In a local volatility model, the instantaneous stock volatility $\sigma(S, t)$ is a function of stock price and future time.

How to build and use a binomial tree with variable local volatility.

The BSM implied volatility of a standard option in a local volatility model is approximately the average of the local volatilities between the initial stock price and the strike.

In the preceding chapter we extended the Black-Scholes-Merton (BSM) model to accommodate a term structure of implied volatilities. In practice, implied volatility varies not only with time but with the level of the underlier. In this chapter we extend the model to encompass a volatility that is a function of both future time and underlier level.

**MODELING A STOCK WITH VARIABLE VOLATILITY**

In the previous chapter we extracted the forward volatilities $\sigma(t)$ of the stock from the term structure of implied volatilities using the equation

$$\Sigma^2(t, T) = \frac{1}{T-t} \int_t^T \sigma^2(s)ds$$

(14.1)

Just as we can imagine a volatility $\sigma(t)$ that varies with time, we can similarly imagine a volatility $\sigma(S, t)$ that varies with future time and with stock price. We refer to this instantaneous volatility $\sigma(S, t)$ as the local volatility, and to option models based on it as local volatility models.

In local volatility models, the realized volatility over any time period will depend on the path that the stock price takes over time. Ultimately we will want to use the local volatility model to determine the value of options.
THE VOLATILITY SMILE

Even if we assume that the local volatility model is an accurate representation of realized volatility, we may still want to use the BSM model and its implied volatility as a quoting convention.

In exploring local volatility models, these are some of the questions that will concern us:

1. Can we find a unique local volatility function or surface \( \sigma(S, t) \) to match the observed implied volatility surface \( \Sigma(S, t, K, T) \)? If we can, that means that we can explain the observed smile by means of a local volatility process for the stock.
2. But is the explanation meaningful? Does the stock actually evolve according to an observable local volatility function? There are, as we will see, many different models that can match the implied volatility surface, but achieving a match doesn’t mean that model is “correct.”
3. What does the local volatility model tell us about the hedge ratios of vanilla options and the values of exotic options? How do the results differ from those of the classic BSM model?

We begin by constructing binomial local volatility models, assuming we have been given a local volatility function. In a subsequent chapter we will determine how to extract the local volatility function from the prices of standard options.

BINOMIAL LOCAL VOLATILITY MODELING

In the previous chapter, we were able to build a closed binomial tree with time-dependent volatility by changing the size of the time steps. When the level of volatility varies with both time and stock price, we can also build a closed tree. There are a number of ways to do this, but this time we will find it easier to use equal time steps.\(^1\)

Assume the risk-neutral evolution of the stock price \( S(t) \) can be described by

\[
\frac{dS}{S} = (r - b)dt + \sigma(S, t)dZ
\]  

(14.2)

\(^1\)This section and much of the chapter are based in part on Emanuel Derman and Iraj Kani, “The Volatility Smile and Its Implied Tree,” Risk 7, no. 2 (February 1994): 32–39.
where \( r \) is the riskless rate, \( b \) is the stock’s continuous dividend yield, \( dZ \) is a standard Wiener process, and \( \sigma(S, t) \) is the local volatility. It follows that the variance of changes in the stock price at any time \( t \) is

\[
(dS)^2 = S^2 \sigma^2(S, t) dt \tag{14.3}
\]

The expected value of \( S \) after a small interval \( dt \) is

\[
F = S e^{(r-b)dt} \tag{14.4}
\]

which is also the forward price of the stock.

Figure 14.1 shows a binomial approximation to the stochastic process over time \( dt \).

