1

Introduction: Uncertainty Analysis and Dependence Modelling

1.1 Wags and Bogsats

‘...whether true or not [it] is at least probable; and he who tells nothing exceeding the bounds of probability has a right to demand that they should believe him who cannot contradict him’. Samuel Johnson, author of the first English dictionary, wrote this in 1735. He is referring to the Jesuit priest Jeronimo Lobo’s account of the unicorns he saw during his visit to Abyssinia in the 17th century (Shepard (1930) p. 200).

Johnson could have been the apologist for much of what passed as decision support in the period after World War II, when think tanks, forecasters and expert judgment burst upon the scientific stage. Most salient in this genre is the book The Year 2000 (Kahn and Wiener (1967)) in which the authors published 25 ‘even money bets’ predicting features of the year 2000, including interplanetary engineering and conversion of humans to fluid breathers. Essentially, these are statements without pedigree or warrant, whose credibility rests on shifting the burden of proof. Their cavalier attitude toward uncertainty in quantitative decision support is representative of the period. Readers interested in how many of these even money bets the authors have won, and in other examples from this period, are referred to (Cooke (1991), Chapter 1).

Quantitative models pervade all aspects of decision making, from failure probabilities of unlaunched rockets, risks of nuclear reactors and effects of pollutants on health and the environment to consequences of economic policies. Such quantitative models generally require values for parameters that cannot be measured or...
assessed with certainty. Engineers and scientists sometimes cover their modesty with churlish acronyms designating the source of ungrounded assessments. ‘Wags’ (wild-ass guesses) and ‘bogsats’ (bunch of guys sitting around a table) are two examples found in published documentation.

Decision makers, especially those in the public arena, increasingly recognize that input to quantitative models is uncertain and demand that this uncertainty be quantified and propagated through the models.

Initially, it was the modellers themselves who provided assessments of uncertainty and did the propagating. Not surprisingly, this activity was considered secondary to the main activity of computing ‘nominal values’ or ‘best estimates’ to be used for forecasting and planning and received cursory attention.

Figure 1.1 shows the result of such an in-house uncertainty analysis performed by the National Radiological Protection Board (NRPB) and The Kernforschungszentrum Karlsruhe (KFK) in the late 1980s (Crick et al. (1988); Fischer et al. (1990)). The models in question predict the dispersion of radioactive material in the atmosphere following an accident in a nuclear reactor. The figure shows predicted lateral dispersion under stable conditions, and also shows wider and narrower plumes, which the modellers are 90% certain will enclose an actual plume under the stated conditions.

It soon became evident that if things were uncertain, then experts might disagree, and using one expert-modeller’s estimates of uncertainty might not be sufficient. Structured expert judgment has since become an accepted method for quantifying models with uncertain input. ‘Structured’ means that the experts are identifiable, the assessments are traceable and the computations are transparent. To appreciate the difference between structured and unstructured expert judgment, Figure 1.2 shows the results of a structured expert judgment quantification of the same uncertainty pictured in Figure 1.1 (Cooke (1997b)). Evidently, the picture of uncertainty emerging from these two figures is quite different.

One of the reasons for the difference between these figures is the following:

The lateral spread of a plume as a function of down wind distance \( x \) is modelled, per stability class, as

\[
\sigma(x) = A x^B.
\]

Figure 1.1 5%, 50% and 95% plume widths (stability D) computed by NRPB and KFK.
Both the constants $A$ and $B$ are uncertain as attested by spreads in published values of these coefficients. However, these uncertainties cannot be independent. Obviously if $A$ takes a large value, then $B$ will tend to take smaller values. Recognizing the implausibility of assigning $A$ and $B$ as independent uncertainty distributions, and the difficulty of assessing a joint distribution on $A$ and $B$, the modellers elected to consider $B$ as a constant; that is, as known with certainty.\(^1\)

The differences between these two figures reflect a change in perception regarding the goal of quantitative modelling. With the first picture, the main effort has gone into constructing a quantitative deterministic model to which uncertainty quantification and propagation are added on. In the second picture, the model is essentially about capturing uncertainty. Quantitative models are useful insofar as they help us resolve and reduce uncertainty. Three major differences in the practice of quantitative decision support follow from this shift of perception.

