CLASSIFICATION OF LOW-FREQUENCY ELECTROMAGNETIC PROBLEMS. POISSON AND LAPLACE EQUATIONS IN INTEGRAL FORM

INTRODUCTION

The first section of this chapter starts with a physical model of an electric circuit. This example allows us to introduce and visualize the following primary research areas of static and quasistatic analyses:

- Electrostatics
- Magnetostatics
- Direct current (DC) flow
- Eddy current quasistatic approximation

Next, we quantify the necessary physical conditions that justify static and quasistatic approximations of Maxwell’s equations. Three major dimensionless parameters encountered in static and quasistatic approximations are as follows:

- The ratio of problem dimensions to the wavelength
- The ratio of charge relaxation time to the wave period
- The ratio of problem dimensions to the skin depth

The end of the first section is devoted to nonlinear electrostatics, which is an important part of semiconductor device analysis with critical analogues to the subject of bimolecular research.
The second section introduces the Poisson and Laplace equations, along with the free-space Green’s function, and briefly outlines the Green’s function technique. We specify Dirichlet, Neumann, and mixed boundary conditions and demonstrate practical examples of each. Special attention is paid to the integral form of the Poisson and Laplace equations, which present the foundation for the boundary element method (BEM). We consider the surface charge density at boundaries as the unknown function and thus utilize the surface charge method (SCM).

We establish the continuity of the potential function at boundaries and mathematically derive the discontinuity condition for the normal potential derivative. This condition provides the framework of almost all specific integral equations for individual static and quasistatic problems of various types studied in the main text.

1.1 CLASSIFICATION OF LOW-FREQUENCY ELECTROMAGNETIC PROBLEMS

Low-frequency electromagnetics finds its applications in many areas of electrical engineering including the fields of power electronics and power lines [1–4], semiconductor devices and integrated circuits [5, 6], alternative energy [7], and nondestructive testing and evaluation [8, 9]. Major biomedical applications include EEG, ECG, and EMG (cf. [10, 11]), biomedical impedance tomography [12–18], and rather new fields such as biomolecular electrostatics [19–22] and magnetic [23–25] and DC [26–30] brain stimulation, among many others.

1.1.1 Physical Model of an Electric Circuit

The bulk of low-frequency electromagnetic problems may be visualized with the help of a static or a quasistatic model of an electric circuit, as shown in Figure 1.1. The model includes three elements:

1. A voltage power source that in the direct current (DC) case generates a constant voltage between its terminals.
2. An electric load that consumes electric power. The load may be modeled as a resistant material of low conductivity.
3. Two finite-conductivity conductors that extend from the source to the load. These wires form a transmission line. In the laboratory, both wires may be arbitrarily bent. However, this is not the case in power electronics and high-frequency circuits.

Figure 1.1a shows the (computed) electric field or electric field intensity, \( \mathbf{E} \), everywhere in space. The subject of electrostatics is the computation of \( \mathbf{E} \) and the associated quantities (surface charges, capacitances) when there is no load attached to the source. In other words, there is no DC flow in the conductors. In this case, the field distribution around the transmission line might be somewhat different from that shown in Figure 1.1a. However, the difference becomes negligibly small when the wires in
FIGURE 1.1 Physical model of an electric circuit depicting (a) Electrostatics and (b) Magnetostatics scenarios produced by direct current flow. Note that the electric field between the two wires decreases when moving from the source to the load. This is not the case when the wires have the infinite conductivity resulting in zero potential drop. This figure was generated using numerical modeling tools developed in the text.
Figure 1.1a are close to ideal—possessing a very large conductivity. The situation becomes more complicated when a dielectric material, which alters the electric field both inside and outside, is present.

**Exercise 1.1:** How would the voltage (or potential) of two wires in Figure 1.1a change under open-circuit conditions (the electrostatic model)?

**Answer:** Both wire surfaces will become strictly equipotential surfaces, say, at 1 and 0 V. There will be no electric field within the wires themselves.

The subject of DC computations is the evaluation of the electric field in conductors themselves and in the surrounding space. This is exactly the problem shown in Figure 1.1a. After the electric field, \( E \), is found, the current density, \( J \), in the conductors is obtained as \( E \) multiplied by the conductivity (see Fig. 1.1b). DC computations deal with finite-conductivity conductors, whereas in electrostatics, any conductor is ideal. At the same time, electrostatics models dielectric materials or insulators. DC computations are typically not intended to do so since there is no current present in insulators. DC computations may deal with quite complicated current distributions in heterogeneous conducting media, for example, human tissues.

**Exercise 1.2:** As far as DC flow is concerned, Figure 1.1a and b has a few simplifications. What is the most significant one?

**Answer:** The electric field distribution and the associated current distribution within the load may be highly nonuniform, at least close to the load terminals.

The subject of magnetostatics is the computation of the magnetic field or magnetic field intensity, \( H \), and the associated quantities (mutual and self-inductances). The magnetic field is due to currents flowing in conductors as shown in Figure 1.1b. Magnetostatics typically deals with external current excitations, which are known a priori (e.g., from DC analysis). The situation complicates when a magnetic material, which alters the magnetic field both inside and outside, is present.

**Exercise 1.3:** After the magnetic field \( H \) and the electric field \( E \) in Figure 1.1b are found, a vector \( P = E \times H \) (also shown in Fig. 1.1b) may be constructed everywhere in space. What is the intuitive feel of this vector?
The subject of eddy current theory (or quasistatic theory) is the effect of a time-varying magnetic field producing alternating currents. According to Faraday’s law of induction, this magnetic field will create a secondary electric field in conductors. In its turn, the secondary electric field will result in certain currents, known as eddy currents. These eddy currents may be excited in a conductor without immediate electrode contacts (which is to say, in a wireless manner). They may also affect the original alternating current distribution (via the skin layer effect). The situation greatly complicates for arbitrary geometries and in heterogeneous conducting media where eddy currents have to cross boundaries between different materials.

Exercise 1.4: As far as the eddy current theory is concerned, Figure 1.1a and b has a few simplifications. What is the most significant one?

Answer: The current distribution in thick metal wire conductors is nonuniform, even at 60 Hz. The current density mostly concentrates within a skin layer close to the conductor’s surface.

