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Mathematical Preliminaries

Advanced mathematical knowledge is required to learn the finite elasto-plasticity theory. The basics of vector and tensor analyses are described in preparation for the explanations of advanced theory in later chapters. Various representations of tensors, for example, the eigenvalues and principal directions, the Cayley–Hamilton theorem, the polar and spectral decompositions, the isotropic function and various differential equations, the time-derivatives, and the integration theorems are described. Component descriptions of vectors and tensors in this chapter are limited to the normalized rectangular coordinate system, that is, the rectangular coordinate system with unit base vectors, while the terms orthogonal, orthonormal and Cartesian are often used instead of rectangular. However, the derived tensor relations hold even in the general curvilinear coordinate system of the Euclidian space described in later chapters.

1.1 Basic Symbols and Conventions

An index appearing twice in a term is summed over the specified range of the index. For instance, we may write

\[ u_r v_r = \sum_{r=1}^{n} u_r v_r, \quad T_{ir} v_r = \sum_{r=1}^{n} T_{ir} v_r, \quad T_{rr} = \sum_{r=1}^{n} T_{rr}, \]  

(1.1)

where the range of index is taken to be 1, 2, \ldots, n. The indices used repeatedly are arbitrary, and thus they are called dummy indices: note that \( u_r v_r = u_s v_s \) and \( T_{ir} v_r = T_{is} v_s \), for example. This is termed Einstein’s summation convention. Henceforth, repeated indices refer to this convention unless specified by the additional remark ‘(no sum)’.

The symbol \( \delta_{ij} (i, j = 1, 2, 3) \) defined in the following equation is termed the Kronecker delta:

\[ \delta_{ij} = 1 \quad \text{for } i = j, \quad \delta_{ij} = 0 \quad \text{for } i \neq j \]  

(1.2)

from which it follows that

\[ \delta_{ir} \delta_{rj} = \delta_{ij}, \quad \delta_{ii} = 3. \]  

(1.3)
Furthermore, the symbol \( \varepsilon_{ijk} \) defined by the following equation is called the alternating (or permutation) symbol or Eddington’s epsilon or Levi-Civita ‘e’ tensor:

\[
\varepsilon_{ijk} = \begin{cases} 
1 & \text{for cyclic permutation of } ijk \text{ from 123,} \\
-1 & \text{for anticyclic permutation of } ijk \text{ from 123,} \\
0 & \text{for others.}
\end{cases}
\] (1.4)

1.2 Definition of Tensor

In this section the definition of an objective tensor is given and, based on it, the criteria for a given physical quantity to be a tensor and its order are provided.

1.2.1 Objective Tensor

Let the set of \( n^m \) functions be described as \( T(p_1, p_2, \ldots, p_m) \) in the coordinate system \( \{O, x_i\} \) with the origin \( O \) and the axes \( x_i \) in \( n \)-dimensional space, where each of the indices \( p_1, p_2, \ldots, p_m \) takes a numerical value \( 1, 2, \ldots, n \). This set of functions is defined as the \( m \)-th-order tensor in \( n \)-dimensional space, if the set of functions is observed in the other coordinate system \( \{O, x_i^\ast\} \) with the origin \( O \) and the axes \( x_i^\ast \) as follows:

\[
T^\ast(p_1, p_2, \ldots, p_m) = Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p_m q_m} T(q_1, q_2, \ldots, q_m)
\] (1.5)

or

\[
T^\ast(p_1, p_2, \ldots, p_m) = \frac{\partial x^\ast_{p_1}}{\partial x_{q_1}} \frac{\partial x^\ast_{p_2}}{\partial x_{q_2}} \cdots \frac{\partial x^\ast_{p_m}}{\partial x_{q_m}} T(q_1, q_2, \ldots, q_m),
\] (1.6)

provided that only the directions of axes are different but the origin is common and relative motion does not exist. Here, \( Q_{ij} \) in equation (1.5) is defined by

\[
Q_{ij} = \frac{\partial x^\ast_i}{\partial x_j}
\] (1.7)

which fulfills

\[
Q_{ir} Q_{jr} = \delta_{ij}
\] (1.8)

because of

\[
Q_{ir} Q_{jr} = \frac{\partial x^\ast_i}{\partial x_r} \frac{\partial x^\ast_r}{\partial x_j}.
\]

Denoting \( T(p_1, p_2, \ldots, p_m) \) by the symbol \( T_{p_1 p_2 \ldots p_m} \), equation (1.6) is expressed as

\[
T_{p_1 p_2 \ldots p_m}^\ast = Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p_m q_m} T_{q_1 q_2 \ldots q_m}
\] (1.9)

that is,

\[
T_{p_1 p_2 \ldots p_m}^\ast = \frac{\partial x^\ast_{p_1}}{\partial x_{q_1}} \frac{\partial x^\ast_{p_2}}{\partial x_{q_2}} \cdots \frac{\partial x^\ast_{p_m}}{\partial x_{q_m}} T_{q_1 q_2 \ldots q_m}.
\] (1.10)
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Noting that
\[ Q_{11Q_1} \cdots Q_{p_{1Q_1}} T_{p_{1p_2} \cdots p_m}^* = Q_{11Q_1} \cdots Q_{p_{1Q_1}} Q_{p_{1Q_1}} \cdots Q_{p_{1Q_1}} T_{q_1q_2 \cdots q_m}^* \]
\[ = (Q_{p_{1Q_1}} Q_{p_{1Q_1}}) (Q_{p_{2Q_2}} Q_{p_{2Q_2}}) \cdots (Q_{p_{mQ_m}} Q_{p_{mQ_m}}) T_{q_1q_2 \cdots q_m}^* \]
\[ = \delta_{q_1q_1} \delta_{q_2q_2} \cdots \delta_{q_mq_m} T_{q_1q_2 \cdots q_m}^* \]

with equation (1.8), the inverse relation of equation (1.9) is given by
\[ T_{q_1q_2 \cdots q_m} = Q_{p_{1Q_1}} Q_{p_{2Q_2}} \cdots Q_{p_{mQ_m}} T_{p_{1p_2} \cdots p_m}^*. \]  

(1.11)

Indices put in a tensor take the dimension of the space in which the tensor exists. The number of indices, which is equal to the number of operators \( Q_{ij} \), is called the order of tensor. For instance, the transformation rule of the first-order tensor, that is, the vector \( v_i \), and the second-order tensor \( T_{ij} \) are given by
\[
\begin{align*}
v_i^* &= Q_{ij} v_j, \\
T_{ij} &= Q_{ij} T_{rs}, \\
T_{ij} &= Q_{ij} T_{rs}^*.
\end{align*}
\]

(1.12)

Consequently, in order to prove that a certain quantity is a tensor, one needs only to show that it obeys the tensor transformation rule (1.9) or that multiplying the quantity by a tensor leads to a tensor by virtue of the quotient rule described in the next section.

The coordinate transformation rule in the form of equation (1.9) or (1.11) is called the \textit{objective transformation}. A tensor obeying the objective transformation rule even between coordinate systems with a relative rate of motion, that is, relative parallel and rotational velocities, is called an \textit{objective tensor}. Vectors and tensors without the time-dimension, for example, force, displacement, rotational angle, stress and strain, are objective vectors and tensors. On the other hand, time-rate quantities, for example, rate of force, velocity, spin and the material-time derivatives of physical quantities, for example, stress and strain, are not objective vectors and tensors in general; they are influenced by the relative rate of motion between coordinate systems. Constitutive equations of materials have to be formulated in terms of objective tensors, since material properties are not influenced by the rigid-body rotation of material and therefore must be described in a form independent of the coordinate systems, as will be explained in Section 5.3.

Tensors obviously fulfill the linearity
\[
\begin{align*}
T_{p_{1p_2} \cdots p_m} (G_{p_{1p_2} \cdots p_l} + H_{p_{1p_2} \cdots p_l}) &= T_{p_{1p_2} \cdots p_m} G_{p_{1p_2} \cdots p_l} + T_{p_{1p_2} \cdots p_m} H_{p_{1p_2} \cdots p_l} \\
T_{p_{1p_2} \cdots p_m} (aA_{p_{1p_2} \cdots p_l}) &= aT_{p_{1p_2} \cdots p_m} A_{p_{1p_2} \cdots p_l}
\end{align*}
\]

(1.13)

where \( a \) is an arbitrary scalar variable. Therefore, the tensor plays the role of linearly transforming one tensor into another and thus it is also called a \textit{linear transformation}. The operation that lowers the order of a tensor by multiplying it by another tensor is called \textit{contraction}.

Denoting by \( e_i^*, e_i^* \), \( \ldots, e_m^* \) the unit base vectors of the coordinate axes \( x_i^*, x_i^* \), \ldots, \( x_m^* \), the quantity \( Q_{ij} \) in equation (1.7) is represented in terms of the base vectors as follows:
\[
Q_{ij} = e_i^* \cdot e_j^*.
\]

(1.14)
noting that

\[ Q_{ij} = \frac{\partial x_i^*}{\partial x_j} = e^*_i \cdot \frac{\partial x^*_j}{\partial x_j} = e^*_i \cdot \frac{\partial x_j}{\partial x_j} = e^*_i \cdot \delta_{ji}, \]

where the coordinate transformation operator \( Q_{ij} \) is interpreted as

\[ Q_{ij} \equiv \cos(\text{angle between } e^*_i \text{ and } e_j) \] (1.15)

which fulfills equations (1.8), that is,

\[ Q_{ir} Q_{jr} = Q_{ri} Q_{rj} = \delta_{ij} \] (1.16)

which can be also verified by

\[ Q_{ir} Q_{jr} = (e^*_i \cdot e_r)(e^*_j \cdot e_r) = e^*_i \cdot (e^*_j \cdot e_r) = \delta_{ij}. \]

The transformation rule for base vectors is given by

\[ e_i = Q_{ir} e^*_r, \quad e^*_i = Q_{ir} e_r, \] (1.17)

noting that

\[ e_i = (e_i \cdot e^*_r)e^*_r, \quad e^*_i = (e^*_i \cdot e_r)e_r. \]

### 1.2.2 Quotient Law

There is a convenient law, referred to as the quotient law, which enables us to judge whether or not a given quantity is tensor and to find its tensorial order as follows: If a set of functions \( T(p_1, p_2, \ldots, p_m) \) becomes \( B_{p_1+p_2+\cdots+p_m} \) \( (m-l) \)-th order tensor lacking the indices \( p_1 \sim p_l \) through multiplying it by \( A_{p_1p_2\cdots p_l} \) \( (lh\text{-order tensor} \ (l \leq m)) \), the set is an \( m \)-th order tensor.

(Proof) This convention is proved by showing that the quantity \( T(p_1, p_2, \ldots, p_m) \) is an \( m \)-th order tensor when the relation

\[ T(p_1, p_2, \ldots, p_m) A_{p_1p_2\cdots p_l} = B_{p_1+p_2+\cdots+p_m} \] (1.18)

holds, which is described in the coordinate system \( \{O-x^*_i\} \) as follows:

\[ T^*(p_1, p_2, \ldots, p_m) A^*_{p_1p_2\cdots p_l} = B^*_{p_1+p_2+\cdots+p_m}. \] (1.19)

Here, \( A_{p_1p_2\cdots p_l} \) is the \( lh \)-th order tensor and \( B_{p_1+p_2+\cdots+p_m} \) is the \( (m-l) \)-th order tensor. Therefore, the following relation holds:

\[ B^*_{p_1+p_2+\cdots+p_m} = Q_{p_1+p_2+\cdots+p_m} A_{p_1p_2\cdots p_l} B_{r_1+r_2+\cdots+r_m} \]
\[ = Q_{p_1+p_2+\cdots+p_m} T(r_1, r_2, \ldots, r_m) A_{r_1r_2\cdots r_l} \]
\[ = Q_{p_1+p_2+\cdots+p_m} Q_{p_1+p_2+\cdots+p_m} T(r_1, r_2, \ldots, r_m) Q_{p_1r_1} Q_{p_2r_2} \cdots Q_{p_lr_l} A^*_{p_1p_2\cdots p_l}. \]

\[ (l+1 \sim m) \quad (1 \sim l) \] (1.20)

Substituting equation (1.19) into the left-hand side of equation (1.20) yields

\[ (T^*(p_1, p_2, \ldots, p_m) - Q_{p_1r_1} Q_{p_2r_2} \cdots Q_{p_lr_l} T(r_1, r_2, \ldots, r_m)) A^*_{p_1p_2\cdots p_l} = 0, \]
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from which it follows that

$$T^* (p_1, p_2, \ldots, p_m) = Q_{p_1r_1} Q_{p_2r_2} \cdots Q_{p_mr_m} T(r_1, r_2, \ldots, r_m).$$  \hfill (1.21)

Equation (1.21) satisfies the definition of tensor in equation (1.5). Therefore, the quantity $T(p_1, p_2, \ldots, p_m)$ is an $m$th-order tensor.

According to the proof presented above, equation (1.18) can be written as

$$T_{p_1p_2\cdots p_m} A_{p_1p_2\cdots p_l} = B_{p_l+1p_l+2\cdots p_m}.$$  \hfill (1.22)

For instance, if the quantity $T(i, j)$ transforms the first-order tensor, that is, vector $v_i$, to the vector $u_i$ by the operation $T(i, j)v_j = u_i$, one can regard $T(i, j)$ as the second-order tensor $T_{ij}$.

1.3 Vector Analysis

In this section some basic rules for vectors are given which are required to understand the representation of tensors in the general coordinate system described in the next chapter.

1.3.1 Scalar Product

The scalar (or inner) product of the vectors $a$ and $b$ is defined by

$$a \cdot b = ||a|| ||b|| \cos \theta = a_i b_i,$$  \hfill (1.23)

where $\theta$ is the angle between the vectors $a$ and $b$, and $|| ||$ means the magnitude, that is,

$$||v|| = \sqrt{v_i v_i} = \sqrt{v \cdot v}.$$  \hfill (1.24)

Here, the following relations hold for the scalar product:

$$a \cdot b = b \cdot a \quad \text{(commutative law)},$$  \hfill (1.25)

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{(distributive law)},$$  \hfill (1.26)

$$s(a \cdot b) = (sa) \cdot b = a \cdot (sb) = (a \cdot b)s,$$  \hfill (1.27)

$$(sa + bb) \cdot c = sa \cdot c + bb \cdot c$$  \hfill (1.28)

for arbitrary scalars $s$, $a$, $b$.

The vector is represented in terms of components with base vectors as follows:

$$v = v_i e_i,$$  \hfill (1.29)

where the components $v_i$ are given by the projection of $v$ onto the base vector $e_i$, that is, their scalar product and thus it follows that

$$v_i = v \cdot e_i, \quad v = (v \cdot e_i) e_i.$$  \hfill (1.30)
1.3.2 Vector Product

The vector (or outer or cross) product of vectors is defined by
\[
\mathbf{a} \times \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \sin \theta \mathbf{n} = a_i \mathbf{e}_i \times b_j \mathbf{e}_j = \varepsilon_{ijk} a_i b_j \mathbf{e}_k
\]
\[
= (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3, \quad (1.31)
\]
where \( \mathbf{n} \) is the unit vector which forms the right-handed bases \((\mathbf{a}, \mathbf{b}, \mathbf{n})\) in this order. It follows for base vectors from equation (1.31) that
\[
\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k. \quad (1.32)
\]
Here, the following equations hold for the vector product:
\[
\mathbf{a} \times \mathbf{a} = \mathbf{0}, \quad (1.33)
\]
\[
\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \quad (1.34)
\]
\[
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \quad (\text{distributive law}), \quad (1.35)
\]
\[
s(\mathbf{a} \times \mathbf{b}) = (s\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times s\mathbf{b}) = (s\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times s\mathbf{b}), \quad (1.36)
\]
\[
(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}, \quad (1.37)
\]
\[
||\mathbf{a} \times \mathbf{b}||^2 + (\mathbf{a} \cdot \mathbf{b})^2 = (||\mathbf{a}|| ||\mathbf{b}||)^2. \quad (1.38)
\]

1.3.3 Scalar Triple Product

The scalar triple product of vectors is defined by
\[
[\mathbf{abc}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \varepsilon_{ijk} a_i b_j c_k, \quad (1.39)
\]
fulfilling
\[
[\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}] = -[\mathbf{bac}] = -[\mathbf{cba}] = -[\mathbf{acb}]. \quad (1.40)
\]
Denoting the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) by \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \), it follows from equation (1.39) that
\[
[\mathbf{v}_i \mathbf{v}_j \mathbf{v}_k] = \varepsilon_{ijk} [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3], \quad (1.41)
\]
noting the fact that the term on the right-hand side of this equation is \(+[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3], -[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \)
and 0 when indices \( i, j, k \) are even and odd permutations and two of indices coincide with each other, respectively.

Here, the following equations hold for the scalar triple product.
\[
[\mathbf{e}, \mathbf{e}, \mathbf{e}_k] = \varepsilon_{ijk}, \quad (1.42)
\]
\[
[s\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{a}, s\mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{b}, sc] = s[\mathbf{abc}], \quad (1.43)
\]
\[
[\mathbf{a} + \mathbf{b}, \mathbf{c}, \mathbf{d}] = a[\mathbf{acd}] + b[\mathbf{bcd}], \quad (1.44)
\]
\[
[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = [\mathbf{abc}]^2, \quad (1.45)
\]
\[
[\mathbf{abc}]\mathbf{d} = [\mathbf{bcd}]\mathbf{a} + [\mathbf{adc}]\mathbf{b} + [\mathbf{abd}]\mathbf{c}, \quad (1.46)
\]
\[
[\mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d}, \mathbf{e} \times \mathbf{f}] = [\mathbf{abd}]\mathbf{cef} - [\mathbf{abc}]\mathbf{def}. \quad (1.47)
\]
1.3.4 Vector Triple Product

The vector product is defined by \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \). Setting \( \mathbf{v} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \), it follows from equation (1.31) that

\[
\begin{align*}
v_1 &= a_2(b \times c)_3 - a_3(b \times c)_2 = a_2(b_1 c_2 - b_2 c_1) - a_3(b_3 c_1 - b_1 c_3) \\
&= (a_1 c_1 + a_2 c_2 + a_3 c_3)b_1 - (a_1 b_1 + a_2 b_2 + a_3 b_3)c_1 \\
&= (\mathbf{a} \cdot \mathbf{c})b_1 - (\mathbf{a} \cdot \mathbf{b})c_1.
\end{align*}
\]

\( v_2 \) and \( v_3 \) can be represented analogously. Collecting these equations yields \( \mathbf{v} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \), and then one has the formula of the vector product:

\[
\begin{align*}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= -(\mathbf{b} \cdot \mathbf{c})\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c} (\neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})) \quad \text{for } n \in \mathbb{R}.
\end{align*}
\]

Equation (1.48)

noting

\[
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -[(\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}]
\]

to derive the second equation in equation (1.48), while the first equation in equation (1.48) is exploited to the second term in this equation. Besides, equation (1.48) can be derived easily using equation (1.56) below:

\[
\begin{align*}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= a_1 e_1 \times \epsilon_{pqr} b_q c_r e_p = \epsilon_{pqr} a_i b_q c_r e_i \times e_p = \epsilon_{pqr} a_i b_q c_r \epsilon_{ipk} e_k \\
&= \epsilon_{pqr} \epsilon_{ipk} a_i b_q c_r e_k = (\delta_{qk} \delta_{ri} - \delta_{qi} \delta_{rk}) a_i b_q c_r e_k = a_i c_r b_k e_k - a_i b_k c_r e_k. 
\end{align*}
\]

By virtue of equation (1.48), we have

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}
\]

because of

\[
\begin{align*}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\
\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) &= (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \\
\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}
\end{align*}
\]

Furthermore, it follows, noting equation (1.48), that

\[
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})] = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})
\]

(1.51)

\[
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\|\mathbf{a}\|\|\mathbf{b}\|)^2 - (\mathbf{a} \cdot \mathbf{b})^2.
\]

(1.52)

noting the following equation obtained by \( \mathbf{x} \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{x} \times \mathbf{c}) \cdot \mathbf{d} \) with \( \mathbf{x} = \mathbf{a} \times \mathbf{b} \):

\[ (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}] \cdot \mathbf{d} = \{\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\} \cdot \mathbf{d}. \]

Moreover, we can derive

\[
(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \left\{ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}) \mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \mathbf{d} = [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d} \\
((\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}) \mathbf{b} - ((\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b}) \mathbf{a} = [\mathbf{cda}]\mathbf{b} - [\mathbf{cdb}]\mathbf{a}
\right\}
\]

from equation (1.48), leading to

\[
[\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d} = [\mathbf{cda}]\mathbf{b} - [\mathbf{cdb}]\mathbf{a},
\]

(1.53)
which in the special case of $c = b$ and $d = c$ reduces to

$$ (a \times b) \times (b \times c) = [abc]b $$

An arbitrary vector $v$ is represented by

$$ v = \frac{[bev]a + [cav]b + [abv]c}{[abc]} $$

by solving the following equation for $v$:

$$ [abv]c - [abc]v = [cva]b - [cvb]a, $$

which is obtained from equation (1.53) with the replacement of $d$ by $v$.

The left- and right-hand sides of equation (1.51) are described in component form as follows:

$$ (a \times b) \cdot (c \times d) = \varepsilon_{ijk} a_j b_k \varepsilon_{rs} c_r d_s, $$

$$ (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) = \delta_{jr} a_j c_r \delta_{ks} b_k d_s - \delta_{js} a_j d_s \delta_{kr} b_k c_r = (\delta_{jr} \delta_{ks} - \delta_{js} \delta_{kr}) a_j b_k c_r d_s. $$

Equating these equations, we obtain

$$ \varepsilon_{ijk} \varepsilon_{irs} = \varepsilon_{jki} \varepsilon_{rsi} = \delta_{jr} \delta_{ks} - \delta_{js} \delta_{kr}. $$

Setting $r = j$, equation (1.56) reduces to

$$ \varepsilon_{ijk} \varepsilon_{ijr} = 2 \delta_{ks}. $$

Furthermore, setting $s = k$, equation (1.57) reduces to

$$ \varepsilon_{ijk} = 2 \delta_{kk} = 3! = 6. $$

### 1.3.5 Reciprocal Vectors

The three vectors $a', b', c'$ defined by the following equations, are called the **reciprocal vectors** of the primary vectors $a, b, c$:

$$ a' = \frac{b \times c}{[abc]}, \quad b' = \frac{c \times a}{[abc]}, \quad c' = \frac{a \times b}{[abc]} $$

The inverse expression of equation (1.59) is given by

$$ a = \frac{b' \times c'}{[a'b'c']}, \quad b = \frac{c' \times a'}{[a'b'c']}, \quad c = \frac{a' \times b'}{[a'b'c']}, $$

noting equation (1.54). These vectors fulfill the relations:

$$ a \cdot a' = b \cdot b' = c \cdot c' = 1, $$

$$ a' \cdot b = a' \cdot c = b' \cdot c = b' \cdot a = c' \cdot a = c' \cdot b = 0, $$

so that the normality relations $a' \perp b$, $a' \perp c$, $b' \perp c$, $c' \perp a$, $c' \perp b$ hold. In the particular case of the primary triad $(a, b, c)k$ where $i, j, k$ are the orthonormal unit vectors, the reciprocal triad is given by $(1/a)i$, $(1/b)j$, $(1/c)k$.
Using the reciprocal vectors, we can write an arbitrary vector $v$ from equations (1.55) and (1.59) as

$$v = (v \cdot a')a + (v \cdot b')b + (v \cdot c')c. \tag{1.64}$$

It is known that the vector is represented by the assembly of primary vectors with the projections onto directions of the reciprocal vectors.

Further, it follows from equations (1.50) and (1.59) that

$$a \times a' + b \times b' + c \times c' = 0. \tag{1.65}$$

1.3.6 Tensor Product

A second-order tensor is constructed by the dyadic (or tensor) product of two vectors, designated as $a \otimes b$ and possessing the following property for another vector $c$:

$$a \otimes bc = a(b \cdot c) \tag{1.66}$$

Therefore, $a \otimes bc$ is the vector possessing the direction of $a$, while $b$ is projected in the direction of $c$ resulting in a scalar. Here, we have

$$(a \otimes b)^T = b \otimes a, \tag{1.67}$$

$$a \otimes (b + c) = a \otimes b + a \otimes c. \tag{1.68}$$

$$b \otimes c - c \otimes b) = a \times (b \times c) = (a \cdot c)b - (a \cdot b)c. \tag{1.69}$$

The vector product is represented in the direct notation of the alternating tensor as follows:

$$a \times b = \varepsilon : (a \otimes b), \quad e_i \times e_j = \varepsilon : (e_i \otimes e_j) \tag{1.70}$$

because of $a \times b = \varepsilon_{ijk} a_j b_k e_i = \varepsilon_{ijk} a_1 (e_j \cdot e_1) b_k (e_k \cdot e_1) = (\varepsilon_{ijk} e_i \otimes e_j \otimes e_k) \cdot (a_1 e_r \otimes b_k e_s)$, where $\varepsilon = \varepsilon_{ijk} e_i \otimes e_j \otimes e_k$ and $\varepsilon : T$ for $\varepsilon_{ijk} T_{jk}$.

1.4 Tensor Analysis

Various tensors and their algebra are addressed in this section. They are used often throughout this book.

1.4.1 Properties of Second-Order Tensor

The general definition of a tensor was given in Section 1.2. Here, based on that definition, basic properties of second-order tensors are described below.

Two tensors $A$ and $B$ are same when they yield the same transformation of an arbitrary vector $v$, that is,

$$A = B \text{ when } Av = Bv. \tag{1.71}$$
The following equations hold by virtue of the linear transformation in equation (1.13):

\[ T(\alpha a + \beta b) = \alpha (Ta) + \beta (Tb), \quad (1.72) \]
\[ A + B = B + A, \quad (1.73) \]
\[ s(AB) = (sA)B = A(sB), \quad (1.74) \]
\[ (A + B)v = Av + Bv, \quad (1.75) \]
\[ (AB)v = A(Bv) \quad (1.76) \]
\[ A(B + C) = AB + AC, \quad (A + B)C = AC + BC, \quad (1.77) \]
\[ A(BC) = (AB)C, \quad (1.78) \]

where \(a, b\) and \(s\) are arbitrary scalar variables.

The magnitude of a tensor is given by

\[ ||T|| = \sqrt{T_{rs} T_{rs}} = \sqrt{T : T} = \sqrt{\text{tr}(T T^T)}. \quad (1.79) \]

Here, \(\text{tr}(\cdot)\) is called the \textit{trace} and is defined by the scalar product with the identity tensor, that is,

\[ \text{tr}T \equiv T : I = T_{ii}, \quad (1.80) \]

which fulfills

\[ \text{tr}T^T = \text{tr}T, \quad \text{tr}(AB) = \text{tr}(BA), \quad \text{tr}(AB) = A : B^T, \quad \text{tr}(a \otimes b) = a \cdot b, \quad (1.81) \]

where \(I\) is the identity tensor, that is, \(I = \delta_{ij} e_i \otimes e_j\).

Powers of tensors are defined as follows:

\[ T^0 = I, \quad T^n = T \cdots T. \quad (1.82) \]

These obey the rules of \textit{exponentiation}:

\[ T^m T^n = T^{m+n} = T^n T^m, \quad (aT)^n = a^n T^n, \quad (T^m)^n = T^{mn} \quad (1.83) \]

for arbitrary integers \(m\) and \(n\).

The zero tensor \(O\) and the identity tensor \(I\) transform an arbitrary vector \(v\) to the zero vector \(0\) and the vector \(v\) to itself, respectively, that is,

\[ Ov = 0, \quad Iv = v. \quad (1.84) \]

\subsection*{1.4.2 Tensor Components}

Equation (1.29) describes a vector in terms of base vectors. We also require the expression for a tensor referring to base vectors. We obtain the following expression of the second-order tensor in terms of the components on the base vectors.

\[ T = T_{ij} e_i \otimes e_j, \quad T_{ij} = e_i \cdot Te_j \quad (1.85) \]
based on
\[ T = TI = T(\delta_{ij}e_i \otimes e_j) = Te_j \otimes e_j = (Te_j \cdot e_i) e_i \otimes e_j \tag{1.86} \]
noting the expression \( v = (v \cdot e_i)e_i \) for arbitrary vector \( v \) which is chosen as \( v = Te_j \) in equation (1.86) which is the expression for a second-order tensor in terms of base vectors.