In our binomial approximation, the forward price is simply the probability-weighted average of the two possible stock prices \( S_u \) and \( S_d \) in the \( q \)-measure, so that

\[
F = qS_u + (1-q)S_d \tag{14.5}
\]

Solving for \( q \), we obtain

\[
q = \frac{F - S_d}{S_u - S_d} \tag{14.6}
\]
In our binomial approximation the variance of changes in S is then

\[ \text{Var}[dS] = q(S_u - F)^2 + (1 - q)(S_d - F)^2 \]  \tag{14.7} \]

In the limit \( dt \to 0 \), Equation 14.3 and Equation 14.7 must agree, so

\[ S^2 \sigma^2(S, t) dt = q(S_u - F)^2 + (1 - q)(S_d - F)^2 \]  \tag{14.8} \]

Substituting the formula for \( q \) from Equation 14.6 into Equation 14.8, we see that

\[ S^2 \sigma^2(S, t) dt = (S_u - F)(F - S_d) \]  \tag{14.9} \]

We can rearrange Equation 14.9 to express the up and down prices relative to the node with price \( S \) as

\[ S_u = F + \frac{S^2 \sigma^2(S, t) dt}{F - S_d} \]  \tag{14.10a} \]

\[ S_d = F - \frac{S^2 \sigma^2(S, t) dt}{S_u - F} \]  \tag{14.10b} \]

Thus, for any binomial step like that in Figure 14.1, with an initial node \( S \) and two subsequent nodes \( S_u \) and \( S_d \) relative to it, if we know \( S, F, \) and \( S_d \), we can calculate \( S_u \) consistent with the volatility \( \sigma(S, t) \); conversely, if we know \( S, F, \) and \( S_u \), we can calculate \( S_d \).

Figure 14.1 displays one step in a binomial tree. In order to create additional steps, we will first construct the center of the tree, and then build out the upper and lower branches of the tree in a way that is consistent with the local volatility surface \( \sigma(S, t) \) via Equation 14.10. This will produce a tree with all the appropriate local volatilities. We can then go back to each tree node and use Equation 14.6 to solve for the risk-neutral probabilities. Once we have these, we can value any derivative security of the stock price by the usual process of backward induction on the tree.

We start by making the central spine of the tree consistent with the Cox-Ross-Rubinstein (CRR) approach as described in the preceding

---

2Formally, for a discrete random variable \( x \), the variance of \( x \) is \( \text{Var}[x] = \text{E}[(x - \text{E}[x])^2] \). In our current model, \( \text{E}[dS] = q(S_u - S) + (1 - q)(S_d - S) = F - S \). The variance of \( dS \) can then be found as \( \text{Var}[dS] = q((S_u - S) - \text{E}[dS])^2 + (1 - q)((S_d - S) - \text{E}[dS])^2 \). For more on discrete random variables, variance, and expectations operators, see Miller (2014).
chapter. Starting with the initial node with price $S_0$ at the root of the tree, the central node $S$ of every level with an odd number of nodes is chosen to be equal to the initial price $S_0$. For the other levels, those with an even number of nodes, the two central nodes connected to the previous level’s central node $S$ are given by

$$S_u = S e^{\sigma(S,t)\sqrt{dt}}$$
$$S_d = S e^{-\sigma(S,t)\sqrt{dt}}$$

(14.11)

where $\sigma(S, t)$ is the local volatility at the stock price $S$ at future time $t$. This procedure specifies the spine of the tree.

At each level, from these central nodes, we can sequentially build out the up nodes above the spine by using Equation 14.10a, and the down nodes below the spine by using Equation 14.10b.

This initial choice of $S_0$ for the central spine of the tree is arbitrary. We could, for example, have chosen the central spine to correspond to the forward stock price at each level, or to any other price. Assuming that the forward stock price at a level with an odd number of nodes is given by $F_t$, Equation 14.11 for the subsequent level with an even number of nodes would be replaced by

$$S_u = F_t e^{\sigma(F_t,t)\sqrt{dt}}$$
$$S_d = F_t e^{-\sigma(F_t,t)\sqrt{dt}}$$

(14.12)

This guarantees that the local volatility at $F_t$ is in fact $\sigma(F_t, t)$.

Let’s illustrate the method by building a simple tree.