Finally, the load in Figure 1.1 may be a basic semiconductor element, a diode, for example. The internal diode behavior at reverse and small forward-bias voltages is still modeled by electrostatic equations, but those equations will be nonlinear. At large forward-bias voltages, DC theory is applied, which also becomes nonlinear.

1.1.2 Starting Point of Static/Quasistatic Analysis

In order to quantitatively explain various static and quasistatic approximations, we need to start with the full set of Maxwell’s equations, which include electric field, \( \mathbf{E} \), measured in V/m; magnetic field, \( \mathbf{H} \), measured in A/m; volumetric electric current density, \( \mathbf{J} \), of free charges with the units of A/m\(^2\); and the (volume or surface) electric charge density, \( \rho \), of free charges with the units of C/m\(^3\) or C/m\(^2\). Permittivity, \( \varepsilon \), measured in F/m and permeability, \( \mu \), measured in H/m may vary in space. Maxwell’s equations are then given as

\[
\varepsilon \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J} \quad (1.1)
\]

Faraday’s law

\[
\mu \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E} \quad (1.2)
\]
Gauss’ law for electric fields
\[ \nabla \cdot \mathbf{E} = \rho \]  (1.3)

Gauss’ law for magnetic fields (no magnetic charges)
\[ \nabla \cdot \mu \mathbf{H} = 0 \]  (1.4)

Continuity equation for the electric current
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \]  (1.5)

The continuity equation (1.5) for the electric current is not independent; it is directly obtained from Equations 1.3 and 1.1 keeping in mind that the divergence of the curl of any vector field is always 0 and that the medium is locally homogeneous. Electric current is related to the electric field by a local form of Ohm’s law
\[ \mathbf{J} = \sigma \mathbf{E} \]  (1.6)

where \( \sigma \) is the medium conductivity with the units of S/m.

1.1.3 Electrostatic, Magnetostatic, and DC Approximations

Certain approximations can be made when analyzing objects subject to electromagnetic excitation. Consider an object under study of a certain size, \( D \), with a given electromagnetic excitation at a frequency of \( f = \omega / (2\pi) \) and the corresponding wavelength, \( \lambda \), as shown in Figure 1.2; we assume that \( \lambda \) is the shortest wavelength in the object material. The necessary condition for both electrostatic and magnetostatic approximations and for the DC approximation is the condition [32]
\[ D \ll \lambda, \quad \lambda \equiv \frac{c}{f}, \quad c = \frac{1}{\sqrt{\mu \varepsilon}} \]  (1.7)

The time derivative in Equations 1.1 and 1.2 may be approximated as \( \partial / \partial t \propto f \). The spatial derivatives may be approximated as \( \partial / \partial x \propto \partial / \partial y \propto \partial / \partial z \propto 1 / D \). Therefore,

![Figure 1.2](image)

**FIGURE 1.2** Illustration of electrostatic and magnetostatic approximations.
Equation 1.7 rewritten in the form $f \ll c/D$ suggests that the two terms with time derivatives in both Equations 1.1 and 1.2 are much smaller than the terms with spatial derivatives and can therefore be entirely neglected. The local speed of light, $c$, plays the role of a proportionality constant when such a comparison is made. This process of neglecting all terms with time derivatives in Equations 1.1–1.5 is the electrostatic and/or magnetostatic approximation.

1.1.4 Static Versus Parametric Quasistatic Analysis

1.1.4.1 Parametric Dependence on Time The word “static” used in the previous text is somewhat confusing. It often means not only the true steady-state problem but also a large number of problems where the time dependence is present only parametrically, through time-dependent excitation conditions or otherwise. An example is current in a thin wire subject to a time-varying voltage. The current remains the same along the wire (follows a static pattern), which is then simply multiplied by a time-varying factor. At the same time, the absolute operating frequency may still be quite high—on the order of tens or hundreds of kHz or so. Therefore, the “static” approximation often also implies a low-frequency parametric approximation.

1.1.4.2 Radiation Conditions Any oscillating electromagnetic system eventually emits radio waves. They can be extremely weak, but they do exist at large distances of $r$, with $r \geq \lambda$. This effect is not described by the parametric quasistatic analysis.

Example 1.1: Two electrodes attached to a human body are separated by $D = 37.2$ cm. The electrodes source and sink a total current of $i(t) = I_0 \cos 2\pi ft$, where $f = 10$ kHz, $I_0 = 1$ mA. Determine whether or not Equation 1.7 is satisfied, that is, whether or not the solution for the current distribution within the body may be approximately given by the product $J(r) \cos 2\pi ft$, where $J(r)$ is the solution of the steady-state problem with the injection current, $I_0$.

Solution: We model the human body as a mass of muscle tissue with parameters $\varepsilon = 2.6 \times 10^4 \varepsilon_0$, $\mu = \mu_0$ at 10 kHz [31], where $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m and $\mu_0 = 1.257 \times 10^{-6}$ H/m are permittivity and permeability of vacuum, respectively. In this case, the wavelength, $\lambda$, within the body is 186 m. The ratio $D/\lambda$ is thus equal to 0.002. This reasonably small quantity allows us to consider the product $J(r) \cos 2\pi ft$ to be a viable solution, though a more accurate and sophisticated analysis might be necessary or desired.

Electrostatic and DC approximations are studied in Parts II and III of this text.

1.1.5 Eddy Current (Quasistatic) Approximation

The eddy current approximation is a true quasistatic approximation, which cannot be obtained through the multiplication of a static solution by the time-varying factor. It
relates to media with a large or significant conductivity. Typical examples are metals, seawater, human body tissues, soils, and similar materials. The eddy current approximation in its most general form only affects Ampere’s law (1.1). This approximation will be explained with reference to Figure 1.3. In terms of the general eddy current approximation, we assume that the conduction current from Equation 1.6 dominates the displacement current in Ampere’s law, that is, for a periodic excitation

\[ |J| = |\sigma E| = \sigma |E| \gg \left| e \frac{\partial E}{\partial t} \right| \approx \varepsilon \omega |E| \]

It is seen from Equation 1.8 that the following inequality should be satisfied:

\[ \frac{\varepsilon \omega}{\sigma} \ll 1 \text{ or } \omega \tau \ll 1, \quad \tau = \frac{\varepsilon}{\sigma} \]

where \( \omega \) is the angular frequency of interest and the constant \( \tau \) is known as the charge relaxation time. Inequality (1.9) means that only the displacement current is neglected, that is, only the time derivative in Ampere’s law is neglected. Faraday’s law of induction still remains intact.