### 1.4.3 Transposed Tensor

The tensor \( T^T \) satisfying the following equation for arbitrary vectors \( a \) and \( b \) is defined as the transposed tensor of a tensor \( T \):
\[ a \cdot Tb = b \cdot T^T a \tag{1.87} \]

The relations \( T^T = T \) and \( T^{TT} = -T \) hold for a symmetric tensor and a skew-symmetric tensor, respectively (see Section 1.4.6).

In the rectangular Cartesian coordinate system, the particular selection of \( a = e_i, b = e_j \) in equation (1.87) reads:
\[
\begin{align*}
  e_i \cdot Te_j &= e_i \cdot T_{ir}e_r \otimes e_s e_j = T_{ij} \\
  e_j \cdot T^T e_i &= e_j \cdot (T^T)_{ir} e_r \otimes e_s e_i = T^T_{ji}
\end{align*}
\]

Designating \( T^T_{ij} \equiv (T^T)_{ij} \) for the sake of simplicity. Equating these components, we have
\[ T^T_{ij} = T_{ji} \tag{1.88} \]

It follows that
\[ T^T = (T_{ij}e_i \otimes e_j)^T = T_{ji}e_j \otimes e_i \tag{1.89} \]

The transposed tensor of \( T = T_{ij}e_i \otimes e_j \) can also be written \( T^T = T_{ji}e_j \otimes e_i \) in the rectangular Cartesian coordinate system. However, the transposed tensor of the mix of covariant and contravariant bases and components in the general curvilinear coordinate system is given by a tensor which possesses the same components but exchanged base vectors, as will be addressed in Chapter 2.

The following equations hold for a transposed tensor:
\[ ||T^T|| = ||T||, \tag{1.90} \]
\[ \text{tr}T^T = \text{tr}T, \quad \text{tr}(AB)^T = \text{tr}(B^TA^T), \tag{1.91} \]
\[ v^T = I^T v \quad (v_j T_{ji} = T_{ji}v_j), \tag{1.92} \]
\[ a \cdot Tb = T^T a \cdot b \quad (a_i(T_{ir}b_r) = (T_{ir}a_i)b_r), \tag{1.93} \]
\[ a \cdot b^T = Ta \cdot b \quad (a_i(b_rT_{ri}) = (T_{ri}a_i)b_r), \tag{1.94} \]
1.4.4 Inverse Tensor

A tensor $T$ fulfilling $\det T \neq 0$ is called a non-singular tensor, for which there exists a tensor, called the inverse tensor and designated by $T^{-1}$, satisfying the relation

$$TT^{-1} = T^{-1}T = I,$$

where the components are denoted as $T_{ij}^{-1} = (T^{-1})_{ij}$ for the sake of simplicity. It follows that:

$$(AB)^{-1} = B^{-1}A^{-1}$$

because of

$$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I.$$
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Therefore, an orthogonal transformation leaves the magnitude of a vector unchanged. This fact, along with equation (1.100), means an orthogonal transformation also leaves the angle formed by vectors unchanged.

The magnitude of the orthogonal tensor is given from equation (1.102) as

$$||Q|| = \sqrt{3},$$

(1.105)

noting that $||Q|| = \sqrt{\text{tr}(QQ^T)} = \sqrt{\text{tr}I}$.

As well as leaving the scalar product of two vectors unchanged (see equation (1.100)), an orthogonal transformation also leaves the trace of two tensors unchanged, that is,

$$\text{tr}((QAQ^T)(QBQ^T)) = \text{tr}(AB).$$

(1.106)

Furthermore, in addition to leaving the magnitude of a vector unchanged (see equation (1.104)), an orthogonal transformation also leaves the magnitude of a tensor unchanged, that is,

$$||QTQ^T|| = ||T||,$$

(1.107)

noting equations (1.81), (1.102) and

$$\sqrt{\text{tr}(QTQ^T(QTQ^T)^T)} = \sqrt{\text{tr}(QTQ^TQT^TQT^T)} = \sqrt{\text{tr}(TT^T)}.$$}

Then, the following tensor possessing the components $Q_{ij}$ in equation (1.14) satisfies equation (1.16), the direct notation of which is identical to equation (1.102) and thus it is an orthogonal tensor:

$$Q = Q_{ij}e_i \otimes e_j = Q_{ij}e^*_i \otimes e^*_j = e_i \otimes e^*_i$$

(1.108)

noting that

$$Q_{ij}e_i \otimes e_j = e_i \otimes (e^*_j \cdot e_j)e_j = e_i \otimes e^*_i = (e_i \cdot e^*_j)e^*_i \otimes e^*_j = Q_{ij}e^*_i \otimes e^*_j$$

by use of equation (1.17). Furthermore, because of

$$e_i = e_i \delta_{ir} = e_r \otimes e^*_r e^*_i, \quad e^*_i = e^*_r \delta_{ir} = e^*_r \otimes e_r e_i,$$

one has the expressions

$$e_i = Q e^*_i, \quad e^*_i = Q^T e_r.$$  

(1.109)

An orthogonal tensor is expressed in matrix form as follows:

$$[Q_{ij}] = \begin{bmatrix}
\cos \phi \cos \theta - \cos \phi \cos \theta \sin \phi & \cos \phi \sin \theta + \cos \phi \cos \sin \phi & \sin \phi \sin \theta \\
-\sin \phi \cos \theta - \cos \phi \cos \theta \sin \phi & \sin \phi \sin \theta + \cos \phi \cos \sin \phi & \cos \phi \sin \theta \\
\sin \phi \sin \theta & \sin \phi \cos \theta & \cos \phi 
\end{bmatrix},$$

(1.110)

where $\theta, \phi, \varphi$ are the Euler angles shown in Figure 1.1.
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Figure 1.1 Rotation of coordinate system shown by Euler angles

The rotational transformation tensor $Q^{(i)}$ around the base vector $e_i$ is given from equation (1.110) as follows:

$$Q^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix},$$
$$Q^{(2)} = \begin{bmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{bmatrix},$$
$$Q^{(3)} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.\quad (1.111)$$

Hereinafter, consider only $Q^{(3)}$ and denote it simply by $Q$, that is,

$$Q = Q_{11}e_1 \otimes e_1 + Q_{12}e_1 \otimes e_2 + Q_{21}e_2 \otimes e_1 + Q_{22}e_2 \otimes e_2 + e_3 \otimes e_3$$
$$= \cos \theta e_1 \otimes e_1 + \sin \theta e_1 \otimes e_2 - \sin \theta e_2 \otimes e_1 + \cos \theta e_2 \otimes e_2 + e_3 \otimes e_3$$
$$= (e_1 \otimes e_1 + e_2 \otimes e_2) \cos \theta + (e_1 \otimes e_2 - e_2 \otimes e_1) \sin \theta + e_3 \otimes e_3\quad (1.112)$$

or

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad (1.113)$$

for which we can write

$$\{e^*_1\} = \begin{bmatrix} Q_{11}e_1 \\ Q_{21}e_1 \\ Q_{31}e_1 \end{bmatrix}, \quad \{e^*_2\} = \begin{bmatrix} Q_{12}e_1 \\ Q_{22}e_1 \\ Q_{32}e_1 \end{bmatrix}, \quad \{e^*_3\} = \begin{bmatrix} Q_{13}e_1 \\ Q_{23}e_1 \\ Q_{33}e_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}\quad (1.114)$$

(see Figure 1.2). Equation (1.112) is referred to as the canonical expression for orthogonal tensors.
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The orthogonal tensor causing a $\pi$-rotation is given by

$$Q^\pi_i = 2e_i \otimes e_i - I \text{ (no sum),}$$

that is,

$$Q_1^\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad Q_2^\pi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_3^\pi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(1.116)

by which an arbitrary vector $v$ is $\pi$-rotated around the axis $e_i$.

The orthogonal tensor causing a reversal rotation, that is, reflection, is given by

$$Q^R_i = I - 2e_i \otimes e_i \text{ (no sum),}$$

that is,

$$Q_1^R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_2^R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_3^R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(1.118)

by which an arbitrary vector $v$ is reflected in the plane perpendicular to the base vector $e_i$.

1.4.6 Tensor Decompositions

Several types of tensor decomposition are often used for convenience, as will be described below.
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Symmetric and skew-symmetric tensors

The tensor \( T \) is additively decomposed into the \textit{symmetric tensor} \( S \) and the \textit{skew-symmetric (or anti-symmetric) tensor} \( \Omega \) as follows:

\[
T = S + \Omega,
\]

where

\[
S \equiv \text{sym}[T] \equiv \frac{1}{2}(T + T^T), \quad \Omega \equiv \text{ant}[T] \equiv \frac{1}{2}(T - T^T)
\]

which satisfy

\[
S^T = S, \quad \Omega^T = -\Omega, \quad a \cdot Sb = b \cdot Sa, \quad a \cdot (\Omega b) = -b \cdot (\Omega a)
\]

noting equation (1.87). Equation (1.119) is called the \textit{Cartesian decomposition}. Here, it follows that

\[
SS^T = S^2, \quad \text{tr}(S\Omega) = \text{tr}(S\Omega^T) = 0
\]

because of \( \text{tr}(S\Omega) = \text{tr}((S\Omega)^T) = \text{tr}(\Omega^T S) = -\text{tr}(\Omega S) = -\text{tr}(S\Omega) \), noting equation (1.91).

The skew-symmetric tensor fulfills

\[
\text{tr}\Omega = 0, \quad a \cdot (\Omega a) = 0,
\]

exploiting equations (1.81)_1, (1.87), and (1.121)_2.

By virtue of equation (1.124), it follows that

\[
\text{tr}(AB) = \text{tr}(\text{sym}[A]\text{sym}[B]) + \text{tr}(\text{ant}[A]\text{ant}[B]).
\]

Spherical and deviatoric tensors

The tensor \( T \) can decomposed as follows:

\[
T = T_m + T',
\]

\[
T_m \equiv T_m I, \quad T_m \equiv \frac{1}{3}\text{tr}T, \quad T' \equiv T - T_m I \quad (\text{tr}T' = 0),
\]

where \( T_m \) and \( T' \) are called the \textit{spherical (or mean) part} and the \textit{deviatoric part}, respectively, of the tensor \( T \). A prime (\( \cdot \)' ) is used for the deviatoric part throughout this book.

Normal and tangential tensors: Projection tensors

The vector \( v \) can be decomposed as follows:

\[
v = v_n + v_t,
\]
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where

\[ v_{n} \equiv P_{n}v, \quad v_{t} \equiv P_{t}v \]

with

\[
\begin{align*}
P_{n} & \equiv n \otimes n \\
P_{t} & \equiv I - P_{n} = I - n \otimes n
\end{align*}
\]

\[ (1.130) \]

\[ (1.131) \]

\( n \) being an arbitrary unit vector. \( P_{n} \) and \( P_{t} \) are called the second-order normal projection tensor and tangential projection tensor, respectively, and fulfill

\[ P_{n}^{2} = P_{n}, \quad P_{n}P_{t} = 0. \]

\[ (1.132) \]

\( v_{n} \) and \( v_{t} \) are the projection of the vector \( v \) onto the vector \( n \) and the orthogonal projection of \( v \) onto the plane normal to the vector \( n \), respectively. Here, \( v_{n} \) and \( v_{t} \) are also interpreted as the vector decompositions into the normal part and the tangential part of the plane normal to the vector \( n \).

Similarly, the second-order tensor \( T \) can be decomposed as follows:

\[ T = T_{n} + T_{t}, \]

\[ (1.133) \]

where

\[ T_{n} \equiv \mathcal{P}_{n}T, \quad T_{t} \equiv \mathcal{P}_{t}T \]

\[ (1.134) \]

with

\[
\begin{align*}
\mathcal{P}_{n} & \equiv N \otimes N \\
\mathcal{P}_{t} & \equiv \mathcal{T} - \mathcal{P}_{n} = \mathcal{T} - N \otimes N
\end{align*}
\]

\[ (1.135) \]

\( N \) being an arbitrary second-order unit tensor and \( \mathcal{T} = \delta_{ik}\delta_{jl}e_{i} \otimes e_{j} \otimes e_{k} \otimes e_{l} \) being the fourth-order identity tensor which will be described in Section 1.5.4. The fourth-order tensors \( \mathcal{P}_{n} \) and \( \mathcal{P}_{t} \) are called also the normal-projection tensor and the tangential-projection tensor, respectively, and fulfill

\[ \mathcal{P}_{n}^{2} = \mathcal{P}_{n}, \quad \mathcal{P}_{n}\mathcal{P}_{t} = 0. \]

\[ (1.136) \]

1.4.7 Axial Vector

The anti-symmetric tensor fulfills the following properties as known by comparing equations (1.85) and (1.122) regarding \( a \) and \( b \) as the base vectors.

1. Corresponding non-diagonal components have the same absolute value but opposite sign.
2. Diagonal components are zero and thus eigenvalues are zero. Therefore, the general anti-symmetric tensor \( \Omega \) is given by

\[
\begin{bmatrix}
\Omega_{ij}
\end{bmatrix} = \begin{bmatrix}
0 & \Omega_{12} & \Omega_{13} \\
0 & 0 & \Omega_{23} \\
\text{ant.} & &
\end{bmatrix}.
\]

\[ (1.137) \]
The anti-symmetric tensor thus possesses only three components, and we can infer that it can be related to a vector uniquely. We examine it below.

The axial vector $\omega$ is defined as the vector which fulfills the following equation for the skew-symmetric tensor $\Omega$ and an arbitrary vector $a$:

$$\Omega a = \omega \times a.$$  \hfill (1.138)

Choosing $a$ as the base vector $e_j$ and making the scalar product with $e_i$, we find

$$\Omega_{ij} = e_i \cdot \Omega e_j = e_i \cdot \omega \times e_j = \varepsilon_{pqr} (e_i)_p \omega_q (e_j)_r = \varepsilon_{pqr} \delta_{ip} \delta_{jq} \delta_{jr} = \varepsilon_{ijr} \omega_q = -\varepsilon_{ijr} \omega_q,$$

that is,

$$\Omega_{12} = -\omega_3, \quad \Omega_{23} = -\omega_1, \quad \Omega_{13} = \omega_2.$$  \hfill (1.139)

Then $\Omega$ is expressed as follows:

$$\Omega = -\varepsilon \omega = -\varepsilon_{ijk} e_i \otimes e_j \otimes e_k$$  \hfill (1.140)

that is,

$$[\Omega_{ij}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ 0 & 0 & -\omega_1 \\ \text{ant.} & 0 \end{bmatrix},$$  \hfill (1.141)

noting that

$$\Omega = -\varepsilon \omega = \varepsilon_{ijk} e_i \otimes e_j \otimes e_k.$$  \hfill (1.142)

The inverse relation of equation (1.140) is given by

$$\omega = -\frac{1}{2} \varepsilon \Omega = -\frac{1}{2} \Omega_{ij} e_i \times e_j = -\frac{1}{2} e_{ij} \Omega_{ij} e_i$$  \hfill (1.143)

because of $\omega_p e_p = \delta_{pq} \omega_q e_p = \varepsilon_{ijk} \delta_{ij} \omega_k e_p / 2 = -\varepsilon_{ijk} \Omega_{ij} e_p / 2 = -\Omega_{ij} e_i \times e_j / 2$ by virtue of equation (1.57).

It is follows from equation (1.140) that

$$\Omega = \mathbf{1} \times \omega = \omega \times \mathbf{1}$$  \hfill (1.144)

because of

$$-\varepsilon_{ijk} \omega_k e_i \otimes e_j = \begin{cases} e_i \otimes \varepsilon_{ijk} \omega_k e_j = e_i \otimes e_j \times \omega_k e_k = e_j \otimes e_i \times \omega \\
-\varepsilon_{ijk} \omega_k \otimes e_j = e_j \omega_k \times e_k \otimes e_j = \omega \times e_j \otimes e_j \end{cases},$$

using equations (1.32) and (1.201) shown later.

It follows in view of equation (1.48)2 that

$$(u \otimes v - v \otimes u)a = (v \cdot a)u - (u \cdot a)v = (v \times u) \times a.$$  \hfill (1.145)

Thus, by virtue of equation (1.138), $v \times u$ is the axial vector of the anti-symmetric second-order tensor $u \otimes v - v \otimes u$. 
Figure 1.3  Axial vector in case of rotation

The relation of the skew-symmetric tensor and the axial vector is illustrated in Figure 1.3 for the rotation phenomenon, regarding \( \omega \) as the angular velocity and \( \mathbf{v} \) as the peripheral velocity.

Now consider rotation in the Cartesian coordinate system. The time-differentiation of the base vector \( \mathbf{e}_i^* \) in equation (1.109) leads to

\[
\mathbf{\dot{e}}_i^* = \dot{Q}^T \mathbf{e}_i + Q^T \mathbf{\dot{e}}_i = \dot{Q}^T Q \mathbf{e}_i^* + Q^T \mathbf{\dot{e}}_i. \tag{1.146}
\]

If we put \( \mathbf{\dot{e}}_i = \mathbf{0} \) in equation (1.146), we have the variation of \( \mathbf{e}_i^* \) observed from the coordinate system with the base \( \{\mathbf{e}_i\} \), that is,

\[
\mathbf{\dot{e}}_i^* = \mathbf{\Omega} \mathbf{e}_i^*, \tag{1.147}
\]

where

\[
\mathbf{\Omega} \equiv \dot{Q}^T Q = \dot{\mathbf{e}}_i^* \otimes \mathbf{e}_i^*, \quad \Omega_{ij} = (\mathbf{\dot{e}}_i^* \cdot \mathbf{e}_j)(\mathbf{\dot{e}}_j^* \cdot \mathbf{e}_i). \tag{1.148}
\]

\( \mathbf{\Omega} \) designates the spin of the base vector \( \mathbf{e}_i^* \) observed from the coordinate system with the base \( \{\mathbf{e}_i\} \), bearing in mind that only the direction is changeable in the base vector because of the unit vector. \( \mathbf{\dot{e}}_i^* \) can be rewritten as

\[
\mathbf{\dot{e}}_i^* = \omega \times \mathbf{e}_i^*, \tag{1.149}
\]

by virtue of the property in equation (1.138), where \( \omega \) is the angular velocity of \( \mathbf{e}_i^* \), which is related to \( \mathbf{\Omega} \) as

\[
\omega = -\frac{1}{2} \mathbf{\varepsilon} \mathbf{\Omega} = -\frac{1}{2} \Omega_{ij} \mathbf{e}_i \times \mathbf{e}_j = -\frac{1}{2} \delta_{irs} \Omega_{rs} \mathbf{e}_i, \tag{1.150}
\]

noting equation (1.143), where \( \Omega_{rs} \) is specified by equation (1.148)_2 (see Figure 1.4).
Figure 1.4 Rotation of coordinate system with base \{e_i^*\} (illustrated in two-dimensional state $e_3 = e_3^*$)

1.4.8 Determinant

The determinant is defined by

$$\det T = |T_{ij}| = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = \frac{1}{3!} \varepsilon_{abc} \varepsilon_{pqr} T_{ap} T_{bq} T_{cr} \tag{1.151}$$

where the last term is divided by $3!(=6)$ which is the total number of permutations of thee numbers. Equation (1.151) can be rewritten as

$$\det T = \frac{1}{3} T_{ap} (\text{cof} T)_{ap}, \quad \det T = \frac{1}{3} \text{tr} \{T(\text{cof} T)^T\}, \tag{1.152}$$

where cof $T$ is the cofactor defined by

$$\text{(cof} T)_{ap} = \frac{1}{2} \varepsilon_{abc} \varepsilon_{pqr} T_{bq} T_{cr} \tag{1.153}$$

The cofactor is defined through multiplying the minor determinant lacking the $i$th row and $j$th column components by the sign $(-1)^{i+j}$.

Now, note that the following equation satisfies equation (1.152).

$$\frac{1}{3} (\det T) I = \frac{1}{3} T (\text{cof} T)^T, \quad \frac{1}{3} (\det T) \delta_{ij} = \frac{1}{3} T_{ip} (\text{cof} T)_{jp}. \tag{1.154}$$

Equation (1.154) can be rewritten as

$$T (\text{cof} T)^T = (\det T) I, \quad T_{ip} (\text{cof} T)_{jp} = (\det T) \delta_{ij} \tag{1.155}$$
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from which we have

\[ T = (\det T)(\text{cof } T)^{-T}, \quad T_{ij} = (\det T)(\text{cof } T)_{ji}^{-1}, \]  

(1.156)

\[ \text{cof } T = (\det T)T^{-T}, \quad (\text{cof } T)_{ij} = (\det T)T_{ji}^{-1} \]  

(1.157)

and

\[
\begin{bmatrix}
\begin{matrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{matrix}
\end{bmatrix}^{-1} = \frac{(\text{cof } T)^{T}}{\det T}, \quad T_{ij}^{-1} = \frac{(\text{cof } T)_{ji}}{\det T}.
\]

(1.158)

Equation (1.158) is represented for the 2 \times 2 and 3 \times 3 matrices as follows:

\[
\begin{bmatrix}
T^{-1} = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\end{bmatrix},
\]

(1.160)

\[
\begin{bmatrix}
T^{-1} = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{bmatrix}
\end{bmatrix}.
\]

(1.158)'

We obtain the following equations:

\[
\begin{align*}
\det[-T] &= -\det T, \quad \det(sT) = s^3\det T, \quad \det T = \det T^T, \\
\det[AB] &= \det A\det B, \quad \det[T^t] = (\det T)^t, \quad \det(\exp T) = \exp(\det T), \\
\det[a \otimes b] &= 0, \\
\det[T^{-1}] &= (\det T)^{-1}
\end{align*}
\]

(1.159)

(1.160)

(1.161)

by virtue of the following:

\[
\begin{align*}
\det[-T] &= \epsilon_{pqrs}T_{1p}(-1)T_{2q}(-1)T_{3r} = -\epsilon_{pqrs}T_{1p}T_{2q}T_{3r}, \\
\det[sT] &= \epsilon_{pqrs}T_{1p}T_{2q}sT_{3r} = s^3\epsilon_{pqrs}T_{1p}T_{2q}T_{3r}, \\
\det T &= \frac{1}{3!}\epsilon_{abc}\epsilon_{pqrs}T_{ap}T_{bq}T_{cr} = \frac{1}{3!}\epsilon_{abc}\epsilon_{pqrs}T_{pa}T_{qb}T_{rc} = \frac{1}{3!}\epsilon_{abc}\epsilon_{pqrs}T_{pa}T_{qb}T_{rc}, \\
\det[AB] &= \epsilon_{pqrs}(A_{1a}B_{ap})(A_{2b}B_{bq})(A_{3c}B_{cr}) = \epsilon_{pqrs}A_{1a}A_{2b}A_{3c}B_{ap}B_{bq}B_{cr} = A_{1a}A_{2b}A_{3c}\epsilon_{abc}\det B,
\end{align*}
\]

\[
\begin{align*}
\det(\exp T) &= \begin{vmatrix}
\exp(T_{11}) & 0 & 0 \\
0 & \exp(T_{22}) & 0 \\
0 & 0 & \exp(T_{33})
\end{vmatrix} = \exp(T_{11})\exp(T_{22})\exp(T_{33})
\end{align*}
\]

(spectral representation),

\[
\begin{align*}
\det(a \otimes b) &= \epsilon_{ijk}(a_1b_i)(a_2b_j)(a_3b_k) = a_1a_2a_3\epsilon_{ijk}b_ib_jb_k = (a_1a_2a_3)(b \times b) \cdot b, \\
\det T \det[T^{-1}] &= \det[TT^{-1}] = 1.
\end{align*}
\]

An alternative proof of equation (1.159) of will be given later by applying equation (1.371) for the derivative of the determinant.
The following equations hold for the cofactor, noting equation (1.157) together with equation (1.161).

\[
\begin{align*}
\text{cof}(T) &= s^2 \text{cof} T \\
(\text{cof} T)^T &= \text{cof}(T^T), \quad (\text{cof} T)^{-1} = \text{cof}(T^{-1}), \quad (\text{cof} T)^{-T} = \text{cof}(T^{-T}) \\
\text{cof}(AB) &= \text{cof}(A)\text{cof}(B) \\
\text{tr}(\text{cof} T) &= \frac{1}{2}(\text{tr}^2 T - \text{tr} T^2) = II \\
\det(\text{cof} T) &= 1
\end{align*}
\]

\text{(1.162)}

noting
\[
\text{tr}(\text{cof} T) = \frac{1}{2}\varepsilon_{abc}\varepsilon_{apq} T_{bq} T_{cr} = \frac{1}{2}(\delta_{bq}\delta_{cr} - \delta_{br}\delta_{cq}) T_{bq} T_{cr} = \frac{1}{2}(T_{qq} T_{rr} - T_{rq} T_{qp}).
\]

II is called the second principal invariant as will be described in Section 1.6.1.

The vector product in equation (1.31) and the scalar triple product in equation (1.39) are described in terms of the determinant as follows:

\[
a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} e_1 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} e_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} e_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},
\]

\text{(1.163)}

\[
||a \times b|| = \sqrt{(a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2},
\]

\text{(1.164)}

\[
[abc] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a \cdot e_1 & a \cdot e_2 & a \cdot e_3 \\ b \cdot e_1 & b \cdot e_2 & b \cdot e_3 \\ c \cdot e_1 & c \cdot e_2 & c \cdot e_3 \end{vmatrix},
\]

\text{(1.165)}

\[
(a \times b) \cdot (c \times d) = \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix},
\]

\text{(1.166)}

\[
[abc][pqr] = \begin{vmatrix} a_1 & a_2 & a_3 & p_1 & q_2 & r_3 \\ b_1 & b_2 & b_3 & p_2 & q_2 & r_3 \\ c_1 & c_2 & c_3 & p_3 & q_2 & r_3 \end{vmatrix} = \begin{vmatrix} a \cdot p & a \cdot q & a \cdot r \\ b \cdot p & b \cdot q & b \cdot r \\ c \cdot p & c \cdot q & c \cdot r \end{vmatrix},
\]

\text{(1.167)}

noting equation (1.159). The following equation is derived as a special case of equation (1.167) for the three vectors \(v_1, v_2, v_3\):

\[
v^2 = [v_1 v_2 v_3]^2 = \begin{vmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & v_1 \cdot v_3 \\ v_2 \cdot v_1 & v_2 \cdot v_2 & v_2 \cdot v_3 \\ v_3 \cdot v_1 & v_3 \cdot v_2 & v_3 \cdot v_3 \end{vmatrix},
\]

\text{(1.168)}

where

\[
v = [v_1 v_2 v_3] = \varepsilon_{ijk}(v_1)_i (v_2)_j (v_3)_k = \varepsilon_{ijk}(v_1 \cdot e_i) (v_2 \cdot e_j) (v_3 \cdot e_k).
\]

\text{(1.169)}
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is the volume of the parallelepiped formed by the line-elements \( v_1, v_2, v_3 \). Introducing the symbol

\[ v_{ij} \equiv v_i \cdot v_j, \]  
(1.170)
equation (1.168) can be written as

\[ v^2 = \begin{vmatrix} v_{11} & v_{12} & v_{13} \\ v_{22} & v_{23} & \text{sym.} \\ v_{33} \end{vmatrix} = \det[v_{ij}]. \]  
(1.171)

It follows from equation (1.167) that

\[ \det T = [Te_1 Te_2 Te_3] = Te_1 \cdot (Te_2 \times Te_2), \]  
(1.172)
noting that

\[ \det T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} (e_1 \cdot Te_1) & (e_1 \cdot Te_2) & (e_1 \cdot Te_3) \\ (e_2 \cdot Te_1) & (e_2 \cdot Te_2) & (e_2 \cdot Te_3) \\ (e_3 \cdot Te_1) & (e_3 \cdot Te_2) & (e_3 \cdot Te_3) \end{bmatrix}. \]

Applying equations (1.159) to equations (1.102) and (1.121), the determinants of the orthogonal tensor and the skew-symmetric tensor are given as

\[ \det Q = \det Q^T = \pm 1 \]  
(1.173)
and

\[ \det \Omega = 0 \]  
(1.174)

1.4.9 On Solutions of Simultaneous Equation

The linear inhomogeneous simultaneous equation in the unknown variables \( x_r \),

\[ T_{ir} x_r = y_i \]  
(1.175)
(where the \( y_i \) are not all zero), possesses a unique solution only if the determinant of the coefficient \( T_{ij} \) is not zero, that is,

\[ \det[T_{ij}] \neq 0, \]  
(1.176)
noting equation (1.158).