- First of all, the representation of uncertainty via expert judgment, or some other method is seen as a scientific activity subject to methodological rules every bit as rigorous as those governing the use of measurement or experimental data.

- Second, it is recognized that an essential part of uncertainty analysis is the analysis of dependence. Indeed, if all uncertainties are independent, then their propagation is mathematically trivial (though perhaps computationally

\(^1\)This is certainly not the only reason for the differences between Figures 1.1 and 1.2. There was also ambivalence with regard to what the uncertainty should capture. Should it capture the plume uncertainty in a single accidental release, or the uncertainty in the average plume spread in a large number of accidents? Risk analysts clearly required the former, but meteorologists are more inclined to think in terms of the latter.
challenging). Sampling and propagating independent uncertainties can easily be trusted to the modellers themselves. However, when uncertainties are dependent, things become much more subtle, and we enter a domain for which the modellers’ training has not prepared them.

- Finally, the domains of communication with the problem owner, model evaluation, and so on, undergo significant transformations once we recognize that the main purpose of models is to capture uncertainty.

1.2 Uncertainty analysis and decision support: a recent example

A recent example serves to illustrate many of the issues that arise in quantifying uncertainty for decision support. The example concerns transport of *Campylobacter* infection in chicken processing lines. The intention here is not to understand *Campylobacter* infection, but to introduce topics covered in the following chapters. For details on *Campylobacter*, see Cooke et al. (Appearing); Van der Fels-Klerx et al. (2005); Nauta et al. (2004).

*Campylobacter* contamination of chicken meat may be responsible for up to 40% of *Campylobacter*-associated gastroenteritis and for a similar proportion of deaths. A recent effort to rank various control options for *Campylobacter* contamination has led to the development of a mathematical model of a processing line for chicken meat (these chickens are termed ‘broilers’).

A typical broiler processing line involves a number of phases as shown in Figure 1.3. Each phase is characterized by transfers of *Campylobacter* colony forming units from the chicken surface to the environment, from the environment back to the surface and from the faeces to the surface (until evisceration), and the destruction of the colonies. The general model, applicable with variations in each processing phase, is shown in Figure 1.4.

Given the number of *Campylobacter* on and in the chickens at the inception of processing, and given the number initially in the environment, one can run the model with values for the transfer coefficients and compute the number of *Campylobacter* colonies on the skin of a broiler and in the environment at the end of each phase. Ideally, we would like to have field measurements or experiments

![Figure 1.3 Broiler chicken processing line.](image-url)
to determine values for the coefficients in Figure 1.4. Unfortunately, these are not feasible. Failing that, we must quantify the uncertainty in the transfer coefficients, and propagate this uncertainty through the model to obtain uncertainty distributions on the model output.

This model has been quantified in an expert judgment study involving 12 experts (Van der Fels-Klerx et al. (2005)). Methods for applying expert judgments are reviewed in Chapter 2. We may note here that expert uncertainty assessments are regarded as statistical hypotheses, which may be tested against data and combined with a view to optimizing performance of the resulting ‘decision maker’.

The experts have detailed knowledge of processing lines, but owing to the scarcity of measurements, they have no direct knowledge of the transfer mechanisms defined by the model. Indeed, use of environmental transport models is rather new in this area, and unfamiliar. Uncertainty about the transfer mechanisms can be large, and, as in the dispersion example discussed in the preceding text, it is unlikely that these uncertainties could be independent. Combining possible values for transfer and removal mechanism independently would not generally yield a plausible picture. Hence, uncertainty in one transfer mechanism cannot be addressed independently of the rest of the model.