Neglecting displacement currents implies that the wave propagation mechanism is lost; we no longer permit transmission of electromagnetic waves. Instead, a diffusion equation will be obtained with a formally infinite propagation speed of small perturbations.

**Example 1.2:** A human body is subject to a 20 kHz AC magnetic field generated by an external coil (see Fig. 1.3). Determine whether or not Equation 1.9 is satisfied, that is, if the displacement current can be neglected compared to the conduction current.

**Solution:** We model the human body as a mass of muscle tissue with parameters \( \varepsilon = 1.6 \times 10^4 \varepsilon_0, \quad \sigma = 0.35 \text{ S/m at } 20 \text{ kHz} \) [31]. We use the value \( \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \) and obtain the following value of the ratio in Equation 1.9:
which is a reasonably small number. Still, a more accurate analysis might be required.

General eddy current (quasistatic) approximation is studied in Part IV of this text.

1.1.6 Eddy Current Approximation in a Weakly Conducting Medium
("Thin Limit" Condition)

The general eddy current approximation with nonzero surface charges approaches original Maxwell’s equations in terms of complexity. An important and simpler case is related to media of a lower conductivity. Metals are highly conducting materials. Therefore, the skin effect is becoming dominant even at low frequencies. Human tissues, on the other hand, have a conductivity that is six to seven orders of magnitude smaller. Therefore, they could be considered as a weakly conducting medium in the following sense. In a weakly conducting medium, the induced eddy currents are small in the typical range of frequencies of interest. Their own (secondary or internal) magnetic field is also small as compared to the known external large magnetic field in Faraday’s law (1.2). Therefore, its time derivative in Equation 1.2 can be neglected, whereas the known and much larger time derivative of the external field is kept. It will be shown in Chapter 11 of Part III that such an approximation leads to the Poisson equation for the electric potential of eddy currents, similar to the Poisson equations used in electrostatics.

Physically, the earlier assumption means that the skin layer depth, \( \delta \), is very large compared to the object size, \( D \):

\[
\delta = \sqrt{\frac{2}{\omega \mu \sigma}} \gg D
\]

(1.10)

![FIGURE 1.4](image)

(a) \( i(t) = I_0 \cos 2\pi ft \)

(b) \( i(t) = I_0 \cos 2\pi ft \)

FIGURE 1.4 (a) Eddy current approximation in a highly conducting medium and (b) eddy current approximation in a weakly conducting medium. Oscillating curves outline the total magnetic field within a conductor.
Otherwise, the secondary magnetic field would eventually cancel the primary one (the skin layer effect), which is only possible when these two fields are comparable in magnitude and neither of them can be neglected. The eddy current approximation in a weakly conducting medium is illustrated in Figure 1.4a. Figure 1.4a illustrates the case of a highly conducting medium. Oscillating curves in these figures outline the total magnetic field within the conducting material.

### Example 1.3:

A. An aluminum wire has the diameter of \( D = 1 \text{ cm} \) and is characterized by a conductivity of \( \sigma = 4.0 \times 10^7 \text{ S/m} \). Alternating current at 60 Hz flows through the wire. Determine whether or not the weakly conducting media approximation (1.10) is satisfied.

B. Repeat for a human muscle at 20 kHz characterized by conductivity \( \sigma = 0.35 \text{ S/m} \) [31]. The muscle size \( D \) is 0.1 m.

**Solution:** We use the value \( \mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \) for both materials and obtain

\[
\frac{D}{\delta} = 1.38 \quad \text{for aluminum at 60 Hz}
\]

\[
\frac{D}{\delta} = 0.024 \quad \text{for human muscle at 20 kHz}
\]

Clearly, the second case allows us to use the weakly conducting media approximation (the “thin limit condition”), whereas the first case does not.

The eddy current approximation in weakly conducting media is thoroughly studied in Part IV of this text.

### 1.1.7 Nonlinear Electrostatic Approximation for Semiconductors and Biomolecular Electrostatics

In semiconductor physics, junctions between two semiconductors or between a semiconductor and a metal are the primary subject of research. The physical sizes of these junctions in any direction are very small compared to RF frequencies under interest. Exceptions may be terahertz frequencies and optical waves. The same is valid for semiconducting biological objects such as cell membranes. Therefore, Equation 1.7 is directly applicable, which leads us to an electrostatic Poisson equation. In this case, however, the Poisson equation obtained is nonlinear, with a right-hand side dependent on the electric potential itself.
Example 1.4: Figure 1.5 shows a sample junction structure of a general-purpose 1N4148 Si switching diode. An important pn-junction parameter is the width of the depletion region, $W$, which appears between p- and n-doped semiconductors with doping concentrations, $N_A$ and $N_D$, respectively. It is given by (see Chapter 14)

$$W = \sqrt{\frac{2\varepsilon (N_A + N_D)}{q N_A N_D}} V_T \ln \left( \frac{N_A N_D}{n_i^2} \right)$$  \hspace{1cm} (1.13)

where $V_T$ is the thermal voltage of 0.026 V, $n_i$ is intrinsic carrier concentration, $n_i \approx 1 \times 10^{10}$ cm$^{-3}$ for Si, and $q$ is electron charge. Given $N_{D0} = N_{A0} = 10^{16}$ cm$^{-3}$, estimate the width of the depletion region and compare this value with a wavelength in Si at 1 GHz.

Solution: The relative dielectric constant of Si is 11.86. Substitution of all values into Equation 1.13 gives

$$W \approx 0.5 \text{ \ensuremath{\mu m}}$$  \hspace{1cm} (1.14)

which is indeed much smaller than the wavelength of 8.7 cm at 1 GHz in Si. Other geometry dimensions in Figure 1.4 also satisfy Equation 1.7.
The nonlinear electrostatic approximation with application to a semiconductor pn-junction is studied in Part V of this text. The nonlinear Poisson equation of the pn-junction theory studied and solved there is very similar to the Poisson–Boltzmann equation used in a continuum representation of biomolecular electrostatics [19–21]. The differences may include an additional but already known function on the right-hand side and somewhat different boundary conditions.