On the other hand, the necessary and sufficient condition for the linear homogeneous simultaneous equation

\[ T_{ir} x_r = 0 \]  
(1.177)
to possess a non-zero (non-trivial) solution is given by

\[ \det[T_{ij}] = 0 \]  
(1.178)
as is known from \( x = T^{-1}0 = (\text{cof} T)^T 0 / \det T \) obtained by virtue of equation (1.158); this solution is not unique because all scalar multiplications of any solution are also solutions.
### 1.4.10 Scalar Triple Products with Invariants

The following formulae of the scalar triple products related to the principal invariants hold:

\[
[Ta \ b \ c] + [a \ Tb \ c] + [a \ b \ Tc] = \text{tr}T[abc] = I[abc]
\]

\[
[a \ Tb \ Tc] + [Ta \ b \ Tc] + [Ta \ Tb \ c] = \text{tr}(\text{cof}T)[abc] = II[abc]
\]

\[
[Ta \ Tb \ Tc] = \detT[abc] = III[abc]
\]

(1.179)

where I, II, III are principal invariants of \(T\), that is,

\[
I \equiv \text{tr}T, \quad II \equiv (\text{tr}^2T - \text{tr}T^2)/2, \quad III \equiv \detT
\]

(see Section 1.6.1).

The proof for equation (1.179) is given below (cf. Chadwick, 1976; Kyoya, 2008).

**Proof of equation (1.179)\(_1\):**

We have the relation

\[
[Ta, \ b, \ c] + [a, \ Tb, \ c] + [a, \ b, \ Tc] = a_i b_j c_k ([Te_i, \ e_j, \ e_k] + [e_i, \ Te_j, \ e_k] + [e_i, \ e_j, \ Te_k])
\]

\[
= e_{ijk} a_i b_j c_k ([Te_1, \ e_2, \ e_3] + [e_1, \ Te_2, \ e_3] + [e_1, \ e_2, \ Te_3])
\]

\[
= [abc]([Te_1, \ e_2, \ e_3] + [e_1, \ Te_2, \ e_3] + [e_1, \ e_2, \ Te_3]).
\]

(1.180)

making use of equation (1.43) and noting that

\[
[Te_i, \ e_j, \ e_k] + [e_i, \ Te_j, \ e_k] + [e_i, \ e_j, \ Te_k] = e_{ijk} ([Te_1, \ e_2, \ e_3] + [e_1, \ Te_2, \ e_3] + [e_1, \ e_2, \ Te_3]).
\]

(1.181)

Further, the terms in parentheses in the last equation in equation (1.180) are given by

\[
[Te_1, \ e_2, \ e_3] + [e_1, \ Te_2, \ e_3] + [e_1, \ e_2, \ Te_3] = [T_{11}e_1, \ e_2, \ e_3] + [e_1, \ T_{22}e_2, \ e_3] + [e_1, \ e_2, \ T_{33}e_3]
\]

\[
= T_{11}[e_1 e_2 e_3] + T_{22}[e_1 e_2 e_3] + T_{33}[e_1 e_2 e_3]
\]

\[
= T_{11} + T_{22} + T_{33} = \text{tr}T.
\]

(1.182)

Equation (1.179)\(_1\) is obtained by substituting equation (1.182) into equation (1.180).

**Proof of equation (1.179)\(_2\):**

Similarly to equation (1.180), we have first

\[
[a, \ Tb, \ Tc] + [Ta, \ b, \ Tc] + [Ta, \ Tb, \ c]
\]

\[
= e_{ijk} a_i b_j c_k ([e_1, \ Te_2, \ Te_3] + [Te_1, \ e_2, \ Te_3] + [Te_1, \ Te_2, \ e_3])
\]

\[
= [abc]([e_1, \ Te_2, \ Te_3] + [Te_1, \ e_2, \ Te_3] + [Te_1, \ Te_2, \ e_3]).
\]

(1.183)
Here, applying equations (1.43) and (1.181), the terms in parentheses are given by
\[
[e_1, Te_2, Te_3] + [Te_1, e_2, Te_3] + [Te_1, Te_2, e_3]
\]
\[
= T_{r_3}T_{s_3}[e_1, e_2, e_3] + T_{r_3}T_{s_3}[e_2, e_3] + T_{r_3}T_{s_3}[e_3, e_3]
\]
\[
= T_{r_3}T_{s_3}e_{r_3s_3} + T_{r_3}T_{s_3}e_{r_3s_3} + T_{r_3}T_{s_3}e_{r_3s_3}
\]
\[
= (T_{s_3}T_{r_3} - T_{r_3}T_{s_3}) + (T_{r_3}T_{s_3} - T_{s_3}T_{r_3}) + (T_{s_3}T_{r_3} - T_{r_3}T_{s_3})
\]
\[
= T_{s_3}T_{r_3} + T_{r_3}T_{s_3} + T_{s_3}T_{r_3} - T_{r_3}T_{s_3}
\]
\[
= (T_{s_3}T_{r_3} + T_{r_3}T_{s_3} - T_{r_3}T_{s_3} - T_{s_3}T_{r_3})/2
\]
leading to
\[
[e_1, Te_2, Te_3] + [Te_1, e_2, Te_3] + [Te_1, Te_2, e_3] = ((trT)^2 - tr^2T)/2.
\]  
Equation (1.179) is obtained by substituting equation (1.185) into equation (1.183).

Proof of equation (1.179): 
Changing to the component-based representations of the three vectors and then applying equations (1.43) and (1.181), we have
\[
[Te_1, Te_2, Te_3] = \epsilon_{ijk}a_{ij}b_{jk}c_{ik}[Te_1, Te_2, Te_3] = [abc][Te_1, Te_2, Te_3].
\]  
Equation (1.179) is obtained by substituting equation (1.172) into equation (1.186).

1.4.11 Orthogonal Transformation of Scalar Triple Product
Consider the set of three vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) which are arrayed as a right-handed set in this order fulfilling \( [\mathbf{a}, \mathbf{b}, \mathbf{c}] > 0 \) and the vectors \( \mathbf{a^*}, \mathbf{b^*}, \mathbf{c^*} \) transformed from \( \mathbf{a}, \mathbf{b}, \mathbf{c} \), respectively, by arbitrary orthogonal tensor \( Q \), fulfilling
\[
\mathbf{a^*} = Q\mathbf{a}, \quad \mathbf{b^*} = Q\mathbf{b}, \quad \mathbf{c^*} = Q\mathbf{c}
\]
\[
\mathbf{a} = Q^T\mathbf{a^*}, \quad \mathbf{b} = Q^T\mathbf{b^*}, \quad \mathbf{c} = Q^T\mathbf{c^*}
\]  
By virtue of equation (1.179), we find that
\[
[a^*b^*c^*] = [Q\mathbf{a}Q\mathbf{b}Q\mathbf{c}] = \det Q[abc]
\]  
which gives
\[
[abc] = [a^*b^*c^*] \quad \text{for } \det Q = 1
\]
\[
[abc] = -[a^*b^*c^*] \quad \text{for } \det Q = -1
\]  
noting equation (1.173). Therefore, the signs of \( [abc] \) and \( [a^*b^*c^*] \) are same and opposite for \( \det Q = 1 \) and \( \det Q = -1 \), respectively. In other words, the scalar triple product is transformed...
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to same as and opposite of the right- and left-handed set for \( \det Q = 1 \) and \( \det Q = -1 \), respectively. The orthogonal tensors with the plus and the minus determinants are called the proper orthogonal tensor and the improper orthogonal tensor, respectively.

The transformation of the scalar triple product of the orthogonal base vectors is described from equation (1.188) as

\[
[e^*_i e^*_j e^*_k] = \det Q [e_i e_j e_k]
\]

(1.190)

for the orthogonal tensor \( Q \) in equation (1.109) with equation (1.159).

1.4.12 Pseudo Scalar, Vector and Tensor

Directions and senses of quantities such as force, displacement and force are determined naturally. They obey the transformation rule in equation (1.9) and possess a plane of symmetry perpendicular to their direction such that quantities with different signs possess opposite senses with respect to the plane. On the other hand, quantities such as a surface, a volume, an angular moment and a spin do not possess such a sense. These quantities do not obey the transformation rule in equation (1.9) and possess a plane of symmetry parallel to their direction. Their sense is chosen by the right-hand screw rule. These quantities are called pseudo scalars, vectors, or tensors. Several examples are shown below.

The transformation rule for the scalar triple product is given by

\[
[a^* b^* c^*] = [Qa Qb Qc] = \det Q [abc],
\]

(1.191)

noting equation (1.188). The scalar product changes its sign for \( \det Q = -1 \) and thus it is called the pseudo scalar. Volume is regarded as a pseudo scalar.

The transformation of the vector product is given by

\[
a^* \times b^* = (\det Q)Q (a \times b),
\]

(1.192)

noting that

\[
(a^* \times b^*) = \varepsilon_{ijk} a^*_i b^*_j = \varepsilon_{ijk} Q_{jp} Q_{kq} a_p b_q = Q_{ij} Q_{kl} \varepsilon_{ijk} Q_{jp} Q_{kq} a_p b_q = Q_{ij} (\varepsilon_{ijk} Q_{kp} Q_{lj} Q_{pq}) a_p b_q = Q_{ij} \varepsilon_{spq} a_p b_q = \det Q Q_{ij} (a \times b),
\]

The vector obeying the transformation rule in equation (1.192) is called a pseudo vector or axial vector. A surface vector is regarded as an axial vector.

The transformation rule for the permutation symbol \( \varepsilon \) is given from equation (1.190) as follows:

\[
\varepsilon^*_{ijk} = (\det Q)\varepsilon_{ijk}.
\]

(1.193)

Thus, \( \varepsilon \) is a third-order pseudo tensor and plays the role of the third-order isotropic tensor as will be described later in equation (1.211).
1.5 Tensor Representations

Various notations are used to represent tensors in continuum mechanics. They are collectively shown in this section.

1.5.1 Tensor Notations

The tensor $T$ can be expressed as

$$T = T_{p_1 p_2 \cdots p_m} e_{p_1} \otimes e_{p_2} \cdots \otimes e_{p_m}$$  \hspace{1cm} (1.194)

which is called the indicial (or component) notation with base, where $e_1, e_2, \ldots, e_m$ are the unit base vectors of the coordinate axes $x_1, x_2, \ldots, x_m$. The pertinence of the transformation of $T$ to another coordinate system with the base vectors $e^*_1, e^*_2, \ldots, e^*_m$ of the coordinate axes $x^*_1, x^*_2, \ldots, x^*_m$ is confirmed as follows:

$$T = T_{p_1 p_2 \cdots p_m} e_{p_1} \otimes e_{p_2} \cdots \otimes e_{p_m} = Q_{r_1 p_1} Q_{r_2 p_2} \cdots Q_{r_m p_m} T_{p_1 p_2 \cdots p_m} e^*_{r_1} \otimes e^*_{r_2} \cdots \otimes e^*_{r_m},$$  \hspace{1cm} (1.195)

by substituting equation (1.17), where equation (1.195) designates equation (1.9).

In addition to the component notation with base (equation (1.194)), tensors can be written in indicial notation $T_{p_1 p_2 \cdots p_m}$; in matrix notation $[T]$, $[T]_{ij}$; or in symbolic (or direct) notation $T$.

The matrix notation holds only for a vector or a second-order tensor or for a fourth-order tensor if it is formally expressed by two indices. For instance, the stress–strain relation can be represented in matrix notation by expressing the stress and the strain of second-order tensors as a form of vector and the stiffness coefficient in a fourth-order tensor as a form of second-order tensor, which is called the Voigt notation. Various contractions exist in the operation of higher-order tensors and thus the symbolic notation is not useful in general. In other words, the symbolic notation is limited to the representation of tensors of order lower than fourth. On the other hand, the component notation holds always without adopting special rules. However, it should be emphasized that equations described in symbolic notation hold in any coordinate system including not only Cartesian but also curvilinear coordinate systems, even if they are formulated or derived in the Cartesian coordinate system.

1.5.2 Tensor Components and Transformation Rule

The components of vector $\mathbf{v}$, second-order tensor $\mathbf{T}$ and fourth-order tensor $\mathbf{F}$ can be written as follows:

$$v_i = \mathbf{v} \cdot \mathbf{e}_i, \quad T_{ij} = \mathbf{e}_i \cdot \mathbf{T} \mathbf{e}_j, \quad F_{ijkl} = (\mathbf{e}_i \otimes \mathbf{e}_j) : (\mathbf{e}_k \otimes \mathbf{e}_l).$$  \hspace{1cm} (1.196)
noting that
\[ v_i = v_i e_i \cdot e_i, \quad T_{ij} = e_i \cdot T_{ij} e_j \otimes e_j, \]
\[(e_i \otimes e_j) \mathbf{Z}_{pqrs} e_p \otimes e_q \otimes e_r \otimes e_s (e_k \otimes e_l) = (e_i \cdot e_j) (e_j \cdot e_q) \mathbf{Z}_{pq} (e_r \cdot e_s) (e_k \cdot e_l) = \delta_{ij} \delta_{kl} \delta_{pq} \delta_{rs}.\]

Noting equation (1.195) and then introducing the notation
\[
\begin{align*}
(Q \llbracket T \rrbracket)_{p_1 p_2 \cdots p_m} & = Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p_m q_m} T_{q_1 q_2 \cdots q_m} \\
(Q^T \llbracket T \rrbracket)_{p_1 p_2 \cdots p_m} & = Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p_m q_m} T_{q_1 q_2 \cdots q_m},
\end{align*}
\]
for the component transformation, let equations (1.9) and (1.11) be expressed formally in symbolic notation for convenience as follows:
\[ T^* = Q \llbracket T \rrbracket, \quad T = Q^T \llbracket T^* \rrbracket \]
(1.198)

The objective transformation rules for vector and the second-order tensor are as follows:
\[ \mathbf{v}^* = Q \mathbf{v}, \; \; (\mathbf{v} = Q^T \mathbf{v}^*), \mathbf{T}^* = Q \mathbf{T} Q^T (\mathbf{T} = Q^T \mathbf{T}^* Q) \]
(1.199)

Here, it should be emphasized that these equations do not express relations between different vectors or tensors but should be regarded as relations between components of vectors or tensors described by the two coordinate systems with bases \{e_i\} and \{e_i^*\).

1.5.3 Notations of Tensor Operations

The notation for first- to fourth-order tensors presented in this section will be used throughout this book. For vectors a, b, c and v, second-order tensors A, B and T, the third-order tensor \mathbf{Z} and fourth-order tensor \mathbf{X}, we have:
\[
\begin{align*}
\mathbf{a} \cdot \mathbf{b} & = \text{tr} (\mathbf{a} \otimes \mathbf{b}) \quad \text{for } a_i b_j, \\
\mathbf{a} \times \mathbf{b} & \quad \text{for } \mathbf{e}_{ij} a_i b_j, \\
\mathbf{a} \otimes \mathbf{b} & \quad \text{for } a_i b_j \mathbf{e}_i \cdot \mathbf{b}, \quad \text{for } a_i b_j c_j
\end{align*}
\]
(1.200)

\[
\begin{align*}
\mathbf{T} v & \quad \text{for } T_{ij} v_j, \\
\mathbf{v} \times \mathbf{T} & \quad \text{for } v_i T_{ij}, \\
\mathbf{v} \times \mathbf{v} & \quad \text{for } (\mathbf{v} \times \mathbf{T})_{ij} = \varepsilon_{ijl} v_k T_{lj}, \quad \mathbf{T} \times \mathbf{v} \quad \text{for } (\mathbf{T} \times \mathbf{v})_{ij} = \varepsilon_{ijl} T_{kl} v_l
\end{align*}
\]
(1.201)

\[
\begin{align*}
\mathbf{T} \cdot \mathbf{v} & = v_k T_{kj} e_j \otimes e_j = v_k T_{kj} \varepsilon_{kl} e_i \otimes e_j = \varepsilon_{kl} v_k T_{lj} e_i \otimes e_j = \varepsilon_{kl} v_k T_{lj} e_i \otimes e_j = v_k T_{kj} e_j \otimes e_j = \varepsilon_{kl} v_k T_{lj} e_i \otimes e_j = v_k T_{kj} e_j \otimes e_j = \varepsilon_{kl} v_k T_{lj} e_i \otimes e_j
\end{align*}
\]
(1.202)
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\[
\begin{align*}
\Xi T & \text{ for } \Xi_{ijkl} T_{kl}, \quad T \Xi & \text{ for } T_{ij} \Xi_{ijkl} \quad \{ \text{ (1.203) } \} \\
\Xi : T & \text{ for } \Xi_{ijkl} T_{jk}, \quad T : \Xi & \text{ for } T_{ij} \Xi_{ijkl} \\
\Xi T & \text{ for } \Xi_{ijkl} T_{ln}, \quad T \Xi & \text{ for } T_{ij} \Xi_{ijkl} \\
\Xi : T & \text{ for } \Xi_{ijkl} T_{kl}, \quad T : \Xi & \text{ for } T_{ij} \Xi_{ijkl} \quad \{ \text{ (1.204) } \}
\end{align*}
\]

\[
\begin{align*}
(\Xi : \mathbf{A}) : \mathbf{B} = \mathbf{B} : \mathbf{A} & \text{ for } \Xi_{ijkl} A_{ij} B_{kl} = B_{ij} \Xi_{ijkl} A_{ij} \\
(\Xi^T : \mathbf{A}) : \mathbf{B} = \mathbf{B} : \Xi & \text{ for } \Xi_{ijkl} A_{ij} B_{kl} = A_{ij} \Xi_{ijkl} B_{kl} \quad \{ \text{ (1.205) } \}
\end{align*}
\]

\[
\begin{align*}
||v|| = \sqrt{v \cdot v} & \text{ for } \sqrt{v_i v_i}, \\
||T|| = \sqrt{\text{tr}(TT^T)} & \text{ for } \sqrt{T_{ij} T_{ij}} \quad \{ \text{ (1.206) } \}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f^2(T)}{\partial T} & \text{ for } \frac{\partial f^2(T)}{\partial T_{ij} T_{kl}} \quad \{ \text{ (1.207) } \}
\end{align*}
\]

The following symbols for the tensor products of second-order tensors \( \mathbf{A} \), \( \mathbf{B} \), \( \mathbf{C} \) are defined (del Piero, 1979; Steinmann et al., 1997; Kintzel and Başar, 2006; Wang and Dui, 2008):

\[
\begin{align*}
(\mathbf{A} \otimes \mathbf{B})_{ijkl} & = A_{ij} B_{kl} \quad \text{with } \mathbf{A} \otimes \mathbf{B} : \mathbf{C} = \mathbf{A} (\mathbf{B} : \mathbf{C}) \quad (\mathbf{C} : \mathbf{A} \otimes \mathbf{B})_{ijkl} = C_{ijkl} A_{ij} B_{kl} \\
(\mathbf{A} \boxtimes \mathbf{B})_{ijkl} & = A_{ik} B_{lj} \quad \text{with } \mathbf{A} \boxtimes \mathbf{B} : \mathbf{C} = \mathbf{A} \mathbf{B} \mathbf{C}^T \quad (\mathbf{C} : \mathbf{A} \boxtimes \mathbf{B})_{ijkl} = C_{ijkl} A_{ik} B_{lj}
\end{align*}
\]

(1.208)

by which the following expressions hold.

\[
\begin{align*}
\mathbf{A} \mathbf{B} & = \mathbf{A} \mathbf{B} \mathbf{I} = \mathbf{A} \boxtimes \mathbf{I} : \mathbf{B} \\
\mathbf{B} \mathbf{A}^T & = \mathbf{I} \mathbf{A} \mathbf{B}^T = \mathbf{I} \boxtimes \mathbf{A} : \mathbf{B} \\
\mathbf{A} \mathbf{B}^T & = \mathbf{A} \mathbf{B}^T \mathbf{I} = \mathbf{A} \boxtimes \mathbf{I} : \mathbf{B} \\
\mathbf{A}^T \mathbf{B}^T & = \mathbf{I} \mathbf{A}^T \mathbf{B}^T = \mathbf{I} \boxtimes \mathbf{A} : \mathbf{B} \quad \{ \text{ (1.209) } \}
\end{align*}
\]

1.5.4 Operational Tensors

The second-order identity tensor is defined by the following equation, which involves the Kronecker delta:

\[
\mathbf{I} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_j. \quad \{ \text{ (1.210) } \}
\]
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The alternating tensor is defined by the following equation, which involves the alternating symbol:

\[ \varepsilon = \varepsilon_{ijk} e_i \otimes e_j \otimes e_k, \quad \varepsilon_{ijk} = [e_i, e_j, e_k] \] (1.211)

The fourth-order identity tensor \( I \) and the transposing tensor \( \mathcal{I} \) are defined by

\[
I = \delta_{ik} \delta_{jl} e_i \otimes e_j \otimes e_k \otimes e_l = e_i \otimes e_j \otimes e_j \otimes e_i = \mathbf{I} \otimes \mathbf{I},
\]

\[
\mathcal{I} = \delta_{ij} \delta_{kl} e_i \otimes e_j \otimes e_k \otimes e_l = \mathbf{I} \otimes \mathbf{I} \quad (1.212)
\]

Note that \( \mathcal{I} : T = T : \mathcal{I} = T, \mathcal{I} : T = T : \mathcal{I} = T, \mathcal{I} : \mathcal{I} = \mathcal{I} : \mathcal{I} = \mathbf{I} \).

Moreover, the fourth-order tracing identity tensor defined by

\[
\mathcal{I} = \delta_{ij} \delta_{kl} e_i \otimes e_j \otimes e_k \otimes e_l = \mathbf{I} \otimes \mathbf{I} \quad (1.213)
\]

has the operation

\[
\mathcal{I} : T = (\text{tr} T) \mathbf{I} \quad (1.214)
\]

The symmetrizing tensor \( S \) ant the skew-(or anti-) symmetrizing tensor \( A \) are given by

\[
S = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl}) e_i \otimes e_j \otimes e_k \otimes e_l = \frac{1}{2} (\mathcal{I} + \mathcal{I})
\]

\[
A = \frac{1}{2} (\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl}) e_i \otimes e_j \otimes e_k \otimes e_l = \frac{1}{2} (\mathcal{I} - \mathcal{I})
\]

(1.215)

leading to \( S : T = (T + T^T)/2, A : T = (T - T^T)/2 \). The following notation is also used:

\[
S : T = \text{sym}[T] = (T)_s, A : T = \text{ant}[T] = (T)_a
\]

(1.216)

The deviatoric projection tensor is defined by

\[
D = \delta_{ik} \delta_{jl} e_i \otimes e_j \otimes e_k \otimes e_l = \mathbf{I} - \frac{1}{3} (\mathbf{I} \otimes \mathbf{I})
\]

(1.217)

leading to \( D : T = T - (\text{tr} T)/3 \mathbf{I} \).

The following relations hold in these tensors:

\[
\mathcal{I} : \mathcal{I} = \mathcal{I}, \mathcal{I} : \mathcal{I} = \mathcal{I}, \mathcal{I} : \mathcal{I} = \mathcal{I}, \mathcal{I} : \mathcal{I} = \mathcal{I}
\]

(1.218)

\[
S : S = S, D : D = D, D : S = S : D = 0,
\]

(1.219)

noting that

\[
\mathcal{I} : \mathcal{I} = \{(e_i \otimes e_j) \otimes (e_i \otimes e_j)\} = \{(e_p \otimes e_q) \otimes (e_p \otimes e_q)\}
\]

\[
= e_i \otimes e_j \delta_{pq} \otimes e_p \otimes e_q = e_i \otimes e_j \otimes e_j \otimes e_i = \mathcal{I},
\]

\[
\mathcal{I} : \mathcal{I} = \{(e_i \otimes e_j) \otimes (e_j \otimes e_i)\} = \{(e_p \otimes e_q) \otimes (e_q \otimes e_p)\}
\]

\[
= e_i \otimes e_j \delta_{pq} \otimes e_q \otimes e_p = e_i \otimes e_j \otimes e_i \otimes e_j = \mathcal{I}.
\]
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\[ \mathcal{T} : \mathcal{I} = \{ (e_i \otimes e_j) \otimes (e_i \otimes e_j) \} : \{ (e_i \otimes e_j) \otimes (e_i \otimes e_j) \} \]

\[ = e_i \otimes e_j \delta_{ij} \delta_{ij} \otimes e_i \otimes e_j = \mathcal{I}, \]

\[ \mathcal{I} : \mathcal{I} = (\mathcal{I} \otimes \mathcal{I}) : (\mathcal{I} \otimes \mathcal{I}) = (\mathcal{I} \otimes \mathcal{I}) \mathcal{I} \otimes \mathcal{I} = 3 \mathcal{I}, \]

\[ \mathcal{T} : \mathcal{I} = \{ (e_r \otimes e_s) \otimes (e_s \otimes e_r) \} : \{ (e_r \otimes e_s) \otimes (e_s \otimes e_r) \} \]

\[ = e_i \otimes e_j \delta_{ij} \otimes e_r \otimes e_s = \mathcal{I}, \]

\[ \mathcal{I} : \mathcal{I} = \{ (e_i \otimes e_j) \otimes (e_i \otimes e_j) \} : \{ (e_i \otimes e_j) \otimes (e_i \otimes e_j) \} \]

\[ = e_i \otimes e_j \delta_{ij} \otimes e_r \otimes e_s = \mathcal{I}, \]

\[ S : S = \frac{1}{2} (\mathcal{T} + \mathcal{I}) : \frac{1}{2} (\mathcal{T} + \mathcal{I}) = \frac{1}{4} (\mathcal{T} + 2 \mathcal{I} + \mathcal{I}) = S, \]

\[ D : D = \frac{1}{2} (\mathcal{T} - \mathcal{I}) : \frac{1}{2} (\mathcal{T} - \mathcal{I}) = \frac{1}{4} (\mathcal{T} - 2 \mathcal{I} + \mathcal{I}) = D, \]

\[ S : D = \frac{1}{2} (\mathcal{T} + \mathcal{I}) : \frac{1}{2} (\mathcal{T} - \mathcal{I}) = \frac{1}{4} (\mathcal{T} - \mathcal{I}) = 0. \]

Note that the symmetrizing tensor \( S \) can be used instead of the identity tensor \( \mathcal{I} \) for symmetric tensors.

### 1.5.5 Isotropic Tensors

An isotropic tensor is defined as a tensor possessing components which are unchanged by arbitrary rotation of the coordinate system and thus it must satisfy

\[ T = Q \| T \|, \quad (1.220) \]

where use is made of the notation for objective transformation in equation (1.198) for general tensor. As a trivial case, all tensors possessing zero components are isotropic tensors. We consider non-trivial isotropic tensors for the first- to fourth-order tensors for which equation (1.220) is described as

\[ S = S \]

\[ v_i = Q_{ir} v_r \]

\[ T_{ij} = Q_{ir} Q_{jr} T_{rs} \]

\[ T_{ijk} = Q_{ir} Q_{jr} Q_{ks} T_{rst} \]

\[ T_{ijkl} = Q_{ir} Q_{jr} Q_{ks} Q_{ku} T_{rstu} \] \quad (1.221)

Here, note that the permutation of indices does not influence on the values of components by virtue of the isotropy.

Now consider a small rotation of coordinate system given by the following equation (Jeffreys, 1931):

\[ Q_{ij} = \delta_{ij} + \Omega_{ij} \]

(1.222)

with an infinitesimal anti-symmetric tensor \( \Omega_{ij} \), for which one has

\[ Q_{ir} Q_{jr} = (\delta_{ir} + \Omega_{ir}) (\delta_{jr} + \Omega_{jr}) = \delta_{ij} + \Omega_{ij} + \Omega_{ji} + \Omega_{ir} \Omega_{jr} = \delta_{ij} + \Omega_{ir} \Omega_{jr} \equiv 0, \]

\[ \Omega_{ij} \equiv 0. \]
exhibiting the property of the orthogonal tensor, while $\Omega_{ir}\Omega_{jr}$ is infinitesimal in the second order.

