For simple differential equations, it is possible to find closed form solutions. For example, given a function \( g \), the general solution of the simplest equation

\[
y'(t) = g(t)
\]

is

\[
y(t) = \int g(s) \, ds + c
\]

with \( c \) an arbitrary integration constant. Here, \( \int g(s) \, ds \) denotes any fixed antiderivative of \( g \). The constant \( c \), and thus a particular solution, can be obtained by specifying the value of \( y(t) \) at some given point:

\[
y(t_0) = y_0.
\]

**Example 1.1** The general solution of the equation

\[
y'(t) = \sin(t)
\]

is

\[
y(t) = -\cos(t) + c.
\]
If we specify the condition
\[ Y\left(\frac{\pi}{3}\right) = 2, \]
then it is easy to find \( c = 2.5 \). Thus the desired solution is
\[ Y(t) = 2.5 - \cos(t). \]

The more general equation
\[ Y''(t) = f(t, Y(t)) \]  \hspace{1cm} (1.1)
is approached in a similar spirit, in the sense that usually there is a general solution dependent on a constant. To further illustrate this point, we consider some more examples that can be solved analytically. First, and foremost, is the first-order linear equation
\[ Y'(t) = a(t)Y(t) + g(t). \]  \hspace{1cm} (1.2)
The given functions \( a(t) \) and \( g(t) \) are assumed continuous. For this equation, we obtain
\[ f(t, z) = a(t)z + g(t), \]
and the general solution of the equation can be found by the so-called method of integrating factors.

We illustrate the method of integrating factors through a particularly useful case,
\[ Y'(t) = \lambda Y(t) + g(t) \]  \hspace{1cm} (1.3)
with \( \lambda \) a given constant. Multiplying the linear equation (1.3) by the integrating factor \( e^{-\lambda t} \), we can reformulate the equation as
\[ \frac{d}{dt} (e^{-\lambda t} Y(t)) = e^{-\lambda t} g(t). \]
Integrating both sides from \( t_0 \) to \( t \), we obtain
\[ e^{-\lambda t} Y(t) = c + \int_{t_0}^{t} e^{-\lambda s} g(s) \, ds, \]
where
\[ c = e^{-\lambda t_0} Y(t_0). \]  \hspace{1cm} (1.4)
So the general solution of (1.3) is
\[ Y(t) = e^{\lambda t} \left[ c + \int_{t_0}^{t} e^{-\lambda s} g(s) \, ds \right] = ce^{\lambda t} + \int_{t_0}^{t} e^{\lambda (t-s)} g(s) \, ds. \]  \hspace{1cm} (1.5)
This solution is valid on any interval on which \( g(t) \) is continuous.

As we have seen from the discussions above, the general solution of the first-order equation (1.1) normally depends on an arbitrary integration constant. To single out
a particular solution, we need to specify an additional condition. Usually such a condition is taken to be of the form

\[ Y(t_0) = Y_0. \]  

(1.6)

In many applications of the ordinary differential equation (1.1), the independent variable \( t \) plays the role of time, and \( t_0 \) can be interpreted as the initial time. So it is customary to call (1.6) an initial value condition. The differential equation (1.1) and the initial value condition (1.6) together form an initial value problem

\[
\begin{align*}
Y'(t) &= f(t, Y(t)), \\
Y(t_0) &= Y_0.
\end{align*}
\]  

(1.7)

For the initial value problem of the linear equation (1.3), the solution is given by the formulas (1.5) and (1.4). We observe that the solution exists on any open interval where the data function \( g(t) \) is continuous. This is a property for linear equations. For the initial value problem of the general linear equation (1.2), its solution exists on any open interval where the functions \( a(t) \) and \( g(t) \) are continuous. As we will see next through examples, when the ordinary differential equation (1.1) is nonlinear, even if the right-side function \( f(t, z) \) has derivatives of any order, the solution of the corresponding initial value problem may exist on only a smaller interval.