1.1.8 Classification of Quasistatic Electromagnetic Problems and Related Numerical Methods

The previous analysis is summarized in Table 1.1. Here, we also list traditional numerical methods used to solve different low-frequency approximations. Most of the problems are linear. The nonlinear problems typically involve high-voltage electrostatics, nonlinear magnetic materials, and semiconductor materials.

1.1.8.1 Step-by-Step Approach

When applied to the most complicated full-wave case, the BEM and the FEM significantly reuse algorithms developed previously for static problems. Therefore, it makes sense to study these methods for static problems first and then add the required complexity step by step. The exception is the FDTD method, with a formulation that inherently begins with the full-wave problem; while it is primarily applicable to this case, low-frequency modifications exist.

### TABLE 1.1 Schematic classification of low-frequency electromagnetic numerical problems

<table>
<thead>
<tr>
<th>Problem type</th>
<th>Physical condition</th>
<th>Underlying equation(s)</th>
<th>Underlying numerical methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrostatic problems</td>
<td>$D \ll \lambda$</td>
<td>Elliptic Laplace or Poisson equations with Neumann or Dirichlet boundary conditions</td>
<td>FEM, BEM, MoM, HS</td>
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<tr>
<td>Magnetostatic problems</td>
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<td>Direct current problems</td>
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<tr>
<td>(parametric time dependence)</td>
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<tr>
<td>Eddy current (quasistatic)</td>
<td>$\frac{\epsilon \omega}{\sigma} \ll 1$</td>
<td>Parabolic equations with infinite propagation speed</td>
<td>FEM, BEM, MoM</td>
</tr>
<tr>
<td>approximation</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Eddy current approximation in weakly conducting media</td>
<td>$\frac{\epsilon \omega}{\sigma} \ll 1$</td>
<td>Elliptic Laplace or Poisson equations with Neumann or Dirichlet boundary conditions</td>
<td>FEM, BEM, MoM, HS</td>
</tr>
<tr>
<td>Full-wave (radio-frequency) problems</td>
<td>$D \geq \lambda/100$</td>
<td>Hyperbolic Maxwell’s equations or wave equations</td>
<td>FDTD, FEM, BEM, MoM, HS</td>
</tr>
</tbody>
</table>

BEM, boundary element method [33–40];
FDTD, finite-difference time-domain method [41, 42];
FEM, finite-element method [43–46];
HS, hybrid and miscellaneous methods (finite-difference method, finite volume method, method of lines, etc.);
MoM, method of moments (equivalent to BEM).
1.1.8.2 Necessity of Mesh Generation  

Every three-dimensional (3D) numerical electromagnetic solver involving finite elements or boundary elements—static, quasi-static, or full wave—implies the existence of a mesh—a set of small surface patches (e.g., triangles) or small volumetric elements (e.g., tetrahedra). The same is true for applications of the BEM and FEM in other disciplines. Even the FDTD method, which operates with uniform cubical grids in 3D space, often requires mesh(es) for identification of material properties when complicated geometries are considered. Generation, description, and usage of basic triangular surface meshes will be studied in the next chapter.

PROBLEMS

1.1.1 For the circuit in Figure 1.1a, answer the following questions:
   A. Why does the electric field between the two wires decrease when moving from the source to the load?
   B. When does such a decrease become negligibly small?
   C. Will the electric field between the two wires be uniform when no load is present? Hint: You might want to run MATLAB® module E23.m from Chapter 5 and test the case of two parallel cylinders subject to 1 and 0 V, respectively.

1.1.2 In the circuit from Figure 1.1, both wires have an infinite conductivity and the radius of 1 mm.
   A. What is the electric field within the wires (show units)?
   B. What is the current density within the wires (show units)?

1.1.3 Establish the KVL (Kirchhoff’s voltage law) for the circuit in Figure 1.1a.

1.1.4 For the circuit in Figure 1.1b, the magnetic field due to one wire at the cross-section centerline is \( \mathbf{H} \). What is the total magnetic field at the centerline?

1.1.5 Figure 1.6a shows a finite-element model of a 345 kV power tower used by National Grid, MA, United States (front view). Figure 1.6b and c depict the corresponding simulation results obtained with the electrostatic solver Maxwell 3D of ANSYS.
   A. Determine which figure corresponds to the electric potential and which to the magnitude of the electric field.
   B. Provide a detailed justification of your answer.

1.1.6 Given that \( \mathbf{H} = [0, x, 0] \) A/m, compute:
   A. \( \nabla \cdot \mathbf{H} \)
   B. \( \nabla \times \mathbf{H} \)

1.1.7 Repeat the previous problem for \( \mathbf{E} = [x, y, 0] \) V/m.

1.1.8 Determine the divergence of a field shown in Figure 1.7.

1.1.9 Derive the continuity equation for the electric current (1.5) based on the full set of Maxwell’s Equations 1.1–1.4.
FIGURE 1.6  Electrostatic FEM modeling of (a) the geometry and (b), (c) the response of a 345 kV power tower.
FIGURE 1.7 A vector field.
1.1.10 A. List major types of low-frequency electromagnetic approximations.
   B. Attempt to give an example of every problem type.
   C. What are three major dimensionless ratios that enable us to classify between different low-frequency approximations? Show the units for every quantity used in these ratios.

1.1.11 A. Repeat Example 1.1 when the AC frequency changes to 50 kHz.
   B. Repeat Example 1.2 when the AC frequency changes to 50 kHz.
   C. Repeat Example 1.3 Part B when the AC frequency changes to 50 kHz. 
   **Hint:** In every case, use online reference [31].

1.1.12 A 10 × 10 cm printed circuit board (PCB) uses FR4 laminate with a relative permittivity of 4.4 and a dielectric loss tangent of 0.02 at 20 MHz. Only the passive circuit elements (lumped resistances) connected by N metal traces are present. The total current \( i(t) = I_0 \cos 2\pi f t \), where \( f = 1 \) MHz, \( I_0 = 50 \) mA, enters and leaves the board. Determine whether or not the solution for trace currents \( i_n(t), n = 1, \ldots, N \) may be given by the products \( I_{n0} \cos 2\pi f t \) where \( I_{n0}, n = 1, \ldots, N \) are the solutions of the steady-state problem with the total injection current \( I_0 \). Justify your answer.
   **Hint:** Neglect losses in metal traces.