Substituting equation (1.222) into equation (1.221), we have

\[ v_i = (\delta_{ir} + \Omega_{ir})v_r = v_i + \Omega_{ir}v_r, \]
\[ T_{ij} = (\delta_{ir} + \Omega_{ir})(\delta_{jr} + \Omega_{jr})T_{rs} \equiv T_{ij} + \Omega_{ir}T_{rj} + \Omega_{jr}T_{ir}, \]
\[ T_{ijk} = (\delta_{ir} + \Omega_{ir})(\delta_{js} + \Omega_{js})(\delta_{kt} + \Omega_{kt})T_{rst} \]
\[ \sim = T_{ijk} + \Omega_{ir}T_{rjk} + \Omega_{jr}T_{irk} + \Omega_{kr}T_{ijr}, \]
\[ T_{ijkl} = (\delta_{ir} + \Omega_{ir})(\delta_{js} + \Omega_{js})(\delta_{kt} + \Omega_{kt})(\delta_{lu} + \Omega_{lu})T_{rstu} \]
\[ \sim = T_{ijkl} + \Omega_{ir}T_{rjkl} + \Omega_{jr}T_{iklj} + \Omega_{kr}T_{ijlk} + \Omega_{lr}T_{ijkl}, \]

and thus

\[
\begin{align*}
\Omega_{ir}v_r &= 0 \\
\Omega_{ir}T_{rj} + \Omega_{jr}T_{ir} &= 0 \\
\Omega_{ir}T_{rjk} + \Omega_{jr}T_{irk} + \Omega_{kr}T_{ijr} &= 0 \\
\Omega_{ir}T_{rjk} + \Omega_{jr}T_{iklj} + \Omega_{kr}T_{ijlk} + \Omega_{lr}T_{ijkl} &= 0
\end{align*}
\]

(1.223)

for arbitrary $\Omega_{ij}$, where $\Omega_{ij} = 0$ for $i = j$ and $\Omega_{ij} = -\Omega_{ji}$ for $i \neq j$.

**Zero-order (scalar) isotropic tensor**

Any scalar satisfies equation (1.221) and thus is a zero-order isotropic tensor.

**First-order (vector) isotropic tensor**

The linear homogeneous simultaneous equation (1.223) possesses a non-trivial (non-zero) solution of $v_i$ for a given (fixed) $\Omega_{ir}$ because the necessary and sufficient condition, equation (1.178), for the existence of a non-trivial solution is satisfied by virtue of the property of the anti-symmetric tensor shown in equation (1.174), that is, $\det[\Omega_{ir}] = 0$. However, equation (1.223) leads to

\[
\begin{align*}
\Omega_{12}v_2 - \Omega_{31}v_3 &= 0 \\
-\Omega_{12}v_2 + \Omega_{23}v_3 &= 0 \\
\Omega_{31}v_1 - \Omega_{23}v_2 &= 0
\end{align*}
\]

which reduces to $\Omega_{12}(v_2 - v_1) + \Omega_{23}(v_3 - v_2) + \Omega_{31}(v_1 - v_3) = 0$. In order that this equation is satisfied for arbitrary $\Omega_{ij}$, only the trivial solution $v_1 = v_2 = v_3 = 0$ can exist. Consequently, there is no first-order isotropic tensor (vector).
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Second-order isotropic tensor

If \( i = 1, j = 1 \) in equation (1.223)\(_2\), then
\[
2(\Omega_{12}T_{21} + \Omega_{13}T_{31}) = 0,
\]
and thus it follows that
\[
T_{ij} = 0 \text{ for } i \neq j. \tag{2}
\]

Further, if \( i = 1, j = 2 \) in equation (1.223)\(_2\), then
\[
\Omega_{12}(T_{22} - T_{11}) = 0, \tag{3}
\]
noting equation (2), and thus it must follow that
\[
T_{ii} = T_{jj} \text{ (no sum).} \tag{4}
\]
The second-order isotropic tensor satisfying equations (2) and (4) is generally given by
\[
s\delta_{ij}e_i \otimes e_j = sI \tag{1.224}
\]

Third-order isotropic tensor

If \( i = 1, j = 1 \) in equation (1.223)\(_3\), then
\[
\Omega_{12}T_{21k} + \Omega_{13}T_{31k} + \Omega_{12}T_{12k} + \Omega_{13}T_{13k} + \Omega_{21}T_{111} + \Omega_{22}T_{112} + \Omega_{23}T_{113} = 0. \tag{1}
\]

Further, putting \( k = 2 \) and noting that \( \Omega_{22} = 0 \) in this equation, one has
\[
\Omega_{12}(T_{212} + T_{122} - T_{111}) + \Omega_{13}(T_{312} + T_{132}) + \Omega_{23}T_{113} = 0
\]
and thus
\[
\begin{align*}
T_{212} + T_{122} &= T_{111} \\
T_{312} + T_{132} &= 0 \\
T_{113} &= 0
\end{align*}
\]
When using the symmetry, from the last equation, \( T_{ijk} = 0 \) if two of \( i, j, k \) are equal and the third unequal, that is,
\[
T_{ii} = T_{jj} = T_{jjj} = 0 \text{ (no sum, } i \neq j). \tag{3}
\]
Then, substituting this result into the first equation, it follows that
\[
T_{ii} = 0 \text{ (no sum).} \tag{4}
\]

Further, noting that the same permutation must not influence the value of components by the isotropy of \( T_{ijk} \), it follows from equation (2)\(_2\) that
\[
T_{ijk} = T_{kji} = -T_{kij} = -T_{ikj} = -T_{jik} \text{ (} i \neq j \neq k \neq i). \tag{5}
\]
The third-order isotropic tensor satisfying equations (3), (4) and (5) is given generally by the permutation symbol multiplied by arbitrary scalar, that is,

\[ s\varepsilon_{ijk} e_i \otimes e_j \otimes e_k = s\varepsilon \]  

(1.225)

**Fourth-order isotropic tensor**

For a fourth-order isotropic tensor, with its four indices which can only take values 1, 2, 3, at least two of the indices must be equal. We can separate the components into four classes:

(a) two are unequal and the other two equal; (b) three equal; (c) two equal and the other two equal; and (d) all four equal.

If \( i = j = 1, k = 2, l = 3 \) in class (a), equation (1.223) leads to

\[
\Omega_{12} T_{2123} + \Omega_{13} T_{3123} + \Omega_{42} T_{1223} + \Omega_{13} T_{1323} + \Omega_{21} T_{1113} + \Omega_{23} T_{1133} + \Omega_{31} T_{1123} + \Omega_{32} T_{1112} = 0,
\]

that is,

\[
\Omega_{12}(T_{2123} + T_{1223} - T_{1113}) + \Omega_{13}(T_{3123} + T_{1323} - T_{1123}) + \Omega_{23}(T_{1133} - T_{1122}) = 0,
\]

and thus

\[
\begin{cases}
T_{2123} + T_{1223} - T_{1113} = 0 \\
T_{3123} + T_{1323} - T_{1123} = 0 \\
T_{1133} - T_{1122} = 0
\end{cases}
\]  

(1)

If \( i = j = k = 1, l = 2 \) in class (b), equation (1.223) yields

\[
\Omega_{12} T_{2112} + \Omega_{13} T_{3112} + \Omega_{42} T_{1212} + \Omega_{13} T_{1312} + \Omega_{21} T_{1112} + \Omega_{23} T_{1113} + \Omega_{31} T_{1122} + \Omega_{32} T_{1111} = 0,
\]

that is,

\[
\Omega_{12}(T_{2112} + T_{1212} + T_{1122} - T_{1111}) + \Omega_{13}(T_{3112} + T_{1312} + T_{1132}) + \Omega_{23} T_{1113} = 0,
\]

and thus

\[
\begin{cases}
T_{2112} + T_{1212} + T_{1122} - T_{1111} = 0 \\
T_{3112} + T_{1312} + T_{1132} = 0 \\
T_{1113} = 0
\end{cases}
\]  

(2)

The last equation leads to the fact that all components in class (b) are zero, that is,

\[ T_{iij} = T_{iji} = T_{iij} = T_{jii} = T_{jjj} = 0 \] (no sum, \( i \neq j \)).  

(3)

Then equation (1) results in

\[ T_{2123} + T_{1223} = 0 \]
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which, by changing $1 \rightarrow 3$, $2 \rightarrow 1$, $3 \rightarrow 2$, leads to

$$T_{1312} + T_{3112} = 0.$$ 

Substituting this into equation (2)$_2$, one has

$$T_{1132} = 0,$$

so that all components in class (a) are also zero, that is,

$$T_{ii jk} = T_{ij ik} = T_{ij ki} = T_{ji ki} = T_{jk ii} = T_{jk i} = 0 \quad \text{(no sum, } i \neq j \neq k \neq i).$$

(4)

Taking account of equation (4) in equation (2)$_1$, the components in class (d) can be expressed by the components in class (c).

Eventually, we have only to take account of the components in class (c), which are classified into the following three types:

$$\begin{align*}
T_{1122} &= T_{2211} = T_{2233} = T_{3322} = T_{3311} = T_{1133} = \lambda \\
T_{1212} &= T_{2112} = T_{3232} = T_{3232} = T_{3131} = T_{1313} = \mu + \nu \\
T_{1221} &= T_{2112} = T_{2332} = T_{3223} = T_{3113} = T_{1331} = \mu - \nu
\end{align*}$$

(5)

Here, the permutation of indices does not influence the values of components by virtue of the isotropy and thus all components in class (d) are identical so that they can be expressed collectively as

$$T_{1111} = T_{2222} = T_{3333} = \lambda + 2\mu$$

(6)

by substituting equation (5) into equation (2)$_1$.

Let us now formulate the tensor satisfying equation (5). Note the following facts:

1. The tensor with components for $i = j$ and $k = l$ in (5)$_1$ given by $\lambda$ and other components zero is given by multiplication of the tracing identity tensor $I_{ijkl} (= \delta_{ij}\delta_{kl})$ in equation (1.213) by $\lambda$.

2. The tensor possessing components such that the half of the sum of the components for $i = k$, $j = l$ in equation (5)$_2$ and $i = l$, $j = k$ in equation (5)$_3$ ($(T_{1212} + T_{1221})/2$ is given by multiplication of the symmetrizing tensor $S_{ijkl} (= \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$ in equation (1.215)$_1$ by $\mu$.

3. On the other hand, its subtraction $((T_{1212} - T_{1221})/2$, etc.) is given by multiplication of the anti-symmetrizing tensor $A_{ijkl} (= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})/2$ in equation (1.215)$_2$ by $\nu$.

Finally, the fourth-order isotropic tensor is given generally by the sum of these independent tensors as follows:

$$\mathcal{J} = (\lambda I_{ijkl} + \mu S_{ijkl} + \nu A_{ijkl})e_i \otimes e_j \otimes e_k \otimes e_l = \lambda I + \mu S + \nu A$$

(1.226)

The basic fourth-order tensors described in Sections 1.5.4 and 1.5.5 are gathered together in Table 1.1.
Table 1.1 Various fourth-order basic tensors

<table>
<thead>
<tr>
<th>Fourth-order tensors</th>
<th>Remarks</th>
</tr>
</thead>
</table>
| Identity tensor $\mathcal{I} = \delta_{ik} \delta_{jl} e_i \otimes e_j \otimes e_k \otimes e_l = e_i \otimes e_j \otimes e_i \otimes e_j = I \otimes I$ | $\mathcal{I} : T = T$  
$\frac{\partial T}{\partial \mathcal{I}} = \mathcal{I}$ |
| Transposing tensor $\mathcal{I} = \delta_{ij} \delta_{kl} e_i \otimes e_j \otimes e_k \otimes e_l = e_i \otimes e_j \otimes e_j \otimes e_i = I \otimes I$ | $\mathcal{I} : T = T^T$  
$\frac{\partial T}{\partial \mathcal{I}} = \mathcal{I}$ |
| Tracing identity tensor $\mathcal{I} = \delta_{ij} \delta_{kl} e_i \otimes e_j \otimes e_j \otimes e_i = I \otimes I$ | $\mathcal{I} : T = (\text{tr}T)I$ |
| Symmetrizing tensor $\mathcal{S} = \frac{1}{2}(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) e_i \otimes e_j \otimes e_k \otimes e_l = \frac{1}{2}(\mathcal{I} + \mathcal{I})$ | $\mathcal{S} : T = \text{sym}[T]$ |
| Skew-symmetrizing tensor $\mathcal{A} = \frac{1}{2}(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) e_i \otimes e_j \otimes e_k \otimes e_l = \frac{1}{2}(\mathcal{I} - \mathcal{I})$ | $\mathcal{A} : T = \text{ant}[T]$ |
| Deviatoric projection tensor $\mathcal{D} \equiv \left( \delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{ij} \delta_{kl} \right) e_i \otimes e_j \otimes e_k \otimes e_l = \mathcal{I} - \frac{1}{3} \mathcal{I}$ | $\mathcal{D} : T = T - \{(\text{tr}T)/3\}I = T'$ |
| Isotropic tensor $\mathcal{J} \equiv (\lambda \mathcal{I}, \mu \mathcal{S}, \nu \mathcal{A})$ | $\mathcal{J} : I = I$, $\mathcal{J} : I = I$, $\mathcal{J} : I = I$, $\mathcal{J} : I = I$, $\mathcal{J} : I = I$, $\mathcal{J} : I = I$, $\mathcal{J} : I = I$, $\mathcal{J} : I = I$, $\mathcal{J} : I = I$  
$\mathcal{J} : I = 3\mathcal{I}$  
$\mathcal{J} : I = I$, $\mathcal{J} : I = I$, $\mathcal{J} : I = I$, $\mathcal{J} : I = I$  
$\mathcal{S} : S = S$, $\mathcal{D} : D = D$, $\mathcal{D} : S = S$, $\mathcal{D} : D = 0$ |

1.6 Eigenvalues and Eigenvectors

A tensor is expressed in the component notation having only normal components by choosing coordinate base into special directions. In what follows, we consider these special directions for the second-order tensor.

1.6.1 Eigenvalues and Eigenvectors of Second-Order Tensors

The unit vector $e$ fulfilling

$$\textbf{Te} = T\textbf{e}, \quad T_{ij}e_j = Te_i,$$  

(1.227)

that is,

$$(T - T\textbf{I})e = 0, \quad (T_{ij} - T\delta_{ij})e_j = 0,$$  

(1.228)
for the second-order tensor is called the eigenvector (or principal or characteristic or proper vector) and the scalar $T$ is called the eigenvalue (or principal or characteristic or proper value). The prefix ‘eigen-’ comes from German and was first used in this context by David Hilbert.

In order for the linear simultaneous equation (1.228) to possess a non-zero solution, the determinant of the coefficient must be zero as described in equations (1.177) and (1.178), that is,

$$
\det[T - T1] = 0
$$

Equation (1.229), a cubic equation in $T$, is called the characteristic equation of the tensor. The three eigenvectors $e_1, e_2, e_3$ are derived by solving equation (1.228) for each of three solutions $T_1, T_2, T_3$ of $T$ obtained from equation (1.229).

In what follows, it is verified that the eigenvectors are mutually orthogonal for eigenvalues that differ from each other in the second-order real symmetric tensor fulfilling $T = T^T$.

It follows from equation (1.227) that

$$Te_\alpha = T_\alpha e_\alpha \text{ (no sum)} \tag{1.230}$$

for the eigenvectors $e_\alpha$ ($\alpha = 1, 2, 3$) of $T$. Multiplying equation (1.230) by the eigenvectors, we have

$$e_\beta \cdot Te_\alpha = T_\alpha e_\alpha \cdot e_\beta \text{ (no sum).}$$

Subtracting the second equation from the first yields

$$e_\beta \cdot Te_\alpha - e_\alpha \cdot Te_\beta = (T_\alpha - T_\beta) e_\alpha \cdot e_\beta, \tag{1.231}$$

that is,

$$(T_\alpha - T_\beta)e_\alpha \cdot e_\beta = 0 \text{ (no sum)}, \tag{1.232}$$

noting that $e_\beta \cdot Te_\alpha - e_\alpha \cdot Te_\beta = e_\beta \cdot Te_\alpha - T^T e_\alpha \cdot e_\beta = e_\beta \cdot (T - T^T) e_\alpha = 0$ by virtue of the symmetry of the tensor $T$ on the left-hand side of equation (1.231). The following facts can be concluded from equation (1.232).

1. If the three eigenvalues are all different from each other, there will exist three principal directions perpendicular to each other.
2. If two of three eigenvalues are equal, all directions in the plane perpendicular to the principal direction for the other eigenvalue will be principal directions for the two equal eigenvalues.
3. If all three eigenvalues are equal, all directions in the space will be principal directions.

Returning to the second-order tensor including a skew-symmetric tensor, the expansion of the characteristic equation (1.229) of $T$ leads to

$$
\begin{vmatrix}
T_{11} - T & T_{12} & T_{13} \\
T_{21} & T_{22} - T & T_{23} \\
T_{31} & T_{32} & T_{33} - T
\end{vmatrix}
= (T_{11} - T)(T_{22} - T)(T_{33} - T) + T_{12}T_{23}T_{31} + T_{21}T_{32}T_{13}
- (T_{11} - T)T_{23}T_{32} - (T_{22} - T)T_{31}T_{13} - (T_{33} - T)T_{12}T_{21}
$$
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\[ -T^3 + (T_{11} + T_{22} + T_{33}) T^2 - (T_{11}T_{22} + T_{22}T_{33} + T_{33}T_{11}) \\
- T_{12}T_{21} - T_{23}T_{32} - T_{31}T_{13})T + T_{11}T_{22}T_{33} + T_{12}T_{23}T_{31} \\
+ T_{13}T_{23}T_{21} - T_{11}T_{23}T_{32} - T_{22}T_{31}T_{13} - T_{33}T_{12}T_{21} \]

\[ = -T^3 + (T_{11} + T_{22} + T_{33}) T^2 - \frac{1}{2} (T_{11} + T_{22} + T_{33})^2 \\
- (T_{11}^2 + T_{22}^2 + T_{33}^2 + 2(T_{12}T_{21} + T_{23}T_{32} + T_{31}T_{13}))T \\
+ T_{11}T_{22}T_{33} + T_{12}T_{23}T_{31} + T_{13}T_{23}T_{21} - T_{11}T_{23}T_{32} - T_{22}T_{31}T_{13} \\
- T_{33}T_{12}T_{21} = 0 \]

from which the characteristic equation is expressed as

\[ (-\det(T - T I) = T^3 - IT^2 + 2IT - III = 0 \] (1.233)

where

\[
I \equiv T_{11} + T_{22} + T_{33} = T_{ii} = \text{tr}T
\] (1.234)

\[
\begin{bmatrix}
T_{22} & T_{23} \\
T_{32} & T_{33}
\end{bmatrix}
+ \begin{bmatrix}
T_{11} & T_{13} \\
T_{31} & T_{33}
\end{bmatrix}
+ \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
= \text{tr}(\text{cof}T)
\]

\[ = \frac{1}{2} (T_{rs}T_{sr} - T_{rs}T_{sr}) = \frac{1}{2} (\text{tr}T^2 - \text{tr}T^2) \] (1.235)

\[
III \equiv \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{bmatrix}
= \detT = \epsilon_{rs}T_{rs}T_{23}T_{21} = \frac{1}{6} \text{tr}^3T - \frac{1}{2} \text{tr}T \text{tr}T^2 + \frac{1}{3} \text{tr}T^3
\] (1.236)

The direct notation by the traces in equation (1.236) will be shown later in equation (1.297) based on the Cayley–Hamilton theorem in 1.6.6.

The characteristic equation (1.233) is derived above by expanding the determinant for components. On the other hand, it can be derived in the direct notation as follows (Chadwick, 1976; Kyoya, 2008). Multiply equation (1.229) by \([abc]\),

\[ \det(T - \lambda I)[abc] = 0, \]

and transform it using equation (1.179), into

\[ [(T - \lambda I)a(T - \lambda I)b(T - \lambda I)c] = 0. \]

This equation leads to

\[
[(T - \lambda I)a(T - \lambda I)b(T - \lambda I)c] = -[abc])^3 + ([Ta b c] + [a Tb c] + [a b Tc])\lambda^2 \\
- ([a Tb Tc] + [Ta b Tc] + [Ta Tbc])\lambda + ([Ta Tb Tc]) \lambda \\
= -[abc]](\lambda^3 - (\text{tr}T)\lambda^2 + \frac{1}{2}((\text{tr}T)^2 - \text{tr}T^2)\lambda - \detT = 0
\]
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by virtue of equation (1.179). Then we have
\[ \lambda^3 - (\text{tr}T)\lambda^2 + \frac{1}{2}[(\text{tr}T)^2 - \text{tr}^2T]\lambda - \det T = 0 \]
which is nothing more than the characteristic equation in equation (1.233).

On the other hand, the characteristic equation (1.233) is expressed by means of the eigenvalues as follows:
\[ (T - T_1)(T - T_2)(T - T_3) = 0. \quad (1.237) \]
Comparing equations (1.233) and (1.237), we can express the coefficients I, II, and III as
\[ I = T_1 + T_2 + T_3 \]
\[ II = T_1T_2 + T_1T_3 + T_2T_3 \]
\[ III = T_1T_2T_3 \quad (1.238) \]
Equation (1.238) can also be derived by inserting \( T_{11} = T_1, T_{22} = T_2, T_{33} = T_3, T_{12} = T_{23} = T_{31} = 0 \) into equations (1.234)–(1.236). Since I, II, and III are symmetric functions of eigenvalues, they are invariant under coordinate system rotation and are called the principal invariants.

The following invariants are called moments.
\[ \text{I} = \text{tr}T, \quad \text{II} = \text{tr}^2T, \quad \text{III} = \text{tr}^3T \quad (1.239) \]
The principal invariants are described in terms of these moments from equations (1.234)–(1.236) as follows:
\[ \begin{align*}
I &= \text{I} \\
II &= \frac{1}{2}(\text{I}^2 - \text{II}) \\
III &= \frac{1}{6}I^3 - \frac{1}{2}\text{I}\text{II} + \frac{1}{3}\text{III}
\end{align*} \quad (1.240) \]
The characteristic equation (1.233) can be obtained directly exploiting the formula for the scalar triple product in equation (1.179). Multiplying equation (1.229) by the triple scalar product of arbitrary three independent vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) yields (Chadwick, 1976; Kyoya, 2008; Bertram, 2008)
\[ \det(T - TI)[\mathbf{abc}] = 0, \quad (1.241) \]
which is transformed using equation (1.179), to
\[ [(T - TI)a, (T - TI)b, (T - TI)c] = 0. \quad (1.242) \]
Here, expanding equation (1.242) and applying equation (1.179), one has
\[ [(T - TI)a, (T - TI)b, (T - TI)c] = -T^3[\mathbf{abc}] + T^2([\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}]) \]
\[ -T([\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}]) + [\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}] \]
\[ = -[\mathbf{abc}][T^3 - (\text{tr}T)T^2 + \frac{1}{2}((\text{tr}T)^2 - \text{tr}^2T)T - \det(T)] = 0. \]
Because of the arbitrariness of $a, b, c$ the following equality must hold:

$$T^3 - (\text{tr}T)T^2 + \frac{1}{2}\text{tr}(\text{cof}T)T - \text{det}T = 0$$

(1.243)

which is nothing more than the characteristic equation (1.233).

Next, consider the deviatoric tensor $T'$. The characteristic equation of $T'$ is given by replacing $T$ by $T'$ in equation (1.233) as follows:

$$T'^3 - II'T' - III' = 0$$

(1.244)

noting that

$$I' \equiv \text{tr}T' = 0,$$

(1.245)

where

$$II' \equiv \frac{1}{2}\text{tr}T'^2 = \frac{1}{2}T'^{rs}T'^{sr}$$

$$= \frac{1}{2}(T'^{12} + T'^{23} + T'^{31}) + T'^{12} + T'^{23} + T'^{31}$$

$$= \frac{1}{6}[(T_1 - T_2)^2 + (T_2 - T_3)^2 + (T_3 - T_1)^2] + T'^{12} + T'^{23} + T'^{31}$$

(1.246)

$$III' \equiv \text{det}T' = \frac{1}{3}\text{tr}T'^3 = \frac{1}{3}T'^{rs}T'^{sr}T'^{rt}$$

$$= T'^{1i}T'^{2j}T'^{3k} - T'^{1i}T'^{2k}T'^{3j} - T'^{2j}T'^{3k}T'^{1i} - T'^{2k}T'^{3j}T'^{1i} + 2T'^{1i}T'^{2j}T'^{3k}$$

$$= T'^{1i}T'^{2j}T'^{3k} = \frac{1}{3}(T'^{13} + T'^{23} + T'^{33}).$$

(1.247)

### 1.6.2 Spectral Representation and Elementary Tensor Functions

By choosing as base vectors the principal directions $e_p$ of the tensor $T$, the identity tensor is represented by $I = e_p \otimes e_p$. $T$ can be represented by multiplying it by $T$ itself, that is, $T = TI = Te_p \otimes e_p$ as follows:

$$T = \sum_{p=1}^{3} T_p e_p \otimes e_p$$

(1.248)
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where \( T_P \) and \( e_P \) are the eigenvalues and eigenvectors, respectively, fulfilling

\[
T e_P = T_P e_P \quad (P = 1, 2, 3; \text{ no sum}).
\]

Equation (1.248) is called the spectral decomposition (or spectral representation).

If the tensor \( \tilde{T} \) with eigenvectors \( \tilde{e}_P \) possesses the same eigenvalues as those of the tensor \( T \), we may write

\[
\tilde{T} \tilde{e}_P = T_P \tilde{e}_P \quad \text{(no sum),}
\]

where the orthogonal tensor \( \tilde{Q} \) between the eigenvectors of these tensors is given by

\[
\tilde{Q}_{PQ} = \tilde{e}_P \cdot \tilde{e}_Q, \quad \tilde{Q} = e_P \otimes \tilde{e}_P, \quad \tilde{e}_P = \tilde{Q}^T e_P.
\]

Then

\[
T = \tilde{Q} \tilde{T} \tilde{Q}^T, \quad \tilde{T} = \tilde{Q}^T T \tilde{Q}
\]

because of

\[
T = \sum_{P=1}^{3} T_P e_P \otimes e_P = \sum_{P=1}^{3} T_P \tilde{Q} \tilde{e}_P \otimes \tilde{Q} \tilde{e}_P = \tilde{Q} \sum_{P=1}^{3} T_P \tilde{e}_P \otimes \tilde{e}_P \tilde{Q}^T,
\]

noting equation (1.94). Therefore, tensors having identical eigenvalues can be related by an orthogonal transformation: they are called similar tensors. The orthogonal transformation rule (1.199) and relation (1.252) for similar tensors are of mutually opposite forms. The magnitudes of similar tensors are identical to each other by equation (1.107).

Furthermore, it follows for the \( n \)th power of the tensor \( T \) that

\[
T^n e_P = T_P^n e_P,
\]

by repeated application of \( T \) to equation (1.227), that is,

\[
T^n e = T^n e = T^{n-1} T e = T T^{n-2} T e = T^2 T^{n-3} T e = \cdots = T^n e.
\]

Equation (1.254) means that the eigenvalues of \( T^n \) are \( T_P^n \), where \( T_P \) are the eigenvalues of \( T \) and the principal directions of the tensor \( T^n \) are identical to those of \( T \). Tensors that have an identical set of principal directions are called coaxial or said to fulfill coaxiality. The second-order tensor function \( f(T) \) of only \( T \) is coaxial with \( T \) and the eigenvalues are given by \( f(T_P) \), and thus it follows that

\[
f(T) = \sum_{P=1}^{3} f(T_P) e_P \otimes e_P.
\]
The general second-order tensor function can be defined by extending equation (1.256) as follows. Applying the Maclaurin expansion

\[ f(T) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} T^n = f(0) + f'(0)T + \frac{f''(0)}{2!}T^2 + \frac{f'''(0)}{3!}T^3 + \cdots \]

(1.257)

to the scalar function \( f(T) \) in equation (1.256), various second-order tensor functions are described explicitly as follows:

\[ \exp T = \sum_{n=0}^{\infty} \frac{1}{n!} T^n = I + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \cdots \]

(1.258)

\[ \ln(I + T) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} T^n = T - \frac{T^2}{2} + \frac{T^3}{3} - \cdots \]

(1.259)

\[ (I + T)^m = \sum_{n=0}^{\infty} \frac{m!}{(m-n)!n!} T^n = I + m T + \frac{m(m-1)}{2!} T^2 + \frac{m(m-1)(m-2)}{3!} T^3 + \cdots \]

(1.260)

\[ T^{-1} = (I + (T - I))^{-1} = I - (T - I) + (T - I)^2 - \cdots \]

(1.261)

\[ a^T = \sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} T^n = I + (\ln a) T + \frac{(\ln a)^2}{2!} T^2 + \frac{(\ln a)^3}{3!} T^3 + \cdots \]

(1.262)

\[ \sin T = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} T^{2n+1} = T - \frac{1}{3!} T^3 + \frac{1}{5!} T^5 - \cdots \]

(1.263)

\[ \cos T = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} T^{2n} = I - \frac{1}{2!} T^2 + \frac{1}{4!} T^4 - \cdots \]

(1.264)

### 1.6.3 Calculation of Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors of tensor must be calculated in order to obtain the spectral representation. Solutions (cf. Carlson and Hoger, 1986; Bruhns, 2003) are as follows.