Our quantification problem has the following features:

- There are no experiments or measurements for determining values.
- There is relevant expert knowledge, but it is not directly applicable.
- The uncertainties may be large and may not be presumed to be independent, and hence dependence must be quantified.
These obstacles will be readily recognized by anyone engaged in mathematical modelling for decision support beyond the perimeter of direct experimentation and measurement. As the need for quantitative decision support rapidly outstrips the resources of experimentation, these obstacles must be confronted and overcome. The alternative is regression to wags and bogsats.

Although experts cannot provide useful quantification for the transfer coefficients, they are able to quantify their uncertainty regarding the number of *Campylobacter* colonies on a broiler in the situation described below taken from the elicitation protocol:

At the beginning of a new slaughtering day, a thinned flock is slaughtered in a ‘typical large broiler chicken slaughterhouse’. . . . We suppose every chicken to be externally infected with $10^5$ *Campylobacters* per carcass and internally with $10^8$ *Campylobacters* per gram of caecal content at the beginning of each slaughtering stage. . . .

**Question A1:** All chickens of the particular flock are passing successively through each slaughtering stage. How many *Campylobacters* (per carcass) will be found after each of the mentioned stages of the slaughtering process each time on the first chicken of the flock?

Experts respond to questions of this form, for different infection levels, by stating the 5%, 50% and 95% quantiles, or percentiles, of their uncertainty distributions. If distributions on the transfer coefficients in Figure 1.4 are given, then distributions per processing phase for the number of *Campylobacter* per carcass (the quantity assessed by the experts) can be computed by Monte Carlo simulation: We sample a vector of values for the transfer coefficients, compute a vector of *Campylobacter* per carcass and repeat this until suitable distributions are constructed. We would like the distributions over the assessed quantities computed in this way to agree with the quantiles given by the combined expert assessments. Of course we could guess an initial distribution over the transfer coefficients, perform this Monte Carlo computation and see if the resulting distributions over the assessed quantities happen to agree with the experts’ assessments. In general they will not, and this trial-and-error method is quite unlikely to produce agreement. Instead, we start with a diffuse distribution over the transfer coefficients, and adapt this distribution to fit the requirements in a procedure called ‘probabilistic inversion’.

More precisely, let $X$ and $Y$ be $n$- and $m$-dimensional random vectors, respectively, and let $G$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. We call $x \in \mathbb{R}^n$ an inverse of $y \in \mathbb{R}^m$ under $G$ if $G(x) = y$. Similarly, we call $X$ a probabilistic inverse of $Y$ under $G$ if $G(X) \sim Y$, where $\sim$ means ‘has the same distribution as’. If $\{Y | Y \in C\}$ is the set of random vectors satisfying constraints $C$, then we say that $X$ is an element of the probabilistic inverse of $\{Y | Y \in C\}$ under $G$ if $G(X) \in C$. Equivalently, and more conveniently, if the distribution of $Y$ is partially specified, then we say that $X$ is a probabilistic inverse of $Y$ under $G$ if $G(X)$ satisfies the partial specification of $Y$. In the current context, the transfer coefficients in Figure 1.4 play the role of $X$, and the assessed quantities play the role of $Y$. 
In our *Campylobacter* example, the probabilistic inversion problem may now be expressed as follows: Find a joint distribution over the transfer coefficients such that the quantiles of the assessed quantities agree with the experts’ quantiles. If more than one such joint distribution exists, pick the least informative of these. If no such joint distribution exists, pick a ‘best-fitting’ distribution, and assess its goodness of fit.

Probabilistic inversion techniques are the subject of Chapter 9.