1.1.13 Is the condition \( \nabla \cdot J = 0 \) for the total current density valid for the eddy current (quasistatic) approximation? What physical sense does it have?

1.1.14 Equation 1.10 is in fact an approximation; the full-wave expression for the skin layer depth may be found to be [32]

\[
\delta = \sqrt{\frac{2}{\omega \mu \sigma}} \left\{ \sqrt{1 + \left( \frac{\varepsilon \omega}{\sigma} \right)^2} - \frac{\varepsilon \omega}{\sigma} \right\}
\]  
(1.15)

Which low-frequency electromagnetic model leads to this approximation?

1.1.15 Repeat Example 1.4 when the semiconductor doping concentrations change to \( N_{D0} = N_{A0} = 10^{14} \) cm\(^{-3}\).

1.2 POISSON AND LAPLACE EQUATIONS, BOUNDARY CONDITIONS, AND INTEGRAL EQUATIONS

The Poisson and Laplace equations of potential theory have been the subject of extensive and excellent mathematical research over many years [47–50]. In this section, we provide a short overview of some basic facts related to their solution via the BEM. Special attention is paid to the accurate formation of integral equations for Dirichlet, Neumann, and mixed boundary conditions. Exact formulations and practical realizations of those integral equations in application to specific problems will be thoroughly considered in the main text.
1.2.1 Poisson Equation in Differential and Integral Forms

1.2.1.1 Differential Form  When the time derivative included in Faraday’s law is omitted, the condition of the curl-free electric field, \( \nabla \times \mathbf{E} = 0 \), is obtained. This allows for the electric field to be represented as a potential field in the form

\[
\mathbf{E}(\mathbf{r}) = -\nabla \varphi(\mathbf{r}), \quad \mathbf{r} = [x, y, z]
\]  (1.16)

where \( \varphi(\mathbf{r}) \) is the electric potential with units of volts. Substitution of this result into Gauss’ law (1.3) and assuming \( \varepsilon = \text{const} \) leads to the Poisson equation in the differential form

\[
\Delta \varphi = -\frac{\rho}{\varepsilon}
\]  (1.17)

\[
\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]  (1.18)

which is none other than the Laplace equation

\[
\Delta \varphi = 0
\]  (1.19)

with a nonzero right-hand side due to distributed volume charges (volume sources). When the charges are concentrated only on the interfaces, which is a common case, the Poisson equation is reduced to the Laplace equation everywhere in space except at the interfaces.

1.2.1.2 Integral Form  Solution to Equation 1.18 in free space (\( \varepsilon = \varepsilon_0 \)) is constructed by superposition: we add up all contributions \( \varphi(\mathbf{r}) \) from point charges \( q \) in infinitesimally small volumes, \( dV' \), at location \( \mathbf{r}' \)

\[
\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|}, \quad q = \rho(\mathbf{r}')dV'
\]  (1.20)

The final result is the sum of all contributions (1.20) or, more precisely, an integral. To within an arbitrary constant, one has

\[
\varphi(\mathbf{r}) = \int_{V} \frac{\rho(\mathbf{r}')dV'}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|}
\]  (1.21)

which is the integral form of the Poisson equation. The most important scenario is the case of charges at interfaces. The remainder of the medium (medium volume) is electrically neutral. If this is the case, one should replace the volume integral in
Equation 1.21 by a surface integral over all interfaces (denoted by $S$) and the volume charge density $\rho$ by a surface charge density $\sigma$. Equation 1.21 is then converted to

$$\varphi(\mathbf{r}) = \int_{S} \frac{\sigma(\mathbf{r}')}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|} dS', \quad \mathbf{r}' \in S$$  \hspace{1cm} (1.22)

The surface charge density $\sigma(\mathbf{r})$ has the units of C/m$^2$. Equation 1.22 is called the single-layer potential in potential theory [51–53]. Using Equation 1.22, the electric field may be calculated everywhere in space (the gradient is always evaluated with regard to $\mathbf{r}$)

$$\mathbf{E}(\mathbf{r}) = - \nabla \varphi(\mathbf{r}) = - \int_{S} \frac{\sigma(\mathbf{r}')}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|} dS', \quad \mathbf{r}' \in S$$  \hspace{1cm} (1.23)

Integrals (1.22) and (1.23) are improper integrals since the integrand contains a singularity.

**Exercise 1.5:** What is the expression for the electric potential if both volume and surface charges are present?

**Answer:** The total may be found as the sum of Equations 1.21 and 1.22.

**Exercise 1.6:** A volumetric charge density $\rho(\mathbf{r})$ is given. What is the analytical solution of the Poisson equation in free space?

**Answer:** The solution is given by Equation 1.21. Unfortunately, this simple problem is rather uncommon. As we will see in later chapters, Equation 1.22 is commonly solved, where the surface charge density is unknown and needs to be found.

### 1.2.1.3 Universal Character of the Poisson Equation

The Poisson or Laplace equations are encountered in all problems of electrostatics, magnetostatics, DC flow, and even in the quasistatics that include eddy current problems in weakly conducting media with a large skin depth. Equations 1.22 and 1.23 are the starting points for all integral equations used in this text. The particular meaning of different quantities may be quite different though. In particular, $\sigma(\mathbf{r})$ means:

- The density of free charges in Parts II and IV of the text (electrostatics, direct current flow, eddy currents in weakly conducting media)
- The density of polarization charges in Part II of the text (electrostatic of dielectrics)
• The apparent magnetic surface charge density at the interfaces between different magnetic materials in Part III of the text (magnetostatics)

Also, the electric field may be replaced by the magnetic field and the electric scalar potential by the magnetic scalar potential or by the magnetic vector potential.

1.2.2 Free-Space Green’s Function

The integration kernel in Equations 1.22 and 1.23

\[ G(r, r') = \frac{1}{4\pi |r - r'|} \]  \hspace{1cm} (1.24)

is called the free-space Green’s function; it satisfies the equation

\[ \Delta G(r, r') = -\delta(r - r') \]  \hspace{1cm} (1.25)

which is the Poisson equation with the right-hand side in the form of a unit point charge (represented by the 3D delta function). Point \( r \) is usually called the observation point; point \( r' \) is the source point or the integration point. In terms of the free-space Green’s functions, Equations 1.22 and 1.23 have the form

\[ \varphi(r) = \frac{1}{\varepsilon_0} \int_S G(r, r') \sigma(r') dS' \]  \hspace{1cm} (1.26)

\[ E(r) = -\frac{1}{\varepsilon_0} \int_S \nabla G(r, r') \sigma(r') dS' \]  \hspace{1cm} (1.27)

Exercise 1.7: Present expressions for the Green’s function and its gradient in Cartesian coordinates.