**Example 1.2** By a direct computation, it is easy to verify that the equation

\[ Y'(t) = -[Y(t)]^2 + Y(t) \]

has a so-called trivial solution \( Y(t) \equiv 0 \) and a general solution

\[ Y(t) = \frac{1}{1 + ce^{-t}} \]  

(1.8)

with \( c \) arbitrary. Alternatively, this equation is a so-called separable equation, and its solution can be found by a standard method such as that described in Problem 4. To find the solution of the equation satisfying \( Y(0) = 4 \), we use the solution formula at \( t = 0 \):

\[ 4 = \frac{1}{1 + c}. \]

\[ c = -0.75. \]

So the solution of the initial value problem is

\[ Y(t) = \frac{1}{1 - 0.75e^{-t}}, \quad t \geq 0. \]

With a general initial value \( Y(0) = Y_0 \neq 0 \), the constant \( c \) in the solution formula (1.8) is given by \( c = Y_0^{-1} - 1 \). If \( Y_0 > 0 \), then \( c > -1 \), and the solution \( Y(t) \) exists for \( 0 \leq t < \infty \). However, for \( Y_0 < 0 \), the solution exists only on the finite interval
\[ t_0 = \log(1 - Y_0^{-1}) \]; the value \( t = \log(1 - Y_0^{-1}) \) is the zero of the denominator in the formula (1.8). Throughout this work, \( \log \) denotes the natural logarithm. \( \square \)

**Example 1.3** Consider the equation
\[ Y'(t) = -[Y(t)]^2. \]

It has a trivial solution \( Y(t) \equiv 0 \) and a general solution
\[ Y(t) = \frac{1}{t + c} \tag{1.9} \]
with \( c \) arbitrary. This can be verified by a direct calculation or by the method described in Problem 4. To find the solution of the equation satisfying the initial value condition \( Y(0) = Y_0 \), we distinguish several cases according to the value of \( Y_0 \). If \( Y_0 = 0 \), then the solution of the initial value problem is \( Y(t) \equiv 0 \) for any \( t \geq 0 \). If \( Y_0 \neq 0 \), then the solution of the initial value problem is
\[ Y(t) = \frac{1}{t + Y_0^{-1}}. \]

For \( Y_0 > 0 \), the solution exists for any \( t \geq 0 \). For \( Y_0 < 0 \), the solution exists only on the interval \( [0, -Y_0^{-1}] \). As a side note, observe that for \( 0 < Y_0 < 1 \) with \( c = Y_0^{-1} - 1 \), the solution (1.8) increases for \( t \geq 0 \), whereas for \( Y_0 > 0 \), the solution (1.9) with \( c = Y_0^{-1} \) decreases for \( t \geq 0 \). \( \square \)

**Example 1.4** The solution of
\[ Y'(t) = \lambda Y(t) + e^{-t}, \quad Y(0) = 1 \]
is obtained from (1.5) and (1.4) as
\[ Y(t) = e^{\lambda t} + \int_0^t e^{\lambda(t-s)} e^{-s} ds. \]

If \( \lambda \neq -1 \), then
\[ Y(t) = e^{\lambda t} \left\{ 1 + \frac{1}{\lambda + 1} \left[ 1 - e^{-(\lambda+1)t} \right] \right\}. \]

If \( \lambda = -1 \), then
\[ Y(t) = e^{-t} (1 + t). \] \( \square \)

We remark that for a general right-side function \( f(t, z) \), it is usually not possible to solve the initial value problem (1.7) analytically. One such example is for the equation
\[ Y' = e^{-t} Y'' \]
In such a case, numerical methods are the only plausible way to compute solutions. Moreover, even when a differential equation can be solved analytically, the solution
formula, such as (1.5), usually involves integrations of general functions. The integrals mostly have to be evaluated numerically. As an example, it is easy to verify that the solution of the problem

\[
\begin{aligned}
Y' &= 2t Y + 1, \quad t > 0, \\
Y(0) &= 1
\end{aligned}
\]

is

\[Y(t) = e^{t^2} \int_0^t e^{-s^2} \, ds + e^{t^2}.
\]

For such a situation, it is usually more efficient to use numerical methods from the outset to solve the differential equation.

