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BASIC $q$-COMBINATORICS AND $q$-HYPERGEOMETRIC SERIES

1.1 INTRODUCTION

The basic $q$-sequences and $q$-functions of the calculus of $q$-hypergeometric series, which facilitate the study of discrete $q$-distributions, are thoroughly presented in this chapter. More precisely, after introducing the notions of a $q$-power, a $q$-factorial, and a $q$-binomial coefficient of a real number, two $q$-Vandermonde’s ($q$-factorial convolution) formulae are derived. Also, two $q$-Cauchy’s ($q$-binomial convolution) formulae are presented as a corollary of the two $q$-Vandermonde’s formulae. Furthermore, the $q$-binomial and the negative $q$-binomial formulae are obtained. In addition, a general $q$-binomial formula is derived and, as limiting forms of it, $q$-exponential and $q$-logarithmic functions are deduced. The $q$-Stirling numbers of the first and second kind, which are the coefficients of the expansions of $q$-factorials into $q$-powers and of $q$-powers into $q$-factorials, respectively, are presented. Also, the generalized $q$-factorial coefficients are briefly discussed. Moreover, the $q$-factorial and $q$-binomial moments, which, apart from the interest in their own, are used as an intermediate step in the calculation of the usual factorial and binomial moments of a discrete $q$-distribution, are briefly presented. Finally, the probability function of a nonnegative integer-valued random variable is expressed in terms of its $q$-binomial (or $q$-factorial) moments.
1.2 $q$-FACTORIALS AND $q$-BINOMIAL COEFFICIENTS

Let $x$ and $q$ be real numbers, with $q \neq 1$, and $k$ be an integer. The number

$$[x]_q = \frac{1 - q^x}{1 - q}$$

is called $q$-number and in particular $[k]_q$ is called $q$-integer. Note that

$$\lim_{q \to 1} [x]_q = x.$$ 

The base (parameter) $q$, in the theory of discrete $q$-distributions, varies in the interval $0 < q < 1$ or in the interval $1 < q < \infty$. In both these cases,

$$[x]_q \preceq [y]_q \text{, if and only if } x \preceq y,$$

respectively.

In particular,

$$[x]_q \preceq 0 \text{, if and only if } x \preceq 0,$$

respectively.

In this book, unless stated otherwise, it is assumed that $0 < q < 1$ or $1 < q < \infty$.

The $k$th-order factorial of the $q$-number $[x]_q$, which is defined by

$$[x]_{k,q} = [x]_q [x-1]_q \cdots [x-k+1]_q, \quad k = 1, 2, \ldots,$$

is called $q$-factorial of $x$ of order $k$. In particular,

$$[k]_q! = [1]_q [2]_q \cdots [k]_q, \quad k = 1, 2, \ldots,$$

is called $q$-factorial of $k$.

The $q$-factorial of $x$ of negative order may be defined as follows. Clearly, the following fundamental property of the $q$-factorial

$$[x]_{r+k,q} = [x]_{r,q} \cdot [x-r]_{k,q}, \quad k = 1, 2, \ldots, r = 1, 2, \ldots,$$

is readily deduced from its definition. Requiring the validity of this fundamental property to be preserved, the definition of the factorial may be extended to zero or negative order. Specifically, it is required that the fundamental property is valid for any integer values of $k$ and $r$. Then, substituting into it $r = 0$, it follows that

$$[x]_{k,q} = [x]_{0,q} \cdot [x]_{k,q},$$

for any integer $k$. This equation, if $x \neq 0, 1, \ldots, k - 1$, whence $[x]_{k,q} \neq 0$, implies

$$[x]_{0,q} = 1,$$

while, if $x = 0, 1, \ldots, k - 1$, reduces to an identity for any value $[x]_{0,q}$ is required to represent. Furthermore, from the fundamental property, with $k$ a positive integer and $r = -k$, it follows that

$$[x]_{-k,q} \cdot [x+k]_{k,q} = 1$$
and, for \( x \neq -1, -2, \ldots, -k \),
\[
[x]_{k,q} = \frac{1}{[x + k]_{k,q}} = \frac{1}{[x + k]_q[x + k - 1]_q \cdots [x + 1]_q}, \quad k = 1, 2, \ldots.
\]

Notice that the last expression, for \( x = 0 \), yields
\[
[0]_{k,q} = \frac{1}{[k]_q!}, \quad k = 1, 2, \ldots.
\]

The \( q \)-binomial coefficient (or Gaussian polynomial) is defined by
\[
\left[ \begin{array}{c} x \\ k \end{array} \right]_q = \frac{[x]_{k,q}}{[k]_q!}, \quad k = 0, 1, \ldots,
\]
and so
\[
\lim_{q \to 1} \left[ \begin{array}{c} x \\ k \end{array} \right]_q = \left( \begin{array}{c} x \\ k \end{array} \right), \quad k = 0, 1, \ldots.
\]

Note that
\[
[x]_{q^{-1}} = \frac{1 - q^{-x}}{1 - q^{-1}} = \frac{q^{-x}}{q^{-1}} \cdot \frac{1 - q^x}{1 - q} = q^{-x+1}[x]_q,
\]
and since
\[
[x]_{k,q^{-1}} = \prod_{j=1}^{k} [x - j + 1]_{q^{-1}} = \prod_{j=1}^{k} q^{-x+j}[x - j + 1]_q = q^{-x+1+2+\cdots+k}[x]_{k,q},
\]
it follows that
\[
[x]_{k,q^{-1}} = q^{-x+k+\frac{k+1}{2}}[x]_{k,q}, \quad [k]_{q^{-1}}! = q^{-\frac{k}{2}}[k]_q!,
\]
and
\[
\left[ \begin{array}{c} x \\ k \end{array} \right]_{q^{-1}} = q^{-k(x-k)}\left[ \begin{array}{c} x \\ k \end{array} \right]_q, \quad k = 0, 1, \ldots.
\]

Using these expressions, a formula involving \( q \)-numbers, \( q \)-factorials, and \( q \)-binomial coefficients in a base \( q \), with \( 1 < q < \infty \), can be converted, with respect to the base, into a similar formula in the base \( p = q^{-1} \), with \( 0 < p < 1 \).

Two useful versions of a triangular recurrence relation for the \( q \)-binomial coefficient, which constitutes a \( q \)-analogue of Pascal’s triangle, are derived in the next theorem.

**Theorem 1.1.** Let \( x \) and \( q \) be real numbers, with \( q \neq 1 \), and let \( k \) be a positive integer. Then, the \( q \)-binomial coefficient \( \left[ \begin{array}{c} x \\ k \end{array} \right]_q \) satisfies the triangular recurrence relation
\[
\left[ \begin{array}{c} x \\ k \end{array} \right]_q = \left[ \begin{array}{c} x - 1 \\ k \end{array} \right]_q + q^{r-k}\left[ \begin{array}{c} x - 1 \\ k - 1 \end{array} \right]_q, \quad k = 1, 2, \ldots, \quad (1.1)
\]
with initial condition \( [x]_0 = 1 \). Alternatively,
\[
\begin{align*}
[x]_k &= q^k [x-1]_k, \\
&= q^k [x-1]_k + [x-1]_{k-1}, \quad k = 1, 2, \ldots .
\end{align*}
\] (1.2)

**Proof.** The \( q \)-factorial of \( x \) of order \( k \), since \( [x]_{k,q} = [x]_q [x-1]_{k-1,q} \) and
\[
[x]_q = [x-k+k]_q = [x-k]_q + q^{x-k}[k], \quad [x-1]_{k-1,q} [x-k]_q = [x-1]_{k,q},
\]
satisfies the triangular recurrence relation
\[
[x]_{k,q} = [x-1]_{k,q} + q^{x-k}[k]_q [x-1]_{k-1,q},
\]
with initial condition \( [x]_{0,q} = 1 \). Thus, dividing both members of it by \([k]_q!\) and using the expression
\[
\begin{align*}
[x]_k &= [x]_{k,q} / [k]_q!, \quad k = 0, 1, \ldots ,
\end{align*}
\]
the triangular recurrence relation (1.1) is readily deduced. Furthermore, replacing the base \( q \) by \( q^{-1} \) and using the relation
\[
\begin{align*}
[x]_k = [k+x-k]_q = [k]]_q + q^{x-k}[x-k]_q,
\end{align*}
\]
which entails for the \( q \)-factorial of \( x \) of order \( k \) the triangular recurrence relation
\[
[x]_{k,q} = [k]_q [x-1]_{k-1,q} + q^k [x-1]_{k,q}.
\]
Hence, dividing both members of it by \([k]_q!\), (1.2) is obtained.
\[\Box\]