#### Eigenvalues

In order to obtain eigenvalues, one must solve the characteristic equation, which is the cubic equation with coefficients which are functions of the invariants. First, let the following form of solution be postulated.

\[ T' = \sqrt{\frac{4\Pi}{3}} \cos \psi \]

(1.265)
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for the eigenvalues of the deviatoric part of tensor $T$. Substituting equation (1.265) into equation (1.244), we have

$$
\left( \frac{4II'}{3} \right)^{3/2} \cos^3 \psi - II' \left( \frac{4II'}{3} \right)^{1/2} \cos \psi - III' = 0
$$

which reduces to

$$
\frac{2}{3\sqrt{3}} II^{3/2} \cos 3\psi - III' = 0,
$$

(1.267)

using the trigonometric formula

$$
\cos^3 \psi = \frac{1}{4} (\cos 3\psi + 3 \cos \psi).
$$

(1.268)

It follows from (1.267) that

$$
\cos 3\psi = \frac{3\sqrt{3}III'}{2II'^{3/2}}.
$$

(1.269)

Bearing in mind that the cosine is a periodic function with the period $2\pi$, the angle $\psi$ is expressed by the equation

$$
\psi_K = \frac{1}{3} \left\{ \cos^{-1} \left( \frac{3\sqrt{3}III'}{2II'^{3/2}} \right) - 2\pi K \right\},
$$

(1.270)

where $K$ is an integer. Substituting equation (1.270) into equation (1.265) and adding the spherical component $I/3$, the eigenvalues of $T$ are as follows:

$$
T_K = \frac{1}{3} I + T_K' = \frac{1}{3} I + \sqrt{\frac{4II'}{3}} \cos \psi_K = \frac{1}{3} I + \sqrt{\frac{4II'}{3}} \cos \left[ \frac{1}{3} \cos^{-1} \left( \frac{3\sqrt{3}III'}{2II'^{3/2}} \right) - 2\pi K \right]
$$

(1.271)

Eigenvectors

Equation (1.248) can be expressed as

$$
T = \sum_{P=1}^{3} T_P E_P
$$

(1.272)

in which the tensor

$$
E_P \equiv e_P \otimes e_P \quad \text{(no sum)}
$$

(1.273)

is called the eigenprojection of $T$ and satisfies

$$
\sum_{P=1}^{3} E_P (= e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) = I,
$$

(1.274)
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\[ E_J E_K = \begin{cases} E_J & \text{for } J = K \\ 0 & \text{for } J \neq K \end{cases} \]

\( \text{tr}(E_J E_K) = \delta_{IJ} \) \hspace{1cm} (1.275)

Here, we have

\[ T E_J = \left( \sum_{K=1}^{3} T_K E_K \right) e_J \otimes e_J = \left( \sum_{K=1}^{3} T_K e_K \otimes e_K \right) e_J \otimes e_J = T_J e_J \otimes e_J \]

\[ E_J T = e_J \otimes e_J \left( \sum_{K=1}^{3} T_K E_K \right) = e_J \otimes e_J \left( \sum_{K=1}^{3} T_K e_K \otimes e_K \right) = e_J \otimes e_J T_J \]

and thus we have also

\[ T E_J = E_J T = T_J E_J \] \hspace{1cm} (no sum).

\( \text{On the other hand, it follows from equation (1.274) that} \)

\[ T - T_K I = \sum_{J=1}^{3} T_J E_J - T_K \sum_{J=1}^{3} E_J \]

\( \text{and thus} \)

\[ T - T_K I = \sum_{J=1}^{3} (T_J - T_K) E_J \] \hspace{1cm} (1.277)

from which we also have

\[ \prod_{K \neq \theta}^{3} \left( T - T_K I \right) = \prod_{K \neq \theta}^{3} \left( \sum_{J=1}^{3} (T_J - T_K) E_J \right) \] \hspace{1cm} (1.278)

where \( \theta \) is 1, 2, or 3. The right-hand side in equation (1.278) can be rewritten as

\[ \prod_{K \neq \theta}^{3} \sum_{J=1}^{3} (T_J - T_K) E_J = \left\{ \prod_{K \neq \theta}^{3} (T_\theta - T_K) \right\} E_\theta , \] \hspace{1cm} (1.279)

noting equation (1.275). Substituting equation (1.279) into equation (1.278), Sylvester’s formula for the eigenprojections is obtained:

\[ E_\theta = \prod_{K \neq \theta}^{3} \frac{T - T_K I}{T_\theta - T_K} \] \hspace{1cm} (1.280)

For instance, the eigenprojection of \( E_2 (\theta = 2) \) is obtained by the above-mentioned method as follows:

\[ E_2 = \prod_{K \neq 2}^{3} \frac{T - T_K I}{T_2 - T_K} = \frac{(T - T_1 I)(T - T_3 I)}{(T_2 - T_1)(T_2 - T_3)} \] \hspace{1cm} (1.281)
1.6.4 Eigenvalues and Vectors of Orthogonal Tensor

It follows from equation (1.102) that

\[
\det(Q^T (Q - I)) = \begin{cases} 
\det(I - Q^T) = \det(-(Q - I)^T) = -\det(Q - I) \\
\det(Q^T)\det(Q - I) = \det(Q - I) 
\end{cases}
\]

by virtue of equation (1.159)\textsubscript{1,3} and (1.173). Then, we have

\[
\det(Q - I) = 0.
\]

By comparing equation (1.282) with equation (1.229), it is evident that one of the eigenvalues of \(Q\) is 1.

Here, choosing \(e_\text{III}\) as the unit principal vector possessing eigenvalue 1 in the orthogonal tensor \(Q\), it follows that

\[
e_\text{III} = Qe_\text{III}.
\]

Further, denoting the orthonormal set of base vectors by \(\{e_3, e_2, e_\text{III}\}\), we have

\[
e_\text{III} \cdot (Q^T e_1) = Qe_\text{III} \cdot e_1 = e_\text{III} \cdot e_1 = 0, \quad e_\text{III} \cdot (Q^T e_2) = Qe_\text{III} \cdot e_2 = e_\text{III} \cdot e_2 = 0
\]

by virtue of equations (1.93)\textsubscript{2} and (1.283). In other words, \(Q^T e_1\) and \(Q^T e_2\) are rotations around the principal base vector \(e_\text{III}\) (see equation (1.109)).

The principal invariants are given by incorporating equations (1.113) into equations (1.234)–(1.236) as follows:

\[
I_Q = II_Q = 1 + 2 \cos \theta, \quad III_Q = 1,
\]

where

\[
I_Q = \text{tr}Q = 1 + 2 \cos \theta \\
II_Q = \frac{1}{2}(\text{tr}^2 Q - \text{tr}Q^2) = \frac{1}{2}(Q_{rr}Q_{ss} - Q_{rs}Q_{sr}) \\
= \frac{1}{2}[2 \cos \theta + 1]^2 - \{\cos^2 \theta + \cos^2 \theta + 1 + 2 \sin \theta (-\sin \theta)] = 1 + 2 \cos \theta \\
III_Q = \det Q = 1
\]

An arbitrary vector rotated from \(v\) on the \((e_1, e_2)\) plane, fulfilling \(v \cdot e_\text{III} = 0\), is expressed by multiplying \(v\) by \(Q^T\) based on equation (1.112) as follows:

\[
Q^T v = v \cos \theta + e_\text{III} \times v \sin \theta,
\]

noting that

\[
Q^T v = [(e_1 \otimes e_1 + e_2 \otimes e_2) \cos \theta - (e_1 \otimes e_2 - e_2 \otimes e_1) \sin \theta + e_\text{III} \otimes e_\text{III}]v \\
= [(e_1(e_1 \cdot v) + e_2(e_2 \cdot v)) \cos \theta - (e_1(e_2 \cdot v) - e_2(e_1 \cdot v)] \sin \theta \\
= v \cos \theta - (v_2 e_1 - v_1 e_2) \sin \theta, \\
-(v_2 e_1 - v_1 e_2) = v_1 e_2 - v_2 e_1 = e_\text{III} \times (v_1 e_1 + v_2 e_2) = e_\text{III} \times v.
\]
1.6.5 Eigenvalues and Vectors of Skew-Symmetric Tensor and Axial Vector

The characteristic equation of the skew-symmetric tensor $\Omega$ is given by equation (1.233) as

$$
\Omega^3 - I_\Omega \Omega^2 + II_\Omega \Omega - III_\Omega = 0,
$$

(1.287)

denoting the eigenvalue by $\Omega$, where the principal invariants are given as

$$
\begin{align*}
I_\Omega &= \text{tr}\Omega = 0 \\
II_\Omega &= -\frac{1}{2} \text{tr}\Omega^2 = -\frac{1}{2} \Omega_{rs} \Omega_{sr} = \omega_1^2 + \omega_2^2 + \omega_3^2 \geq 0 \\
III_\Omega &= \det\Omega = 0
\end{align*}
$$

(1.288)

noting equation (1.141). Equation (1.287) reduces to

$$
\Omega (\Omega^2 + \omega_1^2 + \omega_2^2 + \omega_3^2) = 0.
$$

(1.289)

Therefore, the skew-symmetric tensor possesses only one real eigenvalue 0. This is self-evident because the diagonal components of a skew-symmetric tensor are zero. There exists only one principal direction having zero eigenvalue.

Choosing $e_1$ as the principal direction of $\Omega$ possessing zero eigenvalue, $\Omega$ is represented as

$$
\Omega = \Omega_{32} (e_3 \otimes e_2 - e_2 \otimes e_3).
$$

(1.290)

noting $\Omega_{11} = \Omega_{22} = \Omega_{33} = 0$ and

$$
\begin{align*}
\Omega_{21} &= (e_2 \cdot (\Omega e_1)) = e_2 \cdot (0e_1)) = \Omega_{12} = 0 \\
\Omega_{11} &= (e_3 \cdot (\Omega e_1)) = e_3 \cdot (0e_1)) = \Omega_{13} = 0
\end{align*}
$$

It follows that

$$
\begin{align*}
\Omega a - \Omega_{23} e_1 \times a &= \Omega_{32} (e_3 \otimes e_2 - e_2 \otimes e_3)a + \Omega_{32} e_1 \times a \\
&= \Omega_{32}[(a \cdot e_2)e_3 - (a \cdot e_3)e_2] + e_1 \times [(a \cdot e_1)e_1 + (a \cdot e_2)e_2 + (a \cdot e_3)e_3] \\
&= \Omega_{23}[(a \cdot e_3)e_2 - (a \cdot e_2)e_3 + (a \cdot e_2)e_3 - (a \cdot e_3)e_2] = 0.
\end{align*}
$$

(1.291)

Then the axial vector $\omega$ for the skew-symmetric tensor $\Omega$ can be represented as

$$
\omega = \Omega_{23} e_1,
$$

(1.292)

exploiting equations (1.138) and (1.291). It follows from equations (1.290) and (1.292) that

$$
\Omega \omega = \Omega_{23} e_1 = 0.
$$

(1.293)
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1.6.6 Cayley–Hamilton Theorem

Multiplication to the characteristic equation of equation (1.233) by the principal vector \( \mathbf{e} \) with the aid of equation (1.254) yields

\[
T^3 - \mathbf{I}T^2 + \mathbf{II}T - \mathbf{III} = \mathbf{O}
\]  

(1.294)

Equation (1.294) is referred to as the Cayley–Hamilton theorem.

By virtue of equation (1.294), any power function of \( T \) can be expressed as a polynomial in \( T^2, T, I \) with coefficients consisting of the invariants. For instance, the fourth-power of \( T \) reduces to

\[
T^4 = (I T^2 - I I T + I I I)T = IT^3 - I I T^2 + I I I T = I((IT^2 - I I T + I I I) - I IT^2 + I I I T)
\]

(1.295)

and the inverse of \( T \) is expressed from equation (1.294) by

\[
T^{-1} = (T^2 - I T + I I I) / I I I.
\]

(1.296)

1.7 Polar Decomposition

A second-order tensor \( P \) which is symmetric and fulfills

\[
\mathbf{P} \cdot \mathbf{v} > 0
\]

for an arbitrary vector \( \mathbf{v}(\neq 0) \), is called a positive-definite tensor. For an eigenvalue and direction of \( \mathbf{P} \) given by \( P_f \) and \( \mathbf{e}_f \), respectively, it follows that

\[
P \mathbf{e}_f \cdot \mathbf{e}_f = P_f \mathbf{e}_f \cdot \mathbf{e}_f = P_f ||\mathbf{e}_f||^2 > 0 \text{ (no sum)}.
\]

(1.299)

Thus the eigenvalues of a positive-definite tensor are positive. Taking account of this fact in equation (1.238)_3, we have

\[
\det \mathbf{P} = I I I > 0.
\]

(1.300)

Now, consider the non-singular second-order tensor \( T \) which fulfills \( \det \mathbf{T} \neq 0 \), and thus \( \mathbf{T} \mathbf{v} \neq \mathbf{0} \) holds for an arbitrary vector \( \mathbf{v}(\neq 0) \), noting equations (1.175) and (1.176). The symmetric tensors \( T^T \mathbf{T} \) and \( T T^T \) fulfill

\[
\begin{align*}
T^T \mathbf{T} \mathbf{v} \cdot \mathbf{v} \\
T T^T \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot T^T \mathbf{T} \mathbf{v} = \mathbf{T} \mathbf{v} \cdot \mathbf{T} \mathbf{v} = ||\mathbf{T} \mathbf{v}||^2 > 0,
\end{align*}
\]

(1.301)

using equation (1.93). Thus \( T^T \mathbf{T} \) and \( T T^T \) are positive-definite tensors. Therefore, they possess the same positive eigenvalues, and thus they are mutually similar tensors described in equation
(1.252). Denoting their eigenvalues by $\lambda_2^\alpha$, and the unit principal direction vectors of $T^TT$ and $TT^T$ by $N^{(\alpha)}$ and $n^{(\alpha)} (\alpha = 1, 2, 3)$, respectively, we can write the spectral representations

$$T^TT = \sum_{\alpha=1}^{3} \frac{\lambda_2^\alpha}{\lambda_1^\alpha} N^{(\alpha)} \otimes N^{(\alpha)} \{1.302\}$$

$$TT^T = \sum_{\alpha=1}^{3} \frac{\lambda_2^\alpha}{\lambda_1^\alpha} n^{(\alpha)} \otimes n^{(\alpha)} \{1.303\}$$

Further, let the new positive-definite tensors be defined as follows:

$$U = (T^TT)^{1/2}(T^TT)^{-1}$$

$$V = (TT^T)^{1/2}(TT^T)^{-1}$$

It follows that

$$UN^{(\alpha)} = \lambda_2^\alpha N^{(\alpha)}$$

$$Vn^{(\alpha)} = \lambda_2^\alpha n^{(\alpha)} \{\text{no sum}\} \{1.304\}$$

and then $U$ and $V$ can be written in the spectral representation as follows:

$$U = \sum_{\alpha=1}^{3} \frac{\lambda_2^\alpha}{\lambda_1^\alpha} N^{(\alpha)} \otimes N^{(\alpha)} \{1.305\}$$

$$V = \sum_{\alpha=1}^{3} \frac{\lambda_2^\alpha}{\lambda_1^\alpha} n^{(\alpha)} \otimes n^{(\alpha)} \{1.306\}$$

Now let the following tensor $R$ be introduced.

$$R = TU^{-1}, \quad R = V^{-1}T \{1.307\}$$

for which, noting equation (1.98), one has

$$RR^T = (TU^{-1})(TU^{-1})^T = TU^{-1}U^{-1}T^TT = T(U^2)^{-1}T = TT^{-1}T^TT^T = I \{1.308\}$$

Then, $R$ is an orthogonal tensor. It follows from equation (1.308) that

$$T = RU = VR$$

Now, let us prove that the decomposition in equation (1.308) is unique. Assume that there exists a different decomposition

$$T = \bar{R} \bar{U} = \bar{V} \bar{R} \{1.309\}$$

It follows from equations (1.308) and (1.309) that

$$U^2 = T^TT = (\bar{R} \bar{U})^T(\bar{R} \bar{U}) = \bar{U}^2$$

$$V^2 = TT^T = (\bar{V} \bar{R})(\bar{V} \bar{R})^T = \bar{V}^2 \{1.310\}$$

and

$$\bar{R} = T\bar{U}^{-1} = PU^{-1} = R \{1.311\}$$
Thus, we have \( \bar{U} = U, \bar{V} = V, \bar{R} = R \). It is concluded that the decomposition in equation (1.308) is unique. Equation (1.308) is called the polar decomposition, and \( RU \) and \( VR \) are called the right and left polar decompositions, respectively. From equation (1.308) one has
\[
U = R^T V R, \quad V = R U R^T. \tag{1.312}
\]
Substituting equation (1.305) into equation (1.312), it follows that
\[
V = R U R^T = R \sum_{\alpha=1}^{3} \lambda_\alpha N^{(\alpha)} \otimes N^{(\alpha)} R^T = \sum_{\alpha=1}^{3} \lambda_\alpha R N^{(\alpha)} \otimes R N^{(\alpha)}.
\]
The last quantity in this equation must coincide with equation (1.305) and thus one obtains
\[
n^{(\alpha)} = R N^{(\alpha)}, \quad N^{(\alpha)} = R^T n^{(\alpha)} \tag{1.313}
\]
from which \( R \) can be represented as
\[
R = \sum_{\alpha=1}^{3} n^{(\alpha)} \otimes N^{(\alpha)}. \tag{1.314}
\]
The following expressions are obtained by substituting equations (1.305) and (1.314) into equation (1.308):
\[
T = \sum_{\alpha=1}^{3} \lambda_\alpha n^{(\alpha)} \otimes N^{(\alpha)}, \quad T^{-1} = \sum_{\alpha=1}^{3} \frac{1}{\lambda_\alpha} N^{(\alpha)} \otimes n^{(\alpha)}. \tag{1.315}
\]

### 1.8 Isotropy

Isotropic materials are widely used in engineering practice. The definition of isotropic material and the representation theorem for isotropic tensor-valued tensor functions of single variables are given below.

#### 1.8.1 Isotropic Material

A function \( f \) of tensors \( T, S, \cdots \) is called an isotropic function if it satisfies the equation
\[
Q \llbracket f(T, S, \cdots) \rrbracket = f(Q \llbracket T \rrbracket, Q \llbracket S \rrbracket, \cdots), \tag{1.316}
\]
where the symbol in equation (1.198) is used. If \( f \) is a scalar, it is an invariant, and if it is a tensor, it is called an isotropic tensor-valued tensor function.

Isotropic material is defined to exhibit an identical mechanical response that is independent of the chosen direction of material element or of the coordinate system by which the response is described. Here, the input/output tensor-valued variables related to the mechanical response are the stress and the strain in the elastic material and the stress rate and the strain rate in inelastic (viscoelastic, plastic and viscoplastic) materials which are described in the rate type.

The elastic constitutive equation is generally given by
\[
f(\sigma, \varepsilon, H_i) = 0, \tag{1.317}
\]
where the stress tensor and strain tensor are denoted by the symbols $\sigma$ and $\varepsilon$, respectively, and the scalar- and tensor-valued internal variables are collectively denoted by the symbol $H_i$. When the following equation holds by giving coordinate transformations only for stress and strain tensors in the function $f$, it describes the constitutive equation of isotropic elastic material:

$$f(Q\sigma Q^T, Q\varepsilon Q^T, H_i) = Qf(\sigma, \varepsilon, H_i)Q^T$$

(1.318)

For inelastic materials, the stress rate tensor $\dot{\sigma}$ and the strain rate tensor $\dot{\varepsilon}$ are related through the stress and the internal variables so that the constitutive equation is generally given by

$$f(\dot{\sigma}, \sigma, H_i, \dot{\varepsilon}) = 0, \quad (1.319)$$

When the following equation holds by giving coordinate transformations only for stress (rate) and strain rate tensors in the function $f$, it describes constitutive equations of isotropic materials:

$$f(Q\dot{\sigma}Q^T, Q\sigma Q^T, H_i, QdQ^T) = Qf(\dot{\sigma}, \sigma, H_i, d)Q^T$$

(1.320)

In the plastic constitutive equation formulated incorporating the yield and/or plastic potential function, the isotropy holds if the yield and/or plastic potential function is given by the scalar function of stress invariants and scalar internal variables. Then, denoting the yield and plastic potential functions by $f$, the following equation must hold in isotropic materials:

$$f(Q\sigma Q^T, H_i) = Qf(\sigma, H_i)Q^T$$

(1.321)

Therefore, $f$ is limited to a function of stress invariants and scalar variables $H_i$.

### 1.8.2 Representation Theorem of Isotropic Tensor-Valued Tensor Function

The general representation of the isotropic tensor-valued tensor function of a single second-order tensor is given below, to which isotropic elastic constitutive equations belong.

Letting $A$ and $T$ be symmetric second-order tensors, consider the following isotropic tensor function $A$ of $T$:

$$A = f(T), \quad (1.322)$$

where $f$ fulfills

$$f(QTQ^T) = Qf(T)Q^T. \quad (1.323)$$

Introduce the coordinate system with the bases $e_1, e_2, e_3$ which are the eigenvectors of the tensor $T$ so that equations (1.248) and (1.249) hold. Then adopt the following three particular types of orthogonal tensor possessing only diagonal components:

$$Q_1 = e_1 \otimes e_1 - e_2 \otimes e_2 - e_3 \otimes e_3$$
$$Q_2 = -e_1 \otimes e_1 + e_2 \otimes e_2 - e_3 \otimes e_3$$
$$Q_3 = -e_1 \otimes e_1 - e_2 \otimes e_2 + e_3 \otimes e_3$$

(1.324)

where $Q_1, Q_2, Q_3$ give $\pi$-rotations in the anticlockwise direction around the axes $e_1, e_2, e_3$, respectively. Denoting these principal base vectors by $e_P$ and the orthogonal tensors
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in equation (1.324) collectively by \( Q_p \), it follows that

\[
Q_p = Q_p^T, \tag{1.325}
\]

\[
Q_p e_p = Q_p^T e_p = e_p \quad \text{(no sum)}. \tag{1.326}
\]

Equation (1.326) means that the fixed base vectors \( e_p \) are also eigenvectors of the orthogonal tensor \( Q_p \), keeping in mind that \( e_p \) are the eigenvectors of the tensor \( T \).

Furthermore, from equations (1.248) and (1.324) we have that

\[
Q_p T Q_p^T = T \quad \text{(no sum)} \tag{1.327}
\]

or

\[
Q_p T = T Q_p, \tag{1.328}
\]

noting that

\[
Q_p \sum_{s=1}^{3} \lambda_s e_s \otimes e_s Q_p^T = Q_p \sum_{s=1}^{3} \lambda_s e_s \otimes e_s Q_p
\]

\[
= \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + \lambda_3 e_3 \otimes e_3 = \sum_{p=1}^{3} \lambda_p e_p \otimes e_p.
\]

Substituting equation (1.327) into equation (1.322) leads to

\[
f(Q_p T Q_p^T) = f(T) = A. \tag{1.329}
\]

From equation (1.323), on the other hand, one has

\[
f(Q_p T Q_p^T) = Q_p A Q_p^T. \tag{1.330}
\]

From equations (1.329) and (1.330) the commutative law

\[
A Q_p = Q_p A \tag{1.331}
\]

is derived, and further, noting equation (1.326), the following relation is obtained:

\[
Q_p A e_p = A Q_p e_p = A e_p \quad \text{(no sum)}, \tag{1.332}
\]

which means that \( A e_p \) is the eigenvector of \( Q_p \) and thus has the same direction as \( e_p \). Then, denoting the eigenvalue of \( A \) for the eigenvector \( e_p \) by \( \alpha_p \), one can write

\[
A e_p = \alpha_p e_p \quad \text{(no sum)}. \tag{1.333}
\]

It can be concluded that the tensor \( A \) has the same eigenvectors as the tensor \( T \), leading to coaxiality, so that they are described in the spectral representation in a common eigenvector basis. Therefore, the eigenvalues \( \alpha_1, \alpha_2, \alpha_3 \) of the tensor \( A \) can be represented by the eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) of the tensor \( T \), that is,

\[
\alpha_p = \alpha_p(\lambda_1, \lambda_2, \lambda_3). \tag{1.334}
\]
Regarding the eigenvalues, we may consider three cases. First, suppose all three eigenvalues of \( T \) are different: \( \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1 \). Assume the following linear simultaneous equation:

\[
\alpha_i = \phi_0 + \phi_1 \lambda_i + \phi_2 \lambda_i^2,
\]

that is,

\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix} =
\begin{bmatrix}
1 & \lambda_1 & \lambda_1^2 \\
1 & \lambda_2 & \lambda_2^2 \\
1 & \lambda_3 & \lambda_3^2
\end{bmatrix}
\begin{bmatrix}
\phi_0 \\
\phi_1 \\
\phi_2
\end{bmatrix},
\]

where \( \phi_0, \phi_1, \phi_2 \) are scalar functions of \( \lambda_1, \lambda_2, \lambda_3 \). If we regard equation (1.335) as a simultaneous equation for the unknown values \( \phi_0, \phi_1, \phi_2 \), the Vandermonde determinant of the coefficients \( \lambda_1, \lambda_2, \lambda_3 \) is not zero, that is,

\[
\begin{vmatrix}
1 & \lambda_1 & \lambda_1^2 \\
1 & \lambda_2 & \lambda_2^2 \\
1 & \lambda_3 & \lambda_3^2
\end{vmatrix} = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \neq 0. \tag{1.337}
\]

Therefore, \( \phi_0, \phi_1, \phi_2 \) are uniquely determined by \( \lambda_1, \lambda_2, \lambda_3 \) and thus \( \alpha_1, \alpha_2, \alpha_3 \), as was described in equations (1.175) and (1.176) in Section 1.4.9. Equation (1.335) thus provides the exact relation of the eigenvalues \( \alpha_1, \alpha_2, \alpha_3 \) to \( \lambda_1, \lambda_2, \lambda_3 \).

Since \( \alpha_1, \alpha_2, \alpha_3 \) and \( \lambda_1, \lambda_2, \lambda_3 \) are the eigenvalues of the tensor \( A = f(T) \) and \( T \), respectively, and these tensors possess the same principal directions, the following representation holds from equation (1.335):

\[
\begin{bmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{bmatrix} = \phi_0 \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} + \phi_1 \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix} + \phi_2 \begin{bmatrix}
\lambda_1^2 & 0 & 0 \\
0 & \lambda_2^2 & 0 \\
0 & 0 & \lambda_3^2
\end{bmatrix}, \tag{1.338}
\]

or

\[
\sum_{p=1}^3 \alpha_p \mathbf{e}_p \otimes \mathbf{e}_p = \phi_0 \sum_{p=1}^3 \mathbf{e}_p \otimes \mathbf{e}_p + \phi_1 \sum_{p=1}^3 \lambda_p \mathbf{e}_p \otimes \mathbf{e}_p + \phi_2 \left( \sum_{p=1}^3 \lambda_p \mathbf{e}_p \otimes \mathbf{e}_p \right)^2,
\]

namely,

\[
f(T) = \phi_0 \mathbf{I} + \phi_1 T + \phi_2 T^2. \tag{1.340}
\]

where \( \phi_0, \phi_1, \phi_2 \) can be expressed by the invariants I, II, III of \( T \) in equation (1.238), because they are scalar functions.

Now suppose two of three eigenvalues are same: \( \lambda_1 \neq \lambda_2 = \lambda_3 \). Assume the following linear simultaneous equation:

\[
\begin{align*}
\alpha_1(\lambda_1, \lambda_2) &= \phi_0(\lambda_1, \lambda_2) + \phi_1(\lambda_1, \lambda_2)\lambda_1, \\
\alpha_2(\lambda_1, \lambda_2) &= \phi_0(\lambda_1, \lambda_2) + \phi_1(\lambda_1, \lambda_2)\lambda_2.
\end{align*} \tag{1.341}
\]

The Vandermonde determinant is not zero, that is,

\[
\begin{vmatrix}
1 & \lambda_1 \\
1 & \lambda_2
\end{vmatrix} = -\lambda_1 + \lambda_2 \neq 0. \tag{1.342}
\]
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Equation (1.341) for the spectral expression is represented in the direct tensor notation as follows:

\[ f(T) = \phi_0 I + \phi_1 T, \]  

(1.343)

where \( \phi_0, \phi_1 \) are functions of the first and second invariants, \( I \) and \( II \) of \( T \).

Finally, suppose all three eigenvalues are equal: \( \lambda_1 = \lambda_2 = \lambda_3 \). Then

\[ \alpha_1(\lambda_1) = \phi_0(\lambda_1), \]  

(1.344)

or in direct tensor notation,

\[ f(T) = \phi_0 I, \]  

(1.345)

where \( \phi_0 \) is function of the first invariant of \( T \).