1.1 GENERAL SOLVABILITY THEORY

Before we consider numerical methods, it is useful to have some discussions on properties of the initial value problem (1.7). The following well-known result concerns the existence and uniqueness of a solution to this problem.

**Theorem 1.5** Let \( D \) be an open connected set in \( \mathbb{R}^2 \), let \( f(t, y) \) be a continuous function of \( t \) and \( y \) for all \((t, y)\) in \( D \), and let \((t_0, Y_0)\) be an interior point of \( D \). Assume that \( f(t, y) \) satisfies the Lipschitz condition

\[|f(t, y_1) - f(t, y_2)| \leq K |y_1 - y_2| \quad \text{all } (t, y_1), (t, y_2) \text{ in } D \tag{1.10}
\]

for some \( K \geq 0 \). Then there is a unique function \( Y(t) \) defined on an interval \([t_0 - \alpha, t_0 + \alpha]\) for some \( \alpha > 0 \), satisfying

\[Y'(t) = f(t, Y(t)), \quad t_0 - \alpha \leq t \leq t_0 + \alpha,
\]

\[Y(t_0) = Y_0.
\]

The Lipschitz condition on \( f \) is assumed throughout the text. The condition (1.10) is easily obtained if \( \partial f(t, y) / \partial y \) is a continuous function of \((t, y)\) over \( \overline{D} \), the closure of \( D \), with \( D \) also assumed to be convex. (A set \( D \) is called convex if for any two points in \( D \) the line segment joining them is entirely contained in \( D \). Examples of convex sets include circles, ellipses, triangles, parallelograms.) Then we can use

\[K = \max_{(t, y) \in \overline{D}} \left| \frac{\partial f(t, y)}{\partial y} \right|,
\]

provided this is finite. If not, then simply use a smaller \( D \), say, one that is bounded and contains \((t_0, Y_0)\) in its interior. The number \( \alpha \) in the statement of the theorem depends on the initial value problem (1.7). For some equations, such as the linear equation given in (1.3) with a continuous function \( g(t) \), solutions exist for any \( t \), and we can take \( \alpha \) to be \( \infty \). For many nonlinear equations, solutions can exist only in
bounded intervals. We have seen such instances in Examples 1.2 and 1.3. Let us look at one more such example.

Example 1.6 Consider the initial value problem

\[ Y'(t) = 2t[Y(t)]^2, \quad Y(0) = 1. \]

Here

\[ f(t, y) = 2ty^2, \quad \frac{\partial f(t, y)}{\partial y} = 4ty, \]

and both of these functions are continuous for all \((t, y)\). Thus, by Theorem 1.5 there is a unique solution to this initial value problem for \(t\) in a neighborhood of \(t_0 = 0\). This solution is

\[ Y(t) = \frac{1}{1 - t^2}, \quad -1 < t < 1. \]

This example illustrates that the continuity of \(f(t, y)\) and \(\partial f(t, y)/\partial y\) for all \((t, y)\) does not imply the existence of a solution \(Y(t)\) for all \(t\).

\[ \square \]

1.2 STABILITY OF THE INITIAL VALUE PROBLEM

When numerically solving the initial value problem (1.7), we will generally assume that the solution \(Y(t)\) is being sought on a given finite interval \(t_0 \leq t \leq b\). In that case, it is possible to obtain the following result on stability. Make a small change in the initial value for the initial value problem, changing \(Y_0\) to \(Y_0 + \epsilon\). Call the resulting solution \(Y_\epsilon(t)\).

\[ Y_\epsilon'(t) = f(t, Y_\epsilon(t)), \quad t_0 \leq t \leq b, \quad Y_\epsilon(t_0) = Y_0 + \epsilon. \quad (1.11) \]

Then, under hypotheses similar to those of Theorem 1.5, it can be shown that for all small values of \(\epsilon\), \(Y(t)\) and \(Y_\epsilon(t)\) exist on the interval \([t_0, b]\), and moreover,

\[ \|Y_\epsilon - Y\|_\infty \equiv \max_{t_0 \leq t \leq b} |Y_\epsilon(t) - Y(t)| \leq c \epsilon \quad (1.12) \]

for some \(c > 0\) that is independent of \(\epsilon\). Thus small changes in the initial value \(Y_0\) will lead to small changes in the solution \(Y(t)\) of the initial value problem. This is a desirable property for a variety of very practical reasons.