**Remark 1.1.** The lack of uniqueness of \( q \)-analoyues of expressions and formulae. The lack of uniqueness, due to the presence of powers of \( q \) in pseudo-isomorphisms as
\[
[x+y]_q = [x]_q + q^x[y]_q \quad \text{and} \quad [x+y]_q = q^y[x]_q + [y]_q,
\]
where \( 0 < q < 1 \) or \( 1 < q < \infty \), should be remarked from the very beginning of the presentation of the basic \( q \)-sequences, \( q \)-functions and \( q \)-formulae. It should also be noticed that the two formulae may be considered as equivalent in the sense that any of these implies the other by replacing the base \( q \) by \( q^{-1} \). In this framework, the existence of two versions of the \( q \)-analogue of Pascal’s triangle, which may be considered as equivalent, is attributed to the lack of uniqueness.
The particular cases of the $q$-binomial coefficients $\binom{n}{k}_q$ and $\binom{n+k-1}{k}_q$, with $n$ and $k$ positive integers, admit $q$-combinatorial interpretations, which are deduced in the following theorem, starting from a generating function of a number of partitions of an integer into parts of restricted size. Recall that a partition of a positive integer $m$ into $k$ parts is a nonordered collection of positive integers, $\{r_1, r_2, \ldots, r_k\}$, with $r_1 \geq r_2 \geq \cdots \geq r_k \geq 1$, for $k = 1, 2, \ldots, m$, whose sum equals $m$. In a partition of $m$ into $k$ parts, let $k_i \geq 0$ be the number of parts that are equal to $i$, for $i = 1, 2, \ldots, m$. Then,

$$k_1 + 2k_2 + \cdots + mk_m = m, \quad k_1 + k_2 + \cdots + k_m = k.$$ 

**Theorem 1.2.** The $q$-binomial coefficient $\binom{n}{k}_q$, for $n$ and $k$ positive integers, equals the $k$-combinations of the set $\{1, 2, \ldots, n\}$, $\{m_1, m_2, \ldots, m_k\}$, weighted by $q^{m_1 + m_2 + \cdots + m_k - \left(\begin{array}{c}k+1 \\frac{k}{2}\end{array}\right)}$,

$$\sum_{1 \leq m_1 < m_2 < \cdots < m_k \leq n} q^{m_1 + m_2 + \cdots + m_k - \left(\begin{array}{c}k+1 \\frac{k}{2}\end{array}\right)} = \binom{n}{k}_q. \quad (1.3)$$

Also, the $q$-binomial coefficient $\binom{n+k-1}{k}_q$, for $n$ and $k$ positive integers, equals the $k$-combinations of the set $\{1, 2, \ldots, n\}$ with repetition, $\{r_1, r_2, \ldots, r_k\}$, weighted by $q^{r_1 + r_2 + \cdots + r_k - k}$,

$$\sum_{1 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq n} q^{r_1 + r_2 + \cdots + r_k - k} = \binom{n+k-1}{k}_q. \quad (1.4)$$

**Proof.** Let $p(m, k; n)$ denotes the number of partitions of $m$ into $k$ parts, each of which is less than or equal to $n$, and consider its bivariate generating function

$$g_n(t, q) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} p(m, k; n) t^k q^m,$$

for fixed $n$. Clearly, using the definition of a partition of $m$ into $k$ parts, each of which is less than or equal to $n$, it may be expressed as

$$g_n(t, q) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left( \sum_{k_1 + k_2 + \cdots + k_m = m} q^{k_1 + 2k_2 + \cdots + mk_m} \right),$$

where in the inner sum the summation is extended over all integer solutions $k_i \geq 0$, for $i = 1, 2, \ldots, n$, and $k_i = 0$, for $i = n + 1, n + 2, \ldots, m$, of the equations $k_1 + 2k_2 + \cdots + mk_m = m$ and $k_1 + k_2 + \cdots + k_m = k$. Since this inner sum is summated over all $k = 0, 1, \ldots, m$ and $m = 0, 1, \ldots$, it follows that

$$g_n(t, q) = \prod_{i=1}^{n} \left( \sum_{k_i=0}^{\infty} \frac{t^{k_i} q^{k_i}}{(1 - t q^{i})^{-1}} \right) = \prod_{i=1}^{n} (1 - t q^i)^{-1}.$$
Furthermore, for the sequence of univariate generating functions
\[ g_{n,k}(q) = \sum_{m=k}^{\infty} p(m,k;n)q^m, \quad k = 0, 1, \ldots, \]
on using the relation \( g_n(t, q) = \sum_{k=0}^{\infty} g_{n,k}(q)t^k \) and since
\[ (1 - t q^{n+1}) g_n(t, q) = (1 - t) g_n(t, q), \]
we deduce the following first-order recurrence relation
\[ g_{n,k}(q) = 1 - q^{n+k-1} \frac{1}{1 - q^k} g_{n,k-1}(q), \quad k = 1, 2, \ldots, \quad g_{n,0}(q) = 1. \]

Applying it repeatedly, we find the expression
\[ g_{n,k}(q) = \sum_{m=k}^{\infty} p(m,k;n)q^m = q^k \binom{n + k - 1}{k}_q. \]

Since the number \( p(m,k;n) \), of partitions of \( m \) into \( k \) parts, each of which is less than or equal to \( n \), equals the number of solutions in positive integers of the equation \( r_1 + r_2 + \cdots + r_k = m \), with \( 1 \leq r_k \leq r_{k-1} \leq \cdots \leq r_1 \leq n \), or equivalently (by replacing \( r_{k-i+1} \) with \( r_i \), for \( i = 1, 2, \ldots, k \)), with \( 1 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq n \), the last expression may also be written in the form
\[ \sum_{1 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq n} q^{r_1 + r_2 + \cdots + r_k} = q^k \binom{n + k - 1}{k}_q, \]
which readily implies (1.4).

Furthermore, replacing \( n + k - 1 \) by \( n \) in (1.4) and then setting \( m_i = r_i + i - 1 \), for \( i = 1, 2, \ldots, k \), the inequalities \( 1 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq n - k + 1 \) are transformed into \( 1 \leq m_1 < m_2 < \cdots < m_k \leq n \) and \( r_1 + r_2 + \cdots + r_k = m_1 + m_2 + \cdots + m_k - \binom{k}{2} \). Consequently, expression (1.4) is transformed into
\[ \sum_{1 \leq m_1 < m_2 < \cdots < m_k \leq n} q^{m_1 + m_2 + \cdots + m_k} = q^{k+1} \binom{n}{k}_q, \]
from which (1.3) is deduced. \( \square \)

**Remark 1.2.** An alternative expression of a \( q \)-binomial coefficient. Expression (1.4) of the \( q \)-binomial coefficient \( \binom{n+k-1}{k}_q \), may be written alternatively as follows. Let \( k_i \geq 0 \) be the number of the (bound) variables \( r_1, r_2, \ldots, r_k \), that are equal to \( i \), for \( i = 1, 2, \ldots, n \). Then, \( r_1 + r_2 + \cdots + r_k = k_1 + 2k_2 + \cdots + nk_n \), with \( k_1 + k_2 + \cdots + k_n = k \), and
\[ \{(r_1, r_2, \ldots, r_k) : 1 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq n\} = \{(k_1, k_2, \ldots, k_n) : k_i \geq 0, i = 1, 2, \ldots, n, k_1 + k_2 + \cdots + k_n = k\}. \]
Consequently, expression (1.4) may be, alternatively, expressed as

$$
\sum_{k_1 + k_2 + \cdots + k_n = k} q^{k_1 + 2k_2 + \cdots + nk_n - k} = \binom{n + k - 1}{k}_q.
$$

(1.5)

A few interesting combinatorial and probabilistic examples, in which \(q\)-numbers and \(q\)-binomial coefficients naturally emerged, are presented next. Specifically, in the following example, a number theoretic random variable is defined in a sequence of independent and identically distributed Bernoulli trials, and its probability function is expressed by a \(q\)-number.