In fact, the best fit produced with the model in Figure 1.4 was not very good. It was not possible to find a distribution over the transfer coefficients, which, when pushed through the model, yielded distributions matching those of the experts. On reviewing the experts’ reasoning, it was found that the ‘best’ expert (see Chapter 2) in fact recognized two types of transfer from the chicken skin to the environment. A rapid transfer applied to *Campylobacter* on the feathers, and a slow transfer applied to *Campylobacter* in the pores of the skin. When the model was extended to accommodate this feature, a satisfactory fit was found. The second model, developed after the first probabilistic inversion, is shown in Figure 1.5.

Distributions resulting from probabilistic inversion typically have dependencies. In fact, this is one of the ways in which dependence arises in uncertainty analysis. We require tools for studying such dependencies. One simple method is to simply compute rank correlations. Notions of correlation and their properties are discussed in Chapter 3. For now it will suffice simply to display in Table 1.1 the rank correlation matrix for the transfer coefficients in Figure 1.5, for the scalding phase.

![General model (2)](image)

**Figure 1.5** Processing phase model after probabilistic inversion.
Table 1.1  Rank correlation matrix of transfer coefficients, scalding phase.

<table>
<thead>
<tr>
<th>Variable</th>
<th>$a_{extA}$</th>
<th>$a_{extB}$</th>
<th>$ca$</th>
<th>$b$</th>
<th>$ce$</th>
<th>$aint$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{extA}$</td>
<td>1.00</td>
<td>0.17</td>
<td>-0.60</td>
<td>-0.04</td>
<td>0.03</td>
<td>0.00</td>
</tr>
<tr>
<td>$a_{extB}$</td>
<td>0.17</td>
<td>1.00</td>
<td>-0.19</td>
<td>-0.10</td>
<td>-0.06</td>
<td>0.00</td>
</tr>
<tr>
<td>$ca$</td>
<td>-0.60</td>
<td>-0.19</td>
<td>1.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>$b$</td>
<td>-0.04</td>
<td>-0.10</td>
<td>0.01</td>
<td>1.00</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>$ce$</td>
<td>0.03</td>
<td>-0.06</td>
<td>0.02</td>
<td>0.02</td>
<td>1.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$aint$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 1.1 shows a pronounced negative correlation between the rapid transfer from the skin ($a_{extA}$) and evacuation from the chicken ($ca$), but other correlations are rather small. Correlations of course do not tell the whole story. Chapter 7 discusses visual tools for studying dependence in high-dimensional distributions. One such tool is the cobweb plot. In a cobweb plot, variables are represented as vertical lines. Each sample realizes one value of each variable. Connecting these values by line segments, one sample is represented as a jagged line intersecting all the vertical lines. Plate 1 shows 2000 such jagged lines and gives a picture of the joint distribution. In this case, we have plotted the quantiles, or percentiles, or ranks of the variables rather than the values themselves. The negative rank correlation between $a_{extA}$ and $ca$ is readily visible if the picture is viewed in colour: The lines hitting low values of $a_{extA}$ are red, and the lines hitting values of $ca$ are also red.

We see that the rank dependence structure is quite complex. Thus, we see that low values of the variable $ce$ ($c_{env}$, the names have been shortened for this graph) are strongly associated with high values of $b$, but high values of $ce$ may occur equally with high and low values of $b$. Correlation (rank or otherwise) is an average association over all sample values and may not reveal complex interactions. In subsequent chapters, we shall see how cobweb plots can be used to study dependence and conditional dependence. One simple illustration highlights their use in this example.

Suppose, we have a choice of accelerating the removal from the environment $ce$ or from the chicken $ca$; which would be more effective in reducing Campylobacter transmission? To answer this, we add two output variables: $a1$ (corresponding to the elicitation question given in the preceding text) is the amount on the first chicken of the flock as it leaves the processing phase and $a2$ is the amount on the last chicken of the flock as it leaves the processing phase. In Figure 1.6, we have conditionalized the joint distribution by selecting the upper 5% of the distribution for $ca$; in Figure 1.7, we do the same for $ce$.

We easily see that the intervention on $ce$ is more effective than that on $ca$, especially for the last chicken.