Answer:

\[ G(r, r') = \frac{1}{4\pi |r - r'|} = \frac{1}{4\pi \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \]  \hspace{1cm} (1.28)

\[ \nabla G(r, r') = -\frac{1}{4\pi |r - r'|^3} = -\frac{(x-x') \hat{x} + (y-y') \hat{y} + (z-z') \hat{z}}{4\pi \left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{3/2}} \]
1.2.3 Green’s Function Technique

The Green’s function technique is a method to obtain a solution to partial differential equation via a superposition of the responses of individual sources in the form of a delta function (the impulse response). The analog of the Green’s function in circuit analysis is the circuit transfer function. Green’s function has a great practical value when we consider more complicated problems with an infinite ground plane, an infinite dielectric substrate, periodic structures, and other cases. Green’s functions for particular geometries must always satisfy certain boundary conditions. These functions may become quite involved and are usually given as infinite series [32, 48]. The static Green’s function of free space in Equation 1.24 is the simplest case. It satisfies the boundary condition of zero field at infinity.

Example 1.5: Assume that an electrostatic setup is located above an infinite conducting ground plane at \( z = 0 \) in Cartesian coordinates. The specific conductivity value of the ground plane is irrelevant for electrostatics. Establish Equations 1.22 and 1.23 for this particular case.

Solution: For every charge \( q \) located in the upper half-space at \( \mathbf{r}' = (x', y', z') \), the effect of the ground plane is taken into account by imposing an image charge \(-q\) located in the lower half-space at \( \mathbf{r}_i = (x', y', -z') \). This combination satisfies the ground plane boundary condition (tangential \( E \)-field is zero) studied next. The method of image charges is very popular in electrostatics [47–50] and even in full-wave electromagnetics such as antenna theory [32, 54]. Therefore, instead of Equation 1.24, the Green’s function will have the form

\[
G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} - \frac{1}{4\pi|\mathbf{r} - \mathbf{r}_i'|}
\]  

(1.29)

At the same time, Equations 1.26 and 1.27 remain exactly the same. This is the key point of the Green’s function technique.

1.2.4 Boundary Conditions for the Poisson and Laplace Equations

The Poisson and Laplace equations are augmented with the appropriate boundary conditions for the problem under interest. The boundary conditions for the Poisson equation are of four types [51–53]:

1. **Dirichlet boundary conditions**, when the unknown solution \( \varphi(\mathbf{r}) \) of the Poisson equation is given at boundaries or interfaces \( S \)

\[
\varphi(\mathbf{r}) = \varphi_{\text{spec}}(\mathbf{r}), \quad \mathbf{r} \in S
\]  

(1.30)
The given potential (or voltage) \( \phi_{\text{spec}}(r) \) is constant for every conductor but may vary from conductor to conductor.

2. **Neumann boundary conditions**, when the normal derivative of the unknown solution \( \phi(r) \) of the Poisson equation is prescribed at the boundaries or interfaces \( S \). This normal derivative is none other than the projection of the electric field \( E(r) \) onto the surface normal vector \( n(r) \)

\[
\frac{\partial \phi(r)}{\partial n} \equiv \mathbf{n}(r) \cdot \nabla \phi(r) = -\mathbf{n}(r) \cdot \mathbf{E}(r) = -E_n(r) \quad r \in S
\]  

(1.31)

The argument \( r \) is often omitted either for \( n(r) \), for \( E(r) \), or for both. It is important to emphasize from the very beginning that the normal derivative is different at two opposite sides of a boundary carrying surface charges. Therefore, either \( \partial \phi(r)/\partial n \) is prescribed on one side or a relation between \( \partial \phi(r)/\partial n \) on both sides is given.

3. **Mixed boundary conditions**, which imply that Dirichlet boundary conditions are given on some boundaries/interfaces and Neumann boundary conditions are given elsewhere.

4. **Robin boundary conditions**, which imply that a combination of \( \phi(r) \) and \( \partial \phi(r)/\partial n \) are given at the same boundaries. Robin boundary conditions include mixed boundary conditions as a particular case.

**Exercise 1.8:** A conducting object with surface \( S \) is subject to a 1 V surface voltage. The ground is at infinity. Which boundary condition(s) should be used?

**Answer:** We should use Dirichlet boundary conditions (1.30) with \( \phi_{\text{spec}}(r) = 1 \text{ V} \), \( r \in S \).

**Exercise 1.9:** A homogeneous conducting object in air has two electrodes at ±1 V attached to it. Which boundary condition(s) should be used?

**Answer:** We should use Dirichlet boundary conditions (1.30) with \( \phi_{\text{spec}}(r) = \pm 1 \text{ V} \) at the electrode surfaces. We should use Neumann boundary conditions \( \partial \phi(r)/\partial n = 0 \) for the remainder of the object’s surface, on its inner side. Since the current density within the object is proportional to the electric field, these conditions, according to Equation 1.31, are equivalent to the statement that electric current cannot cross the object’s surface and flow into air.

It has been proven that a solution satisfying one of the three types of the boundary conditions listed previously (Neumann, Dirichlet, and mixed) is unique [49, 51–53].
1.2.5 Integral Equations in Terms of Surface Charge Density at Boundaries

Integral equations of the BEM can be obtained, for example, by substitution of Equations 1.22 and 1.23 into the appropriate boundary conditions. The integral equations may have many different forms and may involve different unknowns [51–53]. Here, we will consider the surface charge density $\sigma(r)$ at the boundaries as an unknown function (see Fig. 1.8). Such approach is referred to as a method of moments (MoM) [33–35] or as the SCM. Depending on the boundary conditions involved, we distinguish between Fredholm integral equations of the first kind and Fredholm integral equations of the second kind.