**Example 1.1. Bernoulli trials and number theory.** Consider a sequence of independent Bernoulli trials, with constant failure probability \(q\), and let \(X\) be the number of failures until the occurrence of the first success. Clearly, the random variable \(X\) follows a geometric distribution with probability function

$$
P(X = x) = (1 - q)q^x, \quad x = 0, 1, \ldots,
$$

where \(0 < q < 1\). Also, consider a fixed positive integer \(n\) and let

$$
C_x = \{x + kn : k = 0, 1, \ldots\}, \quad x = 0, 1, \ldots, n - 1.
$$

Clearly, each of the possible values of the random variable \(X\), \(\{0, 1, \ldots\}\), belongs to one of these \(n\) congruence classes (pairwise disjoint sets), modulo \(n\), \(\{C_0, C_1, \ldots, C_{n-1}\}\). Furthermore, consider a sequence of independent Bernoulli trials, with constant failure probability \(q\), and let \(X_n\) be the index of the congruence class in which the number of failures, until the occurrence of the first success, belongs. Since the random variable \(X_n\) assumes the value \(x\) if and only if \(X\) belongs to \(C_x\), its probability function,

$$
P(X_n = x) = P(X \in C_x) = \sum_{k=0}^{\infty} (1 - q)q^{x+kn} = \frac{(1 - q)q^x}{1 - q^n},
$$

may be expressed as

$$
P(X_n = x) = \frac{q^x}{[n]_q}, \quad x = 0, 1, \ldots, n - 1, \quad 0 < q < 1.
$$

Note that the limiting probability function, as \(q \to 1\),

$$
\lim_{q \to 1} P(X_n = x) = \frac{1}{n}, \quad x = 0, 1, \ldots, n - 1,
$$

is the discrete uniform probability function on the set \(\{0, 1, \ldots, n - 1\}\). For this reason, the distribution of \(X_n\) is called **discrete \(q\)-uniform distribution**.
It is worth noting that the probability function $P(X = x|X \leq n - 1)$, of a right truncated geometric distribution, since

$$P(X = x|X \leq n - 1) = \frac{P(X = x, X \leq n - 1)}{P(X \leq n - 1)} = \frac{P(X = x)}{P(X \leq n - 1)},$$

for $x = 0, 1, \ldots, n - 1$, and

$$P(X \leq n - 1) = \sum_{x=0}^{n-1} (1 - q)q^x = 1 - q^n,$$

is readily deduced as

$$P(X = x|X \leq n - 1) = \frac{q^x}{[n]_q}, \quad x = 0, 1, \ldots, n - 1, \quad 0 < q < 1.$$

An interesting combinatorial interpretation of the $q$-binomial coefficient for $x = n$ a positive integer and $q$ a power of a prime number, is given in the next example.

Example 1.2. \textit{Number of subspaces of a vector space.} Let $V(n; q)$ be a vector space of dimension $n$ over a finite field of $q = p^m$ elements, where $p$ is a prime number and $m$ is a positive integer. The vector space $V(n; q)$ contains $q^n$ vectors. A subspace $S(k; q)$ of $V(n; q)$, of dimension $k \leq n$, contains $k$ linearly independent vectors that are selected from $V(n; q)$. The number $G_{n,k}(q)$ of $k$-dimensional subspaces, $S(k; q)$, of an $n$-dimensional vector space $V(n; q)$ is of interest.

Let us first determine the number $H_{n,k}(q)$ of ordered $k$-tuples $(a_1, a_2, \ldots, a_k)$ of linearly independent vectors in $V(n; q)$. The first nonzero vector $a_1$, of an ordered $k$-tuple, can be selected from the $q^n - 1$ nonzero vectors of $V(n; q)$. Any nonzero vector $a_1$ spans (generates) a one-dimensional subspace $S(1; q)$ of $V(n; q)$ containing $q$ vectors. Therefore, excluding these $q$ vectors, the second vector $a_2$ can be selected from the $q^n - q$ vectors, which are linearly independent of $a_1$. The ordered pair $(a_1, a_2)$ spans a two-dimensional subspace $S(2; q)$ of $V(n; q)$ containing $q^2$ vectors. Thus, excluding these $q^2$ vectors, the third vector $a_3$ can be selected from the $q^n - q^2$ vectors, which are linearly independent of $a_1, a_2$. In general, the $r$th vector $a_r$ can be selected from the $q^n - q^{r-1}$ vectors, which are linearly independent of $a_1, a_2, \ldots, a_{r-1}$, for $r = 3, 4, \ldots, k$. Consequently, applying the multiplication principle, it follows that

$$H_{n,k}(q) = \prod_{r=1}^{k} (q^n - q^{r-1}).$$

Note that each ordered $k$-tuple $(a_1, a_2, \ldots, a_k)$ of linearly independent vectors in $V(n; q)$ spans a $k$-dimensional subspace $S(k; q)$. Inversely, any $k$-dimensional subspace $S(k; q)$ is spanned by

$$H_{k,k}(q) = \prod_{r=1}^{k} (q^k - q^{r-1})$$
ordered $k$-tuples $(a_1, a_2, \ldots, a_k)$ of linearly independent vectors in $V(n; q)$. The evaluation of this number is carried out the same way as above. Therefore, the number $G_{n,k}(q)$ of $k$-dimensional subspaces of an $n$-dimensional vector space $V(n; q)$ is given by the quotient

$$G_{n,k}(q) = \frac{H_{n,k}(q)}{H_{k,k}(q)} = \frac{\prod_{r=1}^{k} (q^n - q^{r-1})}{\prod_{r=1}^{k} (q^k - q^{r-1})} = \frac{(-1)^k q^{k \choose 2}}{\prod_{r=1}^{k} (1 - q^{n-r+1})}$$

and so

$$G_{n,k}(q) = \frac{[n]_{k,q}}{[k]_q!} = \left[ \frac{n}{k} \right]_q.$$

In the following example, the $q$-binomial coefficient $\left[ \frac{n+k-1}{k} \right]_q$ is obtained as a generating function of a number of partitions of an integer into parts of restricted size, with variable (indeterminate) $q$, in addition to the generating function deduced in the proof of Theorem 1.2.

**Example 1.3.** Partitions of integers into parts of restricted size. Let us consider the number $P(m; k; n)$, of partitions of $m$ into at most $k$ parts, each of which is less than $n$. It is closely connected to the number $p(m; k; n)$ of partitions $m$ into (exactly) $k$ parts, each of which is less than or equal to $n$, which was discussed in the proof of Theorem 1.2. Specifically, since the size of a part in a partition is a positive integer, the restriction that a size is less than $n$ is equivalent to the restriction that it is less than or equal to $n - 1$ and so $P(m; k; n) = \sum_{r=0}^{k} p(m, r; n - 1)$. The bivariate generating function

$$h_n(t, q) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} P(m, k; n)t^k q^m,$$

for fixed $n$, on introducing the expression $p(m; k; n) = \sum_{r=0}^{k} p(m, r; n - 1)$ and subsequently interchanging the order of summation of the two inner sums,

$$h_n(t, q) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \sum_{r=0}^{k} p(m, r; n - 1)t^k q^m$$

$$= \sum_{m=0}^{\infty} \sum_{r=0}^{m} \left( \sum_{k=r}^{\infty} t^{k-r} \right) p(m, r; n - 1)t^r q^m,$$

is expressed in terms of the bivariate generating function of $p(m, r; n - 1)$ as

$$h_n(t, q) = (1 - t)^{-1} \sum_{m=0}^{\infty} \sum_{r=0}^{m} p(m, r; n - 1)t^r q^m = (1 - t)^{-1} g_{n-1}(t, q)$$

and so

$$h_n(t, q) = \prod_{i=0}^{n-1} (1 - tq^i)^{-1}.$$
Also, for the sequence of univariate generating functions

\[ h_{n,k}(q) = \sum_{m=k}^{\infty} P(m, k; n)q^m, \quad k = 0, 1, \ldots, \]

on using the relation \( h_n(t, q) = \sum_{k=0}^{\infty} h_{n,k}(q)t^k \) and since

\[(1 - tq^n)h_n(tq, q) = (1 - t)h_n(t, q),\]

we deduce the first-order recurrence relation

\[ h_{n,k}(q) = \frac{1 - q^{n+k-1}}{1 - q^k}h_{n,k-1}(q), \quad k = 1, 2, \ldots, h_{n,0}(q) = 1.\]

Applying it repeatedly, we get the expression

\[ h_{n,k}(q) = \sum_{m=k}^{\infty} P(m, k; n)q^m = \left[ \frac{n + k - 1}{k} \right]_q. \]

Therefore, the number \( P(m, k; n) \), of partitions of \( m \) into at most \( k \) parts, each of which is less than \( n \), is the coefficient of \( q^m \) in the expansion of the \( q \)-binomial coefficient \( \left[ \frac{n + k - 1}{k} \right]_q \) into powers of \( q \).

