1
Introduction

1.1 Prerequisites

A very concise description of this book is that it presents a methodology to predict and explain the distribution of forces and deflections that develop within a loaded structure.

For a layman who is unfamiliar with structural analysis, this description requires further explanations of many important points, namely: What is a structure? What are the forces within the structure? What are loads? Why and how do they get distributed?

We will not try to address these questions. For their answers a basic book on structural analysis, for example, Coates et al. (1988); Hibbeler (2008) or Marti (2013) will provide the necessary knowledge on the concepts used to describe structural behaviour – equilibrium, compatibility and constitutive relations – as well as the variables involved – forces, displacements, stresses and strains.

For all but the simplest problems, the mathematical equations used to describe the relations between these structural variables cannot be solved in a closed form. Of the various techniques that are used to obtain approximate solutions of these equations we will focus our attention on the application of a particular technique, the finite element method (FEM). Though it is possible to gain an understanding of FEM concepts solely from the information that will be presented in this text, it is more convenient to start with a basic book on finite element procedures, for example, Fish and Belytschko (2007).

We will therefore assume that the reader has a basic knowledge of the problems of structural analysis, namely of the fundamental equations of solid mechanics and, at least, some understanding of the procedures involved in the application of the FEM, most probably using a conventional displacement based formulation.

Such a reader, probably with an engineering background in the context of aeronautical, civil or mechanical engineering applications, given a title which includes ‘equilibrium’
and ‘finite elements’ might rather wonder: ‘Why another book? The finite element method is well known and it provides solutions that satisfy equilibrium. Doesn’t it?’ The fact is that in most cases it doesn’t, since only an approximate form of equilibrium is achieved by displacement based finite element formulations.

Our text presents a way to obtain solutions that are different from the ‘usual’ ones, because they exactly satisfy equilibrium. Nevertheless, since they normally omit the strict enforcement of compatibility conditions, it is not possible to say a priori which will be better. They just fail in different ways.

We believe that exploiting the complementarity of the two approaches allows for an interpretation of the results that is more profound than what is possible with a single type of analysis, naturally providing the tools for the assessment of their quality.

That, in the end, is our goal. Explaining in detail how equilibrated solutions can be obtained is just a step towards it.

1.2 What Is Meant by Equilibrium? Weak to Strong Forms

We expect the reader to understand what is meant by a free body, and being in a state of equilibrium, that is, the forces and their moments sum to zero. However, although checks on equilibrium at the global or overall level of a structure, for example, as represented by its finite element model, are commonly undertaken, deeper investigations into local levels of equilibrium become more problematic.

In FEM there are various shades of meaning, and perhaps expectation, when considering local equilibrium. The concept of a free body normally starts at the level of an infinitesimal element in a continuum (i.e. strong equilibrium between body forces and stresses), which is itself a mathematical abstraction – since we ignore the microscopic structure of the material.

Then the concept moves to the level of a single finite element, and then it may move back to another mathematical abstraction – a node of an element, where we invoke the concept of nodal forces (i.e. corresponding to a weak form of equilibrium between statically equivalent forces).

It is relevant here to note that the concept of a nodal force may not be explicitly mentioned in texts on finite elements, and we are aware of commercially available software where nodal forces are not available to the user, but only stress contours and tables of stresses at particular points!

In practice some confusion exists, and engineers may be unaware of the ‘subtle’ distinctions between these different levels of equilibrium of free bodies, and their significance to the analysis of a finite element model. We frequently hear of engineers who look blank when advised that local equilibrium is usually violated – they appear to have a firm conviction that equilibrium is being satisfied in all necessary aspects. Their first response might be: ‘Does it matter if there are local violations?’

An appropriate reply might be: ‘It all depends on your needs and how well you know the distribution of the loads.’ This is a matter of judgement, but we would advise that engineers, when faced with many uncertainties, can proceed with more confidence knowing that their analysis provides complete equilibrium. Local violations can be regarded as residual loads that are equilibrated by the errors in stress, and such loads are made orthogonal to the displacements allowed by a conforming model. By refining the model, the solutions converge, even when residual loads persist.
Our starting point is the fact that conventional finite element analyses ‘provide solutions that equilibrate the equivalent nodal forces’, where the adjective *equivalent* plays a central role that is often disregarded in the more basic introductions to the FE method.

