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Introduction

Stochastic differential equations (SDEs) are basically differential equations with an additional stochastic term. The deterministic term, which is common to ordinary differential equations, describes the ‘average’ dynamical behaviour of the phenomenon under study and the stochastic term describes the ‘noise’, i.e. the random perturbations that influence the phenomenon. Of course, in the particular case where such random perturbations are absent (deterministic case), the SDE becomes an ordinary differential equation.

As the dynamical behaviour of many natural phenomena can be described by differential equations, SDEs have important applications in basically all fields of science and technology whenever we need to consider random perturbations in the environmental conditions (environment taken here in a very broad sense) that affect such phenomena in a relevant manner.

As far as I know, the first SDE appeared in the literature in Uhlenbeck and Ornstein (1930). It is the Ornstein–Uhlenbeck model of Brownian motion, the solution of which is known as the Ornstein–Uhlenbeck process. Brownian motion is the irregular movement of particles suspended in a fluid, which was named after the botanist Brown, who first observed it at the microscope in the 19th century. The Ornstein–Uhlenbeck model improves Einstein treatment of Brownian motion. Einstein (1905) explained the phenomenon by the collisions of the particle with the molecules of the fluid and provided a model for the particle’s position which corresponds to what was later called the Wiener process. The Wiener process and its relation with Brownian motion will be discussed on Chapters 3 and 4.

Although the first SDE appeared in 1930, we had to wait till the mid of the 20th century to have a rigorous mathematical theory of SDEs by Itô (1951). Since then the theory has developed considerably and been applied to physics, astronomy, electronics, telecommunications, civil engineering, chemistry, seismology, oceanography, meteorology, biology, fisheries, economics, finance, etc. Using SDEs, one can study phenomena like the dispersion of a pollutant in water or in the air, the effect of noise on the transmission of telecommunication
signals, the trajectory of an artificial satellite, the location of a ship, the thermal noise in an electric circuit, the dynamics of a chemical reaction, the control of an insulin delivery device, the dynamics of one or several populations of living beings when environmental random perturbations affect their growth rates, the optimization of fishing policies for fish populations subject to random environmental fluctuations, the variation of interest rates or of exchange rates, the behaviour of stock prices, the value of a call or put financial option or the risk immunization of investment portfolios or of pension plans, just to mention a few examples.

We will give special attention to the modelling issues, particularly the translation from the physical phenomenon to the SDE model and back. This will be illustrated with several examples, mainly in biological or financial applications. The dynamics of biological phenomena (particularly the dynamics of populations of living beings) and of financial phenomena, besides some clear trends, are frequently influenced by unpredictable components due to the complexity and variability of environmental or market conditions. Such phenomena are therefore particularly prone to benefit from the use of SDE models in their study and so we will prioritize examples of application in these fields. The study of population dynamics is also a field to which the author has dedicated a good deal of his research work. As for financial applications, it has been one of the most active research areas in the last decades, after the pioneering works of Black and Scholes (1973), Merton (1971), and Merton (1973). The 1997 Nobel prize in Economics was given to Merton and Scholes (Black had already died) for their work on what is now called financial mathematics, particularly for their work on the valuation of financial options based on the stochastic calculus this book will introduce you to. In both areas, there is a clear cross-fertilization between theory and applications, with the needs induced by applications having considerably contributed to the development of the theory.

This book is intended to be read by both more mathematically oriented readers and by readers from other areas of science with the usual knowledge of calculus, probability, and statistics, who can skip the more technical parts. Due to the introductory character of this presentation, we will introduce SDEs in the simplest possible context, avoiding clouding the important ideas which we want to convey with heavy technicalities or cumbersome notations, without compromising rigour and directing the reader to more specialized literature when appropriate. In particular, we will only study stochastic differential equations in which the perturbing noise is a continuous-time white noise. The use of white noise as a reasonable approximation of real perturbing noises has a great advantage: the cumulative noise (i.e. the integral of the noise) is the Wiener process, which has the nice and mathematically convenient property of having independent increments.

The Wiener process, rigorously studied by Wiener and Lévy after 1920 (some literature also calls it the Wiener–Lévy process), is also frequently named
Brownian motion in the literature due to its association with the Einstein’s first description of the Brownian motion of a particle suspended in a fluid in 1905. We personally prefer not to use this alternative naming since it identifies the physical phenomenon (the Brownian motion of particles) with its first mathematical model (the Wiener process), ignoring that there is an improved more realistic model (the Ornstein–Uhlenbeck process) of the same phenomenon. The ‘invention’ of the Wiener process is frequently attributed to Einstein, probably because it was thought he was the first one to use it (although at the time not yet under the name of ‘Wiener process’). However, Bachelier (1900) had already used it as a (not very adequate) model for stock prices in the Paris Stock Market.

