E_i \) can be derived from a scalar potential \( \varphi \) according to

\[
E_i = -\partial_i \varphi .
\]

(1.132)
Differentiation reveals that the curl of $E_i$ vanishes,

$$\epsilon^{ijk}\partial_j E_k = \epsilon^{ijk}\partial_j \partial_k \psi = 0,$$

which is due to the antisymmetry of $\epsilon^{ijk}$. Again, this equation is a mathematical identity, a simple example of a Bianchi identity.

### 1.3.4 Gauge Approach and Constitutive Relations

The gauge approach towards electrodynamics defines the properties of gauge fields, which represent the electromagnetic field, and depends on matter fields. However, it does not depend on a metric structure of spacetime. In contrast to this, the constitutive relations do depend on a metric structure of spacetime, as can already be seen from the constitutive relations of vacuum

---

**Figure 1.6** Interrelations between the axiomatic approach (rectangular frames) and the gauge field approach (elliptic frames) of classical electrodynamics. Both approaches yield the Maxwell equations. The gauge approach requires the knowledge of constitutive relations which represent the fourth axiom of the axiomatic approach. However, the first, second and third axiom of the axiomatic approach can be obtained from the gauge approach. Electric charge conservation, the first axiom of the gauge approach, represents the gauge symmetry of electrodynamics by means of the Noether theorem.
that involve the metric $g_{ij}$, compare Equations (1.36) and (1.37). Thus, also in the gauge approach the constitutive relations have to be postulated as an axiom in some way. One should note that, according to Equations (1.86) and (1.87), the gauge potentials are directly related to the field strengths $E_i$ and $B_i$. The excitations $D_i$ and $H_i$ are part of the inhomogeneous Maxwell equations which, within the gauge approach, are derived as equations of motion from an action principle (see Equations (1.109) and (1.110)). Since the action itself involves the gauge potentials, one might wonder how it is possible to obtain equations of motion for the excitations rather than for the field strengths. The answer is that during the construction of the Lagrangian density (1.97) from the gauge potentials the constitutive relations are implicitly used. Figure 1.6 summarizes the interrelations between the axiomatic approach and the gauge approach.

1.4 Solutions of Maxwell Equations

From the axiomatic approach and from the gauge field approach to classical electrodynamics the Maxwell Equations (1.5), (1.16), (1.30) and (1.35) have been obtained. Appropriate constitutive relations of the form (1.36) and (1.37) or, more generally, (1.38) and (1.39) make Maxwell equations a set of determined partial differential equations. Within the limits of classical physics these equations model the interaction between electromagnetic sources $\rho$, $J^i$, and the electromagnetic field, represented by $(E_i, B^i)$ and $(D_i, H_i)$. In order to explicitly formulate a specific problem it is necessary to impose physically meaningful initial and boundary conditions that lead to a well-defined boundary value problem. The solution of such a boundary value problem, in turn, determines a unique solution of Maxwell equations.

For the solution of an electromagnetic boundary value problem it is often advantageous to first rewrite Maxwell equations as (second-order) wave equations. This is straightforward as long as the constitutive relations are of a simple form. Clearly, the solution of wave equations has been studied in many branches of physics and mathematics for a long time and a variety of corresponding solution procedures exist [23].

In the following, the tensor notation of the previous sections will no longer be used and it will be replaced by the more conventional vector notation of ordinary vector analysis that has been adopted by many authors of standard textbooks (see for example [1, 5, 6, 8, 24]). The widespread use of this notation is the reason it will be adopted here as well. Hence, covariant vectors $E_i$, $H_i$ are replaced by ordinary vectors in three-dimensional space, $E_i \rightarrow E$, $H_i \rightarrow H$. Also the contravariant vector densities $B^i$, $D^i$ and $J^i$ are replaced by ordinary vectors, $B^i \rightarrow B$, $D^i \rightarrow D$ and $J^i \rightarrow J$, while the notation for the scalar density $\rho$ remains unchanged. It is obvious that due to this transition information on the geometric properties of the electromagnetic quantities is lost. For example, integration along lines or over surfaces is no longer defined in a natural way since line integrals $\int_c E \cdot dt$ or surface integrals $\int_S B \cdot da$ now require a metric structure of space which is represented by the scalar product. Formally, also the expressions $\int_c D \cdot dt$ or $\int_S H \cdot da$ can be formed but the physical interpretation of these integrals is not clear and certainly needs explanation since the electric excitation $D$ or the magnetic excitation $H$ are not natural integrands of line- or surface-integrals, respectively.

---

7 Constitutive relations are usually not of a simple form if they introduce material parameters that are space or time dependent, if they mix electric and magnetic fields, or if they introduce nonlinearities, for example. In such cases analytic solutions of Maxwell equations often cannot be found and it is necessary to apply numerical methods directly to the Maxwell equations.
This indicates that it is important to keep the limitations of the vector notation in mind in order to construct mathematical expressions of physical objects in a meaningful way.

In the following text, electromagnetic quantities that are defined in the time domain will be considered, but these will quickly be transformed to the frequency domain (see Section 1.4.1.3) and to reciprocal space (see Section 1.4.1.4). It is convenient to print the corresponding Fourier transforms using different fonts. Then it is not necessary to always keep the arguments \( r, t, \omega \) or \( k \) in parentheses next to the symbols of the fields to indicate which domain or space they belong to. The letters of the different fonts are introduced in Table 1.1. Greek letters remain unaltered but this should not lead to confusion since usually they do not appear isolated in an equation.

### 1.4.1 Wave Equations

#### 1.4.1.1 Decoupling of Maxwell Equations

Rewriting Maxwell Equations (1.5), (1.16), (1.30) and (1.35) in vector notation yields the familiar expressions

\[
\nabla \cdot \mathbf{D}(r, t) = \rho(r, t),
\]

\[
\nabla \times \mathbf{H}(r, t) - \frac{\partial \mathbf{D}}{\partial t}(r, t) = \mathbf{J}(r, t),
\]

\[
\nabla \cdot \mathbf{B}(r, t) = 0,
\]

\[
\nabla \times \mathbf{E}(r, t) + \frac{\partial \mathbf{B}}{\partial t}(r, t) = 0.
\]

To decouple the Maxwell equations, constitutive relations are assumed which characterize a homogeneous, isotropic medium. These are of the form

\[
\mathbf{D}(r, t) = \varepsilon \mathbf{E}(r, t),
\]

\[
\mathbf{B}(r, t) = \mu \mathbf{H}(r, t).
\]
with constant parameters \( \varepsilon \) and \( \mu \). Then the curl operator \( \nabla \times \) is applied to Equations (1.135) and (1.137) and the results are combined. This yields

\[
\nabla \times \nabla \times \mathbf{E}(r, t) + \varepsilon \mu \frac{\partial^2 \mathbf{E}(r, t)}{\partial t^2} = -\mu \frac{\partial \mathbf{J}(r, t)}{\partial t},
\]

(1.140)

\[
\nabla \times \nabla \times \mathbf{B}(r, t) + \varepsilon \mu \frac{\partial^2 \mathbf{B}(r, t)}{\partial t^2} = \mu \nabla \times \mathbf{J}(r, t).
\]

(1.141)

These equations can be transformed into standard wave equations if the identity (C.6) is used, together with the constitutive relations (1.138) and (1.139) and the Maxwell Equations (1.134) and (1.136). This yields

\[
\Delta \mathbf{E}(r, t) - \varepsilon \mu \frac{\partial^2 \mathbf{E}(r, t)}{\partial t^2} = \frac{1}{\varepsilon} \nabla \rho(r, t) + \mu \frac{\partial \mathbf{J}(r, t)}{\partial t},
\]

(1.142)

\[
\Delta \mathbf{B}(r, t) - \varepsilon \mu \frac{\partial^2 \mathbf{B}(r, t)}{\partial t^2} = -\mu \nabla \times \mathbf{J}(r, t).
\]

(1.143)

Due to the constitutive relations (1.138) and (1.139) two analogous equations are valid for \( \mathbf{D}(r, t) \) and \( \mathbf{H}(r, t) \) which furnish no additional information. Therefore, six scalar equations are obtained for six unknown field components. Equations (1.142) and (1.143) constitute inhomogeneous wave equations\(^8\) with phase velocity

\[
c = \frac{1}{\sqrt{\varepsilon \mu}}.
\]

(1.144)

In a vacuum (\( \rho(r, t) = 0, \mathbf{J}(r, t) = 0 \)) the inhomogeneous terms on the right-hand sides vanish. Then the homogeneous wave equations

\[
\Delta \mathbf{E}(r, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}(r, t)}{\partial t^2} = 0,
\]

(1.145)

\[
\Delta \mathbf{B}(r, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}(r, t)}{\partial t^2} = 0.
\]

(1.146)

are obtained

**1.4.1.2 Equations of Motion for the Electromagnetic Potentials**

It has already been seen that the field strengths \( \mathbf{E}(r, t) \) and \( \mathbf{B}(r, t) \) can be derived from the scalar potential \( \varphi(r, t) \) and the vector potential \( \mathbf{A}(r, t) \) via

\[
\mathbf{E}(r, t) = -\nabla \varphi(r, t) - \frac{\partial \mathbf{A}(r, t)}{\partial t},
\]

(1.147)

\[
\mathbf{B}(r, t) = \nabla \times \mathbf{A}(r, t).
\]

(1.148)

If the electromagnetic field is expressed by means of \( \varphi(r, t) \) and \( \mathbf{A}(r, t) \) the homogeneous Maxwell Equations (1.136) and (1.137) are recognized as geometric identities which are