Effectively there is equilibrium of equivalent nodal forces in the solutions provided by most FE programs. We will discuss in detail what that means and we will conclude that, in most cases, there are no nodal forces as such. Energetically consistent nodal forces are defined, which are required to produce the same work as the real forces and stresses for all displacements considered. But, in general, this is not sufficient to guarantee equilibrium in a strong, or pointwise, sense.

This happens because only a finite subset of the possible displacements can be included in a given model and the solution space is generally infinite, therefore equilibrium is imposed on an average, or weak form. Generally

the solutions provided by displacement based FE models do not enforce the equilibrium conditions at every point of the domain and/or its boundary.

Our objective is to present in this book a methodology whose models produce solutions that strongly verify all equilibrium conditions. As always there is a drawback for every new approach. In this case the gain in terms of equilibrium will imply a loss in terms of compatibility, which will only be imposed in a weak form.

We will not pretend that these equilibrium formulations are always better than their displacement based counterparts, as each formulation locally enforces one set of conditions, while imposing a weak form of the other.

1.3 What Do We Gain From Strong Forms of Equilibrium?

The complementary nature of these formulations is, in our opinion, the strongest reason for considering solutions obtained from *both approaches*. It does not matter which one is considered first, as the different approximate solutions that they produce are complementary, in the sense that they satisfy complementary equations in a strong and in a weak form.

As we will show, this complementarity can be used in a natural way to assess the quality of the solutions, and to drive a mesh adaptation process, deciding where it is important to have more, or fewer elements. From a practical point of view it is also relevant to point out that equilibrium solutions have the advantage of being immediately usable as a safe basis for design of ductile structures, when the Static Theorem of Limit Analysis can be invoked (fib, 2013; Marti, 2013; Nielsen and Hoang, 2010).

In particular, equilibrium solutions give us a more rational way of accounting for stress concentrations, especially when they arise due to mathematical singularities where the structural geometry has been simplified, for example, at re-entrant corners.
Displacement models tend to pollute local equilibria while seeking infinite values of stress, while equilibrium models continue to provide a statically admissible field in the neighbourhood of a mathematical singularity. Such a stress field represents a redistribution of stress near the singularity, and in this respect it is similar to the behaviour of the real structure, which may adapt itself by local yielding while maintaining stress equilibrium.

On a historical note we recall that, before the introduction of computer techniques, methods requiring strong equilibrium as their starting point formed the basis for most structural design procedures.

Considering the design of arches and their thrust lines, Figure 1.1, the analysis of statically indeterminate trusses and frames by Maxwell-Mohr methods, Figures 1.2 and 1.3, and also in variational methods such as the Trefftz method, we find that most ‘historical’ methods of analysis, in essence, sought to explain how the transmission of forces through the structure may be achieved. The reason for this can be explained by the fact that material resistance is intuitively related to the level of stress within the structure – displacements do not appear to be as important – leading to the application, implicit or explicit, of the Static Theorem of Limit Analysis, already mentioned.

Furthermore, force, or flexibility, methods can lead to better conditioned systems of equations, and fewer of them (Argyris and Kelsey, 1960; Henderson and Maunder, 1969). These were critical factors when solutions were calculated with manual or semi-manual methods (Kurrer, 2008).

![Figure 1.1](image)

**Figure 1.1** Robert Hooke (1676): ‘As hangs the flexible line, so, but inverted will stand the rigid arch’ (Heyman, 1982). The chain adapts its shape so that internal tensions balance the concentrated vertical forces, and transfer them to the ground/supports. The reflection of the shape of the chain is a thrust line, the trajectory of the compressive stress resultants. Source: Adapted from the hanging chain, Robert Hooke (1676), requoted from Heyman 1982.
Introduction

1.4 What Paths Have Been Followed to Achieve Strong Forms of Equilibrium?

Generalizing the approach that is used for the analysis of frame structures using the force method, the obvious solution for the analysis of continua is to combine...
approximations that verify equilibrium in a strong sense so that the generalized relative
displacements corresponding to the hyperstatic forces are zero. In the 1960s and
70s, as a consequence of the historical relevance given to equilibrium in technical
culture, there was considerable effort devoted to obtaining such finite element methods
(Argyris and Kelsey, 1960; Fraeijs de Veubeke, 1965; Gallagher, 1975; Robinson,
1973).

This approach was practically abandoned, an exception being the work of Kaveh (2004,
2014), mainly due to the difficulty in setting up such approximations, and to the superior
computational efficiency of direct stiffness assembly procedures (Przemieniecki, 1985).
We will only briefly address this approach in this book.