**SAMPLE PROBLEM**

**Question:**

Suppose the current value of a stock is $S_0 = $100. Assume that the local volatility is independent of future time $t$ and varies only with the stock price according to

$$\sigma(S) = \max \left[ 0.1 - \frac{S - S_0}{S_0}, 0.01 \right]$$

(14.13)
As shown in Figure 14.2a, near the current stock price local volatility decreases by one percentage point for every 1% increase in the stock price. To ensure that volatility remains positive, we arbitrarily set a minimum local volatility of 1%. Assume dividends and the risk-less rate are zero. Construct the first three levels of a binomial tree with $\Delta t = 0.01$.

Answer:
Figure 14.2b shows a diagram of the tree.
We use the notation $S_{ij}$ to denote the absolute position of nodes on the tree, and the notation $S$, $S_u$, and $S_d$, to denote relative positions on a binomial fork like that in Figure 14.1.

At the root of the tree at node $S_{00}$, $S = $ $100$ and $\sigma(S) = 10\%$. At the next level, the node $S_{11}$ is up relative to $S_{00}$, and the node $S_{10}$ is down relative to $S_{00}$. Their prices are the same as they would be for a standard Cox-Ross-Rubinstein tree:

\[
S_{11} \equiv S_u = Se^{\sigma(S,t)\sqrt{\Delta t}} = 100e^{0.10 \sqrt{0.01}} = 101.01
\]
\[
S_{10} \equiv S_d = Se^{-\sigma(S,t)\sqrt{\Delta t}} = 100e^{-0.10 \sqrt{0.01}} = 99.00
\]

Because dividends and the riskless rate are both zero, the forward price for node $S_{00}$ is equal to the initial price of $100$. The risk-neutral probability of an up move is

\[
q_{00} = \frac{F - S_d}{S_u - S_d} = \frac{100.00 - 99.00}{101.01 - 99.00} = 0.4975 \quad (14.14)
\]

At the third level there are three nodes, $S_{22}$, $S_{21}$, and $S_{20}$. The central node $S_{21}$ at the third level is set equal to the initial price of $100$. Using Equation 14.10, we can find the stock prices at the nodes above and below it.

Consider the node $S_{11}$, whose local volatility is 0.09 and whose forward price is $F = S_{11} = 101.01$. Its relative down node is the central node $S_{21}$. Its relative up node is $S_{22}$, whose price, from Equation 14.10a, is

\[
S_{22} \equiv S_u = 101.01 + \frac{101.01^2 \times 0.09^2 \times 0.01}{101.01 - 100} = 101.83
\]

The risk-neutral probability of going from $S_{11}$ to $S_{22}$ is then:

\[
q_{11} = \frac{101.01 - 100}{101.83 - 100} = 0.5503
\]

(continued)
(continued)

Similarly, consider the node $S_{10}$, whose local volatility is 11% and whose forward price is $F = S_{10} = 99.00$. Its relative up node is the central node $S_{21}$. Its relative down node is $S_{20}$, whose price, from Equation 14.10b, is given by

$$S_{20} \equiv S_d = 99.00 - \frac{99.00^2 \times 0.11^2 \times 0.01}{100 - 99.00} = 97.81$$

The risk-neutral probability of going from $S_{10}$ to $S_{21}$ is then

$$q_{10} = \frac{99.00 - 97.81}{100 - 97.81} = 0.5448$$

The tree of resultant prices and risk-neutral probabilities is therefore as shown in Figure 14.2c.

![Figure 14.2c](c)

**FIGURE 14.2c**  Local Volatility Tree

The unconditional risk-neutral probabilities for moving from the root of the tree to the final three nodes are then

$$q_{22} = 0.4975 \times 0.5503 = 0.2738$$

$$q_{21} = 0.4975 \times (1 - 0.5503) + 0.5025 \times 54.48 = 0.4975$$

$$q_{20} = 0.5025 \times (1 - 0.5448) = 0.2287$$
In this simple local volatility tree, because volatility increases as stock prices decline, the down moves are larger than the up moves and the terminal prices are negatively skewed.