### 1.3 \( q \)-VANDERMONDE’S AND \( q \)-CAUCHY’S FORMULAE

Two versions of a \( q \)-Vandermonde’s (\( q \)-factorial convolution) formula are derived in the next theorem.

**Theorem 1.3.** Let \( n \) be a positive integer and let \( x, y, \) and \( q \) be real numbers, with \( q \neq 1 \). Then,

\[ [x + y]_{n,q} = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^{(n-k)(x-k)}[x]_{k,q}[y]_{n-k,q}. \] (1.6)

Alternatively,

\[ [x + y]_{n,q} = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^{k(n-k)}[x]_{k,q}[y]_{n-k,q}. \] (1.7)

**Proof.** Let us consider the sequence of sums

\[ s_n(x, y; q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^{(n-k)(x-k)}[x]_{k,q}[y]_{n-k,q}, \quad n = 1, 2, \ldots, \]

with initial value

\[ s_1(x, y; q) = q^y[y]_q + [x]_q = [x + y]_q. \]
Using the triangular recurrence relation (1.1), with \( x = n \), the sequence \( s_n(x, y; q) \), \( n = 1, 2, \ldots \), may be expressed as

\[
s_n(x, y; q) = \sum_{k=0}^{n-1} \binom{n}{k} q^{(n-k)(x-k)} [x]_{k,q} [y]_{n-k,q} + \sum_{k=1}^{n} \binom{n-1}{k-1} q^{(n-k)(x-k+1)} [x]_{k,q} [y]_{n-k,q}.
\]

Replacing \( k - 1 \) by \( k \) in the last sum and then using the relation

\[
q^{x-k} [x]_{k,q} [y]_{n-k,q} + [x]_{k+1,q} [y]_{n-1-k,q} = [x + y - n + 1]_q [x]_{k,q} [y]_{n-1-k,q},
\]

it follows that

\[
s_n(x, y; q) = \sum_{k=0}^{n-1} \binom{n-1}{k} q^{(n-1-k)(x-k)} q^{y-k} [x]_{k,q} [y]_{n-k,q} + \sum_{k=0}^{n-1} \binom{n-1}{k} q^{(n-1-k)(x-k)} [x]_{k+1,q} [y]_{n-1-k,q}
\]

\[
= [x + y - n + 1]_q \sum_{k=0}^{n-1} \binom{n-1}{k} q^{(n-1-k)(x-k)} [x]_{k,q} [y]_{n-1-k,q}.
\]

Hence, the sequence \( s_n(x, y; q) \), \( n = 1, 2, \ldots \), satisfies the first-order recurrence relation

\[
s_n(x, y; q) = [x + y - n + 1]_q s_{n-1}(x, y; q), \quad n = 1, 2, \ldots,
\]

with initial condition \( s_1(x, y; q) = [x + y]_q \). Clearly, applying it successively, it follows that \( s_n(x, y; q) = [x + y]_{n,q} \) and so (1.6) is shown. Furthermore, interchanging \( x \) by \( y \) and then replacing \( k \) by \( n - k \), formula (1.6) is rewritten in the form (1.7).

Note that the alternative formula (1.7) may be derived, independently of (1.6), by following the steps of derivation of (1.6) and using the triangular recurrence relation (1.2) instead of (1.1) and

\[
[x]_{k,q} [y]_{n-k,q} + q^{y-n+k+1} [x]_{k+1,q} [y]_{n-1-k,q} = [x + y - n + 1]_q [x]_{k,q} [y]_{n-1-k,q}
\]

instead of

\[
q^{y-k} [x]_{k,q} [y]_{n-k,q} + [x]_{k+1,q} [y]_{n-1-k,q} = [x + y - n + 1]_q [x]_{k,q} [y]_{n-1-k,q}.
\]

The proof of the theorem is thus completed. \( \square \)

Two versions of a \( q \)-Cauchy’s \((q\text{-binomial convolution})\) formula, which by virtue of

\[
\binom{x}{k}_q = \frac{[x]_{k,q}}{[k]_q!}, \quad k = 0, 1, \ldots,
\]

may be considered as reformulations of the corresponding two versions of the \( q \)-Vandermonde’s \((q\text{-factorial convolution})\) formula, are stated in the following corollary of Theorem 1.3.
Corollary 1.1. Let $n$ be a positive integer and let $x$, $y$, and $q$ be real numbers, with $q \neq 1$. Then,

$$
\binom{x + y}{n}_q = \sum_{k=0}^{n} q^{(n-k)(x-k)} \binom{x}{k}_q \binom{y}{n-k}_q.
$$

(1.8)

Alternatively,

$$
\binom{x + y}{n}_q = \sum_{k=0}^{n} q^{k(y-n+k)} \binom{x}{k}_q \binom{y}{n-k}_q.
$$

(1.9)

Two versions of a negative $q$-Vandermonde’s formula are obtained in the next theorem.

Theorem 1.4. Let $n$ be a positive integer and let $x$, $y$, and $q$ be real numbers, with $q \neq 1$. Then,

$$
[x + y]_{-n,q} = \sum_{k=0}^{\infty} \left[ -\frac{n}{k} \right] q^{-(n+k)(x-k)} [x]_{k,q} [y]_{-n-k,q},
$$

(1.10)

with $|q^{-(x+y+1)}| < 1$. Alternatively,

$$
[x + y]_{-n,q} = \sum_{k=0}^{\infty} \left[ -\frac{n}{k} \right] q^{k(y+n+k)} [x]_{k,q} [y]_{-n-k,q},
$$

(1.11)

with $|q^{x+y+1}| < 1$.

Proof. Let us set

$$
s_q(x, y; q) = \sum_{k=0}^{\infty} \left[ -\frac{n}{k} \right] q^{-(n+k)(x-k)} [x]_{k,q} [y]_{-n-k,q},
$$

for $n = 1, 2, \ldots$. Note that, the first term of this sequence,

$$
s_1(x, y; q) = \sum_{k=0}^{\infty} \left[ -\frac{1}{k} \right] q^{-(k+1)(x-k)} [x]_{k,q} [y]_{-k-1,q}
$$

$$
= \sum_{k=0}^{\infty} (-1)^k q^{-(k+1)x + \binom{k+1}{2}} [x]_{k,q} [y]_{-k-1,q},
$$

since

$$
[x + y + 1]_{x+k,q} [y]_{-k-1,q} = q^{-x+k} [x]_{x+k,q} [y]_{-k-1,q} + [x]_{k+1,q} [y]_{-k-1,q},
$$

may be written as

$$
s_1(x, y; q) = \frac{1}{[x + y + 1]_q} \left\{ \sum_{k=0}^{\infty} (-1)^k q^{-kx + \binom{k}{2}} [x]_{k,q} [y]_{-k,q}
$$

$$
- \sum_{k=0}^{\infty} (-1)^{k+1} q^{-(k+1)x + \binom{k+1}{2}} [x]_{k+1,q} [y]_{-k-1,q} \right\}
$$
and so \( s_1(x, y; q) = 1/[x + y + 1]_q \). Using the triangular recurrence relation (1.1), with \( x = -n \), the sequence \( s_n(x, y; q), n = 1, 2, \ldots \), may be expressed as

\[
    s_n(x, y; q) = \sum_{k=0}^{\infty} \left[ -\frac{n-1}{k} \right] q^{-(n+k)(x-k)}[x]_{k,q}[y]_{-n-k,q} + \sum_{k=1}^{\infty} \left[ -\frac{n-1}{k-1} \right] q^{-(n+k)(x-k+1)}[x]_{k,q}[y]_{-n-k,q}.
\]