Example 1.7 The problem

\[ Y'(t) = -Y(t) + 1, \quad 0 \leq t \leq b, \quad Y(0) = 1 \quad (1.13) \]

has the solution \(Y(t) \equiv 1\). The perturbed problem

\[ Y_\epsilon'(t) = -Y_\epsilon(t) + 1, \quad 0 \leq t \leq b, \quad Y_\epsilon(0) = 1 + \epsilon \]
has the solution \( Y_\varepsilon(t) = 1 + \varepsilon e^{-t} \). Thus
\[
Y(t) - Y_\varepsilon(t) = -\varepsilon e^{-t},
\]
\[
|Y(t) - Y_\varepsilon(t)| \leq |\varepsilon|, \quad 0 \leq t \leq b.
\]
The problem (1.13) is said to be stable.

Virtually all initial value problems (1.7) are stable in the sense specified in (1.12); but this is only a partial picture of the effect of small perturbations of the initial value \( Y_0 \). If the maximum error \( \| Y_\varepsilon - Y \|_\infty \) in (1.12) is not much larger than \( \varepsilon \), then we say that the initial value problem (1.7) is well-conditioned. In contrast, when \( \| Y_\varepsilon - Y \|_\infty \) is much larger than \( \varepsilon \) i.e., the minimal possible constant \( \varepsilon \) in the estimate (1.12) is large, then the initial value problem (1.7) is considered to be ill-conditioned. Attempting to numerically solve such a problem will usually lead to large errors in the computed solution. In practice, there is a continuum of problems ranging from well-conditioned to ill-conditioned, and the extent of the ill-conditioning affects the possible accuracy with which the solution \( Y \) can be found numerically, regardless of the numerical method being used.

**Example 1.8** The problem
\[
Y'(t) = \lambda |Y(t) - 1|, \quad 0 \leq t \leq b, \quad Y(0) = 1
\]
(1.14)
has the solution
\[
Y(t) = 1, \quad 0 \leq t \leq b.
\]
The perturbed problem
\[
Y_\varepsilon'(t) = \lambda [Y_\varepsilon(t) - 1], \quad 0 \leq t \leq b, \quad Y_\varepsilon(0) = 1 + \varepsilon
\]
has the solution
\[
Y_\varepsilon(t) = 1 + \varepsilon e^{\lambda t}, \quad 0 \leq t \leq b.
\]
For the error, we obtain
\[
Y(t) - Y_\varepsilon(t) = -\varepsilon e^{\lambda t},
\]
(1.15)
\[
\max_{0 \leq t \leq b} |Y(t) - Y_\varepsilon(t)| = \begin{cases} |\varepsilon|, & \lambda \leq 0, \\ |\varepsilon| e^{\lambda b}, & \lambda \geq 0. \end{cases}
\]
If \( \lambda < 0 \), the error \( |Y(t) - Y_\varepsilon(t)| \) decreases as \( t \) increases. We see that (1.14) is well-conditioned when \( \lambda \leq 0 \). In contrast, for \( \lambda > 0 \), the error \( |Y(t) - Y_\varepsilon(t)| \) increases as \( t \) increases. And for \( \lambda b \) moderately large, say \( \lambda b \geq 10 \), the change in \( Y(t) \) is quite significant at \( t = b \). The problem (1.14) is increasingly ill-conditioned as \( \lambda \) increases.