With the same concern of prioritizing simple contexts in order to more effectively convey the main ideas, we will deal first with unidimensional SDEs. But, of course, if one wishes to study several variables simultaneously (e.g. the value of several financial assets in the stock market or the size of several interacting populations), we need multidimensional SDEs (systems of SDEs). So, we will also present afterwards how to extend the study to the multidimensional case; with the exception of some special issues, the ideas are the same as in the unidimensional case with a slightly heavier matrix notation.

We assume the reader to be knowledgeable of basic probability and statistics as is common in many undergraduate degree studies. Of course, sometimes a few more advanced concepts in probability are required, as well as basic concepts in stochastic processes (random variables that change over time). Chapter 2 intends to refresh the basic probabilistic concepts and present the more advanced concepts in probability that are required, as well as to provide a very brief introduction to basic concepts in stochastic processes. The readers already familiar with these issues may skip it. The other readers should obviously read it, focusing their attention on the main ideas and the intuitive meaning of the concepts, which we will convey without sacrificing rigour.

Throughout the remaining chapters of this book we will have the same concern of conveying the main ideas and intuitive meaning of concepts and results, and advise readers to focus on them. Of course, alongside this we will also present the technical definitions and theorems that translate such ideas and intuitions into a formal mathematical framework (which will be particularly useful for the more mathematically trained readers).

Chapter 3 presents an example of an SDE that can be used to study the growth of a biological population in an environment with abundant resources and random perturbations that affect the population growth rate. The same model is known as the Black–Scholes model in the financial literature, where it is used to model the value of a stock in the stock market. This is a nice illustration of the universality of mathematics, but the reason for its presentation is to introduce the reader to the Wiener process and to SDEs in an informal manner.

Chapter 4 studies the more relevant aspects of the Wiener process. Chapter 5 introduces the diffusion processes, which are in a certain way generalizations
of the Wiener process and which are going to play a key role in the study of SDEs. Later, we will show that, under certain regularity conditions, diffusion processes and solutions of SDEs are equivalent.

Given an initial condition and an SDE, i.e. given a Cauchy problem, its solution is the solution of the associated stochastic integral equation. In a way, either in the deterministic case or in the case of a stochastic environment, a Cauchy problem is no more than an integral equation in disguise, since the integral equation is the fulcrum of the theoretical treatment. In the stochastic world, it is the integral version of the SDE that truly makes sense since derivatives, as we shall see, do not exist in the current sense (the derivatives of the stochastic processes we deal with here only exist in a generalized sense, i.e. they are not proper stochastic processes). Therefore, for the associated stochastic integral equations to have a meaning, we need to define and study stochastic integrals. That is the object of Chapter 6. Unfortunately, the classical definition of Riemann–Stieltjes integrals (alongside trajectories) is not applicable because the integrator process (which is the Wiener process) is almost certainly of unbounded variation. Different choices of intermediate points in the approximating Riemann–Stieltjes sums lead to different results. There are, thus, several possible definitions of stochastic integrals. Itô’s definition is the one with the best probabilistic properties and so it is, as we shall do here, the most commonly adopted. It does not, however, satisfy the usual rules of differential and integral calculus. The Itô integral follows different calculus rules, the Itô calculus; the key rule of this stochastic calculus is the Itô rule, given by the Itô theorem, which we present in Chapter 6. However, we will mention alternative definitions of the stochastic integral, particularly the Stratonovich integral, which does not have the nice probabilistic properties of the Itô integral but does satisfy the ordinary rules of calculus. We will discuss the use of one or the other calculus and present a very useful conversion formula between them. We will also present the generalization of the stochastic integral to several dimensions.

Chapter 7 will deal with the Cauchy problem for SDEs, which is equivalent to the corresponding stochastic integral equation. A main concern is whether the solution exists and is unique, and so we will present the most common existence and uniqueness theorem, as well as study the properties of the solution, particularly that of being a diffusion process under certain regularity conditions. We will also mention other results on existence and uniqueness of the solutions under weaker hypotheses. We end with the generalization to several dimensions. This chapter also takes a first look at how to perform Monte Carlo simulations of trajectories of the solution in order to get a random sample of such trajectories, which is particularly useful in applications.

Chapter 8 will study the Black–Scholes model presented in Chapter 3, obtaining the explicit solution and looking at its properties. Since the solutions under the Itô and the Stratonovich calculi are different (even on relevant qualitative properties), we will discuss the controversy over which calculus, Itô or
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Stratonovich, is more appropriate for applications, a long-lasting controversy in the literature. This example serves also as a pretext to present, in Chapter 9, the author’s result showing that the controversy makes no sense and is due to a semantic confusion. The resolution of the controversy is explained in the context of the example and then generalized to a wide class of SDEs.