For the case where \( T \) is invertible, substituting equation (1.296) into \( T^2 \) in equation (1.340), we have

\[ f(T) = \psi_0 I + \psi_1 T + \psi_{-1} T^{-1}, \]  

(1.346)

where

\[ \psi_0 = \phi_0 - \phi_2 II, \quad \psi_1 = \phi_1 + \phi_2 I, \quad \psi_{-1} = \phi_2 III, \]  

(1.347)

\[ \phi_0 = \psi_0 + \frac{II}{III} \psi_{-1}, \quad \phi_1 = \psi_1 - \frac{I}{III} \psi_{-1}, \quad \phi_2 = \frac{I}{III} \psi_{-1}. \]  

(1.348)

In the particular case where \( f \) is a linear function of \( T \), equation (1.340) reduces to

\[ f(T) = \lambda (\text{tr}T)I + 2\mu T \]  

(1.349)

where \( \lambda \) and \( \mu \) are the material constants, called the Lamé constants, regarding \( f \) and \( T \) as the stress and strain, respectively, in the linear elastic constitutive equation. Equation (1.349) can be rewritten as

\[ f(T) = CT, \]  

(1.350)

where

\[ C \equiv \lambda I + \mu S \quad (C_{ijkl} \equiv \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})). \]  

(1.351)

Equation (1.350) is also obtained by multiplying the fourth-order isotropic tensor in equation (1.226) by \( T \), noting that the term with the anti-symmetrizing tensor \( A \) vanishes because of the symmetry of the tensor \( T \).

Equation (1.340) is expressed in the eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) of the tensor \( T \) as follows:

\[
\begin{align*}
  f(\lambda_1) &= \phi_0 + \phi_1 \lambda_1 + \phi_2 \lambda_1^2 \\
  f(\lambda_2) &= \phi_0 + \phi_1 \lambda_2 + \phi_2 \lambda_2^2 \\
  f(\lambda_3) &= \phi_0 + \phi_1 \lambda_3 + \phi_2 \lambda_3^2 
\end{align*}
\]  

(1.352)

By solving equation (1.352) for \( \phi_0, \phi_1, \phi_2 \) and substituting the solutions into equation (1.340), the following equation, called the Lagrange–Sylvester equation, is obtained
For the second-order isotropic tensor-valued tensor function of a single tensor considered above, the representation theorem of the symmetric isotropic second-order tensor-valued tensor function \( f \) of two tensors \( A \) and \( B \) is shown below (Spencer, 1971):

\[
f(A, B) = \varphi_0 I + \varphi_1 A + \varphi_2 B + \varphi_3 A^2 + \varphi_4 B^2 + \varphi_9 (AB + BA)
+ \varphi_6 (A^2B + BA^2) + \varphi_7 (AB^2 + B^2A) + \varphi_8 (A^2B^2 + B^2A^2),
\]

where \( \varphi_0, \varphi_1, \ldots, \varphi_9 \) are scalar-valued homogeneous functions of invariants:

\[
\begin{align*}
\text{tr} A, \text{tr} A^2, \text{tr} AA^3, \text{tr} B, \text{tr} B^2, \text{tr} B^3 \quad &\quad \left\{ \begin{array}{c}
\text{tr}(AB), \text{tr}(AB^2), \text{tr}(A^2B), \text{tr}(A^3B^2) \end{array} \right. \\
\end{align*}
\]

1.9 Differential Formulae

This section provides various differential formulae which are often used in continuum mechanics.

1.9.1 Partial Derivatives

Several partial derivatives which will appear in later chapters are listed as follows.

1.

\[
\frac{\partial T^a}{\partial T} = \sum_{k=1}^{n} T^{a-1} \otimes T^{q-k} = I \otimes T^{a-1} + T \otimes T^{q-2} + \cdots + T^{q-2} \otimes T + T^{a-1} \otimes I,
\]

\[
\left( \frac{\partial T^a}{\partial T} \right)_{ijkl} = \sum_{k=1}^{n} (T^{a-1})_{lk} (T^{q-k})_{lj}
\]

which will be derived later by the directional derivative in Section 1.9.2.
Examples:
\[ \frac{\partial T}{\partial T} = I \otimes I, \quad \frac{\partial T_{ij}}{\partial T_{kl}} = \delta_{ik}\delta_{j\ell} = \delta_{ik}\delta_{j\ell} \]
\[ \frac{\partial T^2}{\partial T} = I \otimes T + T \otimes I, \quad \frac{\partial T_{ij}}{\partial T_{kl}} = \delta_{ik}\delta_{j\ell} + T_{ik}\delta_{j\ell} \]
\[ \frac{\partial T^3}{\partial T} = I \otimes T^2 + T \otimes T + T^2 \otimes I, \quad \frac{\partial (T_{ij}T_{j\ell})}{\partial T_{kl}} = \delta_{ik}\delta_{j\ell} + T_{ik}T_{j\ell} + T_{ji}T_{\ell k} \]
\[ (1.357) \]

For a symmetric tensor \( T = T^T \), one can write
\[ \frac{\partial T}{\partial T} = S, \quad \frac{\partial T_{ij}}{\partial T_{kl}} = \frac{1}{2} (\delta_{ik}\delta_{j\ell} + \delta_{ik}\delta_{j\ell}). \] \[ (1.357)' \]

2.
\[ \frac{\partial (T_{ijkl}T_{ik}T_{j\ell}T_{k\ell}T_{\ell m})}{\partial T_{ij}} (1.358) \]
\[ = \delta_{i\ell} T_{ijkl}T_{ik}T_{j\ell}T_{k\ell}T_{\ell m} + \delta_{i\ell} T_{ijkl}T_{ik}T_{j\ell}T_{k\ell}T_{\ell m} + \ldots \]
\[ = n(T_{ijkl})_{ij} \]
\[ (1.358) \]

which will be derived later by the directional derivative in Section 1.9.2.

Examples:
\[ \frac{\partial T}{\partial T} = \frac{\partial T^2}{\partial T} = 2T, \quad \frac{\partial T^3}{\partial T} = 3(T^2)^T = 3(T^T)^2. \] \[ (1.359) \]

3.
\[ \frac{\partial T}{\partial T} = D, \quad \frac{\partial T_{ij}}{\partial T_{kl}} = D_{ijkl} \] \[ (1.360) \]

4.
\[ \frac{\partial ||T||}{\partial T} = \frac{T'}{||T||} = T', \quad \frac{\partial \sqrt{T_{ij}T_{ji}}}{\partial T_{ij}} = T'_{ij} \] \[ (1.361) \]
5. \[
\frac{\partial \mathbf{T}'}{\partial \mathbf{T}'} = \frac{1}{||\mathbf{T}'||} (\mathbf{I} - \mathbf{t}' \otimes \mathbf{t}'), \quad \frac{\partial t'_{ij}}{\partial T'_{kl}} = \frac{1}{\sqrt{T'_{pq} T'_{pq}}} (\delta_{ik} \delta_{jl} - t'_{ij} t'_{kl}) \tag{1.362}
\]
\[
\left( \frac{\partial \mathbf{T}'}{\partial \mathbf{T}'} = \frac{\partial}{\partial \mathbf{T}'} \left( \frac{\mathbf{T}'}{||\mathbf{T}'||} \right) \right) = \frac{\overline{\mathbf{T}}}{||\mathbf{T}'||} - \mathbf{T}' \otimes \frac{1}{||\mathbf{T}'||^2} \mathbf{t}'.
\]

6. \[
\frac{\partial \mathbf{T}'}{\partial \mathbf{T}} = \frac{1}{||\mathbf{T}'||} (\overline{\mathbf{T}} - \frac{1}{3} \mathbf{I} - \mathbf{t}' \otimes \mathbf{t}'), \quad \frac{\partial t'_{ij}}{\partial T_{kl}} = \frac{1}{||\mathbf{T}'||} \left( \overline{\mathbf{T}}_{ijkl} - \frac{1}{3} \mathbf{I}_{ijkl} - t'_{ij} t'_{kl} \right) \tag{1.363}
\]
\[
\left( \frac{\partial \mathbf{T}'}{\partial \mathbf{T}} = \frac{\partial}{\partial \mathbf{T}} \left( \frac{\mathbf{T}'}{||\mathbf{T}'||} \right) \right) = \left\{ \frac{1}{||\mathbf{T}'||} (\overline{\mathbf{T}} - \mathbf{t}' \otimes \mathbf{t}') \right\} : \mathbf{D} = \left\{ \frac{1}{||\mathbf{T}'||} (\overline{\mathbf{T}} - \frac{1}{3} \mathbf{I}) \right\}.
\]

7. \[
\frac{\partial \cos 3\theta}{\partial \mathbf{t}'} = 3\sqrt{6} \mathbf{t}', \quad \frac{\partial \cos 3\theta}{\partial t'_{ij}} = 3\sqrt{6} \mathbf{t}'_{ir} \mathbf{t}'_{rj} \tag{1.364}
\]
\[
\frac{\partial \cos 3\theta}{\partial \mathbf{T}} = \frac{\partial \cos 3\theta}{\partial \mathbf{t}'} : \frac{\partial \mathbf{t}'}{\partial \mathbf{T}} = 3\sqrt{6} \overline{\mathbf{t}} : \mathbf{D} = \left\{ \frac{1}{||\mathbf{T}'||} (\overline{\mathbf{T}} - \mathbf{t}' \otimes \mathbf{t}') \right\} : \left\{ \overline{\mathbf{T}} - \frac{1}{3} \mathbf{I} \right\}.
\]

where \[
\cos 3\theta \equiv \sqrt{6} \mathbf{t}' \mathbf{t}'^3,
\] which often appears in yield conditions depending on the third invariant of deviatoric stress, that is, the intermediate principal stress as observed in frictional materials (Hashiguchi, 2009).

8. \[
\frac{\partial \cos 3\theta}{\partial \mathbf{T}} = -\frac{3\sqrt{6}}{||\mathbf{T}'||} \left( \frac{1}{3} ||\mathbf{t}'||^2 \mathbf{I} + \frac{1}{\sqrt{6}} \cos 3\theta \mathbf{t}' - \mathbf{t}'^2 \right),
\]
\[
\frac{\partial \cos 3\theta}{\partial T_{ij}} = -\frac{3\sqrt{6}}{\sqrt{T'_{pq} T'_{pq}}} \left( \frac{1}{3} t'_{ir} t'_{rj} \delta_{ij} + \frac{1}{\sqrt{6}} \cos 3\theta t'_{ij} - t'_{ij} t'_{rj} \right) \tag{1.366}
\]
\[
\left( \frac{\partial \cos 3\theta}{\partial \mathbf{T}} = \frac{\partial \cos 3\theta}{\partial \mathbf{t}'} : \frac{\partial \mathbf{t}'}{\partial \mathbf{T}} = 3\sqrt{6} \overline{\mathbf{t}} : \frac{1}{||\mathbf{T}'||} \left( \overline{\mathbf{T}} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} - \mathbf{t}' \otimes \mathbf{t}' \right) \right)
\]
\[
= \frac{3\sqrt{6}}{||\mathbf{T}'||} \left\{ \mathbf{t}'^2 - \frac{1}{3} (\mathbf{t}' \mathbf{t}'^2) \mathbf{I} - (\mathbf{t}' \mathbf{t}'^3) \right\}.
\]

9. \[
\frac{\partial \mathbf{T}^{-1}}{\partial \mathbf{T}} = -\mathbf{T}^{-1} \otimes \mathbf{T}^{-1} = -\mathbf{T}^{-1} \overline{\mathbf{T}} \mathbf{T}^{-1}, \quad \frac{\partial T_{ij}^{-1}}{\partial T_{kl}^{-1}} = -T_{ik}^{-1} T_{lj}^{-1} = -T_{ir}^{-1} \overline{T}_{rskl} T_{sj}^{-1} \tag{1.367}
\]
\[
\left( \frac{\partial(T^{-1}T_{rs})}{\partial T_{kl}} = \frac{\partial T^{-1}_{ir}}{\partial T_{kl}} T_{rs} + T^{-1}_{ir} \delta_{ik} \delta_{jl} = 0 \sim \frac{\partial T^{-1}_{ir}}{\partial T_{kl}} T_{rs} T_{ij} + T^{-1}_{ir} \delta_{ik} \delta_{jl} T_{ij} = 0 \right) \sim \frac{\partial T^{-1}_{ij}}{\partial T_{kl}} + T^{-1}_{ik} T^{-1}_{lj} = 0
\]

which will be derived later by the directional derivative in Section 1.9.2.

10. \[
\hat{T}^{-1} = -T^{-1} \hat{T} T^{-1}, \quad \hat{T}^{-1}_{ij} = -T^{-1} \hat{T}_{ij} T^{-1}_{ij} \quad (1.368)
\]

\[
(\hat{I} = (T^{-1}T)^* = \hat{T}^{-1}T + T^{-1}\hat{T} = 0) \left( \frac{\partial T^{-1}_{ij}}{\partial T_{kl}} \hat{T}_{ij} = -T^{-1}_{ik} \hat{T}_{kl} T^{-1}_{lj} \right)
\]

which will be derived later by the directional derivative in Section 1.9.2.

11. \[
\frac{\partial \text{tr} (\text{cof} T)}{\partial T} = \frac{\partial H}{\partial T} = (\text{tr} T)I - T^T, \quad \frac{\partial \text{tr} (\text{cof} T)}{\partial T_{ij}} = T_{ij} \delta_{ij} - T_{ij} \quad (1.369)
\]

\[
\left( \frac{\partial \text{tr} (\text{cof} T)}{\partial T_{ij}} = \frac{\partial}{\partial T_{ij}} \epsilon_{abc} \epsilon_{aqr} T_{bq} T_{cr} = 2 \frac{1}{2} \epsilon_{abc} \epsilon_{aqr} \delta_{iq} \delta_{jr} T_{cr} = \epsilon_{abc} \epsilon_{aqr} T_{cr} \right) = (\delta_{ij} \delta_{cr} - \delta_{qr} \delta_{ij}) T_{cr} = T_{ij} \delta_{ij} - T_{ij}
\]

noting equations (1.56), (1.153) and (1.235).

12. \[
\frac{\partial (\text{cof} T)_{ij}}{\partial T_{kl}} = \epsilon_{ikq} \epsilon_{jlr} T_{qr} \quad (1.370)
\]

\[
\left( \frac{\partial (\text{cof} T)_{ij}}{\partial T_{kl}} = \frac{\partial}{\partial T_{kl}} \frac{1}{2} \epsilon_{ipq} \epsilon_{jrs} T_{pr} T_{qs} = \frac{1}{2} \left( \epsilon_{ipq} \epsilon_{jrs} \delta_{kp} \delta_{iq} T_{pr} + \epsilon_{ipq} \epsilon_{jrs} \delta_{kp} \delta_{iq} T_{pr} \right) \right).
\]

13. \[
\frac{\partial \text{det} T}{\partial T} = \frac{\partial H}{\partial T} = \text{cof} T = (\text{det} T)T^{-T}, \quad \frac{\partial \text{det} T}{\partial T_{ij}} = \frac{\partial H}{\partial T_{ij}} = (\text{cof} T)_{ij} = (\text{det} T)T_{ji}^{-1} \quad (1.371)
\]

\[
\left( \frac{\partial}{\partial T_{ij}} \epsilon_{abc} \epsilon_{pqr} T_{ap} T_{bq} T_{cr} = \frac{3}{2} \epsilon_{abc} \epsilon_{pqr} \delta_{ip} \delta_{jq} T_{ap} T_{bq} T_{cr} = \frac{1}{2} \epsilon_{ipq} \epsilon_{jrs} T_{pr} T_{qs} \right) = \left. \left( \frac{\partial}{\partial T_{ij}} \epsilon_{abc} \epsilon_{pqr} T_{ap} T_{bq} T_{cr} \right) \right|_{T}
\]

\[
\frac{\partial H}{\partial T} = \left\{ \frac{\partial}{\partial T} \left( \frac{1}{6} tr^3 T - \frac{1}{2} tr T T^2 + \frac{1}{3} tr T^3 \right) \right\} : T
\]

\[
= \left\{ \frac{3}{6} (tr^2 T) I - \frac{1}{2} tr T T^2 - (tr T) T^T + (T^2)^T \right\} : T
\]

\[
= \left\{ \frac{3}{6} tr^2 T T^T - \frac{1}{2} (tr T) T T^2 - (tr T) T T^2 + tr T T^3 \right\},
\]

\[
= \frac{3}{6} tr^3 T - \frac{3}{2} (tr T) T T^2 + tr T^3 = 3 III = III T^{-T} : T
\]

noting equations (1.151), (1.153), (1.297) with equation (1.157), while equation (1.371) will be also derived later by the directional derivative in 1.9.2. It follows from equation (1.371) that

$$\frac{\partial \sqrt{\det T}}{\partial T} = \frac{1}{2} \frac{\sqrt{\det T}}{\partial T}$$

(1.372)

Besides, applying equation (1.371), equation (1.159) can be proved by the following alternative method:

$$(\det e^{A^t}) = (\det e^{A^t})(e^{A^t})^{-T} : (e^{A^t}) = (\det e^{A^t})(\det e^{A^t})^{-T} : (e^{A^t})$$

$$= (\det e^{A^t})\text{tr}\{(e^{A^t})^{-1}(e^{A^t})A\} = (\det e^{A^t})\text{tr}A$$

$$\sim (\det e^{A^t})\text{tr}A \sim \ln(\det e^{A^t}) = \text{tr}(A) \sim \det[\exp(A^t)] = \exp[\text{tr}(A^t)]$$

$$\sim \det(\exp T) = \exp(\text{tr}T),$$

(1.159)

noting $$(e^{A^t}) = e^{T^t}A$$ by $d[(1/n!)(Tr^n)/dt = [(1/(n−1)!)(Tr^{n−1}]T$$ for $n = 0, 1, 2, \ldots, \infty$ in equation (1.258). This is one of the formulae concerning the exponential map or tensor exponential function used widely in numerical analysis for finite strain elastoplasticity. Other useful formulae are:

$$\exp(A + B) = (\exp A)(\exp B) \text{ if } AB = BA, \quad \exp(mA) = (\exp A)^m$$

$$\exp(BAB^{-1}) = B(\exp A)B^{-1}$$

(1.372)

where $m$ is a positive or negative integer. The former two equations can be proved readily by the spectral representation used for equation (1.159) or by substituting $A + B$ and $mA$ into $T$ in the series expansion equation (1.258) with the binomial theorem (Hirsch and Small, 1974):

$$(A + B)^n = n! \left( \frac{A^i}{i!} \frac{A^j}{j!} \right) \sim \exp(A + B) = \sum_{r=0}^{n} \sum_{r+j+n}^{n} \left( \frac{A^i}{i!} \frac{A^j}{j!} \right)$$

under the infinite series for the first equation. The last equation can be proved by substituting $BAB^{-1}$ into $T$ in equation (1.258), taking into account that $(BAB^{-1})^n = BA^kB^{-1}$.

14.

$$\frac{\partial \ln \sqrt{\det T}}{\partial T} = \frac{1}{2} \frac{\sqrt{\det T}}{\partial T}^{-1}, \quad \frac{\partial \ln \sqrt{\det T}}{\partial T_{ij}} = \frac{1}{2} \frac{\sqrt{\det T}}{\partial T_{ij}}^{-1}$$

(1.373)

$$\left( \frac{\partial \ln \sqrt{\det T}}{\partial T} = \frac{1}{\sqrt{\det T}} \frac{\partial \sqrt{\det T}}{\partial T} = \frac{1}{\sqrt{\det T}} \frac{1}{2 \sqrt{\det T} \sqrt{\det T}} (\det T)^{i_{k_j}} \right)$$

15.

$$\frac{\partial \exp T}{\partial T} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{n!} \frac{1}{T^{n-k}} \sim \exp T^{n-k}, \quad \left( \frac{\partial \exp T}{\partial T} \right) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left( \frac{1}{n!} \frac{1}{T^{n-k}} \right)^{i_{k_j}}$$

(1.374)
\[
\left( \partial \left( I + T + \frac{1}{2} T^2 + \frac{1}{6} T^3 + \ldots \right) \right) = I \otimes I + \frac{1}{2} (T \otimes I + I \otimes T) \\
\quad + \frac{1}{3!} (T^2 \otimes I + T \otimes T + I \otimes T^2) + \ldots
\]

noting equations (1.258) and (1.356) ((1.357)).

For later use, the partial derivatives of a scalar function of various second-order tensor invariants are given below. Noting equations (1.235), (1.359), (1.361), (1.366), (1.369), (1.371) and (1.372) shown in Section 1.9.1, one obtains:

\[
\frac{\partial f(I, II, III)}{\partial T} = \frac{\partial f}{\partial I} I + \frac{\partial f}{\partial II} (T + \frac{1}{2} (T \otimes I + I \otimes T)) + \frac{\partial f}{\partial III} (T^2 + \frac{1}{6} (T^2 \otimes I + T \otimes T + I \otimes T^2)) + \ldots
\]

which is rewritten in view of equation (1.296) as

\[
\frac{\partial f(I, II, III)}{\partial T} = \left( \frac{\partial f}{\partial I} + \frac{\partial f}{\partial II} T \right) I - \frac{\partial f}{\partial II} T^T + \frac{\partial f}{\partial III} (\text{det}(T))T^{-T}
\]

\[
\frac{\partial f(I, II, III)}{\partial T} = \left( \frac{\partial f}{\partial I} + \frac{\partial f}{\partial II} T \right) I - \frac{\partial f}{\partial II} T^T + \frac{\partial f}{\partial III} (\text{det}(T))T^{-T}
\]

\[
\frac{\partial f(I, II, III)}{\partial T} = \frac{\partial f}{\partial I} I + 2 \frac{\partial f}{\partial II} T^T + 3 \frac{\partial f}{\partial III} T^{2T},
\]

\[
\frac{\partial f(I, ||T||, \cos 3\theta)}{\partial T} = \frac{\partial f}{\partial I} I + \frac{\partial f}{\partial ||T||} ||T||^2 + \frac{\partial f}{\partial T} \frac{\partial \cos 3\theta}{\partial T}
\]

\[
= \frac{\partial f}{\partial I} I + \frac{\partial f}{\partial ||T||} ||T||' + \frac{\partial f}{\partial \cos 3\theta} \frac{1}{\sqrt{6}} \cos 3\theta (\cos 3\theta - \frac{1}{3} (||T||^2 I + \frac{1}{\sqrt{6}} \cos 3\theta T^T - T^2))
\]

1.9.2 Directional Derivatives

The derivative of a scalar-valued function \( f(x) \) of a scalar \( x \) is defined by

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

Then, we can write formally as

\[
f'(x; s) = \lim_{h \to 0} \frac{f(x + hs) - f(x)}{h} = \left. \frac{\partial f}{\partial x} \right|_{x=0} + \left. \frac{\partial f}{\partial x} \right|_{x=0} \frac{\partial (x + hs)}{\partial h}
\]

(1.380)
Equation (1.380) can be extended for the scalar-valued vector function as follows:

\[ f'(v; a) \equiv \lim_{h \to 0} \frac{f(v + ha) - f(v)}{h} = \frac{\partial f(v + ha)}{\partial h} \bigg|_{h=0} = \frac{\partial f(v)}{\partial v} \cdot a \quad (1.381) \]

which is referred to as the directional derivative. Analogously, we have the following equations for the vector \( f, v, a \) and the second-order tensors \( F, T, A \).

\[ f'(v; a) \equiv \lim_{h \to 0} \frac{f(v + ha) - f(v)}{h} = \frac{\partial f(v + ha)}{\partial h} \bigg|_{h=0} = \frac{\partial f(v)}{\partial v} \cdot a, \quad (1.382) \]

\[ F'(v; a) \equiv \lim_{h \to 0} \frac{F(v + ha) - F(v)}{h} = \frac{\partial F(v + ha)}{\partial h} \bigg|_{h=0} = \frac{\partial F(v) \cdot a}{\partial v} \quad (1.383) \]

\[ F'(T; A) \equiv \lim_{h \to 0} \frac{F(T + hA) - F(A)}{h} = \frac{\partial F(T + hA)}{\partial h} \bigg|_{h=0} = \frac{\partial F(T) \cdot A}{\partial T} \quad (1.384) \]

Here, the following relations are fulfilled.

\[ \lim_{h \to 0} \frac{f(v + ha) - f(v)}{h} = s \lim_{h \to 0} \frac{f(v + ha) - f(v)}{h}, \quad (1.385) \]

\[ \lim_{h \to 0} \frac{F(v + ha) - F(v)}{h} = s \lim_{h \to 0} \frac{F(v + ha) - F(v)}{h}, \quad (1.386) \]

\[ \lim_{h \to 0} \frac{F(T + hA) - F(T)}{h} = s \lim_{h \to 0} \frac{F(T + hA) - F(T)}{h}, \quad (1.387) \]

and

\[ \lim_{h \to 0} \frac{f(v + h(a + b)) - f(v)}{h} = \lim_{h \to 0} \frac{f(v + h(a + b)) - f(v + ha)}{h} + \lim_{h \to 0} \frac{f(v + ha) - f(v)}{h} = \frac{\partial f(v)}{\partial v} \cdot b + \frac{\partial f(v)}{\partial v} \cdot a, \quad (1.388) \]

\[ \lim_{h \to 0} \frac{F(v + h(a + b)) - F(v)}{h} = \lim_{h \to 0} \frac{F(v + h(a + b)) - F(v + ha)}{h} + \lim_{h \to 0} \frac{F(v + ha) - F(v)}{h} = \frac{\partial F(v)}{\partial v} \cdot b + \frac{\partial F(v)}{\partial v} \cdot a, \quad (1.389) \]

\[ \lim_{h \to 0} \frac{F(T + h(A + B)) - F(T)}{h} = \lim_{h \to 0} \frac{F(T + h(A + B)) - F(T + hA)}{h} + \lim_{h \to 0} \frac{F(T + hA) - F(T)}{h} = \frac{\partial F(T)}{\partial T} \cdot B + \frac{\partial F(T)}{\partial T} \cdot A. \quad (1.390) \]
Therefore, we can be convinced that the directional derivatives of tensor functions are the linear transformations obeying the scalar multiplication and distribution laws. The above-mentioned equations for the directional derivatives are often used to derive the partial derivatives of tensor functions.

In what follows, the partial derivatives for some second-order tensor functions will be derived by exploiting the directional derivative in equation (1.384), which have been derived already along the ordinary tensor derivatives in 1.9.1.

\[
\frac{\partial T^n}{\partial T} : A = \left. \frac{\partial (T + hA)^n}{\partial h} \right|_{h=0} = \left. \frac{\partial}{\partial h} \left( T^n + h \sum_{i=1}^{n} T_i^{-1} A T_i^{-1} + h^2 \sum_{i=1}^{n} \cdots + \cdots \right) \right|_{h=0}
\]

\[
= \sum_{i=1}^{n} T_i^{-1} A T_i^{-1} = \sum_{i=1}^{n} T_i^{-1} \otimes T_i^{-1} : A, \quad (1.391)
\]

\[
\frac{\partial \text{tr} T^n}{\partial T} : A = \left. \frac{\partial \text{tr}(T + hA)^n}{\partial h} \right|_{h=0} = \left. \frac{\partial}{\partial h} \text{tr}(T^n + h(A T_i^{-1} + T A T_i^{-1} + \cdots + T^n A)) \right|_{h=0}
\]

\[
+ h^2 (\cdots + \cdots) = \text{tr}(nT^{n-1}A) = n(T^{n-1})^T : A, \quad (1.392)
\]

\[
\frac{\partial ((T + hA)^{-1}) (T + hA)}{\partial h} \bigg|_{h=0} = \left. \frac{\partial (T + hA)^{-1}}{\partial h} \right|_{h=0} (T + hA) + (T + hA)^{-1} \left. \frac{\partial (T + hA)}{\partial h} \right|_{h=0}
\]

\[
= \frac{\partial (T + hA)^{-1}}{\partial h} \bigg|_{h=0} T + T^{-1} A = 0 \rightarrow \frac{\partial T^{-1}}{\partial T} : A
\]

\[
= \frac{\partial (T + hA)^{-1}}{\partial h} \bigg|_{h=0} = -T^{-1} A T^{-1} = - T^{-1} \otimes T^{-1} : A, \quad (1.393)
\]

\[
\frac{\partial \text{det} T}{\partial T} : A = \left. \frac{\partial \text{det}(T + hA)}{\partial h} \right|_{h=0} = \left. \frac{\partial (\text{det}T \text{det}(I + hT^{-1}A))}{\partial h} \right|_{h=0}
\]

\[
= \left. \frac{\partial (\text{det}Th^2 \text{det}[T^{-1}A - (-h^{-1})I])}{\partial h} \right|_{h=0}
\]

\[
= \left. \frac{\partial [\text{det}Th^3(-(-h^{-1})^3 + (-h^{-1})^2I - (-h^{-1})II + III)]}{\partial h} \right|_{h=0}
\]

\[
= \left. \frac{\partial [\text{det}(1 + hI + h^2 II + h^3 III)]}{\partial h} \right|_{h=0} = (\text{det}T)I
\]

\[
= (\text{det}T)\text{tr}(T^{-1}A) = (\text{det}T)T^{-T} : A, \quad (1.394)
\]

designating principal invariants of the tensor \(T^{-1}A\) as \(I, II, III\) and noting equation (1.233) and that the eigenvalue of \(T^{-1}A\) is \(-h^{-1}\). We obtain equations (1.356), (1.358), (1.367) and (1.371).
from equations (1.391), (1.392), (1.393) and (1.394), respectively. The directional derivative is useful to derive the partial derivatives of tensor functions more concisely compared with the method in component forms.