For the more general initial value problem (1.7) and the perturbed problem (1.11), one can show that
\[
Y(t) - Y_\varepsilon(t) \approx -\varepsilon \exp \left( \int_{t_0}^t g(s) \, ds \right)
\]
(1.16)
with
\[ g(t) = \left. \frac{\partial f(t, y)}{\partial y} \right|_{y = Y(t)} \]
for \( t \) sufficiently close to \( t_0 \). Note that this formula correctly predicts (1.15), since in that case
\[ f(t, y) = \lambda (y - 1), \]
\[ \frac{\partial f(t, y)}{\partial y} = \lambda, \]
\[ \int_0^t g(s) \, ds = \lambda t. \]
Then (1.16) yields
\[ Y(t) - Y_\epsilon(t) \approx -ce^{\lambda t}, \]
which agrees with the earlier formula (1.15).

**Example 1.9** The problem
\[ Y'(t) = -[Y(t)]^2, \quad Y(0) = 1 \]  \hspace{1cm} (1.17)
has the solution
\[ Y(t) = \frac{1}{t + 1}. \]
For the perturbed problem,
\[ Y'_\epsilon(t) = -[Y_\epsilon(t)]^2, \quad Y_\epsilon(0) = 1 + \epsilon, \]  \hspace{1cm} (1.18)
we use (1.16) to estimate \( Y(t) - Y_\epsilon(t) \). First,
\[ f(t, y) = -y^2, \]
\[ \frac{\partial f(t, y)}{\partial y} = -2y, \]
\[ g(t) = -2Y(t) = -\frac{2}{t + 1}, \]
\[ \int_0^t g(s) \, ds = -2 \int_0^t \frac{ds}{s + 1} = -2 \log(1 + t) = \log(1 + t)^{-2}, \]
\[ \exp \left[ \int_0^t g(s) \, ds \right] = e^{\log(1 + t)^{-2}} = \frac{1}{(t + 1)^2}. \]
For \( t \geq 0 \) sufficiently small, substituting into (1.16) gives
\[ Y(t) - Y_\epsilon(t) \approx \frac{-\epsilon}{(1 + \epsilon)^2}. \]  \hspace{1cm} (1.19)
This indicates that (1.17) is a well-conditioned problem.

In general, if

\[
\frac{\partial f(t, Y(t))}{\partial y} \leq 0, \quad t_0 \leq t \leq b,
\]

then the initial value problem is generally considered to be well-conditioned. Although this test depends on \( Y(t) \) over the interval \([t_0, b]\), one can often show (1.20) without knowing \( Y(t) \) explicitly; see Problems 5, 6.

### 1.3 DIRECTION FIELDS

Direction fields serve as a useful tool in understanding the behavior of solutions of a differential equation. We notice that the graph of a solution of the equation

\[
Y' = f(t, Y)
\]

is such that at any point \((t, y)\) on the solution curve, the slope is \( f(t, y) \). The slopes can be represented graphically in direction field diagrams. In MATLAB\textsuperscript{®}, direction fields can be generated by using the `meshgrid` and `quiver` commands.

**Example 1.10** Consider the equation \( Y' = Y \). The slope of a solution curve at a point \((t, y)\) on the curve is \( y \), which is independent of \( t \). We generate a direction field diagram with the following MATLAB code:

First draw the direction field:

\[
[t, y] = meshgrid(-2:0.5:2,-2:0.5:2);
\]
Figure 1.2  The direction field of the equation \( Y' = 2tY^2 \) and the solution \( Y = 1/(1 - t^2) \)

dt = ones(9); \% Generates a matrix of 1's.
dy = y;
quiver(t,y,dt,dy);
Then draw two solution curves:
    hold on
    t = -2:0.01:1;
y1 = exp(t); y2 = -exp(t);
plot(t,y1,t,y2)
    text(1.1,2.8,'\textit{Y=\text{e}^\text{t}}','FontSize',14)
    text(1.1,-2.8,'\textit{Y=-\text{e}^\text{t}}','FontSize',14)
    hold off
The result is shown in Figure 1.1.