In the 1960s Fraeijs de Veubeke et al. proposed and developed finite element for-
mulations providing solutions that verify equilibrium in the strong sense using two
different approaches: to work with a stress potential, for example the Airy stress function,
and to assemble elements where an internally equilibrated stress approximation is
assumed in such a way that their boundary tractions are in equilibrium, that is, are codiffusive.

It appears that these equilibrium models did not find favour – maybe due to their rela-
tive complexity in implementation, and difficulties encountered in the application of
boundary conditions. In any event, the belief in displacement models seems to have
grown, together with the idea that the quality of solutions could be judged entirely by
post-processing procedures such as those proposed by Zienkiewicz & Zhu in the late

However, the value of having complementary solutions for error estimation was
still recognized, and a lot of research has been expended into a variety of ways to
recover enhanced solutions from displacement models, for example, the supercon-
vergent patch recovery (SPR) methods (Zienkiewicz and Zhu, 1987, 1992) and the
error in constitutive relation from P. Ladeveze et al. (Ladevèze and Leguillon, 1983;
Ladevèze and Maunder, 1996). These recovery methods, with the exception of the
approach proposed by Ladevèze, only lead to a better approximation of strong
equilibrium.

Since the 1990s there has been a renewed interest and a renaissance in FEM, with
hybrid formulations that directly enforce a strong form of equilibrium (Almeida and

1.5 Industrial Perspectives

Two different perspectives are particularly relevant when considering the industrial
application of analyses based on equilibrium finite elements:

• verification and validation in ‘simulation governance’ (Szabó and Actis, 2012);
• the design and/or assessment of structures, explicitly or implicitly based on the Static
  Theorem of Limit Analysis.

In this Section we briefly discuss how these aspects can be considered.
1.5.1 Simulation Governance

With the ever wider reliance by industry on finite element analyses to justify compliance with codes of practice or statutory requirements, it is recognized that more formal verification and validation procedures should be adopted.

General procedures have been defined in a rather pithy, but memorable way, to address two questions (Roache, 1998):

i) ‘Am I solving the equations right?’
ii) ‘Am I solving the right equations?’

The ‘equations’ refer to the mathematical model which is assumed to describe the physical behaviour; in our case this corresponds to the equations of elasticity. Then (i) concerns verifying that the solution matches the mathematical model. It may fail to do so because of the intrinsic inaccuracy of the numerical model (for us the finite element approximations that are assumed), combined with other numerical aspects, for example, numerical instabilities and ill-conditioning that may be present.

Point (ii) questions the validity of the mathematical model to adequately represent physical reality, for example, are potential non-linearities in behaviour properly accounted for; do the boundary conditions and loads reflect actual conditions; do the constitutive relations properly match those of the real materials?

Thus two complementary approaches to finite element modelling that satisfy compatibility or equilibrium, and which can deliver opposite bounds to quantities of interest, are clearly advantageous, both for the peace of mind of the engineer, and as a means of providing evidence to satisfy statutory requirements on quality control.

1.5.2 Equilibrium in Structural Design and Assessment

Finite element models can evolve as the design of a structure or a device develops, but with different and evolving aims, particularly in the civil engineering context where structures tend to be ‘one-off’ as opposed to the mass produced artefacts of mechanical engineering. For the design of one-off civil engineering structures, high accuracy in the output of quantities of interest is not normally required, since the variability of materials, construction processes, and loading regimes (including the troublesome question of usually indeterminate residual stresses) do not generally justify this. However, a strong sense of equilibrium is important from the simplest initial models at early stages of design to the refined models necessary to justify the final design.

Wittingly, or unwittingly, designers rely on what Heyman (1995) has termed ‘the master safe theorem’, that is, if any equilibrium state can be found, that is, one for which a set of internal forces is in equilibrium with the external loads, and, further, for which every internal portion of the structure satisfies a strength criterion, then the structure is safe.

It is interesting to note that Wren must have had a rather similar basis for his designs: ‘The design must be regulated by the art of staticks, or invention of the centres of gravity, and the duly poising of all parts to equiponderate; without which, a fine design will fail and prove abortive. Hence I conclude, that all designs must, in the first place, be brought to this test, or rejected.’ (Addis, 2007).