\(^8\) Since in Equations (1.142) and (1.143) the wave operator \( \Delta - \varepsilon \mu \frac{\partial^2}{\partial t^2} \) acts on three-dimensional vectors it is clear that these equations are vector wave equations. However, equations of the type (1.140) and (1.141) are often also called vector wave equations, even though the differential operator \( -\left( \nabla \times \nabla \times \right) - \varepsilon \mu \frac{\partial^2}{\partial t^2} \) does not always coincide with the wave operator. It does coincide with the wave operator if it acts on a divergence free vector field.
automatically fulfilled. Then the remaining inhomogeneous Maxwell Equations (1.134) and (1.135) determine the electromagnetic field. Replacing within the inhomogeneous Maxwell equations the excitations $\mathbf{D}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t)$ by means of the constitutive relations (1.138) and (1.139) and Equations (1.147) and (1.148) by $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ yields

$$\Delta \phi(\mathbf{r}, t) + \frac{\partial (\nabla \cdot \mathbf{A}(\mathbf{r}, t))}{\partial t} = -\frac{\rho(\mathbf{r}, t)}{\varepsilon}, \quad (1.149)$$

$$\Delta \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} - \nabla \left( \nabla \cdot \mathbf{A}(\mathbf{r}, t) + \frac{1}{c^2} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} \right) = -\mu \mathbf{J}(\mathbf{r}, t). \quad (1.150)$$

These are four scalar equations for the four unknown field components $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$. Since electrodynamics is invariant under the gauge transformations

$$\delta_\epsilon \phi(\mathbf{r}, t) = -\frac{\partial \epsilon(\mathbf{r}, t)}{\partial t}, \quad (1.151)$$

$$\delta_\epsilon \mathbf{A}(\mathbf{r}, t) = \mathbf{A}'(\mathbf{r}, t) - \mathbf{A}(\mathbf{r}, t) = \nabla \epsilon(\mathbf{r}, t), \quad (1.152)$$

with an arbitrary function $\epsilon(\mathbf{r}, t)$ it is possible to simplify Equations (1.149) and (1.150) by the choice of a particular gauge. Common gauges are the Coulomb gauge

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0$$

and the Lorenz gauge

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) + \frac{1}{c^2} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} = 0. \quad (1.154)$$

The Coulomb gauge leads to

$$\Delta \phi(\mathbf{r}, t) = -\frac{\rho(\mathbf{r}, t)}{\varepsilon}, \quad (1.155)$$

$$\Delta \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} - \frac{1}{c^2} \frac{\partial (\nabla \phi(\mathbf{r}, t))}{\partial t} = -\mu \mathbf{J}(\mathbf{r}, t), \quad (1.156)$$

while the Lorenz gauge yields a scalar and a vector wave equation,

$$\Delta \phi(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \phi(\mathbf{r}, t)}{\partial t^2} = -\frac{\rho(\mathbf{r}, t)}{\varepsilon}, \quad (1.157)$$

$$\Delta \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = -\mu \mathbf{J}(\mathbf{r}, t). \quad (1.158)$$

### 1.4.1.3 Maxwell Equations in the Frequency Domain and Helmholtz Equations

Time harmonic fields with sinusoidal time dependency can be expressed as

$$\mathbf{F}_{\text{sinus}}(\mathbf{r}, t) = \text{Re} \left[ \mathbf{F}(\mathbf{r}, \omega) e^{i\omega t} \right]. \quad (1.159)$$

This is a special case of the Fourier representation of a field with arbitrary time dependency,

$$\mathbf{F}(\mathbf{r}, t) = \text{Re} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{F}(\mathbf{r}, \omega) e^{i\omega t} d\omega \right]. \quad (1.160)$$
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In the time harmonic case it is straightforward to pass to the frequency domain and to write the Maxwell equations as

\[
\nabla \cdot D(r, \omega) = \rho(r, \omega), \\
\nabla \times H(r, \omega) - j \omega D(r, \omega) = J(r, \omega), \\
\nabla \cdot B(r, \omega) = 0, \\
\n\nabla \times E(r, \omega) + j \omega B(r, \omega) = 0.
\]

(1.161)  
(1.162)  
(1.163)  
(1.164)

The vector wave Equations (1.140) and (1.142) of the electric field \( E \), for example, convert in the frequency domain to vector Helmholtz equations

\[
\nabla \times \nabla \times E(r, \omega) - k^2 E(r, \omega) = -j \omega \mu J(r, \omega), \\
\Delta E(r, \omega) + k^2 E(r, \omega) = \frac{1}{\varepsilon} \nabla \rho(r, \omega) + j \omega \mu J(r, \omega),
\]

(1.165)  
(1.166)

with \( k = \omega/c \) the wave number. For the magnetic field \( B \) this yields

\[
\nabla \times \nabla \times B(r, \omega) - k^2 B(r, \omega) = \mu \nabla \times J(r, \omega), \\
\Delta B(r, \omega) + k^2 B(r, \omega) = -\mu \nabla \times J(r, \omega).
\]

(1.167)  
(1.168)

The Helmholtz equations for the scalar and vector potential in the Lorenz gauge assume the form

\[
\Delta \phi(r, \omega) + k^2 \phi(r, \omega) = -\frac{\rho(r, \omega)}{\varepsilon}, \quad \text{(Lorenz gauge)} \\
\Delta A(r, \omega) + k^2 A(r, \omega) = -\mu J(r, \omega). \quad \text{(Lorenz gauge)}
\]

(1.169)  
(1.170)

If compared to Equations (1.166) and (1.168) these equations are more simple since now the source terms involve no derivatives.

### 1.4.1.4 Maxwell Equations in Reciprocal Space

Fields \( F(r, t) \) that are defined in the time domain are transformed to reciprocal space by a spatial Fourier transform according to

\[
F(k, \omega) = \frac{1}{(2\pi)^{3/2}} \int F(r, t) e^{jk \cdot r} \, d^3r.
\]

(1.171)

The inverse transform is given by

\[
F(r, t) = \frac{1}{(2\pi)^{3/2}} \int F(k, \omega) e^{-jk \cdot r} \, d^3k.
\]

(1.172)

The operator \( \nabla \) transforms to multiplication by \( -jk \) in reciprocal space. Therefore, the Maxwell equations in reciprocal space become

\[
-jk \cdot D(k, t) = \rho(k, t), \\
-jk \times H(k, t) - \frac{\partial D(k, t)}{\partial t} = J(k, t), \\
-jk \cdot B(k, t) = 0, \\
-jk \times E(k, t) + \frac{\partial B(k, t)}{\partial t} = 0.
\]

(1.173)  
(1.174)  
(1.175)  
(1.176)
This representation of the Maxwell equations has the advantage that the fields and time derivatives all depend on the same point \( \mathbf{k} \) in reciprocal space. Hence, the partial differential equations of real space become strictly local equations in reciprocal space.

### 1.4.2 Boundary Conditions at Interfaces

At the transition between two media with parameters \((\varepsilon_1, \mu_1, \sigma_1)\) and \((\varepsilon_2, \mu_2, \sigma_2)\) the boundary conditions can be derived from the Maxwell equations by means of integration and the application of Stokes’ theorem. This is a standard procedure which is described in many textbooks (for example \[25, \text{Section 7.3.6.}\]). This yields

\[
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \implies (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{e}_n = 0, \tag{1.177}
\]

\[
\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \implies (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{e}_n = \mathbf{J}_s, \tag{1.178}
\]

\[
\nabla \cdot \mathbf{B} = 0 \implies (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{e}_n = 0, \tag{1.179}
\]

\[
\nabla \cdot \mathbf{D} = \rho \implies (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{e}_n = \rho_s. \tag{1.180}
\]

These boundary conditions are valid both in the time and the frequency domain. The vector \( \mathbf{e}_n \) denotes a normal unit vector that points on the interface between the different media from medium 1 to medium 2 and \( \mathbf{J}_s \) denotes a surface current that may flow on the interface between the media. Accordingly, \( \rho_s \) denotes a surface charge density.

Additionally, there is a boundary condition for the magnetic vector potential \( \mathbf{A} \):

\[
\mathbf{B} = \nabla \times \mathbf{A} \implies (\mathbf{A}_1 - \mathbf{A}_2) \times \mathbf{e}_n = 0. \tag{1.181}
\]

This boundary condition is gauge invariant since it is based on the rotational part of the magnetic vector potential.

### 1.4.3 Dynamical and Nondynamical Components of the Electromagnetic Field

#### 1.4.3.1 Helmholtz’s Vector Theorem, Longitudinal and Transverse Fields

In the study of vector fields \( \mathbf{F} \) the Helmholtz’s vector theorem is a useful tool \[23\]. It states that any vector field \( \mathbf{F} \), which is finite, uniform, continuous and square integrable, may be split into a longitudinal or irrotational part \( \mathbf{F}_\parallel \) and a transverse or rotational part \( \mathbf{F}_\perp \),

\[
\mathbf{F} = \mathbf{F}_\parallel + \mathbf{F}_\perp, \tag{1.182}
\]

where \( \mathbf{F}_\parallel \) and \( \mathbf{F}_\perp \) are implicitly defined by

\[
\nabla \times \mathbf{F}_\parallel = 0, \tag{1.183}
\]

\[
\nabla \cdot \mathbf{F}_\perp = 0. \tag{1.184}
\]

This split is unique. A good discussion of the Helmholtz’s vector theorem is contained in \[26\] where it is stressed that the theorem critically depends on the boundary conditions of the field \( \mathbf{F} \). In the following discussion, it is assumed that the boundary conditions are such that the
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Figure 1.7  Coordinates of a moving point charge $q$. Position and velocity of the point charge are related by $v(t) = \frac{\partial R(t)}{\partial t}$.

Helmholtz’s vector theorem can be applied. The names longitudinal and transverse acquire a clear geometric interpretation in reciprocal space where Equations (1.183) and (1.184) become

$$-jk \times F_\parallel = 0, \quad (1.185)$$
$$-jk \cdot F_\perp = 0, \quad (1.186)$$

that is, $F_\parallel$ is parallel to $k$ and $F_\perp$ is perpendicular to $k$.