1.2.6 Dirichlet Boundary Conditions: Fredholm Integral Equation of the First Kind

Substitution of Equation 1.22 for the electric potential into Dirichlet boundary conditions (1.30) yields an integral equation for the unknown surface charge density $\sigma(r)$ at boundary $S$ in the form

$$
\int_S \frac{\sigma(r')dS'}{4\pi\varepsilon_0|r-r'|} = \varphi_{spec}(r), \quad r \in S \tag{1.32}
$$

Note that observation point $r$ and integration point $r'$ both belong to the object’s boundary $S$ in Figure 1.8; we do not need to solve inside or outside the object. However, after the solution for the surface charge density $\sigma(r)$ at the boundary is obtained, the potential and the field at every point in space may then be computed according to Equations 1.22 and 1.23. Equation 1.32 is known as the (inhomogeneous) Fredholm

![FIGURE 1.8](image)

Derivation of integral equations in terms of surface charge density for different media. The normal vector to surface $S$ is pointing from inside to outside (the outer normal vector).
integral equation of the first kind. The only unknown function is located inside the integral. Equation 1.32 is typical for electrostatics of conductors; it is thoroughly studied in Chapters 4 and 5 of the text.

1.2.6.1 Continuity of Potential across the Boundaries: Single-Layer Potential

An important fact behind integral equation (1.32) is the continuity of the potential across the boundaries. This fact follows from the definition in Equation 1.17. Even if the (electric) field is discontinuous across the boundary, the integral of a finite discontinuous function will still be a continuous function. Another direct proof is based on Equation 1.22; it is given, for example, in Ref. [49]. Therefore, with reference to Figure 1.8, one may write

\[ \phi_1 = \phi_2 \]  

(1.33)

where indexes denote the two values approaching the boundary from medium #1 or medium #2, respectively. Equation 1.33 holds for conducting, dielectric, or magnetic media, wherever the potential function exists. In potential theory, this fact is known as the continuity of the single-layer potential given by Equation 1.22. The term single-layer stands for the potential of charges of single polarity (i.e., not dipoles) in Figure 1.8.

Exercise 1.10: A conducting object with surface \( S \) in Figure 1.8 is subject to a 1 V surface voltage. What is the electric potential and electric field inside and outside the object?

Answer: The electric potential inside the object is constant and equals 1 V. The electric field inside is zero (the gradient of a constant). The potential outside begins at 1 V and then decays to zero at infinity. The electric field begins with a certain nonzero value at the surface and also decays toward zero.

1.2.7 Neumann Boundary Conditions: Fredholm Integral Equation of the Second Kind

This case is more complicated than the previous one. We must be careful since the normal potential derivative (and indeed the normal electric field for electric potential or the normal magnetic field for magnetic scalar potential) becomes discontinuous across boundaries with different material properties on both sides.

1.2.7.1 Discontinuity of Normal Potential Derivative across the Boundary

Consider \( \lim_{r \to r'} \partial \phi(r) / \partial n, \ r' \in S \), that is, when \( r \) approaches surface \( S \) in Figure 1.8 from the inside or the outside. According to Equations 1.31, 1.23, and 1.28 one may write
If \( \mathbf{r} \) were exactly on the surface, the dot product \( \mathbf{n}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}') \) would be equal to zero when \( \mathbf{r} \to \mathbf{r}' \). The integrand will no longer be singular (for smooth surfaces). Unfortunately, this is not the case and the singularity will give a finite contribution into the final integral. Assume that \( \mathbf{r} \) belongs to the \( z \)-axis and consider a small sphere of radius \( R \) with its center on surface \( S \) at \( z = 0 \) in Figure 1.8. The sphere cuts a small part of surface \( S \), which approaches a circle \( S_0 \) with radius \( R \). The entire surface integral in Equation 1.34 is thus divided into two parts: the integral over the small circle and the integral over the rest of \( S \). Within the circle, the charge density is approximately constant and equals \( \sigma(\mathbf{r}) \). The first integral is computed in cylindrical coordinates \( r, \phi, z \) where \( \mathbf{n}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}') = z \) and \( |\mathbf{r} - \mathbf{r}'| = (r^2 + z^2)^{3/2} \). Therefore, Equation 1.34 is transformed to the sum

\[
\lim_{r \to r'} \frac{\partial \phi(\mathbf{r})}{\partial \mathbf{n}} = -\lim_{r \to r'} \int_{S \setminus S_0} \frac{\sigma(\mathbf{r}')}{4\pi \varepsilon_0} \mathbf{n}(\mathbf{r}') \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dS', \quad \mathbf{r} \in S
\]

Elementary integration yields

\[
\lim_{z \to 0} \int_0^R \frac{z r dr}{(r^2 + z^2)^{3/2}} = \text{sign}(z) \int_0^\infty \frac{udu}{(u^2 + 1)^{3/2}} = \text{sign}(z)
\]

The first integral in Equation 1.35 has no singularity when \( S_0 \) reduces to zero. This is because \( \mathbf{r} \) is exactly on the surface \( S \). The final result therefore becomes

\[
\frac{\partial \phi(\mathbf{r})}{\partial \mathbf{n}} \bigg|_{1,2} = -\lim_{S \to S_0} \int_S \frac{\sigma(\mathbf{r}')}{4\pi \varepsilon_0} \mathbf{n}(\mathbf{r}') \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dS' + \frac{\sigma(\mathbf{r})}{2\varepsilon_0}, \quad \mathbf{r} \in S
\]

where the indexes denote two distinct values of the normal derivative when approaching the boundary in Figure 1.8 from medium #1 (from the inside) or from medium #2 (from the outside), respectively. Equation 1.37 is more complicated than the simple result for the potential given by Equation 1.33. It is critical for the bulk of integral equations.

1.2.7.2 Double-Layer Potential

Mathematically, Equation 1.37 means that the integral on its right-hand side would be a discontinuous function of observation point \( \mathbf{r} \) depending on whether \( \mathbf{r} \) is on the integration surface \( S \) or not. This integral is known as a potential of a dipole layer or a double-layer potential [48, 49]. As the
name implies it describes the electric potential of a dipole layer with the surface density, \( \sigma(r) \).