Replacing \( k - 1 \) by \( k \) in the last sum and then using the relation

\[
    q^{-k}[x]_{k,q}[y]_{-n-k,q} + [x]_{k+1,q}[y]_{-n-1-k,q} = [x + y + n + 1]_q[x]_{k,q}[y]_{-n-1-k,q},
\]

it follows that

\[
    s_n(x, y; q) = \sum_{k=0}^{\infty} \left[ -\frac{n-1}{k} \right] q^{-(n+1+k)(x-k)} q^{-k}[x]_{k,q}[y]_{-n-k,q} + \sum_{k=0}^{\infty} \left[ -\frac{n-1}{k} \right] q^{-(n+1+k)(x-k)}[x]_{k+1,q}[y]_{-n-1-k,q} = [x + y + n + 1]_q \sum_{k=0}^{\infty} \left[ -\frac{n-1}{k} \right] q^{-(n+1+k)(x-k)}[x]_{k,q}[y]_{-n-1-k,q}.
\]

Hence, the sequence \( s_n(x, y; q), n = 1, 2, \ldots \), satisfies the first-order recurrence relation

\[
    s_{n+1}(x, y; q) = \frac{s_n(x, y; q)}{[x + y + n + 1]_q}, \quad n = 1, 2, \ldots,
\]

with initial condition \( s_1(x, y; q) = 1/[x + y + 1]_q = [x + y]_{-1,q} \). Consequently, \( s_n(x, y; q) = 1/[x + y + n]_{n,q} = [x + y]_{-n,q} \) and so (1.10) is shown. Furthermore, replacing the base \( b \) by \( q^{-1} \) and using the relations

\[
    [x + y]_{-n,q} = q^{(x+y)n+\binom{n}{2}}[x + y]_{-n,q}, \quad \left[ -\frac{n}{k} \right] q^{-1} = q^{k(n+k)} \left[ -\frac{n}{k} \right] q
\]

and

\[
    [x]_{k,q} = q^{-x+\binom{k+1}{2}}[x]_{k,q}, \quad [y]_{-n-k,q} = q^{y(n+k)+\binom{n+k}{2}}[y]_{-n-k,q},
\]

formula (1.10) may be rewritten in the form (1.11).

Note that the alternative expression (1.11) may be derived, independently of (1.10), by following the steps of derivation of (1.10) and using the triangular recurrence relation (1.2) instead of (1.1) and

\[
    [x]_{k,q}[y]_{-n-k,q} + q^{y+n+k+1}[x]_{k+1,q}[y]_{-n-1-k,q} = [x + y + n + 1]_q[x]_{k,q}[y]_{-n-1-k,q}
\]
instead of
\[ q^{\gamma=k}[x]_{k,q}[y]_{n-k,q} + [x]_{k+1,q}[y]_{n-1-k,q} = [x+y+n+1]_{q} [x]_{k,q} [y]_{n-1-k,q}. \]

The proof of the theorem is thus completed. \( \Box \)

**Remark 1.3.** Additional expressions of the negative \( q \)-Vandermonde’s formula. The two versions of the negative \( q \)-Vandermonde’s formula (1.10) and (1.11) may be rewritten as

\[
\frac{1}{[y]_{n,q}} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} q^{k(y-n+1)} [x]_{k,q} \frac{[x]_{k,q}}{[x+y]_{n+k,q}}, \quad |q^y| < 1 \quad (1.12)
\]

and

\[
\frac{1}{[y]_{n,q}} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} q^{n(x-k)} [x]_{k,q} \frac{[x]_{k,q}}{[x+y]_{n+k,q}}, \quad |q^{-y}| < 1, \quad (1.13)
\]

respectively.

Indeed, the \( q \)-factorial of \( x = -n \) of order \( k \), with \( n \) a positive integer, may be expressed as

\[ [-n]_{k,q} = (-1)^k q^{-nk \binom{k}{2}} [n+k-1]_{k,q}, \]

since

\[ [-j]_q = -q^{-j} [j]_q, \quad j = n, n+1, \ldots, n+k-1, \]

and

\[ n + (n+1) + \cdots + (n+k-1) = nk + (1+2+\cdots+(k-1)) = nk + \binom{k}{2}. \]

Equivalently, the \( q \)-binomial coefficient of \( x = -n \), with \( n \) a positive integer, is expressed as

\[ \left[ -n \right]_q^k = (-1)^k q^{-nk \binom{k}{2}} \left[ n+k-1 \right]_q^k. \]

Expression (1.10), replacing \( x+y+1 \) by \(-y\) and using the last relation together with the relations

\[ [-y-1]_{-n,q} = \frac{1}{[-y+n-1]_{n,q}} = \frac{1}{(-1)^n q^{-ny+\binom{n}{2}} [y]_{n,q}}, \]

and

\[ [-x-y-1]_{-n-k,q} = \frac{1}{[-x-y+n+k-1]_{n+k,q}} \]

\[ = \frac{1}{(-1)^{n+k} q^{-(n+k)(x+y)+\binom{n+k}{2}} [x+y]_{n+k,q}}, \]
may be written as
\[
\frac{1}{q^{-ny+\binom{n}{2}}[y]_{n,q}} = \sum_{k=0}^{\infty} \frac{[n+k-1]}{[n+k]_{k,q}} q^{-nk-\binom{k}{2}-(n+k)(x-k)} [x]_{k,q}.
\]

Finally, using the relation \(\binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + nk\), the last expression, after the cancelations in the exponents of \(q\), reduces to (1.12). Similarly, expression (1.11) can be rewritten as (1.13).

An interesting \(q\)-identity is deduced in the following example, using \(q\)-Vandermonde’s formula (1.6).

**Example 1.4.** Let \(n\) be a positive integer and let \(x, y, \) and \(q\) be real numbers, with \(q \neq 1\). Using \(q\)-Vandermonde’s formula (1.6), show that
\[
\sum_{k=0}^{n} \binom{n}{k} q^{(n-k)(x-k)} \frac{[x]_{k,q}}{[y+k]_{k,q}} = \frac{[x+y+n]_{n,q}}{[y+n]_{n,q}}
\]

and conclude that
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} q^{\frac{n-k}{2}} \frac{[x]_{q}}{[x-k]_{q}} = 1/[x-1]_{q}
\]

and
\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} q^{\frac{k+1}{2}-(n+y+k)} \frac{[y]_{q}}{[y+k]_{q}} = 1/[y+n]_{q}.
\]

Replacing \(y\) by \(y+n\), in \(q\)-Vandermonde’s formula (1.6), we have
\[
\sum_{k=0}^{n} \binom{n}{k} q^{(n-k)(x-k)} [x]_{k,q} [y+n]_{n-k,q} = [x+y+n]_{n,q}.
\]

Multiplying both sides by \([y]_{-n,q}\) and using the recurrence relation
\[
[y]_{-k,q} = [y]_{-n,q} [y+n]_{n-k,q},
\]

we get
\[
\sum_{k=0}^{n} \binom{n}{k} q^{(n-k)(x-k)} [x]_{k,q} [y]_{-k,q} = [x+y+n]_{n,q} [y]_{-n,q}
\]

and since \([y]_{-k,q} = 1/[y+k]_{k,q}\), we deduce the required formula,
\[
\sum_{k=0}^{n} \binom{n}{k} q^{(n-k)(x-k)} \frac{[x]_{k,q}}{[y+k]_{k,q}} = \frac{[x+y+n]_{n,q}}{[y+n]_{n,q}}.
\]
Setting $y = -x$ and since 

$$[-x + k]_{k,q} = (-1)^k q^{\binom{k}{2}-kx} [x-1]_{k,q}, \quad [-x + n]_{n,q} = (-1)^n q^{\binom{n}{2}-nx} [x-1]_{n,q},$$

and 

$$\binom{n-k}{2} = \binom{n}{2} - k(n-k),$$

we get 

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} q^{\binom{n-k}{2}} [x]_{n} q^{\binom{x}{n}} = 1 \quad \binom{x}{n} _{q}.$$

Similarly, substituting $x = -y$, we conclude that 

$$\sum_{k=0}^{n} (-1)^{k} \binom{k+1}{2} - (y+k) \binom{y}{n} q^{\binom{y+k}{n}} = 1 \quad \binom{y+n}{n} _{q}.$$

### 1.4 $q$-BINOMIAL AND NEGATIVE $q$-BINOMIAL FORMULAE

The number $\binom{n}{k} _{q}$ is called $q$-binomial coefficient, since it is the coefficient of the general term of a $q$-binomial ($q$-Newton’s binomial) formula. This formula is derived in the following theorem.