By means of the vector identity (C.6) and the relation

$$\Delta \left( \frac{1}{|r - r'|} \right) = -4\pi \delta(r - r') \quad (1.187)$$

it can be shown that in real space the longitudinal and transverse part of a vector field $F(r, t)$ are given by

$$F_\parallel(r, t) = -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot F(r', t)}{|r - r'|} \, d^3r', \quad (1.188)$$
$$F_\perp(r, t) = \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{F(r', t)}{|r - r'|} \, d^3r', \quad (1.189)$$

respectively. These explicit formulas show that the split $F = F_\parallel + F_\perp$ introduces nonlocal effects: both $F_\parallel(r, t)$ and $F_\perp(r, t)$, considered at a fixed time $t$ and at a specific point $r$, depend on the values of $F(r', t)$ at the same time and at all points $r'$ in space. Conversely, even if $F(r, t)$ is localized in space, that is if it vanishes outside a compact region, the parts $F_\parallel(r, t)$ and $F_\perp(r, t)$ will generally extend over the whole space.

Example 1.1: Consider a point charge $q$ at a position $R(t)$ which moves with velocity $v(t)$ and is observed from a position $r$, (see Figure 1.7).

The charge density $\rho$ and current density $J$ of this point charge are given by

$$\rho(r, t) = q\delta(r - R(t)), \quad (1.190)$$

From a microscopic but still classical point of view macroscopic boundary conditions are the result of the interaction between the electromagnetic field and electrically charged particles, like electrons or protons. Therefore, if fundamental properties of the electromagnetic field are considered, it is possible to focus on these microscopic interactions and there is no need to separately consider macroscopic boundary conditions. For example, the boundary conditions that are imposed by a perfect conductor can be replaced by the interaction between the electromagnetic field and the electrons and protons that represent the electrically charged particles of the perfect conductor.
and

\[ J(r,t) = qv(t)\delta(r - R(t)), \]

respectively. To calculate the corresponding longitudinal current \( J_\parallel(r,t) \), Equation (1.188) is used and the continuity equation

\[ \nabla' \cdot J(r',t) = -\frac{\partial \rho}{\partial t}(r',t) \]

is applied. This yields

\[ J_\parallel(r,t) = \frac{q}{4\pi} \frac{\partial}{\partial t} \int \frac{\delta(r' - R(t))}{|r - r'|} \, d^3r' \]

\[ = \frac{q}{4\pi} \frac{\partial}{\partial t} \left( \frac{1}{|r - R(t)|} \right) \]

\[ = -\frac{q}{4\pi} \frac{\partial}{\partial t} \left( \frac{r - R(t)}{|r - R(t)|^3} \right) \]

\[ = \frac{q}{4\pi} \left[ \frac{v(t)}{|r - R(t)|^3} - \frac{3(r - R(t))[(r - R(t)) \cdot v(t)]}{|r - R(t)|^5} \right]. \]

Accordingly, due to \( J_\perp(r,t) = J(r,t) - J_\parallel(r,t) \), there is also

\[ J_\perp(r,t) = \frac{q}{4\pi} \left[ 4\pi v(t)\delta(r - R(t)) - \frac{v(t)}{|r - R(t)|^3} + \frac{3(r - R(t))[(r - R(t)) \cdot v(t)]}{|r - R(t)|^5} \right]. \]

and it is clearly seen that both \( J_\parallel(r,t) \) and \( J_\perp(r,t) \) extend over the whole space.

### 1.4.3.2 Nondynamical Maxwell Equations as Boundary Conditions in Time

Among the complete set of Maxwell Equations (1.134) to (1.137) the Equations (1.134) and (1.136) are not dynamical equations but rather so-called boundary conditions that determine appropriate initial conditions of the fields. By virtue of the remaining dynamical Maxwell equations they are fulfilled at all times if they are fulfilled at one time. To illustrate this circumstance for the boundary condition (1.136) it is assumed that at some initial time \( t_0 \)

\[ \nabla \cdot B \bigg|_{t_0} = 0. \]

It then needs to be shown that at an infinitesimally later time \( t_0 + dt \)

\[ \nabla \cdot B \bigg|_{t_0 + dt} = 0, \]

that is

\[ \frac{\partial (\nabla \cdot B)}{\partial t} \bigg|_{t_0} = 0. \]

However, this condition immediately follows if the divergence of the dynamical Maxwell Equation (1.137) is taken. Similarly, it is found from Equation (1.135)

\[ \frac{\partial}{\partial t} \left( \nabla \cdot D - \rho \right) \bigg|_{t_0} = - \left( \nabla \cdot J + \frac{\partial \rho}{\partial t} \right) \bigg|_{t_0} = 0, \]
where in the second step the continuity Equation (1.10) has been employed. Therefore, it is sufficient to calculate the solutions of Equations (1.134) and (1.136) at an initial time \( t_0 \) and then solve with these solutions as boundary conditions the dynamical Maxwell Equations (1.135) and (1.137) to obtain the time evolution of the electromagnetic field.

### 1.4.3.3 Longitudinal Part of the Maxwell Equations

The Maxwell Equation (1.134) can be written as

\[
\nabla \cdot D_\parallel (r, t) = \rho(r, t) \quad (1.203)
\]

and relates the longitudinal electric excitation \( D_\parallel \) to the charge density \( \rho \). In reciprocal space this relation becomes

\[
-jk \cdot D_\parallel (k, t) = \rho(k, t) \quad (1.204)
\]

and can easily be solved for \( D_\parallel (k, t) \) to yield

\[
D_\parallel (k, t) = \frac{j\rho(k, t)k}{k^2}. \quad (1.205)
\]

An inverse Fourier transform to real space gives the result

\[
D_\parallel (r, t) = \frac{1}{4\pi} \int \rho(r', t) \frac{r - r'}{|r - r'|^3} d^3r'. \quad (1.206)
\]

This result is quite remarkable since it turns out that the longitudinal electric displacement is completely determined from the instantaneous Coulomb field of the charge distribution. With the constitutive relation of Equation (1.138) the same is true for the longitudinal electric field strength,

\[
E_\parallel (r, t) = \frac{1}{4\pi \varepsilon} \int \rho(r', t) \frac{r - r'}{|r - r'|^3} d^3r'. \quad (1.207)
\]

The fact that \( D_\parallel (r, t) \) and \( E_\parallel (r, t) \) instantly respond to a change of the charge density does, at this point, not necessarily imply that causality is violated since it is required that the complete fields \( D(r, t) \), \( E(r, t) \) are causal.

The longitudinal part of the second inhomogeneous Maxwell Equation (1.135) is given by

\[
-\frac{\partial D_\parallel (r, t)}{\partial t} = J_\parallel (r, t). \quad (1.208)
\]

Taking the divergence of this equation reveals that it reduces to the continuity equation

\[
\frac{\partial \rho(r, t)}{\partial t} + \nabla \cdot J_\parallel (r, t) = 0. \quad (1.209)
\]

Therefore, Equation (1.208) conveys no additional information.

In summary, the longitudinal components of the electromagnetic field are determined from the instantaneous Coulomb field of the electric charge density. It follows that the longitudinal components do not have their own degrees of freedom, they are tied to the degrees of freedom of the electric charge density.
1.4.3.4 Transverse Part of the Maxwell Equations

What is left to investigate are the transverse parts of the Maxwell Equations (1.135) and (1.137),

\[ \nabla \times \mathbf{H}_\perp(r,t) - \frac{\partial \mathbf{D}_\perp(r,t)}{\partial t} = \mathbf{J}_\perp(r,t), \]

\[ \nabla \times \mathbf{E}_\perp(r,t) + \frac{\partial \mathbf{B}_\perp(r,t)}{\partial t} = 0, \]

With simple constitutive relations of the form in Equations (1.138) and (1.139) these equations are easily decoupled, arriving at the transverse part of the wave Equations (1.142) and (1.143),

\[ \partial_\perp \mathbf{E}_\perp(r,t) - \varepsilon \mu \frac{\partial^2 \mathbf{E}_\perp(r,t)}{\partial t^2} = \mu \frac{\partial \mathbf{J}_\perp(r,t)}{\partial t}, \]

\[ \partial_\perp \mathbf{B}_\perp(r,t) - \varepsilon \mu \frac{\partial^2 \mathbf{B}_\perp(r,t)}{\partial t^2} = -\mu \nabla \times \mathbf{J}_\perp(r,t), \]

since

\[ \mathbf{B}_\perp = \mathbf{B}. \]

In the following the transverse index \( \perp \) of the magnetic field strength will be dropped.

From the wave Equations (1.212) and (1.213) it appears that \( \mathbf{E}_\perp \) and \( \mathbf{B} \) are the dynamical quantities of the electromagnetic field with two independent components each. However, one needs to note that \( \mathbf{E}_\perp \) and \( \mathbf{B} \) are not independent of each other. To explicitly show how both quantities are interrelated the dynamical Maxwell Equations (1.210) and (1.211) are rewritten in reciprocal space.