As the preceding example makes clear, there is a systematic way to build a binomial tree with a variable local volatility. Because of the clear intuition they provide, binomial local volatility trees are a good way to understand the principles and consequences of local volatility models, and we will use them as our main pedagogical tool. For efficient numerical computation on a trading desk, trinomial trees or other more general finite difference approximation schemes for the numerical solution to partial differential equations may converge faster to the continuum limit and be easier to calibrate.

THE RELATIONSHIP BETWEEN LOCAL VOLATILITY AND IMPLIED VOLATILITY

We have demonstrated how to build a local volatility tree. Our longer-term goal, though, is to find out what sort of local volatilities will produce a particular observed implied volatility smile.

To examine this, we must value options on a binomial local volatility tree, and calculate their BSM implied volatilities. Consider a tree like the one in the preceding sample problem, with the same local volatility function, but extended to five levels instead of three.

Figure 14.3 shows the five-level price tree that results from the local volatility, the corresponding local volatilities, the $q$-measure transition probabilities between nodes, and the cumulative probabilities of reaching any node, computed from the products of the $q$-measure transition probabilities.

What is the value of a European call option with strike $102$ expiring after five periods? Looking at the terminal levels of the tree, the only node at which the option is in-the-money at expiration is the one with stock price $103.34$. At that node, the option is worth $1.34$. With risk-neutral valuation and an assumed riskless rate of zero, the present value of this payoff is also $1.34$. The risk-neutral expected value of this payoff is this present value multiplied by the cumulative probability 7.52% of reaching that node, $1.34 \times 0.0752 = 0.10$. Because the call expires worthless on all of the other nodes, this $0.10$ is also the value of the option at inception.

The value of the call option depends on the risk-neutral probability that the stock price will be greater than $102$ at expiration. That probability is, in turn, related to the average local volatility the stock price experiences.
between $100 and $102 as it makes its way to being in-the-money at expiration. In Figure 14.2, the average local volatility between $100 and $102, based on Equation 14.13, is \((10\% + 8\%)/2 = 9\%\). We might therefore guess that the value of a call option with strike $102 in the local volatility model with variable local volatility is the same as on a binomial tree with a constant volatility everywhere of 9%.

To test this, let’s construct a second binomial tree with a constant volatility equal to 9%, using the Cox-Ross-Rubinstein approach as shown in Figure 14.4. Note that the prices at each node and the probabilities of reaching those nodes differ from those of the local volatility tree in Figure 14.3. As before, for a call with strike $102, there is only one node at expiration that results in a nonzero payoff, in this case one with a price of $103.67. The value of that payoff is $1.67, and, with zero interest rates, its present value is \(1.67 \times 0.0614 = 0.10\), the same two decimal places as on the local volatility tree.

We remind the reader that a tree with a constant volatility produces an option value that converges to the BSM formula in the limit as the spacing between tree levels approaches zero. In that sense, the constant
Cox-Ross-Rubinstein (CRR) volatility of 9% that matches the local volatility value of $0.10 can be regarded as the implied CRR volatility of the option value. Just as the BSM implied volatility of an option is the volatility you must insert into the BSM formula to produce that particular option's price, so we define the CRR implied volatility as the constant volatility that produces the option's price in the CRR model. In the limit of zero level spacing, as we showed in Chapter 13, the CRR implied volatility approaches the BSM implied volatility. From our example, then, we conclude that the correct CRR implied volatility for valuing the option is approximately the linear average of the local volatility between the current stock price level and the strike price of the option. Similarly, in the continuum limit, we conjecture that the correct BSM implied volatility is approximately the average of the local volatility between stock price and strike.