Example 1.11 Continuing Example 1.6, we use the following MATLAB M-file to generate a direction field diagram and the particular solution \( Y = 1/(1 - t^2) \) in Figure 1.2.

[t,y] = meshgrid(-1:0.2:1,1:0.5:4);
dt = ones(7,11); dy = 2*t.*y.^2;
quiver(t,y,dt,dy);
    hold on
    tt = -0.87:0.01:0.87;
\[ y_1 = 1/(1-t \cdot \cdot 2); \]
\[
\text{plot}(t, y, y, y) \]
\[
\text{hold off}
\]

Note that for large \( y \) values, the arrows in the direction field diagram (Figure 1.2) point almost vertically. This suggests that a solution to the equation may exist only in a bounded interval of the \( t \) axis, which, indeed, is the case. ■

**PROBLEMS**

1. In each of the following cases, show that the given function \( Y(t) \) satisfies the associated differential equation. Then determine the value of \( c \) required by the initial condition. Finally, with reference to the general format in (1.7), identify \( f(t, y) \) for each differential equation.

   (a) \( Y'(t) = -Y(t) + \sin(t) + \cos(t), \quad Y(0) = 2; \]
   \[
   Y(t) = \sin(t) + ce^{-t}.
   \]

   (b) \( Y'(t) = \left[ Y(t) - Y(t)^2 \right]/t, \quad Y(1) = 2; \]
   \[
   Y(t) = t/(t + c), \quad t > 0.
   \]

   (c) \( Y'(t) = \cos^2(Y(t)), \quad Y(0) = \pi/4; \]
   \[
   Y(t) = \tan^{-1}(t + c).
   \]

   (d) \( Y'(t) = Y(t)[Y(t) - 1], \quad Y(0) = 1/2; \]
   \[
   Y(t) = 1/(1 + ce^t).
   \]

2. Use MATLAB to draw direction fields for the differential equations listed in Problem 1.

3. Solve the following problem by using (1.5) and (1.4):

   (a) \( Y'(t) = \lambda Y(t) + 1, \quad Y(0) = 1. \]

   (b) \( Y'(t) = \lambda Y(t) + t, \quad Y(0) = 3. \)

4. Consider the differential equation

   \[ Y'(t) = f_1(t)f_2(Y(t)) \]

   for some given functions \( f_1(t) \) and \( f_2(z) \). This is called a separable differential equation, and it can be solved by direct integration. Write the equation as

   \[ \frac{Y'(t)}{f_2(Y(t))} = f_1(t), \]

   and find the antiderivative of each side:

   \[ \int \frac{Y'(t)}{f_2(Y(t))} \, dt = \int f_1(t) \, dt. \]

   On the left side, change the integration variable by letting \( z = Y(t) \). Then the equation becomes

   \[ \int \frac{dz}{f_2(z)} = \int f_1(t) \, dt. \]
After integrating, replace $z$ by $Y(t)$; then solve for $Y(t)$, if possible. If these integrals can be evaluated, then the differential equation can be solved. Do so for the following problems, finding the general solution and the solution satisfying the given initial condition.

(a) $Y''(t) = t/Y(t), \quad Y(0) = 2$.
(b) $Y''(t) = te^{-Y(t)}, \quad Y(1) = 0$.
(c) $Y''(t) = Y(t)[a - Y(t)], \quad Y(0) = a/2, \quad a > 0$.

5. Check the conditioning of the initial value problems in Problem 1. Use the test (1.20).

6. Check the conditioning of the initial value problems in Problem 4 (a), (b). Use the test (1.20).

7. Use (1.20) to discuss the conditioning of the problem

$$Y''(t) = Y(t)^2 - 5 \sin(t) - 25 \cos^2(t), \quad Y(0) = 5.$$ 

You do not need to know the true solution.

8. Consider the solutions $Y(t)$ of

$$Y'(t) + aY(t) = de^{-bt}$$

with $a, b, d$ constants and $a, b > 0$. Calculate

$$\lim_{t \to \infty} Y(t).$$

*Hint:* Consider the cases $a \neq b$ and $a = b$ separately.