With the constitutive relations (1.138) and (1.139), the relations \( c^2 = 1/(\varepsilon \mu) \), \( \omega = ck \) and the notation \( \hat{k} = k/k \) one obtains, similar to Equations (1.174) and (1.176), the equations

\[ \frac{\partial \mathbf{E}_\perp(k,t)}{\partial t} = -j\omega \hat{k} \times \mathbf{B}(k,t) - \frac{\mathbf{J}_\perp(k,t)}{\varepsilon_0}, \]

\[ \hat{k} \times \frac{\partial \mathbf{B}(k,t)}{\partial t} = -j\omega \mathbf{E}_\perp(k,t). \]

In the sourceless case with \( \mathbf{J}_\perp = 0 \) it is recognized from these equations by addition and subtraction that eigenfunctions of this system are determined from

\[ \frac{\partial}{\partial t} \left( \mathbf{E}_\perp(k,t) - c\hat{k} \times \mathbf{B}(k,t) \right) = j\omega \left( \mathbf{E}_\perp(k,t) - c\hat{k} \times \mathbf{B}(k,t) \right), \]

\[ \frac{\partial}{\partial t} \left( \mathbf{E}_\perp(k,t) + c\hat{k} \times \mathbf{B}(k,t) \right) = -j\omega \left( \mathbf{E}_\perp(k,t) + c\hat{k} \times \mathbf{B}(k,t) \right). \]

To label these eigenfunctions variables \( a(k,t) \) and \( b(k,t) \) are introduced by

\[ a(k,t) := \frac{j}{2N(k)} \left[ \mathbf{E}_\perp(k,t) - c\hat{k} \times \mathbf{B}(k,t) \right], \]

\[ b(k,t) := \frac{j}{2N(k)} \left[ \mathbf{E}_\perp(k,t) + c\hat{k} \times \mathbf{B}(k,t) \right]. \]

The factor \( j/2N(k) \) denotes a normalization coefficient and is in accordance with a common notation that is used in the context of the quantization of the electromagnetic field [27]. In this
context the function \( N(k) \) is related to the energy of a quantum state of the electromagnetic field. For classical purposes the explicit form of \( N(k) \) is not important. Within expressions of the electromagnetic field in real space the function \( N(k) \) will cancel and drop out. It could also be absorbed in the definition of \( a(k, t) \) and \( b(k, t) \).

It is straightforward to solve Equations (1.219) and (1.220) for \( E_{\perp}(k, t) \) and \( B(k, t) \). Since both quantities have to be real it follows that

\[
b(k, t) = -a^*(-k, t), \tag{1.221}
\]

where the asterisk \( * \) denotes complex conjugation. Then,

\[
E_{\perp}(k, t) = -jN(k) \left[ a(k, t) - a^*(-k, t) \right], \tag{1.222}
\]

\[
B(k, t) = -\frac{jN(k)}{c} \left[ \hat{k} \times a(k, t) + \hat{k} \times a^*(-k, t) \right]. \tag{1.223}
\]

Therefore, the transverse electromagnetic field is completely specified by the function \( a(k, t) \). Since \( E_{\perp}(k, t) \) and \( B(k, t) \) are transverse functions it follows that \( a(k, t) \) is a transverse function, too. Hence, it is concluded that \( a(k, t) \) exhibits two degrees of freedom which are the two dynamical components of the electromagnetic field. The function \( a(k, t) \) is said to represent the normal modes of the electromagnetic field. This term indicates that \( a(k, t) \) represents a whole class of electromagnetic excitations which is parameterized by a discrete or continuous set of values for the wave number \( k \).

Inserting Equations (1.222) and (1.223) in the Maxwell equations yields for the time evolution of \( a(k, t) \) the equation

\[
\frac{\partial a(k, t)}{\partial t} - j\omega a(k, t) = -\frac{j}{2\varepsilon N(k)} J_{\perp}(k, t). \tag{1.224}
\]

This equation of motion for the normal modes represents, in fact, the motion of harmonic oscillation: if a new variable \( c(k, t) \) is implicitly introduced via

\[
a(k, t) = c(k, t) - \frac{j}{\omega} \frac{\partial c(k, t)}{\partial t} \tag{1.225}
\]

the familiar equation of motion of a harmonic oscillator follows from Equation (1.224),

\[
\frac{\partial^2 c(k, t)}{\partial t^2} + \omega^2 c(k, t) = \frac{\omega}{2\varepsilon N(k)} J_{\perp}(k, t). \tag{1.226}
\]

From Equations (1.222) and (1.223) one obtains for the fields \( E_{\perp}(r, t) \) and \( B(t, t) \) from a Fourier transformation the expansions

\[
E_{\perp}(r, t) = -\frac{j}{(2\pi)^{3/2}} \int N(k) \left[ a(k, t)e^{-jk \cdot r} - a^*(k, t)e^{jk \cdot r} \right] d^3 k, \tag{1.227}
\]

\[
B(r, t) = -\frac{j}{(2\pi)^{3/2}} \int N(k) \left[ \hat{k} \times a(k, t)e^{-jk \cdot r} - \hat{k} \times a^*(k, t)e^{jk \cdot r} \right] d^3 k. \tag{1.228}
\]

It is recalled that \( a(k, t) \) and \( a^*(k, t) \) are purely complex quantities such that the fields \( E_{\perp}(r, t) \) and \( B(r, t) \) are real.
In the absence of sources, $J_\perp(k, t) = 0$. Then the equation of motion (1.224) yields the solution
\[ a(k, t) = a(k)e^{j\omega t} \] (1.229)
and the expansions of Equations (1.227) and (1.228) turn into expansions in traveling plane waves,
\[ E_\perp\text{free}(r, t) = -\frac{j}{(2\pi)^{3/2}} \int N(k) \left[ a(k)e^{j(\omega t - k \cdot r)} - a^*(k)e^{-j(\omega t - k \cdot r)} \right] d^3k, \] (1.230)
\[ B_\text{free}(r, t) = -\frac{j}{(2\pi)^{3/2}} \int \frac{N(k)}{c} \left[ \hat{k} \times a(k)e^{j(\omega t - k \cdot r)} - \hat{k} \times a^*(k)e^{-j(\omega t - k \cdot r)} \right] d^3k. \] (1.231)
In these expansions of the free fields the functions $a(k, t) = a(k)e^{j\omega t}$ corresponding to different $k$ are completely decoupled. This also holds true if the electromagnetic sources, represented by $J_\perp(k, t)$, are independent of $a(k, t)$, that is independent of the electromagnetic field. However, if the electromagnetic sources do interact with the electromagnetic field the time evolution of $J_\perp(k, t)$ will depend on $a(k, t)$ and, in general, lead to a coupling between $a(k, t)$ with different $k$.

### 1.4.4 Electromagnetic Energy and the Singularities of the Electromagnetic Field

The energy density $w(r, t)$ of the electromagnetic field is given by
\[ w_{\text{em}}(r, t) = \frac{1}{2} \left( \mathbf{E}(r, t) \cdot \mathbf{D}(r, t) + \mathbf{B}(r, t) \cdot \mathbf{H}(r, t) \right) \] (1.232)
and the corresponding energy $W(t)$ is
\[ W_{\text{em}}(t) = \int w_{\text{em}}(r, t) d^3r \] (1.233)
\[ = \frac{1}{2} \int \left( \mathbf{E}(r, t) \cdot \mathbf{D}(r, t) + \mathbf{B}(r, t) \cdot \mathbf{H}(r, t) \right) d^3r. \] (1.234)
This energy, in general, is not a constant of motion if the electromagnetic field interacts with electromagnetic sources.

For further analysis it is useful to split the electromagnetic field energy into a contribution of the longitudinal fields $\mathbf{E}_l, \mathbf{D}_l$ and a contribution of the transverse fields $\mathbf{E}_\perp, \mathbf{D}_\perp, \mathbf{B}$ and $\mathbf{H}$. Clearly, the magnetic part of $W_{\text{em}}(t)$ only involves the transverse fields $\mathbf{B}$ and $\mathbf{H}$. The electric part can be written as
\[ \frac{1}{2} \int \mathbf{E}(r, t) \cdot \mathbf{D}(r, t) d^3r \]
\[ = \frac{1}{2} \int \mathbf{E}^*(\mathbf{k}, t) \cdot \mathbf{D}(\mathbf{k}, t) d^3k \]
\[ = \frac{1}{2} \int (\mathbf{E}^\parallel(\mathbf{k}, t) + \mathbf{E}^\perp(\mathbf{k}, t)) \cdot (\mathbf{D}^\parallel(\mathbf{k}, t) + \mathbf{D}^\perp(\mathbf{k}, t)) d^3k \] (1.235)
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\[ W_{\text{em}}(t) = \frac{1}{2} \int E_{\parallel}(r, t) \cdot D_{\parallel}(r, t) \, d^3r + \frac{1}{2} \int E_{\parallel}(r, t) \cdot D_{\parallel}(r, t) \, d^3r + \frac{1}{2} \int E_{\perp}(r, t) \cdot D_{\perp}(r, t) \, d^3r. \]  

(1.238)

The result is the desired split into longitudinal and transverse contributions to \( W_{\text{em}}(t) \). If constitutive relations of the form are assumed Equation (1.138) and the results (1.206) and (1.207) are taken into account it is easy to see that the longitudinal contribution \( W_{\text{em}}(t) \) to the electromagnetic field energy is just given by the electrostatic Coulomb energy,

\[ W_{\text{em}\parallel}(t) = \frac{1}{2} \int E_{\parallel}(r, t) \cdot D_{\parallel}(r, t) \, d^3r = \frac{1}{8\pi\varepsilon} \int \rho(r, t) \rho(r', t) \frac{1}{|r - r'|} \, d^3r' \cdot d^3r. \]  

(1.239)

For the transverse contribution

\[ W_{\text{em}\perp}(t) = \frac{1}{2} \int (E_{\perp}(r, t) \cdot D_{\perp}(r, t) + B(r, t) \cdot H(r, t)) d^3r \]  

(1.240)

it is not possible to find in real space a result of similar simplicity. However, if the validity of the constitutive relations (1.138) and (1.139) is assumed and a shift to reciprocal space is performed one can write

\[ W_{\text{em}\perp}(t) = \frac{1}{2} \int (E_{\perp}^*(k, t) \cdot E_{\perp}(k, t) + B^*(k, t) \cdot B(k, t)) d^3k. \]  

(1.241)

Inserting into this expression the relations (1.222) and (1.223) it then follows from simple vector algebra that

\[ W_{\text{em}\perp}(t) = \varepsilon \int N^2(k) [a^*(k, t) \cdot a(k, t) + a(-k, t) \cdot a^*(-k, t)] d^3k. \]  

(1.242)

Therefore, the transverse contribution to the electromagnetic energy is completely determined from the normal modes \( a(k, t) \).