Why should this be so? Figure 14.5 depicts various stock price paths. The paths that contribute to positive option payoffs must traverse the region between the initial stock price $S$ and the strike price $K$ in order to finish in-the-money. The paths that finish in-the-money sample the local volatility in this region. This leads to the implied volatility of a standard option being approximately the linear average of the local volatilities between $S$ and $K$. 
The implied volatility $\Sigma(S, t, K, T)$ for a given $S$ and $t$ has two dimensions, one for the time to expiration $T$ and one for the strike $K$. If you think of the time direction as going forward, and the strike direction as going sideways, then our conclusion above is that, when the local volatility $\sigma(S)$ is a function of stock price alone, the implied volatility for an option of strike $K$ is the “sideways” average of the local volatilities between $S$ and $K$. This relationship between implied and local volatilities is reminiscent of Equation 13.52 in Chapter 13, which showed that when local volatility $\sigma(t)$ is a function of time alone, the implied variance for expiration $T$ is an average of forward variances between the $t$ and $T$. It also resembles Equation 13.53 of Chapter 13, which relates the yield to maturity of a bond to the average of forward rates.

When the local volatility $\sigma(S, t)$ is a function of both stock price and time, from Figure 14.5, we conjecture that the implied volatility will still be an average of the local volatility over the path from the initial stock price to the terminal strike.

It's not surprising that a yield is an exact average of forward rates, because the relationship between continuously compounded yields and forward rates is genuinely linear. It is somewhat surprising that the
relationship between implied volatility and local volatility is approximately a linear average, because the BSM option formula and the CRR binomial tree both exhibit a nonlinear dependence on volatility. We will see in subsequent chapters why this approximation is so surprisingly good.

It’s easy to see why the linear average approximation between implied volatility and local volatility should fail. In Figure 14.5, some paths that end up in-the-money take the stock price below the initial price, whereas others take the stock price above the strike price. Thus, the paths that contribute to the option value sample the local volatility at many different stock price levels, not just those between the current stock price and the strike. Nevertheless, for slowly varying local volatilities, most of the paths that end up in-the-money at expiration will spend most of their time between the initial stock price and the strike price, so it is the local volatilities between current stock price and strike price that contribute predominantly to the option value. That’s why the approximation works so well, and why the prices from the two trees in our example were so similar.

Nevertheless, the linear average is only an approximation. There are contributions to the option payoff from paths that go above the strike and below the current price, but, because of the nature of geometric Brownian motion, these paths have lower risk-neutral probabilities than the more direct paths. In a subsequent chapter, we will discover a better averaging approximation.

The Rule of Two: Understanding the Relationship between Local and Implied Volatilities

We illustrated previously that the implied volatility \( \Sigma(S, K) \) of an option is approximately the average of the local volatilities \( \sigma(S) \) encountered over the life of the option between the current underlying price and the strike. We also remarked that this is analogous to regarding yields to maturity for zero coupon bonds as an average over forward rates. For interest rates, because of this averaging, it is common knowledge that forward short-term rates grow twice as fast with future time as yields to maturity grow with maturity. Similarly, if local volatilities \( \sigma(S) \) are a function of stock price alone, then one can show that local volatilities grow approximately twice as fast with stock price as implied volatilities grow with strike. This relationship is often called the rule of two.

In this section we provide another informal proof of the rule of two.\(^3\) Later we’ll prove it more rigorously. We restrict ourselves to the simple case

in which the value of local volatility of an index is independent of future time, and varies linearly with index level, so that

\[ \sigma(S) = \sigma_0 + \beta S \tag{14.15} \]

Because we refer to the variation in future local volatility as the “forward” volatility curve, we can call this variation with future index level the “sideways” volatility curve.

Consider the implied volatility \( \Sigma(S, K) \) of a slightly out-of-the-money call option with strike \( K \) when the index is at \( S \). Any paths that contribute to the option value must pass through the region between \( S \) and \( K \), as shown in Figure 14.5. As we noted, the volatility of these paths is determined primarily by the local volatility between \( S \) and \( K \). Because of this, you can think of the implied volatility for the option of strike \( K \) when the index is at \( S \) as the average of the local volatilities over the shaded region, so that

\[ \Sigma(S, K) \approx \frac{1}{K - S} \int_S^K \sigma(S')dS' \tag{14.16} \]

By substituting Equation 14.15 into Equation 14.16 you can show that

\[ \Sigma(S, K) \approx \sigma_0 + \frac{\beta}{2}(S + K) \tag{14.17} \]

Comparing Equation 14.15 and Equation 14.17, we see that local volatility varies with \( S \) at twice the rate that implied volatility varies with \( S \). Equation 14.17 also shows that the rate of change of implied volatility with \( S \) is equal to the rate of change with \( K \).