### 1.9.3 Taylor Expansion

The Taylor expansion of a scalar-valued scalar function is given by

\[
f(x + h) = \sum_{k=0}^{\infty} \frac{h^k \, d^k f(x)}{k! \, dx^k} = f(x) + h \frac{df(x)}{dx} + \frac{h^2 \, d^2 f(x)}{2! \, dx^2} + \cdots,
\]

(1.395)

defining 0! = 1. Choosing \( x = x_0 \) and \( h = \alpha u \) in equation (1.395) and noting the relation

\[
\left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0} = \left. \frac{d^n f(x_0 + \epsilon u)}{d(\epsilon u)^n} \right|_{\epsilon=0} = \frac{1}{\alpha^n} \frac{d^n f(x_0 + \epsilon u)}{d\epsilon^n} \bigg|_{\epsilon=0},
\]

(1.397)

it follows that

\[
f(x_0 + \alpha u) = \sum_{k=0}^{\infty} \frac{(\alpha u)^k \, d^k f(x_0)}{k! \, dx_0^k} = \sum_{k=0}^{\infty} \frac{(\alpha u)^k}{k!} \frac{1}{\alpha^k} \left. \frac{d^n f(x_0 + \epsilon u)}{d\epsilon^n} \right|_{\epsilon=0} = f(x_0) + \alpha \left. \frac{df(x_0 + \epsilon u)}{d\epsilon} \right|_{\epsilon=0} + \alpha^2 \frac{1}{2!} \left. \frac{d^2 f(x_0 + \epsilon u)}{d\epsilon^2} \right|_{\epsilon=0} + \cdots.
\]

(1.396)

Based on equation (1.396), the Taylor expansion of the tensor-valued tensor function \( F(x_0 + \alpha u) \) is as follows:

\[
F(x_0 + \alpha u) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \left. \frac{d^k F(x_0 + \epsilon u)}{d\epsilon^k} \right|_{\epsilon=0} = F(x_0) + \alpha \left. \frac{dF(x_0 + \epsilon u)}{d\epsilon} \right|_{\epsilon=0} + \alpha^2 \frac{1}{2!} \left. \frac{d^2 F(x_0 + \epsilon u)}{d\epsilon^2} \right|_{\epsilon=0} + \cdots.
\]

(1.397)

The truncation of the Taylor expansion in equation (1.397) reduces to

\[
F(x_0 + \alpha u) - F(x_0) \approx \alpha \left. \frac{dF(x_0 + \epsilon u)}{d\epsilon} \right|_{\epsilon=0},
\]

(1.398)

which reduces in turn to the following equation by choosing \( \alpha = 1 \):

\[
F(x_0 + u) - F(x_0) \approx \left. \frac{dF(x_0 + \epsilon u)}{d\epsilon} \right|_{\epsilon=0}.
\]

(1.399)

The right-hand side of equation (1.399) is the directional derivative shown in equation (1.383), and thus it follows that

\[
F(x_0 + u) - F(x_0) \equiv F'(x_0; u).
\]

(1.400)
The following equation obtained by setting \( F(x_0 + u) = 0 \) in equation (1.400) is used for a convergence process in numerical calculation of the nonlinear algebraic equation \( F(x) = 0 \) (Bonet and Wood, 2008):

\[
F(x_0) + F'(x_0; u) = 0
\]  

which is rewritten by equation (1.383) as follows:

\[
F(x_0) + \frac{\partial F}{\partial x_0} u = 0. \tag{1.402}
\]

Analogously, the following equation holds for the vector function \( f \) by virtue of equation (1.382).

\[
f(x_0) + \frac{\partial f}{\partial x_0} u = 0. \tag{1.403}
\]

### 1.9.4 Time Derivatives in Lagrangian and Eulerian Descriptions

A map that completely describes the positions of all the material particles (or material points or material elements) in a body is called a configuration. For some scalar- or tensor-valued physical quantity \( \psi \), denoting the time by \( t \), the position vector of a material particle in the reference configuration by \( X \) and the position vector of a material particle in the current configuration by \( x \), the distribution of \( \psi \) in the space can be formally described by \( \psi(X, t) \) or \( \psi(x, t) \), where the mapping from \( x \) to \( X \) is generally described as follows:

\[
x = \chi(X), \quad X = \chi^{-1}(x, t) \tag{1.404}
\]

which is taken to describe the motion of the material particle. The representations \( \psi(X, t) \) and \( \psi(x, t) \) are called the Lagrangian (material) description (or representation) and the Eulerian (spatial) description (or representation), respectively.

The velocity of the material particle \( X \) is given by

\[
v = \frac{\partial x(X, t)}{\partial t} \tag{1.405}
\]

in the Lagrangian description and

\[
v = \frac{\partial \chi^{-1}(x, t)}{\partial t} \tag{1.406}
\]

in the Eulerian description.

The rate of the physical quantity \( \psi \) for a particular material particle is gained by observation, moving with material particle, and is referred to as the material-time derivative which is given by the total differentiation with respect to time. The material-time derivative in the Lagrangian description is defined by

\[
\dot{\psi} \equiv \frac{D \psi}{Dt} = \frac{\partial \psi(X, t)}{\partial t} + \frac{\partial \psi(X, t)}{\partial X} \cdot \frac{\partial X}{\partial t} = \lim_{t \to 0} \frac{\psi(X, t + \Delta t) - \psi(X, t)}{\Delta t}. \tag{1.407}
\]
In the Eulerian description it is given by

\[
\dot{\psi} \equiv D\psi \equiv \frac{d\psi(x, t)}{dt} = \frac{\partial \psi(x, t)}{\partial t} + \nabla \cdot \frac{\partial \psi(x, t)}{\partial x} \cdot \frac{\partial x}{\partial t} = \frac{\partial \psi(x, t)}{\partial t} + v(x, t) \cdot \nabla \psi(x, t)
\]

(1.408)

The first term in equation (1.408) signifies the non-steady (or local time derivative) term describing the variation of the quantity at a spatially fixed point \(x\) with time – the spatial-time derivative. The second term signifies the convective term describing the variation attributable to the movement of the material particle under the existence of the gradient \(\frac{\partial \psi(x, t)}{\partial x}\) due to the heterogeneity in the spatial distribution of \(\psi\). Both symbols (* ) and \(D(\cdot)/Dt\) are used widely to specify the material-time derivative.

The material-time derivatives of vectors and second-order tensors in the Eulerian and the Lagrangian description are given from equation (1.408) as

\[
\dot{\mathbf{a}} = \frac{\partial \mathbf{a}(X, t)}{\partial t} = \frac{d\mathbf{a}(x, t)}{dt} = \frac{\partial \mathbf{a}(x, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{a}(x, t)}{\partial x} = \frac{\partial \mathbf{a}(x, t)}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{a}
\]

(1.409)

\[
\dot{T} = \frac{\partial T(X, t)}{\partial t} = \frac{dT(x, t)}{dt} = \frac{\partial T(x, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial T(x, t)}{\partial x} = \frac{\partial T(x, t)}{\partial t} + \mathbf{v} \cdot \nabla T.
\]

(1.410)

The motion of a perfect fluid is independent of the deformation history from the initial configuration and thus only a current spatial flow (motion) of a physical quantity is meaningful. The Eulerian description based on the spatial-time derivative \(\frac{\partial \psi(x, t)}{\partial t}\) is usually employed in fluid mechanics. On the other hand, the constitutive equation of a solid depends on the deformation history except for a perfectly plastic material and thus one has to analyze the deformation of each material particle moving in a space. Then, in solid mechanics the material-time derivative \(\dot{\psi}\) must be used. Further, as will be explained in Section 5.2, the corotational derivative based on the rate of a physical quantity observed by the coordinate system rotating with a material is often used in rate-type or incremental-type constitutive equations of solids, for example, the viscoelastic, the elasto-plastic and the viscoplastic deformation. The material-time and spatial-time derivatives are illustrated in Figure 1.5

In what follows, we examine the Lagrangian and Eulerian descriptions, the spatial-time and material-time derivatives by means of some simple examples.

**Example 1.1: Lagrangian and Eulerian descriptions of velocity** (Belytschko et al., 2000)

Consider the Lagrangian and Eulerian descriptions of the velocity of a material particle prior to the derivations of the spatial- and the material-time derivatives. Here, assume a one-dimension motion in which the mapping \(x = \chi(X, t)\) and \(X = \chi^{-1}(x, t)\) is given by

\[
x(X, t) = (1 - X)t + \frac{1}{2}Xt^2 + X, \quad X(x, t) = \frac{x - t}{t^2 - t + 1}.
\]

(1.411)

The Lagrangian description of velocity is given by

\[
v(X, t) = \frac{\partial x(X, t)}{\partial t} = 1 + X(t - 1).
\]

(1.412)
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On the other hand, the Eulerian description of the velocity of a material particle is given by substituting equation (1.411) into equation (1.412) as follows:

\[ v(x, t) = \frac{1 - x + xt - \frac{1}{2}t^2}{\frac{1}{2}t^2 - t + 1}. \]  

(1.413)

The Eulerian description of velocity of material particle must be calculated though the Lagrangian description as shown above, since the velocity of the material particle has to be calculated for the condition \( X = \text{const.} \).

**Example 1.2:** **Temperature distribution** (Bonet and Wood, 2008) Consider the one-dimensional stretching of a bar in which the mappings \( x = \chi(X, t) \) and \( X = \chi^{-1}(x, t) \) are as follows:

\[ x = (1 + t)X, \quad X = \frac{x}{1 + t}. \]  

(1.414)
66  

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The bar experiences a temperature distribution given by

\[ T = X^2 t^2, \quad T = x^2 /

in the material and spatial descriptions, respectively. The velocity is

\[ \dot{v} = \frac{\partial x(X, t)}{\partial t} = X, \quad \dot{v} = \frac{x}{1 + t} \]

in the material and spatial descriptions, respectively. The material-time derivative of temperature is

\[ \dot{T} = \frac{\partial T(X, t)}{\partial t} = 2Xt, \quad \dot{T} = \frac{2xt}{1 + t} \]

in the material and spatial descriptions, respectively. On the other hand, the spatial-time derivative and the convective term are given by

\[ \frac{\partial T(x, t)}{\partial t} = \frac{(2t + t^2)x}{(1 + t)^2}, \quad \frac{\partial T(x, t)}{\partial x} = \frac{t^2}{1 + t}, \quad \frac{\partial T(x, t)}{\partial x} v = \frac{t^2}{1 + t} \frac{x}{1 + t} = \frac{x^2}{1 + t} \]

leading to

\[ \dot{T} = \frac{\partial T(x, t)}{\partial t} + \frac{\partial T(x, t)}{\partial x} v = \frac{(2t + t^2)x}{(1 + t)^2} + \frac{x^2}{(1 + t)^2} = \frac{2xt}{1 + t} \]

which coincides with equation (1.17) so that the validity of equation (1.407) is confirmed.

**Example 1.3: Infinitesimal shear strain field** Consider the two-dimensional shear deformation in which the mapping \( x = \chi(X, t) \) is given by

\[ \begin{aligned}
  x_1 &= X_1 + aX_2^2 t^2 \\
  x_2 &= X_2 + ct 
\end{aligned} \]

where \( a \) and \( c \) are constants. The inverse mapping \( X = \chi^{-1}(x, t) \) of equation (1.402) is given by

\[ \begin{aligned}
  X_1 &= x_1 - aX_2^2 t^2 = x_1 + a(x_2 - ct)^2 t^2 \\
  X_2 &= x_2 - ct 
\end{aligned} \]

The velocity of a material particle is given by

\[ \begin{aligned}
  v_1 &= \frac{\partial x_1(X_1, X_2, t)}{\partial t} = 2aX_2^2 t \\
  v_2 &= \frac{\partial x_2(X_1, X_2, t)}{\partial t} = c 
\end{aligned} \]

in the Lagrangian description, or by

\[ \begin{aligned}
  v_1 &= 2a(x_2 - ct)^2 t \\
  v_2 &= c 
\end{aligned} \]

in the Eulerian description.
Further, the infinitesimal shear strain $\gamma_{12}$ is

$$\gamma_{12}(X_1, X_2, t) = \frac{x_1 - X_1}{X_2} = \frac{aX_2^2 t^2}{X_2} = aX_2t^2$$ (1.424)

in the Lagrangian description, or

$$\gamma_{12}(x_1, x_2, t) = a(x_2 - ct)t^2$$ (1.425)

in the Eulerian description. Further, the material-time derivative of infinitesimal shear strain rate is given from equation (1.424) by

$$\dot{\gamma}_{12}(X_1, X_2, t) = \frac{\partial \gamma_{12}(X_1, X_2, t)}{\partial t} = 2aX_2t$$ (1.426)

in the Lagrangian description, or

$$\dot{\gamma}_{12}(x_1, x_2, t) = 2a(x_2 - ct)t = 2ax_2t - 2act^2$$ (1.427)

in the Eulerian description, substituting equation (1.421) into equation (1.426).

On the other hand, the spatial-time derivative is given by

$$\frac{\partial \gamma_{12}(x_1, x_2, t)}{\partial x_1}v_1 + \frac{\partial \gamma_{12}(x_1, x_2, t)}{\partial x_2}v_2 = 0 \cdot [2a(x_2 - ct)^2t] + at^2 \cdot c = act^2.$$ (1.429)

Here, it can be seen that the material-time derivative in equation (1.427) is the sum of the spatial time-derivative in equation (1.428) and the convective term as described in equation (1.429).

Only Eulerian variables in the current configuration are usually used in fluid mechanics, but both Lagrangian variables based on the reference configuration and Eulerian variables based on the current configuration are used in solid mechanics. Some basic Lagrangian and the Eulerian variables used in solid mechanics are listed in Table 1.2. The exact definitions of the Lagrangian and Eulerian variables and their physical meanings will be explained in detail in later chapters. Here, note that there exist Eulerian–Lagrangian variables which are based on both the reference and current configurations, called two-point tensors, as represented by the deformation gradient.

The material-time derivative described in this section designates the rate of physical quantity observed by the observer translating in parallel with a certain material point. Further, a convected derivative will be introduced in Chapter 5, which designates the rate observed by the observer not only translating but also deforming and rotating with a certain infinitesimal region in material so that the adoption of the curvilinear coordinate system studied in Chapter 2 is required.
Table 1.2  Basic Lagrangian and Eulerian variables used in constitutive equations

<table>
<thead>
<tr>
<th>Lagrangian variables</th>
<th>Eulerian variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position vector</td>
<td>X</td>
</tr>
<tr>
<td>Strain tensor</td>
<td>Green strain</td>
</tr>
<tr>
<td></td>
<td>$E = \frac{1}{2} (F^T F - G)$</td>
</tr>
<tr>
<td>Strain rate tensor</td>
<td>$\mathbf{D} = \text{sym}[\mathbf{F}^T \mathbf{F}] = \dot{\mathbf{E}}$</td>
</tr>
<tr>
<td>Stress tensor</td>
<td>Second Piola–Kirchhoff stress $\mathbf{S}$</td>
</tr>
<tr>
<td></td>
<td>$\mathbf{F}^{-1} (\mathbf{t} \mathbf{a})/dA = \mathbf{SN}$</td>
</tr>
<tr>
<td></td>
<td>Cauchy stress $\mathbf{\sigma}$</td>
</tr>
<tr>
<td></td>
<td>$\mathbf{t} = \mathbf{\sigma n}$</td>
</tr>
</tbody>
</table>

Note. $\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X}$: deformation gradient (Eulerian–Lagrangian two-point tensor). $\mathbf{G}, \mathbf{g}$: metric tensors based on reference and current configurations, which coincide with the identity tensor in the Cartesian coordinate system. $\mathbf{N}, \mathbf{n}$: unit outward-normal tensors of surface in reference and current configurations. $\mathbf{A}, \mathbf{a}$: surface areas in reference and current configurations. $\mathbf{t}$, force vector applying to current unit area, called the Cauchy stress vector.

1.9.5  Derivatives of Tensor Field

A field which is a function of a position vector $\mathbf{x}$ is called a tensor field. Various derivatives of the tensor field are shown in this section, where use is made of the operator

$$\nabla \equiv \frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial x_i} \mathbf{e}_i,$$  \hspace{1cm} (1.430)

which is called the *nabla* or *del* or *Hamilton operator*.

1) Gradient

Scalar field:

$$\text{grad} \, s = \nabla s = \frac{\partial s}{\partial x_i} \mathbf{e}_i.$$  \hspace{1cm} (1.431)

Vector field:

$$\text{grad} \, \mathbf{v} = \begin{cases} \mathbf{v} \otimes \nabla = v_i \mathbf{e}_i \otimes \frac{\partial}{\partial x_j} \mathbf{e}_j = \frac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j : \text{right form} \\ \nabla \otimes \mathbf{v} = \frac{\partial}{\partial x_i} \mathbf{e}_i \otimes v_j \mathbf{e}_j = \frac{\partial v_j}{\partial x_i} \mathbf{e}_i \otimes \mathbf{e}_j : \text{left form} \end{cases}.$$  \hspace{1cm} (1.432)

Second-order tensor field:

$$\text{grad} \, \mathbf{T} = \begin{cases} \mathbf{T} \otimes \nabla = T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \frac{\partial}{\partial x_k} \mathbf{e}_k = \frac{\partial T_{ij}}{\partial x_k} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k : \text{right form} \\ \nabla \otimes \mathbf{T} = \frac{\partial}{\partial x_i} \mathbf{e}_i \otimes T_{jk} \mathbf{e}_j \otimes \mathbf{e}_k = \frac{\partial T_{jk}}{\partial x_i} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k : \text{left form} \end{cases}.$$  \hspace{1cm} (1.433)
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2) Divergence
Vector field:
\[
\text{div} \mathbf{v} = \nabla \cdot \mathbf{v} (= \mathbf{v} \cdot \nabla) = v_i e_i \cdot \frac{\partial}{\partial x_j} e_j = \frac{\partial v_i}{\partial x_j}.
\] (1.434)

Second-order tensor field:
\[
\text{div} \mathbf{T} = \begin{cases} 
T_{ij} e_i \otimes e_j = \frac{\partial T_{ij}}{\partial x_k} e_i : \text{right form} \\
\nabla \mathbf{T} = \frac{\partial}{\partial x_i} e_i T_{jk} e_j \otimes e_k = \frac{\partial T_{jk}}{\partial x_j} e_i : \text{left form} 
\end{cases}.
\] (1.435)

3) Rotation (or curl)
Vector field:
\[
\text{rot} \mathbf{v} = \begin{cases} 
v \times \nabla = v_i e_i \times \frac{\partial}{\partial x_j} e_j = \varepsilon_{ijk} \frac{\partial v_j}{\partial x_k} e_k : \text{right form}, \\
\nabla \times \mathbf{v} = \frac{\partial}{\partial x_i} e_i \times v_j e_j = \varepsilon_{ijk} \frac{\partial v_j}{\partial x_k} e_k = -v \times \nabla : \text{left form}. 
\end{cases}
\] (1.436)

Second-order tensor field:
\[
\text{rot} \mathbf{T} = \begin{cases} 
T \times \nabla = T_{ij} e_i \times \frac{\partial}{\partial x_k} e_k \\
\nabla \times \mathbf{T} = \frac{\partial}{\partial x_i} e_i T_{jk} e_j \otimes e_k = \varepsilon_{ijk} \frac{\partial T_{jk}}{\partial x_k} e_i \otimes e_k : \text{right form} \\
\n\end{cases},
\] (1.437)

The symbol \( \nabla \) is regarded as a vector, and the scalar product of itself, that is,
\[
\Delta \equiv \nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x_r \partial x_r} e_r \cdot \frac{\partial}{\partial x_s} e_s = \frac{\partial^2}{\partial x_r \partial x_r}.
\] (1.438)

has the meaning of \( \nabla^2 (\cdot) \equiv \text{div} (\text{grad} (\cdot)) \). The symbol \( \Delta \) is called the Laplacian or Laplace operator, which is often used for scalar or vector fields in the following sense:
\[
\Delta s = \frac{\partial^2 s}{\partial x_r \partial x_r}, \quad \Delta \mathbf{v} = \frac{\partial^2 v_r}{\partial x_r \partial x_r} e_r.
\] (1.439)
Various formulae are derived from equations (1.431)–(1.437). Those often appearing in continuum mechanics are shown below. Note that equations (1.31), (1.32) and (1.56) are used for the derivations.

\[
\begin{align*}
\text{grad}(sv) & = \begin{cases} 
v \otimes \text{grad } s + sv \otimes \nabla : & \text{right form} \\
\text{grad } s \otimes v + s \nabla \otimes v : & \text{left form}
\end{cases} \\
\text{grad}(u \cdot v) & = (\text{grad } u) \cdot v + (\text{grad } v) \cdot u \\
\text{div}(sv) & = \mathsf{s} \text{div } v + v \cdot \text{grad } s \\
\text{div}(u \times v) & = v \cdot (\nabla \times u) + u \cdot (\nabla \times v)
\end{align*}
\]

\[
\begin{align*}
\text{div}(sT) & = \begin{cases} 
(sT)\nabla = T \text{grad } s + s(T \nabla) : & \text{right form} \\
\nabla (sT) = T^T \text{grad } s + s(T \nabla) : & \text{left form}
\end{cases} \\
\text{div}(Tv) & = T^T : \text{grad } v + (\nabla T) \cdot v
\end{align*}
\]

\[
\begin{align*}
\text{rot}(u \times v) & = \begin{cases} 
(u \times v) \times \nabla : & \text{right form} \\
-(u \times v) \times \nabla : & \text{left form}
\end{cases}
\end{align*}
\]

because of

\[
\begin{align*}
\text{grad}(sv) & = \begin{cases} 
\frac{\partial (sv_i)}{\partial x_j} e_i \otimes e_j & = \frac{\partial s}{\partial x_j} v_i e_i \otimes e_j + s \frac{\partial v_j}{\partial x_j} e_i \otimes e_j \\
\nabla \otimes (sv) & = \begin{cases} 
\frac{\partial (sv_i)}{\partial x_j} e_i \otimes e_j & = \frac{\partial s}{\partial x_j} v_i e_i \otimes e_j + s \frac{\partial v_j}{\partial x_j} e_i \otimes e_j \\
\frac{\partial s}{\partial x_j} e_i \otimes v_j e_j + s \frac{\partial v_j}{\partial x_j} e_i \otimes e_j & = \text{grad } s \otimes v + s \nabla \otimes v : \text{right form}
\end{cases}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{grad}(u \cdot v) & = \frac{\partial (u \cdot v)}{\partial x_i} e_i = \frac{\partial u_i v_i}{\partial x_i} e_i = \frac{\partial u_i}{\partial x_i} v_i e_i + \frac{\partial v_i}{\partial x_i} u_i e_i \\
\text{div}(sv) & = \frac{\partial (sv_i)}{\partial x_i} v_i = \frac{\partial s}{\partial x_i} v_i + \frac{\partial v_i}{\partial x_i} v_i = \mathsf{s} \text{div } v + \text{grad } s \cdot v, \\
\text{div}(u \times v) & = \frac{\partial (u \times v)_i}{\partial x_i} = \frac{\partial \epsilon_{ijk} u_j v_k}{\partial x_i} = \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} v_k + u_j \epsilon_{ijk} \frac{\partial v_k}{\partial x_i} = \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} v_k + w_{ik} \epsilon_{ijk} \frac{\partial v_i}{\partial x_j}
\end{align*}
\]
\[ \text{Mathematical Preliminaries} \]

\[ \text{div}(sT) = \begin{cases} (sT)\nabla = \frac{\partial(sT_{ij})}{\partial x_j} e_i = \frac{\partial s}{\partial x_j} T_{ij} e_i + s \frac{\partial T_{ij}}{\partial x_j} e_i = T_{ij} e_i \otimes e_j + s \frac{\partial s}{\partial x_j} e_i + s \frac{\partial T_{ij}}{\partial x_j} e_i, \\ = T \text{grad } s + s(T \nabla) \text{: right form} \end{cases} \]

\[ \text{grad}(sT) = \begin{cases} \nabla(sT) = \frac{\partial(sT_{ij})}{\partial x_j} e_i = \frac{\partial s}{\partial x_j} T_{ij} e_i + s \frac{\partial T_{ij}}{\partial x_j} e_j = T_{ij} e_i \otimes e_j, \\ = T^T \text{grad } s + s(T \nabla) \text{: left form} \end{cases} \]

\[ \text{rot}(u \times v) = \begin{cases} (u \times v) \times \nabla = \varepsilon_{ijk} \frac{\partial(u \times v)}{\partial x_j} e_k = \varepsilon_{ijk} \frac{\partial(u \times v_k)}{\partial x_j} e_k, \\ = (\delta_{j\delta_{k\nu}} - \delta_{j\delta_{k\nu}}) \left( \frac{\partial u_{\nu}}{\partial x_j} v_k e_k + u_{\nu} \frac{\partial v_k}{\partial x_j} e_k \right) \\ = \frac{\partial u_{\nu}}{\partial x_j} v_k e_k + u_{\nu} \frac{\partial v_k}{\partial x_j} e_k - u_{\nu} \frac{\partial v_k}{\partial x_j} e_k, \\ = (\text{div } u) v - (\text{div } v) u + (\text{grad } u) v + (\text{grad } v) u \text{: right form} \end{cases} \]

\[ \nabla \times (u \times v) = -(u \times v) \times \nabla \text{: left form} \]

Hereinafter, only the symbols grad(·) and div(·) are used when there is no difference between the right and left forms.

\subsection*{1.9.6 Gauss’s Divergence Theorem}

Consider a scalar- or vector- or tensor-valued physical quantity \( \psi(\mathbf{x}) \) in the zone surrounded by a smooth surface inside a material. Imagine a thin prism cut by the four planes perpendicular to the \( x_2 \)- and \( x_3 \)-axes at infinitesimal intervals from a zone inside the material. The following equation holds for the prism:

\[ \int_{S} \frac{\partial \psi}{\partial x_1} d\mathbf{v} = \int_{S} \frac{\partial \psi}{\partial x_1} dx_2 dx_3 + [\psi]_{x_1}^+ dx_2 dx_3 = (\psi^+ - \psi^-) dx_2 dx_3, \quad (1.441) \]

where \((\cdot)^+\) and \((\cdot)^-\) denote the values of the physical quantity at the maximum and the minimum \( x_1 \)-coordinates, respectively.

The neighborhood of the surface cut by the prism is magnified in Figure 1.6. Consider the infinitesimal rectangular surface ABCD of the prism exposed at the surface at the maximum \( x_1 \)-coordinate and the infinitesimal rectangular section \( PB'CD' \) cut by the plane passing through the point \( A \) and perpendicular to the \( x_1 \)-axis. Then, denoting \( \mathbb{B}' = dx_2, \mathbb{D}' = dx_3 \), the vectors \( \mathbb{A}' \), \( \mathbb{A}' \) are given by

\[ \mathbb{A}' = dx_2 e_2 + dx_3 e_3, \quad \mathbb{A}' = dx_3 e_3 + dx_5 e_1, \quad (1.442) \]
and thus it follows that
\[ n^+ da^+ = \bar{AB} \times \bar{AD} = (dx_2 e_2 + dx_Q e_1) \times (dx_3 e_3 + dx_3 e_1) \]
\[ = dx_2 dx_3 e_2 \times e_3 + dx_Q dx_3 e_1 \times e_3 + dx_3 dx e_2 \times e_1 \]
\[ = dx_2 dx_3 e_1 - dx_Q dx_3 e_2 - dx_3 dx e_3. \] (1.443)

Comparing the components in the base \( e_1 \) on the both sides in equation (1.443), one has
\[ n^+ da^+ = dx_2 dx_3. \] (1.444)

In a similar manner, for the surface of the prism exposed on the surface in the minimum \( x_1 \)-coordinate, one has
\[ n^- da^- = -dx_2 dx_3. \] (1.445)

The general expression for projected area is given in Appendix A.