Next it is investigated under which conditions the electromagnetic energy of a system becomes divergent: from Equation (1.239) for the electrostatic Coulomb energy of the longitudinal fields it is seen that this energy depends on the relative position of the electric sources within a system. It diverges in the limit \( |r - r'| \) in which case two sources become arbitrarily close. This behavior characterizes the Coulomb singularity of classical electrodynamics and is related to the infinite amount of Coulomb energy that is carried by any electric source. It is independent of the kinematical or dynamical state of a source, that is it is independent of velocity and acceleration. In particular, in the time-harmonic case it is independent of frequency. The expression (1.242) for the energy of the transverse fields exhibits no spatial singularity. It is seen from Equations (1.224) and (1.226) that the normal modes \( a(k, t) \) represent oscillations of the electromagnetic field that are driven by the transverse part \( J_{\perp}(k, t) \) of the electric current. Dominant contributions to the energy given by Equation (1.242) occur if the excitation is such that it operates at an eigenfrequency \( \omega \) of the system. In this case of electromagnetic resonance it is known from the elementary solution of the equation of motion of a forced harmonic oscillator that the amplitude of the resulting oscillation will tend to infinity if no loss mechanism is present [22]. Then the energy in Equation (1.242) will tend to infinity as

...
well. It follows that in the lossless case the oscillations of the electromagnetic field become singular at resonance. This is the second type of electromagnetic singularity which leads to diverging electromagnetic energy. Since forced harmonic motion is the only solution of the motion Equation (1.226) this type of singularity is the only one that is contained in Equation (1.242).

As a result it can be stated that the electromagnetic field exhibits two types of singularities that lead to diverging electromagnetic energy densities. These are:

- Coulomb singularities that are related to the mutual position of electric sources;
- electromagnetic resonances that are related to forced oscillations of the electromagnetic field.

### 1.4.5 Coulomb Fields and Radiation Fields

The split of the electromagnetic fields \(\mathbf{D}, \mathbf{H}, \mathbf{E}\) and \(\mathbf{B}\) and the electric current \(\mathbf{J}\) in longitudinal and transverse parts has already been proven to be useful in the study of basic electromagnetic field properties. Unfortunately, it also has some disadvantages. Two major disadvantages which, in fact, are interrelated, are the following:

- The split in longitudinal and transverse parts is not relativistically invariant. A field which is purely longitudinal if observed from a first reference system will contain transverse contributions if observed from a second reference system which is in relative motion to the first one.
- The decomposition in longitudinal and transverse parts introduces nonlocal effects. This has been noted in Section 1.4.3.1 where the explicit form of the decomposition is given by Equations (1.188) and (1.189).

The physical relevance of these disadvantages becomes apparent if one returns to the analysis of the longitudinal and transverse parts of the Maxwell equations, as outlined in Sections 1.4.3.3 and 1.4.3.4. It was found that the longitudinal electric field \(\mathbf{E}_l\) represents the instantaneous Coulomb field of the electric charge density \(\rho\). Since the complete electric field \(\mathbf{E}\), according to Equation (1.142), fulfills a proper wave equation which leads to causal solutions it follows that the transverse electric part \(\mathbf{E}_\perp\) must contain an instantaneous contribution which exactly cancels the instantaneous contribution of \(\mathbf{E}_l\). Indeed, it can be shown that this is the case [27]. At first sight this is surprising since \(\mathbf{E}_\perp\) fulfills the wave Equation (1.121). However, the solution of this wave equation contains instantaneous contributions since the source term \(\mathbf{J}_\perp\) is nonlocal and contains instantaneous contributions as well (compare the short discussion for the general transverse field \(\mathbf{F}_\perp\) after Equation (1.189)). Therefore, the split in longitudinal and transverse electromagnetic fields does not separate physically independent electromagnetic field contributions since longitudinal and transverse electromagnetic fields are inseparably connected to each other.

To illustrate this circumstance, the electromagnetic field of a moving point charge is considered where the validity of the constitutive relations in Equations (1.36) and (1.37) is assumed. At a fixed time the point charge is located at a position \(r^\prime\) where it moves with velocity \(v\), and an observer is positioned at a position \(r\). The unit vector \(e_{r^\prime r} := (r - r^\prime)/|r - r^\prime|\) points from the charge to the observer (see Figure 1.8) and also the abbreviation \(\beta := v/c\) is introduced.
Figure 1.8 A charge \( q \) moves with velocity \( v \) in the presence of an observer. It is assumed that the observer does not move with respect to the inertial system \( xyz \). The electromagnetic field that is generated by the electric charge requires the time \( |r - r'|/c \) to reach the observer. Therefore, the electromagnetic field that is noticed by the observer at a time \( t \) has been generated by the electric charge at the earlier, retarded time \( t_{\text{ret}} = t - |r - r'|/c \).

Then the electromagnetic field, expressed in terms of the field strengths \( E \) and \( B \), that is noticed by the observer is given by [8]

\[
E(r, t) = \frac{q}{4\pi \varepsilon_0} \left[ \frac{(e_{r', x} - \beta)(1 - \beta^2)}{(1 - \beta \cdot e_{r', x})^3 |r - r'|^2} \right]_{\text{ret}} + \frac{q}{4\pi \varepsilon_0} \left[ \frac{e_{r', x} \times (e_{r', x} - \beta) \times \frac{\beta_0}{c^2}}{c(1 - \beta \cdot e_{r', x})^3 |r - r'|} \right]_{\text{ret}},
\]

velocity field (Coulomb field)

\[
B(r, t) = \frac{1}{c} e_{r', x} \times E(r, t) = \frac{q}{4\pi \varepsilon_0} \left[ \frac{(\beta \times e_{r', x})(1 - \beta^2)}{c(1 - \beta \cdot e_{r', x})^3 |r - r'|^2} \right]_{\text{ret}} + \frac{q}{4\pi \varepsilon_0} \left[ \frac{e_{r', x} \times (e_{r', x} \times (e_{r', x} - \beta) \times \frac{\beta_0}{c^2})}{c^2(1 - \beta \cdot e_{r', x})^3 |r - r'|} \right]_{\text{ret}}.
\]

acceleration field (radiation field)

The brackets \( [ \ ]_{\text{ret}} \) indicate that the enclosed quantities have to be taken at the retarded time \( t_{\text{ret}} \) that is introduced in the caption of Figure 1.8. Each of the fields in Equations (1.243) and (1.244) splits nicely into a first part which depends on the velocity of the charge and a second part which depends on both the velocity and the acceleration of the charge.

The velocity fields can also be obtained by a Lorentz transformation of the static fields

\[
E_{\text{static}}(r) = \frac{q}{4\pi \varepsilon} \frac{r - r'}{|r - r'|^3},
\]

\[
B_{\text{static}}(r) = 0.
\]
This suggests the name Coulomb fields for the electromagnetic velocity fields. They constitute the static Coulomb field of a point charge as noticed by an observer which is in relative motion to the charge. The remaining acceleration fields are commonly called radiation fields.

The fields in Equations (1.243) and (1.244) can be split into longitudinal and transverse parts. Since the magnetic field already is purely transverse only the electric field is considered. The longitudinal component \( E_\parallel \) is given by the instantaneous Coulomb field (see Equation (1.207)) and it follows

\[
E_\parallel(r, t) = \frac{q}{4\pi \varepsilon_0} \frac{r - r'(t)}{|r - r'(t)|^3},
\]

(1.248)

remaining part of the Coulomb field

\[
E_\perp(r, t) = \frac{q}{4\pi \varepsilon_0} \left[ \frac{(e_{r',r} - \beta)(1 - \beta^2)}{(1 - \beta \cdot e_{r',r})^3 |r - r'|^2} - \frac{r - r'(t)}{|r - r'(t)|^3} \right]_{\text{ret}}
+ \frac{q}{4\pi \varepsilon_0} \left[ \frac{e_{r',r} \times (e_{r',r} - \beta) \times \frac{e_{r',r}}{|e_{r',r}|}}{c(1 - \beta \cdot e_{r',r})^3 |r - r'|} \right]_{\text{ret}}.
\]

(1.249)

Therefore, the split of the electric field into longitudinal and transverse parts also splits the Coulomb field into two parts and assigns the aforementioned instantaneous field contributions to both the longitudinal and the transverse component of the electric field.

In principle, the radiation part of the electromagnetic field could be isolated if, at a particular time, it were possible to switch off the coupling between electric charges and the electromagnetic field, that is if it were possible to switch off electric charges and their accompanying Coulomb fields. Then the remaining radiation field would be the solution of the sourceless Maxwell equations with \( \rho = 0 \), \( J = 0 \) and nontrivial initial conditions. A solution of the sourceless Maxwell equations commonly is called free electromagnetic field. It is a pure radiation field and fulfills the homogeneous wave Equations (1.145) and (1.146). A free electromagnetic field has no longitudinal components since its longitudinal electric field component in Equation (1.207) vanishes by virtue of \( \rho = 0 \). The solutions for the transverse components are characterized by oscillatory motion, as is recognized from the solutions in Equations (1.230) and (1.231) which, in turn, reflect the solutions of the harmonic oscillator Equations (1.224) and (1.226) for a vanishing transverse current. The solutions in Equations (1.230) and (1.231) also exhibit that for a radiation field the wave vector \( k \), the electric field \( E = E_\perp \) and the magnetic field \( B \) are always mutually orthogonal to each other. It is in this
way that a radiation field propagates electromagnetic energy with phase velocity \( c = 1/\sqrt{\varepsilon\mu} \) through space.