In pithier terms, Ed Wilson quotes: ‘equilibrium is essential, compatibility is optional’ and then emphasizes that stresses in conforming elements do not strongly satisfy equilibrium, so that mesh design needs to be considered in order to achieve acceptable levels of stress (Wilson, 2000).
Of course these quotations leave open what is meant by ‘portion’ or ‘part’, but at the smallest practical level we can take this to mean infinitesimal parts of a continuum, and internal forces to mean stresses. The conventional conforming finite element model only enforces the weaker equilibrium of nodal forces, and so an element serves as a portion, but then we need a strength criterion! A useful and attractive feature of equilibrium finite elements is to transfer their interactions from the mathematical concept of nodal forces to the engineering concept of tractions on interfaces, backed up by fully equilibrated internal stress fields.

These are immediately in a form amenable to comparison with strength criteria. It may be noted that, even when the non-elastic properties of the structural material do not strictly justify the use of the master safe theorem, equilibrium is a first line of defence! Concepts of equilibrium, and their use in design/assessment stages, are embodied in codes of practice for design, but with some warnings when conventional conforming finite element models are used, for example, Eurocodes EN 1990 (Basis of structural design), EN 1992 (Design of concrete structures), and fib Model Code for Concrete Structures 2010.

- **EN 1990 (1.5.6.2)** Global analysis is defined as the ‘determination, in a structure, of a consistent set of either internal forces and moments, or stresses, that are in equilibrium with a particular defined set of actions on the structure, and depend on geometrical, structural and material properties.’

- **EN 1992:** brief reference to the use of finite element methods is made in Section 5 Structural analysis, 5.1.1 General requirements: ‘However, for certain particular elements, the methods of analysis used (e.g. finite element analysis) gives stresses, strains and displacements rather than internal forces and moments. Special methods are required to use these results to obtain appropriate verification.’

- The fib Model Code for Concrete Structures 2010 allows the use of the theory of plasticity in design, and this includes the ‘lower bound (static) theorem’. Verification of designs may be assisted by numerical simulations, including the finite element method: ‘In the case of the most widely used stiffness method, the shape of the displacement field is assumed and equilibrium is satisfied only in integral sense. The internal stresses are lower, compared with an exact solution. The approximations introduced by the finite element formulation only, can be a significant source of errors in numerical analysis.’

One of the most expensive and dramatic examples of a collapse attributed to the use of conforming models must be that of the Sleipner offshore oil production platform in 1991 (Rombach, 2011). A coarse mesh of solid elements led to a transverse shear stress resultant in a cell wall being underestimated by some 50% compared with later estimates of statically admissible internal forces. The latter forces were sufficient to cause local shear failure in a poorly detailed reinforced concrete wall, and consequently to trigger overall collapse during flotation trials.

### 1.6 The Structure of the Book

After this Introduction we begin in Chapter 2 by illustrating the derivation of approximate solutions to some simple examples involving concepts of compatibility and/or equilibrium, without recourse to the use of finite elements. This is followed in Chapter 3
by a summary of the main finite element formulations other than the hybrid equilibrium one, beginning with the more conventional compatible or conforming approach and then proceeding to variations which may involve stronger forms of equilibrium.

Chapters 4 to 6 focus on the main topic of this book, that is, the hybrid equilibrium formulation, its elements and the particular features associated with their possible kinematic instabilities, known as the spurious kinematic modes and which may be likened to pseudo-mechanisms. The approach taken here is based on the duality between static and kinematic quantities, and after presenting general relations based on linear elastic behaviour, we consider specifics for 2D and 3D continua and plate bending.

Chapter 7 places the formulations in the context of a variational basis and appropriate functionals corresponding to potential and complementary potential energies. This aspect is particularly important when considering the existence and convergence of solutions for typical saddle point problems.

In Chapter 8 we consider the complementary nature of compatible and equilibrating finite element solutions, and present methods for recovering compatibility from equilibrium and vice-versa. This leads us to Chapter 9 which describes the roles of complementary solutions in obtaining error estimates of either solution, and explains how these estimates can be used to adapt a finite element mesh to best achieve desired outputs.

In Chapter 10 we apply the hybrid equilibrium formulation to linear dynamic analyses by exploiting Toupin’s principle to account for inertia effects.

Finally Chapter 11 takes a look beyond modelling linear behaviour, and provides some pointers towards analysing various forms of non-linearity with the hybrid approach.

The book contains two appendices: A, with a resumé of the necessary fundamental equations of linear elasticity; and B, containing some insight and guidance on using the computer programs associated with this book.

Please note that the quantities shown in numerical examples are without specific units; we assume that a particular system of units will always be consistent.

References


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