 Autonomous SDEs, in which the coefficients of the deterministic and the stochastic parts of the equation are functions of the state of the process (state that varies with time) but not direct functions of time, are particularly important in applications and, under mild regularity conditions, the solutions are homogeneous diffusion processes, also known as Itô diffusions.

 In Chapter 10 we will talk about the Dynkin and the Feynman–Kac formulas. These formulas relate the expected value of certain functionals (that are important in many applications) of solutions of autonomous SDEs with solutions of certain partial differential equations.

 In Chapter 11 we will study the unidimensional Itô diffusions (solutions of unidimensional autonomous SDEs) on issues such as first passage times, classification of the boundaries, and existence of stationary densities (a kind of stochastic equilibrium or stochastic analogue of equilibrium points of ordinary differential equations). These models are commonly used in many applications. For illustration, we will use the Ornstein–Uhlenbeck process, the solution of the first SDE in the literature.

 In Chapter 12 we will present several examples of application in finance (the Vasicek model, used, for instance, to model interest rates and exchange rates), in biology (population dynamics model with the study of risk of extinction and distribution of the extinction time), in fisheries (with extinction issues and the study of the fishing policies in order to maximize the fishing yield or the profit), and in the modelling of the dynamics of human mortality rates (which are important in social security, pension plans, and life insurance). Often, SDEs, like ordinary differential equations, have no close form solutions, and so we need to use numerical approximations. In the stochastic case this has to be done for the several realizations or trajectories of the process, i.e. for the several possible histories of the random environmental conditions. Since it is impossible to consider all possible histories, we use Monte Carlo simulation, i.e. we do computer simulations to obtain a random sample of such histories. Like in statistics, sample quantities, like, for example, the sample mean or the sample distribution of quantities of interest, provide estimates of the corresponding mean or distribution of such quantities. We will be taking a look at these issues as they come up, reviewing them in a more organized way in Chapter 12 in the context of some applications.

 Chapter 13 studies the problem of changing the probability measure as a way of modifying the SDE drift term (the deterministic part of the equation, which is the average trend of the dynamical behaviour) through the Girsanov theorem. This is a technical issue extremely important in the financial
applications covered in the following chapter. The idea in such applications with risky financial assets is to change its drift to that of a riskless asset. This basically amounts to changing the risky asset average rate of return so that it becomes equal to the rate of return $r$ of a riskless asset. Girsanov theorem shows that you can do this by artificially replacing the true probabilities of the different market histories by new probabilities (not the true ones) given by a probability measure called the equivalent martingale measure. In that way, if you discount the risky asset by the discount rate $r$, it becomes a martingale (a concept akin to a fair game) with respect to the new probability measure. Martin-gales have nice properties and you can compute easily things concerning the risky and derivative assets that interest you, just being careful to remember that results are with respect to the equivalent martingale measure. So, at the end you should reverse the change of probability measure to obtain the true results (results with respect to the true probability measure).

Chapter 14 assumes that there is no arbitrage in the markets and deals with the theory of option pricing and the derivation of the famous Black–Scholes formula, which are at the foundations of modern financial mathematics. Basically, the simple problem that we start with is to price a European call option on a stock. That option is a contract that gives you the right (but not the obligation) to buy that stock at a future prescribed time at a prescribed price, irrespective of the market price of that stock at the prescribed date. Of course, you only exercise the option if it is advantageous to you, i.e. if such a market price is above the option prescribed price. How much should you fairly pay for such a contract? The Black–Scholes formula gives you the answer and, as a by-product, also determines what can be done by the institution with which you have the contract in order to avoid having a loss. Basically, starting with the money you have paid for the contract and using it in a self-sustained way, the institution should buy and sell certain quantities of the stock and of a riskless asset following a so-called hedging strategy, which ensures that, at the end, it will have exactly what you gain from the option (zero if you do not exercise it or the difference between the market value and the exercise value if you do exercise it). We will use two alternative ways of obtaining the Black–Scholes formula. One uses Girsanov theorem and is quite convenient because it can be applied in other more complex situations for which you do not have an explicit expression; in such a case, we can recur to an approximation, the so-called binomial model, which we will also study. We will also consider European put options and take a quick look at American options. Other types of options and generalizations to more complex situations (like dealing with several risky assets instead of just one) will be considered but without going into details. In fact, this chapter is just intended as an introduction which will enable you to follow more specialized literature should you wish to get involved with more complex situations in mathematical finance.
Chapter 15 presents a summary of the most relevant issues considered in this book in order to give you a synthetic final view in an informal way. Since this will prioritize intuition, reading it right away might be a good idea if we are just interested in a fast intuitive grasp of these matters.

Throughout the book, there are indications on how to implement computing algorithms (e.g. for Monte Carlo simulations) using a spreadsheet or R language codes.

From Chapter 4 onwards there are proposed exercises for the reader. Exercises marked with * are for the more mathematically oriented reader. Solutions to exercises can be found in the Wiley companion website to this book.