However, in practice electric charges cannot simply be switched off to neglect the coupling between electric charges and the electromagnetic field. Instead, it is necessary to consider the nonlocal transverse electric current \( J_\perp \) which drives the transverse electromagnetic field components and, in general, extends through the whole of space. Then the concept of a free electromagnetic field turns to an ideal which, nevertheless, is often a useful one. An example is given by the far field of an antenna in free space. In free space the transverse electric current falls off faster in intensity than the electromagnetic field does. Then the electromagnetic field becomes asymptotically free at large distances where it constitutes the common radiation field.

The fact that generally it is not possible to split a given electromagnetic field into a Coulomb part and a radiation part indicates that there are some conceptual difficulties in classical electrodynamics which cannot be resolved. These do not necessarily have to be a matter of concern. If an electromagnetic boundary value problem is solved it is usually solved for the complete fields and it might be of no practical interest to know which part of the solution constitutes a Coulomb field and which part represents a radiation field. However, the fact that there are two different categories of electric fields with different, and often complementary, properties is the reason for many difficulties that are present in the solution of practical problems in electrical engineering.

1.4.6 The Green’s Function Method

The construction of solutions to linear differential equations with specified sources and given boundary conditions belongs to the fundamental problems of the Maxwell theory and other physical field theories. The Green’s function method provides a technique to find these solutions and in this section this method is introduced. For a proper understanding of the Green’s function method it is necessary to be familiar with a number of functional analytic notions such as a linear operator which is self-adjoint and acts between Hilbert spaces with specified inner products. For those readers who are not familiar with these notions, Appendix B provides the necessary functional analytic background.

Formally, a linear differential equation is often expressed as an operator equation of the form

\[
\mathcal{L}_D f = g \tag{1.250}
\]

with a linear differential operator \( \mathcal{L}_D \), a source function \( g \) which is assumed to be known, and an unknown function \( f \). The Green’s function method consists of finding a Green’s function \( G \) such that the unknown function \( f \) is expressed as an integral over the source function \( g \), weighted with the Green’s function. From a physical point of view the Green’s function method is a representation of the superposition principle: the Green’s function is the solution of the given linear differential equation with respect to a unit source which is placed at a specific position. Then the solution with respect to a general source is obtained from the superposition of known solutions of individual unit sources at various positions.
**1.4.6.1 Basic Ideas**

To introduce the Green’s function method a real, self-adjoint differential operator $\mathcal{L}_D$ is considered with

$$ (\mathcal{L}_D f)(r) = g(r). \quad (1.251) $$

In this case, it is explicitly indicated that the functions depend on a variable $r$ which usually represents a position in space. It is also common to have the time $t$ as an additional parameter.

A Green’s function is implicitly defined by

$$ \mathcal{L}_D G(r, r') = \delta(r - r') \quad (1.252) $$

with

$$ \delta_r := \delta(r - r') \quad (1.253) $$

the Dirac delta function which is a generalized function that is defined in the distributional sense [28, 29]. In a Hilbert space $\mathcal{H}$ with an inner product $\langle \ , \ \rangle$ it can be introduced via the relationship

$$ \langle f, \delta_r \rangle = f(r). \quad (1.254) $$

Suppose that a Green’s function has been constructed that fulfills Equation (1.252). Then the expression

$$ \langle \mathcal{L}_D f, G^* \rangle = \langle f, \mathcal{L}_D G^* \rangle \quad (1.255) $$

$$ = \langle f, (\mathcal{L}_D G)^* \rangle \quad (1.256) $$

may be considered. Here the fact that $\mathcal{L}_D$ is self-adjoint and real has been used. Applying Equation (1.252) and noting that the delta function is a real function yields

$$ \langle \mathcal{L}_D f, G^* \rangle = \langle f, \delta \rangle \quad (1.257) $$

$$ = f. \quad (1.258) $$

With Equation (1.251) the solution of the original problem is found as

$$ f = \langle g, G^* \rangle. \quad (1.259) $$

This establishes the Green’s function method for solving differential equations that are represented by a linear, self-adjoint and real differential operator $\mathcal{L}_D$.

**Example 1.2:** Consider as an example the Hilbert space $L^2(\Omega)^m$ with inner product (B.27) and $f, g \in L^2(\Omega)^m$ (see Appendix B). In an analogy to Equations (1.251) and (1.252) a linear differential equation is assumed,

$$ (\mathcal{L}_D f)(r) = g(r). \quad (1.260) $$

For the corresponding Green’s function the ansatz

$$ \mathcal{L}_D \mathcal{G}(r, r') = \delta(r - r') $
is considered where now the Green’s function $\mathbf{G}(\mathbf{r}, \mathbf{r}')$ is represented as a dyadic and $\mathbf{I}$ denotes the unit dyad [30]. With the inner product (B.27) the delta function acts according to

$$\langle f, \delta_r \rangle = \int_{\Omega} \delta(r - r') f(r') d\Omega'$$

(1.262)

$$= f(r).$$

(1.263)

Repeating the steps that led from Equations (1.255) to (1.259) yields the solution of Equation (1.260) in the form

$$f(r) = \int_{\Omega} g(r') \cdot \mathbf{G}(r, r') d\Omega'.$$

(1.264)

### 1.4.6.2 Self-Adjointness of Differential Operators and Boundary Conditions

The general solutions given in Equation (1.259) and (1.264) that have been obtained by means of the Green’s function method look deceptively simple because they only involve weighting the source function $g$ with the Green’s function $G$. However, from the theory of differential equations it is known that boundary conditions play a fundamental role in the determination of a unique solution. Therefore, the information on boundary conditions must have been incorporated in the derivation of Equations (1.259) and (1.264). Indeed, self-adjointness of the real differential operator $\mathbf{L}_D$ has been presupposed,

$$\langle \mathbf{L}_D f, g \rangle = \langle f, \mathbf{L}_D g \rangle.$$

(1.265)

Since in most function spaces of physical interest the inner product is represented by means of integration it follows that self-adjointness is closely connected to ‘generalized partial integration’, that is to the generalized Green’s identity

$$\int_{\Omega} (\mathbf{L}_D f) g^* d\Omega = \int_{\Omega} f (\mathbf{L}_D g)^* d\Omega + \int_{\Gamma = \partial \Omega} J(f, g) d\Gamma$$

(1.266)

where $\Omega$ is the integration volume and $\partial \Omega = \Gamma$ is its boundary. This identity can be written in terms of the inner product as

$$\langle \mathbf{L}_D f, g \rangle = \langle f, \mathbf{L}_D g \rangle + \int_{\Gamma = \partial \Omega} J(f, g) d\Gamma.$$

(1.267)

It follows that $\mathbf{L}_D$ is self-adjoint if and only if the integral on the right-hand side of this equation vanishes. This requirement will pose restrictions on the a priori unknown function $f$ and the Green’s function $G$.

This circumstance is demonstrated by the following general application of the Green’s function method. Consider again a differential equation of the form

$$\mathbf{L}_D f = g$$

(1.268)

and a corresponding Green’s function $G$ which satisfies

$$\mathbf{L}_D G = \delta.$$  

(1.269)

At this point it is not required for $\mathbf{L}_D$ to be self-adjoint. Taking the inner product of Equation (1.268) with $G^*$ and the inner product of the complex conjugate of Equation (1.269) with
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\( f \) yields

\[
\langle \mathcal{L}_D f, G^* \rangle = \langle g, G^* \rangle, \quad (1.270)
\]

\[
\langle f, (\mathcal{L}_D G)^* \rangle = \langle f, \delta^* \rangle = f. \quad (1.271)
\]

Forming the differences of both equations leads to

\[
\langle \mathcal{L}_D f, G^* \rangle - \langle f, (\mathcal{L}_D G)^* \rangle = \langle g, G^* \rangle - f. \quad (1.272)
\]

In the notation of Equation (1.267) it is found that

\[
\langle g, G^* \rangle - f = \int_{\Gamma = \partial \Omega} J(f, G) \, d\Gamma, \quad (1.273)
\]

that is the simple solution of Equation (1.259) is obtained if and only if the boundary integral, which involves the so-called \textit{conjunct} \( J(f, G) \), vanishes. This explicitly shows the relation between self-adjointness of the differential operator \( \mathcal{L}_D \) and boundary conditions of \( f \) and \( G \).

**Example 1.3:** Consider the Helmholtz Equation (1.169) for the magnetic vector potential \( A \) in the Lorenz gauge,

\[
\Delta A(r) + k^2 A(r) = -\mu J(r). \quad \text{(Lorenz gauge)} \quad (1.274)
\]

Up to a factor \( \mu \) the dyadic Green’s function of this equation has to fulfill

\[
\Delta \mathbf{G}^A(r, r') + k^2 \mathbf{G}^A(r, r') = -\mathbf{I} \delta(r - r'). \quad (1.275)
\]

With the inner product (B.27) of \( L^2(\Omega)^3 \) the general Equation (1.272) yields

\[
\int_{\Omega} \left[ (\Delta A(r')) \cdot \mathbf{G}^A(r, r') - A(r') \cdot \Delta \mathbf{G}^A(r, r') \right] \, d^3r' = \int_{\Omega} J(r') \cdot \mathbf{G}^A(r, r') \, d^3r' + A(r). \quad (1.276)
\]

By means of the second vector-dyadic Green’s second theorem (C.21) the integral on the left-hand side can be transformed to a boundary integral. Then,

\[
A(r) = \mu \int_{\Omega} J(r') \cdot \mathbf{G}^A(r, r') \, d^3r' \\
+ \int_{\Gamma} \left[ (e_n \times A(r')) \cdot (\nabla' \times \mathbf{G}^A(r, r')) - (\nabla \times A(r')) \cdot (e_n \times \mathbf{G}^A(r, r')) \right. \\
+ e_n \cdot A(r')(\nabla' \cdot \mathbf{G}^A(r, r')) - e_n \cdot \mathbf{G}^A(r, r')(\nabla' \cdot A(r')) \right] \, d^2r'. \quad (1.277)
\]

Since the boundary integral should vanish it is necessary to think about appropriate boundary conditions. If it is supposed that the boundary is perfectly conducting it follows that

\[
e_n \times A(r)_{r \in \Gamma} = 0, \quad (1.278)
\]

\[
\nabla \cdot A(r)_{r \in \Gamma} = 0, \quad (1.279)
\]
such that the first and last term within the surface integral will vanish. There are also the corresponding boundary conditions
\[ e_n \times G^A(r, r') \bigg|_{r \in \Gamma} = 0, \]  
\[ \nabla \cdot G^A(r, r') \bigg|_{r \in \Gamma} = 0 \]  
of the dyadic Green’s function such that the surface integral completely vanishes. Therefore, the magnetic vector potential can be calculated from the expression
\[ A(r) = \mu \int J(r') \cdot \overline{G^A}(r, r') \, d^3r', \]  
which involves no boundary terms.