**Theorem 1.5.** Let $n$ be a positive integer and let $t$ and $q$ be real numbers, with $q \neq 1$. Then, 

$$\prod_{i=1}^{n} (1 + tq^{i-1}) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k} q^{t^k}.$$  \hspace{1cm} (1.14)

**Proof.** The multiplications and additions in the product $\prod_{i=1}^{n} (1 + tq^{i-1})$ may be carried out by (a) selecting the term $tq^{m_i-1}$ from the factor 

$$a_{m_i}(t; q) = 1 + tq^{m_i-1},$$

for $i = 1, 2, \ldots, k$, with $\{m_1, m_2, \ldots, m_k\} \subseteq \{1, 2, \ldots, n\}$; (b) forming the products 

$$q^{m_1+m_2+\cdots+m_k-k} t^k, \quad \{m_1, m_2, \ldots, m_k\} \subseteq \{1, 2, \ldots, n\};$$

and (c) adding together these products, a summand of the form $b_{n,k}(q)t^k$, for $k = 0, 1, \ldots, n$, is deduced, where, by (1.3), 

$$b_{n,k}(q) = \sum_{1 \leq m_1 < m_2 < \cdots < m_k \leq n} q^{m_1+m_2+\cdots+m_k-k} = q^{\binom{k}{2}} \binom{n}{k} _{q}.$$ 

Hence, according to the summation principle, expression (1.14) is established.

An alternative algebraic derivation of the $q$-binomial formula (1.14) may be carried out as follows. Consider the sequence of sums 

$$s_n(t; q) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k} q^{t^k}, \quad n = 1, 2, \ldots.$$
Using the triangular recurrence relation (1.1), with \( x = n \), together with the relation \( \binom{k}{2} = \binom{k-1}{2} + \binom{k-1}{1} \), we deduce for the sequence \( s_n(t; q) \), \( n = 1, 2, \ldots \), the following first-order recurrence relation:

\[
s_n(t; q) = (1 + tq^{n-1})s_{n-1}(t; q), \quad n = 2, 3, \ldots ,
\]

with initial condition \( s_1(t; q) = 1 + t \). Therefore,

\[
s_n(t; q) = \prod_{i=1}^{n} (1 + tq^{i-1})
\]

and (1.14) is shown.

An extension of the \( q \)-binomial formula (1.14) to a negative \( q \)-binomial formula is given in the next theorem.

**Theorem 1.6.** Let \( n \) be a positive integer and let \( t \) and \( q \) be real numbers, with \( |t| < 1 \) and \( |q| < 1 \) or \( |t| < |q|^{-(n-1)} \) and \( |q| > 1 \). Then,

\[
\prod_{j=1}^{n} (1 - tq^{j-1})^{-1} = \sum_{k=0}^{\infty} \left[ \frac{n + k - 1}{k} \right] q^k.
\]

(1.15)

**Proof.** The general term of the product \( \prod_{j=1}^{n} (1 - tq^{j-1})^{-1} \) may be expanded into a geometric series as follows:

\[
a_j(t; q) = (1 - tq^{j-1})^{-1} = \sum_{k=0}^{\infty} t^k q^{(j-1)k}, \quad j = 1, 2, \ldots , n.
\]

Thus, the multiplications and additions in the product \( \prod_{j=1}^{n} (1 - tq^{j-1})^{-1} \) may be carried out by (a) selecting the term \( t^j q^{(j-1)k_j} \), \( k_j \geq 0 \), from the factor

\[
a_j(t; q) = 1 + tq^{j-1} + t^2 q^{(j-1)2} + \cdots + t^j q^{(j-1)k} + \cdots ,
\]

for \( j = 1, 2, \ldots , n \); (b) forming the products

\[
q^{(k_1+2k_2+\cdots+nk_n)-(k_1+k_2+\cdots+k_n)} t^{k_1+k_2+\cdots+k_n},
\]

and (c) adding together these products for \( k_j \geq 0, j = 1, 2, \ldots , n \), with \( k_1 + k_2 + \cdots + k_n = k \), a summand of the form \( b_{n,k}(q)t^k \), for \( k = 0, 1, \ldots \), is deduced, where, by (1.5),

\[
b_{n,k}(q) = q^{-k} \sum_{k_j \geq 0, j=1,2,\ldots,n} q^{k_1+2k_2+\cdots+nk_n} = \left[ \frac{n + k - 1}{k} \right] q.
\]

Hence, according to the summation principle, expression (1.15) is established.
An alternative algebraic derivation of the negative $q$-binomial formula (1.15) may be carried out as follows. Consider the convergent sequence
\[ s_n(t; q) = \sum_{k=0}^{\infty} \left[ \frac{n+k-1}{k} \right]_q t^k, \quad n = 1, 2, \ldots, \quad |t| < 1, \quad |q| < 1. \]

Using the triangular recurrence relation
\[ \left[ \frac{n+k-1}{k} \right]_q = \left[ \frac{n+k-2}{k} \right]_q + q^{n-1} \left[ \frac{n+k-2}{k-1} \right]_q, \]
it follows that the sequence $s_n(t; q)$, $n = 1, 2, \ldots$, satisfies the first-order recurrence relation
\[ s_n(t; q) = (1 - tq^{n-1})^{-1} s_{n-1}(t; q), \quad n = 2, 3, \ldots, \]
with initial condition $s_1(t; q) = (1 - t)^{-1}$. Therefore,
\[ s_n(t; q) = \prod_{i=1}^{n} (1 - tq^{i-1})^{-1}, \quad n = 1, 2, \ldots, \quad |t| < 1, \quad |q| < 1, \]
and (1.15) is shown for $n = 1, 2, \ldots$, $|t| < 1$ and $|q| < 1$.

The case $s_n(t; q)$, $n = 1, 2, \ldots$, for $|t| < |q|^{-(n-1)}$ and $|q| > 1$, by setting $p = q^{-1}$ and $u = tq^{n-1}$ and using the relation
\[ \left[ \frac{n+k-1}{k} \right]_q = q^{(n-1)k} \left[ \frac{n+k-1}{k} \right]_{q^{-1}} = p^{-(n-1)k} \left[ \frac{n+k-1}{k} \right]_p, \]
reduces to
\[ c_n(u; p) = \sum_{k=0}^{\infty} \left[ \frac{n+k-1}{k} \right]_p u^k, \quad |u| < 1, \quad |p| < 1, \quad n = 1, 2, \ldots, \]
which is exactly the previous case. \hfill \Box

The $q$-binomial coefficients satisfy an orthogonality relation. Specifically, the next theorem is shown.

**Theorem 1.7.** Let $n$ and $k$ be positive integers and let $q$ be a real number, with $q \neq 1$. Then, the following orthogonality relations hold true:
\[ \sum_{r=k}^{n} (-1)^{n-r} q^{\binom{n-r}{2}} \left[ \frac{n}{r} \right]_q \left[ \frac{r}{k} \right]_q = \delta_{n,k}, \quad \sum_{r=k}^{n} (-1)^{r-k} q^{\binom{r-k}{2}} \left[ \frac{n}{r} \right]_q \left[ \frac{r}{k} \right]_q = \delta_{n,k}, \]
where $\delta_{n,k} = 1$, if $k = n$ and $\delta_{n,k} = 0$, if $k \neq n$, is the Kronecker delta.