Adopting equations (1.444) and (1.445) in equation (1.441), the following expression holds for the prism:
\[ \int \frac{\partial \psi}{\partial x_1} dv = \psi^+ n^+_i da^+ - \psi^- (-n^-_i da^-) = \psi^+ n^+_i da^+ + \psi^- n^-_i da^- . \] (1.446)

Then the following equation holds for the whole zone:
\[ \int \frac{\partial \psi}{\partial x_1} dv = \int_a \psi n_i da . \] (1.447)

Similar equations are obtained also for the \( x_2 \)- and \( x_3 \)-directions, and thus it follows that
\[ \int \frac{\partial \psi}{\partial x_i} dv = \int_a \psi n_i da . \] (1.448)

which is called Gauss’s theorem or Gauss’s divergence theorem.
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The following equations for the scalar $s$, the vector $v$ and the tensor $T$ hold from equation (1.448):

\[
\int_{v} \frac{\partial s}{\partial x_i} dv = \int_{a} s n_i da, \quad \int_{v} \text{div} s dv = \int_{a} s n da
\]  \hfill (1.449)

\[
\int_{v} \frac{\partial v}{\partial x_i} dv = \int_{a} v_i n_i da, \quad \int_{v} \text{div} v dv = \int_{a} v \cdot n da
\]  \hfill (1.450)

\[
\int_{v} \frac{\partial T_{ij}}{\partial x_i} dv = \int_{a} T_{ij} n_i da, \quad \int_{v} \nabla T dv = \int_{a} T^{T} n da
\]  \hfill (1.451)

### 1.9.7 Material-Time Derivative of Volume Integration

Suppose that the zone of material occupying the volume $v$ at the current moment ($t = t$) changes to occupy the volume $v + \delta v$ after an infinitesimal time ($t = t + \delta t$). The material-time derivative of the volume integration $\int_{v} \psi(x, t) dv$ of the scalar- or tensor-valued physical quantity $\psi(x, t)$ involved in the volume is given by

\[
\left( \int_{v} \psi dv \right)^* = \lim_{\delta t \to 0} \frac{1}{\delta t} \left\{ \int_{v+\delta v} \psi(x, t + \delta t) dv - \int_{v} \psi(x, t) dv \right\}
\]

\[
= \lim_{\delta t \to 0} \frac{1}{\delta t} \left[ \int_{v} (\psi(x, t + \delta t) - \psi(x, t)) dv + \int_{v+\delta v} \psi(x, t + \delta t) dv \right]. \hfill (1.452)
\]

The integration of the first term on the right-hand side in equation (1.452) is transformed as

\[
\lim_{\delta t \to 0} \frac{1}{\delta t} \int_{v} (\psi(x, t + \delta t) - \psi(x, t)) dv = \int_{v} \frac{\partial \psi(x, t)}{\partial t} dv. \hfill (1.453)
\]

The second term in equation (1.452) describes the effect of the change of volume during the infinite time. Here, the volume increment, $\delta v$, is given by subtracting the volume going out the boundary of the zone from the volume going into the boundary, which is the sum of $dv(= v \cdot n da)\delta t)$ over the whole boundary surface (Figure 1.7). Therefore, using Gauss's divergence theorem (1.448) and ignoring the second-order infinitesimal quantity, the integration of the second term in the right-hand side of equation (1.452) is given by

\[
\lim_{\delta t \to 0} \frac{1}{\delta t} \int_{v} \psi(x, t + \delta t) dv \equiv \lim_{\delta t \to 0} \frac{1}{\delta t} \int_{v} \psi(x, t) dv
\]

\[
= \lim_{\delta t \to 0} \frac{1}{\delta t} \int_{v} \psi(x, t) v_r n_r da \delta t = \int_{v} \psi(x, t) v_r n_r da = \int_{v} \frac{\partial \psi(x, t)}{\partial x_r} dv.
\]

The sum of the first term on the right-hand side of this equation and equation (1.453) is equal to the material-time derivative of $\psi(x, t)$, and thus equation (1.452) is given by

\[
\left( \int_{v} \psi(x, t) dv \right)^* = \int_{v} (\dot{\psi}(x, t) + \psi(x, t) \text{div} v) dv \quad (1.454)
\]
Figure 1.7 Transfer of volume element

which is called the Reynolds transportation theorem, noting

\[
\left( \int_v \psi(x, t) \, dv \right)^* = \int_v \left\{ \frac{\partial \psi(x, t)}{\partial t} + \frac{\partial (\psi(x, t)v_r)}{\partial x_r} \right\} \, dv \\
= \int_v \left\{ \frac{\partial \psi(x, t)}{\partial t} + \frac{\partial \psi(x, t)}{\partial x_r} v_r + \psi(x, t) \frac{\partial v_r}{\partial x_r} \right\} \, dv
\]

and equation (1.408).

Equation (1.454) can be also obtained in the following manner:

\[
\left( \int_v \psi(x, t) \, dv \right)^* = \left( \int_v \psi(X, t) J \, dV \right)^* = \int_v \left( \dot{\psi}(X, t) J + \psi(X, t) \dot{J} \right) \, dV \\
= \int_v \left\{ \dot{\psi}(x, t) + \psi(x, t) \frac{\partial v_r}{\partial x_r} \right\} \, dv,
\]

where \( V \) is the initial volume of the zone and we set \( J \equiv dv/dV \). Here, it follows that \( \dot{J} = J (\partial v_r/\partial x_r) \) (see equation (1.475)).

For the physical quantity \( \psi \) kept constant in a volume element, equation (1.454) leads to

\[
\int_v (\dot{\psi}(x, t) + \psi(x, t) \nabla v) \, dv = 0. \tag{1.455}
\]

The local form of equation (1.455) is given by

\[
\dot{\psi}(x, t) + \psi(x, t) \nabla v = 0, \tag{1.456}
\]

while equation (1.455) is called the weak form.

1.10 Variations and Rates of Geometrical Elements

Variations of line, surface and volume elements and their rates under the deformation are described in this section.
1.10.1 Variations of Line, Surface and Volume

The relation of the current infinitesimal line element \( dx \) to the initial infinitesimal line element \( dX \) is given by

\[
dx = \mathbf{F} dX, \quad dX = \mathbf{F}^{-1} dx,
\]

(1.457)

where

\[
\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}
\]

(1.458)

which is referred to as the deformation gradient. The determinant of the deformation gradient is called the Jacobian and is denoted by

\[
J \equiv \det \mathbf{F} = \det \left[ \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right].
\]

(1.459)

For the deformation gradient tensor \( \mathbf{F} \), we comment briefly on the indices in its transpose, inverse and transposed inverse. The deformation gradient is expressed by \( \mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X} = \partial x_i/\partial X_j \mathbf{e}_i \otimes \mathbf{e}_j / \mathbf{F}_j = \partial x_i/\partial X_j \mathbf{e}_j \otimes \mathbf{e}_j \), where \( \mathbf{X}(= X_j \mathbf{e}_j) \) and \( \mathbf{x}(= x_i \mathbf{e}_i) \) are the position vectors of a material particle in the reference and current states, respectively. The transposed tensor is given by \( \mathbf{F}^T = (\partial x_i/\partial X_j \mathbf{e}_i \otimes \mathbf{e}_j)^T = (\partial x_i/\partial X_j \mathbf{e}_j \otimes \mathbf{e}_j) = (\mathbf{F}_j)^T \mathbf{e}_i \otimes \mathbf{e}_j \), leading to \( F_{ij}^T \equiv (\mathbf{F}^T)_{ij} = \partial x_i/\partial X_j \). On the other hand, the inversed tensor is given by \( \mathbf{F}^{-1} = \partial X_j/\partial x_i \mathbf{e}_j \otimes \mathbf{e}_j = (\partial X_j/\partial x_i) \mathbf{e}_j \otimes \mathbf{e}_j \), leading to \( F_{ij}^{-1} \equiv (\mathbf{F}^{-1})_{ij} = \partial X_j/\partial x_i \).

Further, the transposed inverse tensor is given by \( \mathbf{F}^{-T} = (\partial x_i/\partial X_j \mathbf{e}_i \otimes \mathbf{e}_j)^T = (\partial x_i/\partial X_j \mathbf{e}_j \otimes \mathbf{e}_j) = (\mathbf{F}^{-T})_{ij} \mathbf{e}_j \otimes \mathbf{e}_j \), leading to \( F_{ij}^{-T} \equiv (\mathbf{F}^{-T})_{ij} = \partial x_j/\partial x_i \).

Then, it is summarized that

\[ F_{ij} = \partial x_i/\partial X_j, \quad F_{ij}^T \equiv (\mathbf{F}^T)_{ij} = \partial x_j/\partial X_i, \quad F_{ij}^{-1} \equiv (\mathbf{F}^{-1})_{ij} = \partial X_i/\partial x_j, \quad F_{ij}^{-T} \equiv (\mathbf{F}^{-T})_{ij} = \partial X_j/\partial x_i. \]

The current and the reference infinitesimal volume elements \( dv \) and \( dV \) formed by the infinitesimal line elements \( dx^a, dx^b, dx^c \) and \( dX^a, dX^b, dX^c \) are related by

\[
dv = [dx^a dx^b dx^c] = [\mathbf{F} dX^a dX^b dX^c] = \det \mathbf{F} [dx^a dx^b dx^c] = (\det \mathbf{F}) dV
\]

(1.460)

by virtue of equation (1.179). Then, the relation of the current and the reference infinitesimal volume elements is given by

\[
J = \det \mathbf{F} = \det \mathbf{U} = \det \mathbf{V} = \sqrt{\det \mathbf{C}} = \sqrt{\det \mathbf{b}} = \frac{dv}{dV} = \frac{\rho_0}{\rho}
\]

(1.461)

noting equation (1.159), while \( \mathbf{U} \equiv \sqrt{\mathbf{F}^T \mathbf{F}}, \mathbf{V} \equiv \sqrt{\mathbf{F} \mathbf{F}^T}, \mathbf{C} \equiv \mathbf{F}^T \mathbf{F} \) and \( \mathbf{b} \equiv \mathbf{F} \mathbf{F}^T \) are described in Chapter 4. \( \rho_0 \) and \( \rho \) are the densities in the reference and current configurations, respectively.

The following equalities hold for the infinitesimal volume elements, denoting the infinitesimal reference and current surface element vectors as \( d\mathbf{A} = d\mathbf{X}^a \times d\mathbf{X}^b \) and \( d\mathbf{a} = dx^a \times dx^b \), respectively, and noting equation (1.93):

\[
dv = \begin{cases} 
\mathbf{a} \cdot dx^c, \\
J dV = J d\mathbf{A} \cdot dx^c = J d\mathbf{A} \cdot \mathbf{F}^{-1} dx^c = J \mathbf{F}^{-T} d\mathbf{A} \cdot dx^c.
\end{cases}
\]
Then the following relation holds:

\[
da = J F^{-T} dA = (\text{cof } F) dA, \quad dA = F^T d\mathbf{a} = (\text{cof } F)^{-1} d\mathbf{a}
\]  

(1.462)

noting that

\[
\text{cof } F = J F^{-T}
\]

(1.463)

by virtue of equation (1.157), leading to

\[
da = F^{-T} \mathbf{N} \cdot \mathbf{n} dA, \quad dA = F^T \mathbf{n} \cdot \mathbf{N} d\mathbf{a}/J, \\
\mathbf{n} = F^{-T} \mathbf{N} dA/da, \quad \mathbf{N} = F^T d\mathbf{a}/(dA/J).
\]  

(1.464)

Equation (1.464) is referred to as Nanson’s formula.

Here, the following Euler formula holds for the cofactor

\[
\nabla_X (\text{cof } F) = \frac{\partial (\text{cof } F)}{\partial X} = 0, \quad \frac{\partial (\text{cof } F)_{ap}}{\partial X_p} = \frac{\partial}{\partial X_p} \frac{1}{2} \varepsilon_{abc} \varepsilon_{pqr} F_{bq} F_{cr} = 0,
\]

(1.466)

noting equation (1.153) and that \(X_p\) is not contained in the deformation gradient.

Denote an arbitrary physical quantity in the reference and current states by \(T\) and \(t\), respectively. Their relation can be written as

\[
t(\mathbf{x}, t) d\mathbf{a}(\mathbf{x}, t) = T(\mathbf{X}) dA(\mathbf{X}).
\]  

(1.467)

The substitution of the Nanson’s formula in equation (1.464) into equation (1.467) yields Piola’s transformation formula

\[
T = (\text{cof } F) t.
\]  

(1.468)

Here, we have

\[
\frac{\partial}{\partial \mathbf{x}} = F^{-T} \frac{\partial}{\partial \mathbf{X}} (\nabla_x = F^{-T} \nabla_X),
\]

(1.469)

noting that

\[
\frac{\partial}{\partial X_i} = \frac{\partial X_R}{\partial X_i} \frac{\partial}{\partial X_R} = \frac{\partial X_R}{\partial X_i} \frac{\partial}{\partial X_R}.
\]

It follows from equations (1.463), (1.466), (1.468) and (1.469) that

\[
\frac{\partial T}{\partial \mathbf{X}} = J \frac{\partial t}{\partial \mathbf{x}},
\]

(1.470)

noting that

\[
\frac{\partial T}{\partial \mathbf{X}} = \frac{\partial (\text{cof } F) t}{\partial \mathbf{X}} = (\text{cof } F) \frac{\partial t}{\partial \mathbf{X}} = J F^{-T} \frac{\partial t}{\partial \mathbf{X}}.
\]

Equation (1.470), which we will need in Chapter 6, is referred to as Piola’s formula.

1.10.2 Rates of Changes of Surface and Volume

Differentiating equation (1.457), we have

\[
(d\mathbf{x})^* = \tilde{F} d\mathbf{X}, \quad (d\mathbf{x})^* = d\mathbf{x},
\]

(1.471)
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where

\[ l = \dot{F}F^{-1} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \]  

(1.472)

which is called the *velocity gradient*. It follows from equations (1.434) and (1.461) that

\[(dv)^* = (\text{tr}l)dv = \dot{J}dV,\]

(1.473)

\[\text{tr}l = \text{div} \mathbf{v},\]

(1.474)

\[\dot{J} = J\text{tr}l\]

(1.475)

noting

\[\dot{J} = (\det F)^* = \frac{\partial \det F}{\partial F} : \dot{F} = (\det F)F^{-T} : \dot{F} = (\det F)\text{tr}(F^{-1}\dot{F}) = (\det F)\text{tr}(\dot{F}F^{-1})\]

with equation (1.371).

Further, it follows from equation (1.463) that

\[(\text{cof} F)^* = \tilde{I}\text{cof} F,\]

(1.476)

where

\[\tilde{I} \equiv (\text{tr}l)I - l^T,\]

(1.477)

noting that

\[(\text{cof} F)^* = \dot{J}F^{-T} + \dot{F}^TJ = \dot{J} \text{cof} F + \dot{F}^T F^T \text{cof} F = \frac{\dot{J}}{J} \text{cof} F - F^{-T} \dot{F}^T F \text{cof} F.\]

\(\tilde{I}\) is referred to as the *surface strain rate tensor*. Further, differentiating equation (1.462) and noting equation (1.476), we have

\[(da)^* = (\text{cof} F)^*dA = \tilde{I}(\text{cof} F)dA\]

which is rewritten as

\[(da)^* = \tilde{I}da\]

(1.478)

in the spatial description.

We now express the surface vectors as follows:

\[da \equiv \mathbf{n}da, \quad dA \equiv \mathbf{N}dA,\]

(1.479)

where \(da\) and \(dA\) are infinitesimal areas and \(\mathbf{n}\) and \(\mathbf{N}\) are unit normal vectors of the current and the reference infinitesimal surface vectors, respectively.

Here, noting \(\mathbf{\tilde{n}} \cdot \mathbf{n} = 0\) from \(\mathbf{n} \cdot \mathbf{n} = 1\) for the unit vector \(\mathbf{n}\), it follows that

\[(da)^* = \mathbf{n} \cdot (\mathbf{n}da)^* = \mathbf{n} \cdot (\mathbf{nda})^* - \mathbf{n} \cdot (\mathbf{da})^* = \mathbf{n} \cdot (da)^*.\]

(1.480)
Substituting equation (1.478) into equation (1.480), one obtains the rate of the current infinitesimal area as follows:

\[(\text{da})^\ast = \mathbf{n} \cdot \hat{\text{da}}\]  \hspace{1cm} (1.481)

or

\[(\text{da})^\ast = (\text{trl} - \mathbf{n} \cdot \mathbf{dn})\text{da},\]  \hspace{1cm} (1.482)

where

\[\mathbf{d} \equiv \text{sym}[I],\]  \hspace{1cm} (1.483)

which is called the strain rate, noting equation (1.93). Further, it follows from equations (1.477), (1.481) and (1.482) that

\[\hat{\text{da}} = (\mathbf{n} \text{da})^\ast - \mathbf{n}(\text{da})^\ast\]

\[= \{(\text{trl})\mathbf{I} - \mathbf{l}^T\} \text{nda} - \mathbf{n}\{(\text{trl}) - \mathbf{n} \cdot \mathbf{dn}\}\text{da}.\]

Then the rate of the unit normal of the current surface element is given by

\[\hat{\mathbf{n}} = \{(\mathbf{n} \cdot \mathbf{ln})\mathbf{I} - \mathbf{l}^T\}\mathbf{n} = \{(\mathbf{n} \cdot \mathbf{ln})\mathbf{I} - \mathbf{l}^T\}\mathbf{n}.\]  \hspace{1cm} (1.484)

The variations and rates of line, surface and volume elements are summarized in Table 1.3.

### Table 1.3 Variations and rates of line, surface and volume elements

<table>
<thead>
<tr>
<th>Elements</th>
<th>Variations from reference to current states</th>
<th>Rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line element</td>
<td>(dx = F dX)</td>
<td>((dx)^\ast = \hat{F} dX) \hspace{1cm} (1.485)</td>
</tr>
<tr>
<td></td>
<td>((dx)^\ast = l dX)</td>
<td>(\hat{F} = lF)</td>
</tr>
<tr>
<td>Surface element</td>
<td>(da = (\text{cof} F) dA)</td>
<td>((\text{cof} F)^\ast = \hat{l} \text{cof} F)</td>
</tr>
<tr>
<td></td>
<td>((\text{da})^\ast = (\text{cof} F)^\ast dA)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((\text{da})^\ast = l dA)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((\text{da})^\ast = (\text{trl} - \mathbf{n} \cdot \mathbf{ln}) d\text{da})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\hat{\mathbf{n}} = {(\mathbf{n} \cdot \mathbf{ln})\mathbf{I} - \mathbf{l}^T}\mathbf{n})</td>
<td></td>
</tr>
<tr>
<td>Volume element</td>
<td>(dv = (\text{det} F) dV)</td>
<td>((\text{det} F)^\ast = \text{trl} \text{det} F)</td>
</tr>
<tr>
<td></td>
<td>((dv)^\ast = (\text{det} F)^\ast dV)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\hat{J} = J t l = \frac{(dv)^\ast}{dV})</td>
<td></td>
</tr>
</tbody>
</table>

\[\text{F} \equiv \frac{\partial X}{\partial x}, \quad \text{J} \equiv \text{det} \text{F} = \frac{dv}{dV}, \quad \text{cof} \text{F} = (\text{det} \text{F}) \text{F}^{-T} = J \text{F}^{-T}\]

\[l = \hat{\text{F}} \text{F}^{-1}, \quad \hat{l} = (\text{trl}) \mathbf{I} - \mathbf{l}^T\]
1.11 Continuity and Smoothness Conditions

There exist various mechanical requirements, for example the thermodynamic restriction and the objectivity for constitutive equations. In addition, there exist the mechanical requirements observed in solid materials. Among them, the continuity and the smoothness conditions are violated in many elasto-plasticity models, while their importance for formulation of constitutive equations has not been sufficiently recognized to date. These requirements will be explained below (Hashiguchi, 1993a, 1993b, 1997, 2000).

1.11.1 Continuity Condition

It is observed in experiments that stress rate changes continuously for a continuous change of strain rate. This fact is called the continuity condition and is expressed mathematically as follows (Hashiguchi, 1993a, 1993b, 1997, 2000):

$$\lim_{\delta d \to 0} \sigma(\sigma, H_i; d + \delta d) = \sigma(\sigma, H_i; d)$$

(1.485)

where $H_i$ ($i = 1, 2, 3, \ldots$) collectively denotes scalar-valued or tensor-valued internal state variables. In addition, $\delta(\cdot)$ stands for an infinitesimal variation. The response of the stress rate to the input of strain rate in the current state of stress and internal variables is denoted by $\sigma(\sigma, H_i; d)$. Uniqueness of the solution is not guaranteed in constitutive equations violating the continuity condition, predicting different stresses or strains. The violation of this condition is schematically shown in Figure 1.8. Ordinary elasto-plastic constitutive equations, in which the plastic strain rate is derived from the consistency condition, fulfill the continuity condition. As will be described later, however, no elasto-plastic constitutive equation satisfies it except for the subloading surface model when they are extended to describe the tangential inelastic strain rate (Hashiguchi, 2009).

The concept of the continuity condition was first advocated by Prager (1949). However, a mathematical expression for this condition was not given. The condition was defined as the continuity of strain rate with respect to the input of stress rate by Prager (1949) inversely to the definition given above. However, an identical stress rate directed into the yield surface can induce different strain rates in loading and unloading states in a softening material, as

![Figure 1.8](image_url)
shown in Figure 1.9. Here, it is noteworthy that a stress rate cannot be given arbitrarily since there exists a limitation in strength of materials, although a strain rate can be given arbitrarily. For that reason, Prager’s (1949) notion does not hold in the general loading state, including softening and perfectly plastic states.

### 1.11.2 Smoothness Condition

It is observed in experiments that a stress rate induced by an identical strain rate changes continuously for a continuous change of stress state. This fact is called the smoothness condition and is expressed mathematically as follows (Hashiguchi, 1993a, 1993b, 1997, 2000):

\[
\lim_{\delta \sigma \to 0} \tilde{\sigma}(\sigma + \delta \sigma, H; \mathbf{d}) = \tilde{\sigma}(\sigma, H; \mathbf{d})
\]  
(1.486)
A smooth stress-strain response is not described in a constitutive equation violating the smoothness condition, causing a discontinuous change in the tangent modulus, as illustrated in Figure 1.10 for the elasto-plastic constitutive equations assuming the yield surface enclosing a purely elastic domain where the tangent modulus changes abruptly from the elastic to the elasto-plastic state.

The linear-rate constitutive equation is given by

$$\dot{\varepsilon} = M^{ep}(\sigma, H) d,$$  \hspace{1cm} (1.487)

where the fourth-order tensor $M^{ep}$ is the elasto-plastic modulus, which is a function of the stress and internal variables and can be described generally as

$$M^{ep} = \frac{\partial \varepsilon}{\partial d}.$$  \hspace{1cm} (1.488)

Consequently, equation (1.486) can be rewritten as

$$\lim_{\delta \sigma \to 0} M^{ep}(\sigma + \delta \sigma, H) = M^{ep}(\sigma, H).$$  \hspace{1cm} (1.489)

### 1.12 Unconventional Elasto-Plasticity Models

Ordinary elasto-plastic constitutive equations fulfill the continuity condition, in which plastic strain rate is derived from the consistency condition of yield condition. They are classified into conventional and the unconventional models by Drucker (1988). The interior of the yield surface is assumed to be a purely elastic domain in conventional models. In unconventional models, on the other hand, the interior of the yield surface is not assumed to be a purely elastic domain and thus the plastic strain rate induced by the rate of stress inside the yield surface is described. Unconventional models are classified further into models based on the notion of kinematic hardening, that is, the *multi-surface model* (Mroz, 1967; Iwan, 1967), the *two-surface model* (Dafalias and Popov, 1975; Krieg, 1975) and the *superposed kinematic hardening-single surface model* (Chaboche et al., 1979; Chaboche, 2008) based on the nonlinear kinematic hardening rule (Armstrong and Frederick, 1966), and the model based on the notion that the plastic strain rate develops as a stress approaches the yield surface, that is, the *subloading surface model* (Hashiguchi, 1980, 1989). The former models based on the
kinematic hardening models assume a small yield surface enclosing a purely elastic domain and thus they violate the smoothness condition. On the other hand, only the subloading surface model always satisfies the smoothness condition since it is based on quite natural postulate that the plastic strain rate develops as the stress approaches the yield surface, describing the smooth transition from the elastic to the plastic state, that is, the smooth elastic–plastic transition.

It should be noted that the mechanisms for the development of plastic deformation is substantially different from the mechanism for the development of kinematic hardening which is merely one of various mechanical behavior observed only in plastically-pressure independent metals. Here, remind that the kinematic hardening (Prager, 1956; Armstrong and Frederick, 1966) is merely the simple method proposed primarily to describe the induced anisotropy of plastically-pressure independent metals, while it is inapplicable to the description of the anisotropy of the plastically-pressure dependent, that is, frictional materials (for example, soils, rocks, concretes and friction behavior) to which the rotational hardening rule (Hashiguchi, 2001) has to be adopted. Furthermore, accumulation of plastic deformation cannot be predicted.

\[
\sigma_1, \sigma_2, \sigma_3
\]

**Figure 1.11** Incorporation of tangential inelastic strain rate

**Input:** $\sigma$

**Yield surface**

**Normal-yield surface**

$\delta':$ suddenly occurs

$\delta':$ gradually develops

**Subloading surface model:** Hashiguchi (2005)

**Conventional plasticity model:** Rudnicki and Rice model (1975).
for cyclic loading of stress inside a small yield surface even at a high stress level near the overall yield state by the unconventional models based on the kinematic hardening, since a small yield surface enclosing a purely-elastic domain is assumed in them. Such cyclic loading is observed frequently in the practical engineering phenomena so that the risky mechanical prediction is given by the unconventional models based on the kinematic hardening rule.

Constitutive equations violating the smoothness condition cannot predict a smooth stress–strain curve. Therefore, they cannot pertinently describe softening behavior. In addition, the models violating the smoothness condition violate not only the smoothness but also the continuity conditions, predicting the sudden occurrence of tangential-inelastic strain rate, leading to the violation of the unique solution condition, when the tangential inelastic strain rate $\mathbf{d}'$ (induced by the stress rate tangential to the yield surface) required for the analysis of plastic instability phenomena is incorporated (cf. Hashiguchi and Tsutsumi, 2003, 2007; Hashiguchi and Protasov, 2004; Hashiguchi, 2009) as shown in Figure 1.11. Among the existing constitutive models only the subloading surface model always fulfills the smoothness condition. The subloading surface model is capable of describing the wide classes of elastoplastic deformation of metals (Hashiguchi et al., 2012), soils (Hashiguchi and Chen, 1998; Hashiguchi and Mase, 2007) and further sliding behavior between solids, that is, the friction phenomenon (Hashiguchi et al., 2005; Hashiguchi and Ozaki, 2008) with high accuracy. In addition, the subloading surface model does not require the judgement of yielding, that is, whether or not the yield condition is satisfied and possesses the automatic controlling function to attract the stress to the yield surface in a plastic deformation process, so that numerical analyses would be enforced drastically as will be described in Chapter 8.

Finally, note that the superposed kinematic hardening-single surface model (Chaboche et al., 1979; Chaboche, 2008) nowadays is in fashion in the application to the deformation analysis of metals, and incorporated into commercial finite element method software, because it possesses quite primitive mathematical structures involving a lot of material parameters with ambiguous meanings. It can even be used by the beginners without expertise knowledge of elastoplasticity. However, it lacks generality as it is applicable only to the limited deformation behavior of metals, possessing various insufficiencies in addition to the above-mentioned points. For instance, it abandoned the description of the stagnation of isotropic hardening during a strain cyclic loading observed widely in metals (Chaboche et al., 1979; Hashiguchi et al. 2012). Besides, the yield surface of geomaterials contains the origin of the stress space near its boundary and thus it goes away easily from the origin by the application of deviatoric stress or pressure if it translates, resulting in the invalidity. Needless to say, it cannot be applied to the description of the friction phenomenon. One should recognize these fundamental drawbacks for the steady development of elastoplasticity theory from the scientific aspect.