### 1.2.7.3 Fredholm Integral Equation of the Second Kind

Assume that the object in Figure 1.8 is a conductor in contact with air. Its conductivity is \( \sigma_1 \). The object has some electrodes attached to it, which source or sink electric current with normal current density \( j(r) \) on the electrode surface. Everywhere on the object surface, except the electrode surface, \( \partial \phi(r)/\partial n = 0 \). On the electrode surface, on the other hand, \( \partial \phi(r)/\partial n = -E_{n1}(r) = -j(r)/\sigma_1 \). Using these boundary conditions and Equation 1.37, the corresponding integral equation for the surface charge density is immediately obtained in the form

\[
\frac{\sigma(r)}{2} - \frac{\int_S \frac{\sigma(r')}{4\pi} \mathbf{n}(r') \cdot \frac{r-r'}{|r-r'|^3} dS'}{S} = \begin{cases} 
0 & \text{not on electrode surface} \\
-\varepsilon_0 j(r)/\sigma_1 & \text{on electrode surface} 
\end{cases} \quad r \in S
\]

(1.38)

This integral equation is known as the (inhomogeneous) *Fredholm integral equation of the second kind*. The unknown function is located not only inside the integral but also outside. This circumstance makes it possible to apply a straightforward iterative solution. Equation 1.38 is typical for the bulk of static and quasistatic problems except the electrostatics of conductors.

### Example 1.6

Assume that the object in Figure 1.8 is a nonconducting dielectric with permittivity \( \varepsilon_1 \) in contact with air having permittivity, \( \varepsilon_0 \). Derive an integral equation for the surface charge density (in this case, it will be the polarization charge density studied in Chapter 6) on the air–dielectric boundary given that an external electric field with potential \( \phi_{\text{ext}}(r) \) is applied. The corresponding boundary condition (continuity of the total electric flux density through the boundary with no free surface charges) in Figure 1.8 has the form

\[
\varepsilon_1 E_{n1}(r) = \varepsilon_2 E_{n2}(r) = 0, \quad r \in S
\]

(1.39)

**Solution:** The total field is the combination of the external field and the field of surface charges given by Equations 1.22 and 1.23. The external field is continuous across the boundary. The field of surface charges is not. Using the equality \( \partial \phi(r)/\partial n = -E_{n1,2} \) and plugging Equation 1.37 into boundary condition (1.39), we obtain the required integral equation in the form

\[
(e_1 + e_2) \frac{\sigma(r)}{2} - (e_1 - e_2) \frac{\int_S \frac{\sigma(r')}{4\pi} \mathbf{n}(r') \cdot \frac{r-r'}{|r-r'|^3} dS'}{S} = (e_1 - e_2) \varepsilon_0 E_{\text{ext}}(r), \quad r \in S
\]

(1.40)
1.2.8 Summary of Boundary Conditions for Maxwell’s Equations

Boundary conditions for the Poisson and Laplace equations follow from a general set of boundary conditions covering the full set of Maxwell’s equations. Table 1.2 summarizes these boundary conditions [32, 47–50] as they are to be used in subsequent chapters. All boundary conditions are given with reference to Figure 1.8. Since the boundary conditions do not involve time derivatives, they must be valid in any case: static, quasistatic, or dynamic.

### PROBLEMS

1.2.1 A. Show that the potential \( \phi(r) \) given by Equation 1.20 satisfies the Laplace equation for every \( r \neq r' \).

B. Show that the potential \( \phi(r) \) given by Equation 1.21 satisfies the Laplace equation for every \( r \) such that \( \rho(r) = 0 \).

C. Show that the single-layer potential \( \phi(r) \) given by Equation 1.22 satisfies the Laplace equation for every \( r \) not on surface \( S \).

1.2.2 Using Maxwell’s Equations 1.1–1.4, attempt to carefully formulate conditions under which the magnetic scalar potential \( \psi(r) \) exists so that \( H(r) = -\nabla \psi(r) \).
1.2.3 Assume that an electrostatic setup is located to the right of an infinite conducting ground plane at \( x = h \) in Cartesian coordinates. Present the corresponding Green’s function.

1.2.4 An electrostatic or magnetostatic structure consists of an infinite number of identical objects cloned along the \( x \)-axis (see Fig. 1.9). Establish the free-space Green’s function for this periodic problem.

1.2.5 The method of images can be quite helpful in establishing Green’s functions other than the Green’s function for the infinite planar ground plane in Example 1.5. Assume that a conducting object under study is located within a 90° metal corner reflector shown in Figure 1.10. Assume also an infinite reflector

![Figure 1.9](image1.png) **Figure 1.9** An infinite periodic structure.

![Figure 1.10](image2.png) **Figure 1.10** (a) Theory of a corner reflector and (b) the application of the method of images. Image (b) is the profile of a metallic backplane.
size with the $x$-axis pointing out of the page. Establish the corresponding Green’s function of the problem under study satisfying the boundary condition of no tangential electric field at the metal boundary.

*Hint:* The method of images is illustrated in Figure 1.10a. It assumes three image charges (one for every plane plus one “balancing” image charge). All four charges (the original one plus three images) form two polar charge pairs that cancel the tangential $E$-field on both corner planes. The field outside the corner angle is nonphysical and should be ignored.

1.2.6 Prove Equation 1.36.

1.2.7 It is well known that the single-layer potential given by Equation 1.22 is a continuous function in space everywhere including surface $S$ (see Eq. 1.33). Can you prove this fact mathematically using an approach similar to that from Equations 1.34 to 1.37?

1.2.8 A. Compare Equation 1.38 with Equation 8.12. Are they or are they not identical? What significant differences do you encounter?

B. Compare Equation 1.40 with Equation 6.6. Are they or are they not identical? What significant differences do you encounter?

1.2.9 Assume that the object in Figure 1.8 is a magnetic material with permittivity $\mu_1$ in contact with air having permittivity $\mu_0$. Derive an integral equation for the apparent “magnetic” surface charge density on the material boundary given that an external DC magnetic field with the scalar potential $\psi_{ext}(r)$ is applied. The corresponding boundary condition (continuity of the total magnetic flux density through the boundary) in Figure 1.8 has the form

$$\mu_1 H_{n1}(r) - \mu_2 H_{n2}(r) = 0, \quad r \in S$$

(1.41)

1.2.10 Assume that the object in Figure 1.8 is a conductor in contact with air. Its conductivity is $\sigma_1$. The object has 2 V electrodes attached to it at $\pm 1$ V. Derive the full set of integral equations for the surface charge density, $\sigma(r)$. Accurately define the domain for the observation variable $r$ in every case.

REFERENCES


REFERENCES


REFERENCES