**Proof.** The first of the orthogonality relations (1.16), since
\[ \left[ \frac{n}{r} \right]_q \left[ \frac{r}{k} \right]_q = \frac{[n]_q!}{[r]_q! [n-r]_q!} \cdot \frac{[r]_q!}{[k]_q! [r-k]_q!} = \frac{[n]_q!}{[k]_q! [n-k]_q!} \cdot \frac{[n-k]_q!}{[n-r]_q! [r-k]_q!} = \left[ \frac{n}{k} \right]_q \left[ \frac{n-k}{n-r} \right]_q, \]
may be expressed as

\[
\sum_{r=k}^{n} (-1)^{n-r} q^{\binom{n-r}{2}} \binom{n}{r}_q \binom{r}{k}_q = \binom{n}{k}_q \sum_{r=k}^{n} (-1)^{n-r} q^{\binom{n-r}{2}} \binom{n-k}{n-r}_q
\]

\[
= \binom{n}{k}_q \sum_{j=0}^{n-k} (-1)^j q^{\binom{j}{2}} \binom{n-k}{n-j}_q.
\]

The last sum, by the \(q\)-binomial formula (1.14), equals \(\delta_{n,k}\) and so the first of (1.16) is proved.

The second of the orthogonality relations (1.16), using the identity

\[
\binom{n}{r}_q \binom{r}{k}_q = \binom{n}{k}_q \binom{n-r}{k-r}_q,
\]

can be similarly shown. □

A useful inversion of the \(q\)-binomial formula (1.14), is derived in the following corollary of Theorem 1.5. As a particular case of it another useful formula is deduced.

**Corollary 1.2.** Let \(n\) be a positive integer and let \(t\) and \(q\) be real numbers, with \(q \neq 1\). Then,

\[
\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{n}{k}_q \prod_{r=1}^{k} (1 - tq^{-(r-1)}) = t^n. \tag{1.17}
\]

In particular,

\[
\sum_{k=0}^{n} (-1)^k (1 - q)^k q^{\binom{k}{2}} \binom{n}{k}_q \binom{t}{k}_q = q^{nt}. \tag{1.18}
\]

**Proof.** The \(q\)-binomial formula (1.14), by replacing \(q\) by \(q^{-1}\), \(t\) by \(-t\), \(n\) by \(k\), the dummy variable \(k\) by \(r\) and using the relation

\[
\binom{k}{r}_{q^{-1}} = q^{-r(k-r)} \binom{k}{r}_q,
\]

is expressed as

\[
\prod_{r=1}^{k} (1 - tq^{-(r-1)}) = \sum_{r=0}^{k} (-1)^r q^{\binom{r}{2} - r(k-r)} \binom{k}{r}_q t^r.
\]

Multiplying both members of the last expression by \((-1)^k q^{\binom{k}{2}} \binom{n}{k}_q\), reducing the exponent of \(q\) in the general term of the sum, using the relation

\[
\binom{k-r}{2} = \binom{k}{2} - \binom{r}{2} - r(k-r),
\]

one has

\[
\sum_{r=0}^{k} (-1)^r q^{\binom{r}{2} - r(k-r)} \binom{k}{r}_q t^r.
\]

Multiplying both members of the last expression by \((-1)^k q^{\binom{k}{2}} \binom{n}{k}_q\), reducing the exponent of \(q\) in the general term of the sum, using the relation

\[
\binom{k-r}{2} = \binom{k}{2} - \binom{r}{2} - r(k-r),
\]

one has

\[
\sum_{r=0}^{k} (-1)^r q^{\binom{r}{2} - r(k-r)} \binom{k}{r}_q t^r.
\]

Multiplying both members of the last expression by \((-1)^k q^{\binom{k}{2}} \binom{n}{k}_q\), reducing the exponent of \(q\) in the general term of the sum, using the relation

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\[
\binom{k-r}{2} = \binom{k}{2} - \binom{r}{2} - r(k-r),
\]

one has

\[
\sum_{r=0}^{k} (-1)^r q^{\binom{r}{2} - r(k-r)} \binom{k}{r}_q t^r.
\]
and summing the resulting expression for \( k = 0, 1, \ldots, n \), we get
\[
\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \left[ \begin{array}{c} n \\ k \end{array} \right] q^k \prod_{i=1}^{k} (1 - tq^{-(i-1)}) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^k \sum_{r=0}^{k} (-1)^{k-r} q^{\binom{k-r}{2}} \left[ \begin{array}{c} k \\ r \end{array} \right] t^r .
\]
\[
= \sum_{r=0}^{n} \left( \sum_{k=r}^{n} (-1)^{k-r} q^{\binom{k-r}{2}} \left[ \begin{array}{c} n \\ k \end{array} \right] q^k \left[ \begin{array}{c} k \\ r \end{array} \right] \right) t^r .
\]

Since, by (1.16),
\[
\sum_{k=r}^{n} (-1)^{k-r} q^{\binom{k-r}{2}} \left[ \begin{array}{c} n \\ k \end{array} \right] q^k = \delta_{n,r},
\]
expression (1.17) is readily deduced. The last expression, by replacing \( t \) by \( q' \) and using the relation
\[
\prod_{i=1}^{k} \left( 1 - q'^{-i+1} \right) = (1 - q')^k [t]_{k,q},
\]
reduces to (1.18). \( \square \)

The inverse of the \( q \)-binomial formula (1.17) and its particular case (1.18) are rewritten in the following remark in a form useful in the theory of discrete \( q \)-distributions.

**Remark 1.4.** Another \( q \)-binomial formula. Expressions of (1.17) and (1.18) may be rewritten as
\[
\sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{n-k} \prod_{i=1}^{k} (1 - tq^{i-1}) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^k \prod_{i=1}^{n-k} (1 - tq^{i-1}) = 1 \tag{1.19}
\]
and
\[
\sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{(n-k)(t-k)}(1 - q)^k [t]_{k,q} = 1, \tag{1.20}
\]
respectively.

Indeed, expression (1.17), by replacing \( t \) by \( t^{-1} \) and using the relation
\[
(-1)^k q^{\binom{k}{2}} \prod_{i=1}^{k} (1 - t^{-1} q^{-(i-1)}) = t^{-k} \prod_{i=1}^{k} (1 - tq^{i-1}),
\]
is rewritten as (1.19). Also, expression (1.18), by replacing \( q \) by \( q^{-1} \) and using the relation
\[
(-1)^k (1 - q^{-1})^k q^{\binom{k}{2}} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{-k(n-k)} (1 - q)^k [t]_{k,q}^{-1},
\]
reduces to (1.20).
Some interesting applications of the $q$-binomial and the negative $q$-binomial formulae are worked out in the following examples.