This example illustrates a feature that pervades quantitative decision support, namely, that input parameters of the mathematical models cannot be known with certainty. In such situations, mathematical models should be used to capture and propagate uncertainty. They should not be used to help a bunch of guys sitting...
around a table make statements that should be believed if they cannot be contradicted. In particular, it shows the following:

- Expert knowledge can be brought to bear in situations where direct experiment or measurement is not possible, namely, by quantifying expert uncertainty on variables that the models should predict.

- By utilizing techniques like probabilistic inversion in such situations, models become vehicles for capturing and propagating uncertainty.

- Configured in this way, expert input can play an effective role in evaluating and improving models.

- Models quantified with uncertainty, rather than wags and bogsats, can provide meaningful decision support.

1.3 Outline of the book

This book focuses on techniques for uncertainty analysis, which are generally applicable. Uncertainty distributions may *not* be assumed to conform to any parametric form. Techniques for specifying, sampling and analysing high-dimensional distributions should therefore be non-parametric. Our goal is to present the mathematical
concepts that are essential in understanding uncertainty analysis and to provide the practitioners with tools they will need in applications.

Some techniques, in particular those associated with bivariate dependence modelling, are becoming familiar to a wide range of users. Even this audience will benefit from a presentation focused on applications in higher dimensions. Subjects like the minimal information, diagonal band and elliptical copula will probably be new. Good books are available for bivariate dependence: Dall’Aglio et al. (1991); Doruet Mari and Kotz (2001); Joe (1997); Nelsen (1999). High-dimensional dependence models, sampling methods, post-processing analysis and probabilistic inversion will be new to non-specialists, both mathematicians and modellers.

The focus of this book is not how to assess dependencies in high-dimensional distributions, but what to do with them once we have them. That being said, the uncertainty, which gets analysed in uncertainty analysis is often the uncertainty of experts, and expert judgment deserves brief mention. Expert judgment is treated summarily in Chapter 2. Chapter 2 also introduces the uncertainty analysis, package UNICORN. Each chapter contains UNICORN projects designed to sensitize the reader to issues in dependence modelling and to step through features of the program. The projects in Chapter 2 provide a basic introduction to UNICORN and are strongly recommended. Chapter 3 treats bivariate dependence, focusing

Figure 1.7 Cobweb plot conditional on high $ce$. 

Samples selected: 500

axa  axb  ca  b  ce  aint  a1  a2

0.00  0.10  0.20  0.30  0.40  0.50  0.60  0.70  0.80  0.90  1.00

0.00  0.10  0.20  0.30  0.40  0.50  0.60  0.70  0.80  0.90  1.00

0.00  0.10  0.20  0.30  0.40  0.50  0.60  0.70  0.80  0.90  1.00

0.00  0.10  0.20  0.30  0.40  0.50  0.60  0.70  0.80  0.90  1.00

0.00  0.10  0.20  0.30  0.40  0.50  0.60  0.70  0.80  0.90  1.00

0.00  0.10  0.20  0.30  0.40  0.50  0.60  0.70  0.80  0.90  1.00

0.00  0.10  0.20  0.30  0.40  0.50  0.60  0.70  0.80  0.90  1.00

0.00  0.10  0.20  0.30  0.40  0.50  0.60  0.70  0.80  0.90  1.00
on techniques that are useful in higher dimensions. The UNICORN projects in Chapter 3 introduce the concepts of *aleatory* and *epistemic* uncertainty.