You can also combine Equations 14.15 and 14.17 to write the relationship between implied and local volatility more directly as

\[ \Sigma(S, K) \approx \sigma(S) + \frac{\beta}{2}(K - S) \tag{14.18} \]

**DIFFICULTIES WITH BINOMIAL TREES**

As we have shown, the positions of the nodes of the local volatility tree and the transition probabilities are uniquely determined by forward interest rates, dividend yields, and the local volatility function. But if the local volatility varies too rapidly with stock price or time, then, with finite spacing between tree levels, some nodes may have stock prices that violate the no-arbitrage condition and result in binomial transition probabilities greater than one or less than zero.
Local Volatility Models

As an example, consider a tree with an initial stock price $S_{00} = $100 and $\Delta t = 1$, as shown in Figure 14.6. We have assumed the riskless rate and dividend yield are both zero. Further, assume that the local volatility is 10% for the first two levels, but jumps to 21% on the third level when $S_{21} = $100. This is a very rapid increase in local volatility, and will cause an arbitrage to occur at the next level. Specifically, because the local volatility at $S_{21}$ is so high, the relative up price from node $S_{32}$ is $S_{32} = 123.37$. But $S_{32}$ is the down node relative to $S_{22}$, and yet $123.37 = S_{32} > S_{22} = $122.13. Then $S_{33}$ must lie even higher than $S_{32}$, with the result that the up and down nodes from $S_{22}$ will both lie above $S_{22}$, which, for zero interest rates, is equal to the forward price of $S_{22}$. When both the up and down prices from a node lie above its forward price, there is an arbitrage opportunity.

These sorts of problems can be remedied by taking much smaller time steps, but smaller time steps produce their own difficulties. For any given stock or index there is only a finite number of options, and therefore a finite number of observable implied volatilities. The implied volatility surface is populated coarsely—it is really a grid rather than a surface. If we try to use a coarse implied volatility grid to calibrate a finely grained local volatility tree, we will find that we simply cannot extract enough information from the implied volatilities unless we make assumptions about how to interpolate and extrapolate the implied volatility grid smoothly.

FURTHER READING

There is a large literature on local volatility models. The following is a brief list of suggested articles and books to get you started:

END-OF-CHAPTER PROBLEMS

14-1. The initial price of a stock is $100. Assume that annualized local volatility is known, and varies only with the stock price according to

$$\sigma(S) = \max \left[ 0.11 - 2 \times \left( \frac{S - S_0}{S_0} \right), 0.01 \right]$$

Assume dividends and the riskless rate are zero. Construct the first five levels of a binomial tree with $\Delta t = 0.01$ years. As in the sample problem, use the Cox-Ross-Rubinstein model to construct the central spine of the tree.

14-2. Using the same information as in the previous problem, calculate the value of a European call option with a strike of $102$, which expires after four time steps. With the exception of constant volatility, assume that all of the BSM assumptions hold.

14-3. Calculate the price of a European call option with strike $102$ that expires after four time steps. Use the same information as in the previous two problems, but assume that the riskless rate is $4\%$.

14-4. The initial price of a stock is $200$. The riskless rate and dividends are zero. Construct the first three levels of a binomial tree using the Cox-Ross-Rubinstein model with time step, $\Delta t = 0.01$ years. Assume that the local volatility is $20\%$ for the first two levels. What is the maximum local volatility for the center node of the third level in order for the tree to have no arbitrage-violating nodes? As before, assume that the central spine of the tree is constructed according to the Cox-Ross-Rubinstein model.