**Example 1.4:** Consider the vector Helmholtz Equation (1.165) for the electric field \( E \),
\[ \nabla \times \nabla \times E(r) - k^2 E(r) = -j \omega \mu J(r). \]  
Up to a factor \(-j \omega \mu\) the corresponding Green’s function needs to fulfill
\[ \nabla \times \nabla \times G^E(r, r') - k^2 G^E(r, r') = T_\delta(r, r'). \]  
The general Equation (1.272) yields
\[ \int_{\Omega} \left[ (\nabla \times \nabla \times E(r')) \cdot \overline{G^E}(r, r') - E(r') \cdot (\nabla \times \nabla \times G^E(r, r')) \right] \, d^3r' \]
\[ = -j \omega \mu \int_{\Omega} J(r') \cdot \overline{G^E}(r, r') \, d^3r' - E(r). \]  
From application of the Green’s theorem (C.20) and the identity (C.15) it is found that
\[ E(r) = -j \omega \mu \int_{\Omega} J(r') \cdot \overline{G^E}(r, r') \, d^3r' \]
\[ + \oint_{\Gamma} \left[ (e_n \times E(r')) \cdot (\nabla \times G^E(r, r')) - (\nabla \times E(r')) \cdot (e_n \times G^E(r, r')) \right] \, d^2r'. \]  
If it is supposed again that the interior of the boundary \( \Gamma \) is perfectly conducting the boundary condition
\[ e_n \times E(r) \bigg|_{r \in \Gamma} = 0 \]  
is valid. The corresponding boundary condition of the Green’s function is
\[ e_n \times \overline{G^E}(r, r') \bigg|_{r \in \Gamma} = 0 \]  
and, as a consequence, the surface integral vanishes such that the electric field can be calculated according to
\[ E(r) = -j \omega \mu \int_{\Omega} J(r') \cdot \overline{G^E}(r, r') \, d^3r'. \]
Examples 1.3 and 1.4 show that the Helmholtz equations for the magnetic vector potential and the electric field strength, respectively, form self-adjoint boundary value problems if the fields are defined in a finite domain which is enclosed by a perfectly conducting boundary.

1.4.6.3 General Solutions of Maxwell Equations

In Sections 1.4.1.1 and 1.4.1.3 it has been seen that for homogeneous and isotropic media it is straightforward to decouple the Maxwell equations and to rewrite them in the form of wave equations or, in the time harmonic case, as Helmholtz equations. In the absence of boundaries, that is in free space, and within a homogeneous medium the general solution of the Maxwell equations is given in terms of the solution of the scalar Helmholtz equation. This follows from Equations (1.169) and (1.170) which constitute in free space four independent scalar Helmholtz equations. These have the general structure

\[(\Delta + k^2) f(r, \omega) = -g(r, \omega)\]  \hspace{1cm} (1.290)

and the appropriate Green’s function \(G_0(r, r')\) needs to satisfy

\[(\Delta + k^2) G_0(r, r') = -\delta(r - r') .\]  \hspace{1cm} (1.291)

The solution for \(G_0(r, r')\) is most easily obtained in spherical coordinates, taking advantage of the symmetries of free space. This yields the retarded solution \([8, p. 243]\)

\[G_0(r, r') = \frac{1}{4\pi} e^{-jk|\mathbf{r} - \mathbf{r}'|}.\]  \hspace{1cm} (1.292)

Therefore, in free space the general solution of the Maxwell equations is represented by the equations

\[
\begin{align*}
\varphi(r) &= \int G_0(r, r') \rho(r') \, d^3r' \\
&= \frac{1}{4\pi \varepsilon} \int \frac{e^{-jk|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \rho(r') \, d^3r',
\end{align*}
\]  \hspace{1cm} (1.293)

\[
\begin{align*}
A(r) &= \int G_0(r, r') J(r') \, d^3r' \\
&= \frac{\mu}{4\pi} \int \frac{e^{-jk|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} J(r') \, d^3r',
\end{align*}
\]  \hspace{1cm} (1.294)

which relate the electromagnetic sources \(\rho, J\) to the electromagnetic field, expressed by \(\varphi\) and \(A\) in the Lorenz gauge.

1.4.6.4 Basic Relations Between Electromagnetic Green’s Functions

In general, there are some basic relations between the various electromagnetic Green’s functions. It is recalled from Section 1.4.1.3 that in a linear, isotropic and homogeneous medium the Maxwell equations in the frequency domain can be reduced to Helmholtz equations. For the vector potential \(A(r)\) in the Lorenz gauge, the electric field \(E(r)\) and the magnetic field
The problem of calculating the electric current on a scatterer is a standard problem in electromagnetic theory. A current is generated by primary sources that produce incident electromagnetic fields $E^{\text{inc}}$, $H^{\text{inc}}$. Then the total fields $E$, $H$ in the presence of the scatterer are a superposition of the incident fields and scattered fields $E^{\text{sca}}$, $H^{\text{sca}}$, 

$$E = E^{\text{inc}} + E^{\text{sca}},$$  

$$H = H^{\text{inc}} + H^{\text{sca}}.$$  

---

10 It is a convention to skip in the definitions of the Green’s functions the factors $\mu$ and $-j\omega\mu$ that appear on the right-hand sides of the corresponding Helmholtz equations.
A general method to find from this decomposition an equation for the unknown current on the scatterer consists of three steps which require the surface equivalence principle, the source–field relationships that follow from the solution of the Maxwell equations and the boundary conditions for the total electromagnetic fields [31, 1]:

1. By virtue of the surface equivalence principle the scatterer is replaced by equivalent electromagnetic sources which, a priori, are unknown. If the scatterer is assumed to be perfectly conducting the equivalent electromagnetic sources are represented by an electric current. In general, the equivalent electromagnetic sources will be determined from the incident electromagnetic fields.

2. The equivalent electromagnetic sources generate a scattered electromagnetic field according to the source–field relationships that express a field by the integral over a source, weighted with the appropriate Green’s function. This allows the unknown scattered electromagnetic field, which is usually defined within an entire volume, to be replaced by the unknown equivalent sources, which are usually defined on a boundary surface.

3. On the boundary surface, where the unknown equivalent sources are defined, the boundary conditions of Equations (1.307) and (1.308) for the total fields must be enforced. This relates on the boundary surface the known incident electromagnetic field to integrals over the unknown equivalent sources.

These three steps comprise, in short, how to construct surface integral equations for unknown electromagnetic sources that are induced by primary electromagnetic fields.

1.5.2 The Standard Electric Field Integral Equations of Antenna Theory and Radiating Nonuniform Transmission-Line Systems

Linear antennas constitute the classical antenna prototype and generalize the concept of an idealized, mathematical electric dipole to an actual engineering device. In the development of antenna theory linear antennas have played a dominant role because many of their properties can be modeled by analytic methods. However, even in the simplest realistic cases simplifying approximations have to be made in order to arrive at analytic results [32–35].

There are four standard electric field integral equations which can be used to determine the electric current on cylindrical thin-wire antennas [36–38]. These equations also constitute the basis of generalized transmission-line theories and are summarized below.

1.5.2.1 Pocklington’s Equation

It is remarkable that 10 years after the discovery of electromagnetic radiation by Hertz [39] in 1887 an integral equation for the current distribution along cylindrical wire dipole antennas was published by Pocklington [40] in 1897. Pocklington’s equation constitutes an electric field integral equation that is adapted to cylindrical, thin-wire antennas. To formulate this integral equation one can follow the three-step procedure of the previous section and first introduce a surface current \( J_s \) which is related to a scattered electric field \( E_s \) via the electric Green’s
function $\mathcal{G}^E$.

$$E_{\text{sca}}(r) = -j \omega \mu \int_{\Gamma} \mathcal{G}^E(r, r') J_s(r') \, \mathrm{d}^2 r'. \quad (1.309)$$

Here the antenna surface is denoted by $\Gamma$. The boundary condition for the total electric field $E$ on a perfectly conducting surface is $\mathbf{e}_n \times E = 0$ or, alternatively, $E_t = 0$, with $E_t = E \cdot \mathbf{e}_t$ the projection of $E$ on a given tangential vector $\mathbf{e}_t$. With this boundary condition and Equation (1.307) an electric field integral equation is obtained,

$$j \omega \mu \left[ \int_{\Gamma} \mathcal{G}^E(r, r') J_s(r') \, \mathrm{d}^2 r' \right] \cdot \mathbf{e}_t(r) = E_{\text{inc}}(r). \quad (1.310)$$

This equation simplifies if the antenna geometry is that of a thin cylindrical wire. Then a thin-wire approximation can be performed where azimuthal currents are neglected and the surface current $J_s$ turns to a filamentary current $I$ that flows along the cylinder axis [41, 42]. Furthermore, if the wire is assumed to be straight and, in Cartesian coordinates, directed along the $z$-axis one obtains from Equation (1.310)

$$j \omega \mu \int_{-L/2}^{L/2} \mathcal{G}^E_{zz}(z, z') I(z') \, \mathrm{d} z' = E_{\text{inc}}^z(z) \quad (1.311)$$

with $L$ the length of the antenna. Finally, Equation (1.301) is used to replace the $zz$-component of the electric dyadic Green’s function by the $zz$-component of the dyadic Green’s function for the magnetic vector potential in the Lorenz gauge. This yields Pocklington’s equation in the form

$$- \frac{1}{j \omega \epsilon} \int_{-L/2}^{L/2} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \mathcal{G}^A_{zz}(z, z') I(z') \, \mathrm{d} z' = E_{\text{inc}}^z(z). \quad (1.312)$$