With regard to dependence in higher dimensions, much is not known. For example, we do not know whether an arbitrary correlation matrix is also a rank correlation matrix.\(^2\) We do know that characterizing dependence in higher dimensions via product moment correlation matrices is *not* the way to go. Product moment correlations impose unwelcome constraints on the one-dimensional distributions. Further, correlation matrices must be positive definite, and must be completely specified. In practice, data errors, rounding errors or simply vacant cells lead to intractable problems with regard to positive definiteness. We must design other friendlier ways to let the world tell us, and to let us tell computers, as to which high-dimensional distribution to calculate. We take the position that graphical models are the weapon of choice. These may be Markov trees, vines, independence graphs or Bayesian belief nets. For constructing sampling routines capable of realizing richly complex dependence structures, we advocate regular vines. They also allow us to move beyond discrete Bayesian belief nets without defaulting to the joint normal distribution. Much of this material is new and only very recently available in the literature: Bedford and Cooke (2001a); Cowell et al. (1999); Pearl (1988); Whittaker (1990). Chapter 4 is devoted to this.

Chapter 5 studies graphical models, which have interesting features, but are not necessarily generally applicable in uncertainty analysis. Bayesian belief nets and independence graphs are discussed. The problem of inferring a graphical model from multivariate data is addressed. The theory of regular vines is used to develop non-parametric continuous Bayesian belief nets. Chapter 6 discusses sampling methods. Particular attention is devoted to sampling regular vines and Bayesian belief nets.

Problems in measuring, inferring and modelling high-dimensional dependencies are mirrored at the end of the analysis by problems in communicating this information to problem owners and decision makers. Here graphical tools come to the fore in Chapter 7.

Chapter 8 addresses the problem of extracting useful information from an uncertainty analysis. This is frequently called *sensitivity analysis* (Saltelli et al. (2000)). We explore techniques for discovering which input variables contribute significantly to the output.

Chapter 9 takes up probabilistic inversion. Inverse problems are as old as probability itself, but their application in uncertainty analysis is new. Again, this material is only very recently available in the literature.

The concluding chapter speculates on the future role of uncertainty analysis in decision support.

Each chapter contains mathematical exercises and projects. The projects can be performed with the uncertainty analysis package UNICORN (UNcertainty analysis wIth CORrelatioNs), a light version of which can be downloaded free at

\(^2\) We have recently received a manuscript from H. Joe that purports to answer this question in the negative for dimensions greater than four.
These projects are meant to sensitize the reader to reason with uncertainty, and modelling dependence. Many of these can also be done with popular uncertainty analysis packages that are available as spreadsheet add-ons, such as Crystal Ball and @Risk. However, these packages do not support features such as multiple copula, vine modelling, cobweb plots, iterated and conditional sampling and probabilistic inversion. All projects can be performed with UNICORN Light, and step-by-step guidance is provided. Of course, the users can program these themselves.

In conclusion, we summarize the mathematical issues that arise in ‘capturing’ uncertainty over model input, propagating this uncertainty through a mathematical model, and using the results to support decision making. References to the relevant chapters are given below:

1. The standard product moment (or Pearson) correlation cannot be assessed independently of the marginal distributions, whereas the rank (or Spearman) correlation can (Chapter 3).

2. We cannot characterize the set of rank correlation matrices. We do know that the joint normal distribution realizes a ‘thin’ set of rank correlation matrices (Chapter 4).

3. There is no general algorithm for extending a partially specified matrix to a positive definite matrix (Chapter 4).

4. Even if we have a valid rank correlation matrix, it is not clear how we should define and sample a joint distribution with this rank correlation matrix. These problems motivate the introduction of regular vines (Chapter 4; sampling is discussed in Chapter 6).

5. Given sufficient multivariate data, how should we infer a graphical model, or conditional independence structure, which best fits the data (Chapter 5)?

6. After obtaining a simulated distribution for the model input and output, how can we analyse the results graphically (Chapter 7) and how can we characterize the importance of various model inputs with respect to model output (Chapter 8)?

7. How can we perform probabilistic inversion (Chapter 9)?

This book assumes knowledge of basic probability, statistics and linear algebra. We have put proofs and details in mathematical supplements for each chapter. In this way, the readers can follow the main line of reasoning in each chapter before immersing themselves in mathematical details.