### 1.5.2.2 Hallén’s Equation

Not until 40 years after the publication of Pocklington’s integral equation was a different integral equation for the calculation of antenna currents proposed by Hallén [43]. This integral equation is derived in close analogy to Pocklington’s equation. First, an equivalent surface current $J_s$ is introduced on the boundary of the antenna surface. This current is related to the scattered magnetic vector potential via

$$A_{\text{sca}}(r) = \mu \int_{\Gamma} \mathcal{G}^A(r, r') J_s(r') \, \mathrm{d}^2 r'. \quad (1.313)$$

The boundary condition for the total magnetic vector potential $A$ on a perfectly conducting surface is $\mathbf{e}_n \times A = 0$ or, equivalently, $A_t = 0$. With this boundary condition and the relation

$$A = A_{\text{inc}} + A_{\text{sca}} \quad (1.314)$$

one obtains

$$\mu \left[ \int_{\Gamma} \mathcal{G}^A(r, r') J_s(r') \, \mathrm{d}^2 r' \right] \cdot \mathbf{e}_t(r) = -A_{\text{inc}}^t(r). \quad (1.315)$$
Again, the special case of a thin, straight cylindrical wire which is directed along the $z$-axis is considered. Then, similar to Equation (1.311),

$$\mu \int_{-L/2}^{L/2} G_z^A(z, z') I(z') \, dz' = -A^\text{inc}_z(z).$$  \hspace{1cm} (1.316)

This result looks rather simple but it must be noted that, in practice, the incident electromagnetic field will usually be given in terms of the electric field strength $E^\text{inc}$ rather than in terms of the magnetic vector potential $A^\text{inc}$. In the Lorenz gauge,

$$E^\text{inc}(r) = -\frac{j\omega}{k^2} \left( \nabla (\nabla \cdot A^\text{inc}(z)) + k^2 A^\text{inc}(z) \right)$$  \hspace{1cm} (1.317)

and this second-order partial differential equation needs to be solved in order to obtain $A^\text{inc}$ from $E^\text{inc}$. In the case of a $z$-directed thin-wire antenna Equation (1.317) simplifies to

$$E^\text{inc}_z(z) = -\frac{j\omega}{k^2} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) A^\text{inc}_z(z).$$  \hspace{1cm} (1.318)

This ordinary differential equation has well-known solutions that are given by the sum of a general solution of the homogeneous problem and a special solution of the inhomogeneous problem. It follows that

$$-A^\text{inc}_z(z) = C_1 e^{jkz} + C_2 e^{-jkz} + \frac{k}{2j\omega} \int_{-L/2}^{L/2} \sin(k|z - z'|) E^\text{inc}_z(z') \, dz',$$  \hspace{1cm} (1.319)

where $C_1, C_2$ denote two integration constants that need to be determined from the boundary condition that the antenna current vanishes at the antenna ends. The function $G(z) = \sin(k|z|)$ that appears in the special solution of the inhomogeneous problem is, up to a constant factor, a Green’s function for the differential Equation (1.318). It fulfills $(\partial^2/\partial z^2 + k^2) G(z) = 2k\delta(z)$.

With Equation (1.319) one obtains from Equation (1.316) Hallén’s equation in the form

$$\mu \int_{-L/2}^{L/2} G_z^A(z, z') I(z') \, dz' = C_1 e^{jkz} + C_2 e^{-jkz} + \frac{k}{2j\omega} \int_{-L/2}^{L/2} \sin(k|z - z'|) E^\text{inc}_z(z') \, dz'.$$  \hspace{1cm} (1.320)

Compared with Pocklington’s Equation (1.312) the integral kernel of Hallén’s equation is less singular and thus preferable for numerical evaluations. The integral kernel of Pocklington’s equation exhibits a spatial singularity that is proportional to $1/|r - r'|^3$ while in the case of Hallén’s equation the spatial singularity is proportional to $1/|r - r'|$. However, to determine the integration constants $C_1, C_2$ in Hallén’s integral equation can be cumbersome and in such cases Pocklington’s equation might be the more practical choice.

### 1.5.2.3 Mixed-Potential Integral Equation

The mixed-potential integral equation represents another version of an electric-field integral equation. It is often used in numerical calculations since its integral kernel is proportional to $1/|r - r'|^2$, that is, the singularity of the integral kernel is weaker than in the case of Pocklington’s equation. Additionally, the mixed-potential integral equation does not require integration constants to be determined as in the case of Hallén’s equation.
To derive the mixed-potential integral equation the relation (1.147) for the scattered electromagnetic field in the frequency domain is considered:

\[ E^{\text{sca}}(r) = -\nabla \phi^{\text{sca}} - j\omega A^{\text{sca}}. \]  

(1.321)

If one replaces by means of the Lorenz gauge the scalar potential \( \phi^{\text{sca}} \) by the vector potential \( A^{\text{sca}} \) and uses Equation (1.313) one will be led back to Equation (1.309) and obtain nothing new. Alternatively, it is possible to consider, besides Equation (1.313), the source–field relation

\[ \phi^{\text{sca}}(r) = \mu \int_{\Gamma} G^{\phi}(r, r') \rho_s(r') \, d^2 r' \]  

(1.322)

with \( G^{\phi}(r, r') \) the scalar Green’s function of the scalar Helmholtz equation, (compare Equation (1.169)), and \( \rho_s \) a surface charge density. This surface charge density is related to a surface current \( J_s \) by a continuity equation which, in integral form, reads

\[ j\omega \int_{\Gamma} \rho_s \, dA + \int_{\partial\Gamma} J_s \cdot dA = 0. \]  

(1.323)

If the surface \( \Gamma \) is simply connected Stokes’ theorem (A.25) can be applied to yield the local continuity equation

\[ j\omega \rho_s(r) + (\nabla \times J_s(r)) \cdot e_n = 0 \]  

(1.324)

with \( e_n \) a normal vector on \( \Gamma \). This is a special case of the fundamental continuity Equation (1.10). It follows that Equation (1.321) can be rewritten in the form

\[ E^{\text{sca}}(r) = \frac{1}{j\omega\epsilon} \int_{\Gamma} \left[ (\nabla G^{\phi}(r, r')) (\nabla' \times J_s(r')) \cdot e_n + k^2 G^A(r, r') J_s(r') \right] d^2 r' \]  

(1.325)

This yields the electric field integral equation

\[ \frac{1}{j\omega\epsilon} \int_{\Gamma} \left[ (\nabla G^{\phi}(z, z')) (\nabla' \times J_s(z')) \cdot e_n + k^2 G^A(z, z') J_s(z') \right] d^2 z' = -E^{\text{inc}}_z(r). \]  

(1.326)

Consider again the special case of a \( z \)-directed, cylindrical and straight antenna. In this case Equation (1.324) is not valid since the surface of a cylinder is not simply connected. However, after a thin-wire approximation Equation (1.324) can be replaced by

\[ j\omega q' + \frac{dI}{dz} = 0, \]  

(1.327)

with \( q' \) the electric charge per unit length and \( I \) the total electric current on the antenna, and one finds in analogy to Equation (1.326) the mixed-potential integral equation

\[ \frac{1}{j\omega\epsilon} \int_{-L/2}^{L/2} \left[ \frac{\partial G^{\phi}(z, z')}{\partial z} \frac{\partial I(z')}{\partial z'} + k^2 G^A_{zz}(z, z') I(z') \right] \, dz' = -E^{\text{inc}}_z(z). \]  

(1.328)

In free space \( G^{\phi}(z, z') \) and \( G^A(z, z') \) are the same functions and the mixed-potential integral equation simplifies further.
1.5.2.4 Schelkunoff’s Equation

For completeness another electric field integral equation is mentioned that is known as Schelkunoff’s equation. It requires the condition

\[
\frac{\partial G^\phi(z, z')}{\partial z} = - \frac{\partial G^\phi(z, z')}{\partial z'}
\]

(1.329)

which implies translational invariance of the Green’s function, \(G^\phi(z, z') = G^\phi(|z - z'|)\). This condition is fulfilled for the Green’s function of free space (compare Equation (1.293)), but it will not be valid in general.

If Equation (1.329) holds, the first term in the integral of Equation (1.328) can be integrated by parts,

\[
\int_{-L/2}^{L/2} \frac{\partial G^\phi(z, z')}{\partial z'} \frac{\partial I(z')}{\partial z'} \, dz' = G^\phi(z, z') \frac{\partial I(z')}{\partial z'} \bigg|_{z'=-L/2}^{z'=L/2} - \int_{-L/2}^{L/2} G^\phi(z, z') \frac{\partial^2 I(z')}{\partial z'^2} \, dz'.
\]

(1.330)

Then the mixed-potential integral equation can be rewritten to yield Schelkunoff’s equation,

\[
\frac{1}{j \omega \varepsilon} \int_{-L/2}^{L/2} \left[ G^\phi(z, z') \frac{\partial^2 I(z')}{\partial z'^2} + k^2 G^A_{zz}(z, z') I(z') \right] \, dz' - \frac{1}{j \omega \varepsilon} G^\phi(z, z') \frac{\partial I(z')}{\partial z'} \bigg|_{z'=-L/2}^{z'=L/2} = -E^\text{inc}_z(z).
\]

(1.331)

As in the case of Hallén’s equation the integral kernel of Schelkunoff’s equation is proportional to \(1/|r - r'|\) and, thus, advantageous for numerical evaluation.

References

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