**Example 1.5.** For $r$, $s$, and $n$ positive integers and $q \neq 1$ a real number, show that

$$
\sum_{k=0}^{n} q^{(n-k)(r-k)} \left[ \begin{array}{c} r \\ k \\ q \end{array} \right] q^{s} \left[ \begin{array}{c} n-k \\ n \\ q \end{array} \right] = \left[ \begin{array}{c} r+s \\ n \\ q \end{array} \right]
$$

and conclude that

$$
\sum_{k=0}^{m} q^{(r-k)(r-k-m)} \left[ \begin{array}{c} r \\ k+1 \\ q \end{array} \right] q^{r} \left[ \begin{array}{c} r \\ k \\ q \end{array} \right] = \left[ \begin{array}{c} 2r \\ r+m \\ q \end{array} \right], \quad m = 0, 1, \ldots, r,
$$

and

$$
\sum_{k=0}^{r} q^{(r-k)^2} \left[ \begin{array}{c} r \\ k \\ q \end{array} \right] = \left[ \begin{array}{c} 2r \\ r \\ q \end{array} \right].
$$

The expansion of the identity

$$
\prod_{i=1}^{r} (1 + t q^{i-1}) \prod_{i=1}^{s} (1 + (t q) q^{i-1}) = \prod_{i=1}^{r+s} (1 + t q^{i-1})
$$

into powers of $t$, using the $q$-binomial formula (1.14), yields

$$
\sum_{k=0}^{r} q^{\left( \frac{k}{2} \right)} \left[ \begin{array}{c} r \\ k \\ q \end{array} \right] t^{k} \sum_{j=0}^{s} q^{\left( \frac{j}{2} \right)+rj} \left[ \begin{array}{c} s \\ j \\ q \end{array} \right] t^{j} = \sum_{n=0}^{r+s} q^{\left( \frac{n}{2} \right)} \left[ \frac{r+s}{n} \right] q^{r+s} t^{n}.
$$

Executing the multiplication in the left-hand side, the general term $a_{n} t^{n}$, for all $n = 0, 1, \ldots, r + s$, is formed by multiplying the term $q^{\left( \frac{k}{2} \right)} \left[ \begin{array}{c} r \\ k \\ q \end{array} \right] q^{k}$ of the first factor by the term $q^{\left( \frac{n-k}{2} \right)+r(n-k)} \left[ \begin{array}{c} s \\ n-k \\ q \end{array} \right] q^{r-n+k}$ of the second factor, for all values $k = 0, 1, \ldots, n$. Thus, by the addition principle and using for the exponent of $q$ the relation $\left( \frac{k}{2} \right) + \left( \frac{n-k}{2} \right) = \left( \frac{n}{2} \right) - k(n-k)$, it follows that

$$
a_{n} = q^{\left( \frac{n}{2} \right)} \sum_{k=0}^{n} q^{(n-k)(r-k)} \left[ \begin{array}{c} r \\ k \\ q \end{array} \right] q^{s} \left[ \begin{array}{c} n-k \\ n \\ q \end{array} \right]
$$

and

$$
\sum_{n=0}^{r+s} q^{\left( \frac{n}{2} \right)} \left( \sum_{k=0}^{n} q^{(n-k)(r-k)} \left[ \begin{array}{c} r \\ k \\ q \end{array} \right] q^{s} \left[ \begin{array}{c} n-k \\ n \\ q \end{array} \right] \right) t^{n} = \sum_{n=0}^{r+s} q^{\left( \frac{n}{2} \right)} \left[ \frac{r+s}{n} \right] q^{r+s} t^{n}.
$$

Equating the coefficients of $t^{n}$ of both sides, we find

$$
\sum_{k=0}^{n} q^{(n-k)(r-k)} \left[ \begin{array}{c} r \\ k \\ q \end{array} \right] q^{s} \left[ \begin{array}{c} n-k \\ n \\ q \end{array} \right] = \left[ \begin{array}{c} r+s \\ n \\ q \end{array} \right].
$$
Note that this formula constitutes a particular case, with \( x = r \) and \( y = s \) positive integers, of \( q \)-Cauchy’s (\( q \)-binomial convolution) formula (1.8). Substituting in this formula \( s = r, n = r - m \) and since

\[
\sum_{k=0}^{r-m} q^{(r-k)(r-k-m)} \left[ \begin{array}{c} r-k \\ k \end{array} \right]_q \left[ \begin{array}{c} r \\ k+m \end{array} \right]_q = \left[ \begin{array}{c} 2r \\ r+m \end{array} \right]_q,
\]

we conclude that

\[
\sum_{k=0}^{r-m} q^{(r-k)(r-k-m)} \left[ \begin{array}{c} r-k \\ k \end{array} \right]_q \left[ \begin{array}{c} r \\ k+m \end{array} \right]_q = \left[ \begin{array}{c} 2r \\ r+m \end{array} \right]_q.
\]

Also, setting \( r = s = n \), and since

\[
\sum_{k=0}^{r} q^{(r-k)^2} \left[ \begin{array}{c} r \\ k \end{array} \right]_q = \left[ \begin{array}{c} 2r \\ r \end{array} \right]_q.
\]

Example 1.6. For \( r, s, \) and \( n \) positive integers and \( q \neq 1 \) a real number, show that

\[
\sum_{k=0}^{n} q^{r(n-k)} \left[ \begin{array}{c} r+k-1 \\ k \end{array} \right]_q \left[ \begin{array}{c} s+n-k-1 \\ n-k \end{array} \right]_q = \left[ \begin{array}{c} r+s+n-1 \\ n \end{array} \right]_q.
\]

The expansion of the identity

\[
\prod_{i=1}^{r} (1 - tq^{i-1})^{-1} \prod_{i=1}^{s} (1 - (tq^r)q^{i-1})^{-1} = \prod_{i=1}^{r+s} (1 - tq^{i-1})^{-1}, \quad |t| < 1,
\]

using the negative \( q \)-binomial formula (1.15), yields

\[
\sum_{k=0}^{\infty} \left[ \begin{array}{c} r+k-1 \\ k \end{array} \right]_q \sum_{j=0}^{\infty} \left[ \begin{array}{c} s+j-1 \\ j \end{array} \right]_q q^{r+j} = \sum_{n=0}^{\infty} \left[ \begin{array}{c} r+s+n-1 \\ n \end{array} \right]_q r^n
\]

and

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} q^{r(n-k)} \left[ \begin{array}{c} r+k-1 \\ k \end{array} \right]_q \left[ \begin{array}{c} s+n-k-1 \\ n-k \end{array} \right]_q r^n = \sum_{n=0}^{\infty} \left[ \begin{array}{c} r+s+n-1 \\ n \end{array} \right]_q r^n,
\]

for \( |t| < 1 \). Therefore,

\[
\sum_{k=0}^{n} q^{r(n-k)} \left[ \begin{array}{c} r+k-1 \\ k \end{array} \right]_q \left[ \begin{array}{c} s+n-k-1 \\ n-k \end{array} \right]_q = \left[ \begin{array}{c} r+s+n-1 \\ n \end{array} \right]_q.
\]

Note that this formula constitutes a particular case, with \( x = -r \) and \( y = -s \) negative integers, of \( q \)-Cauchy’s formula (1.8).
Example 1.7. For \( n, m, \) and \( k \) positive integers, with \( k \leq m \leq n \), show that

\[
\sum_{r=k}^{k+n-m} q^{k(n-r)} \binom{r-1}{k-1}_q \binom{n-r}{m-k}_q = \binom{n}{m}_q.
\]

Multiplying both members of the negative \( q \)-binomial formula

\[
\sum_{j=0}^{\infty} \frac{m+j}{m}_q t^j = \prod_{i=1}^{m+1} (1 - t q^{i-1})^{-1},
\]

by \( t^m \) and substituting \( n = m + j \), we get the expansion

\[
\sum_{n=m}^{\infty} \binom{n}{m}_q t^n = t^m \prod_{i=1}^{m+1} (1 - t q^{i-1})^{-1}.
\]

In the same way, we deduce that

\[
\sum_{r=k}^{\infty} \frac{r-1}{k-1}_q t^r = t^k \prod_{i=1}^{k} (1 - t q^{i-1})^{-1}
\]

and

\[
\sum_{j=m-k}^{\infty} \frac{j}{m-k}_q q^{j+i} t^i = t^{m-k} \prod_{i=1}^{m+1} (1 - (t q^k) q^{i-1})^{-1}.
\]

Expanding the identity

\[
t^k \prod_{i=1}^{k} (1 - t q^{i-1})^{-1} \cdot t^{m-k} \prod_{i=1}^{m-k} (1 - (t q^k) q^{i-1})^{-1} = t^m \prod_{i=1}^{m+1} (1 - t q^{i-1})^{-1},
\]

using the preceding three expansions, we get

\[
\left( \sum_{r=k}^{\infty} \frac{r-1}{k-1}_q t^r \right) \cdot \left( \sum_{j=m-k}^{\infty} \frac{j}{m-k}_q q^{j+i} t^i \right) = \sum_{n=m}^{\infty} \binom{n}{m}_q t^n
\]

and

\[
\sum_{n=m}^{k+n-m} \left( \sum_{r=k}^{k+n-m} q^{k(n-r)} \frac{r-1}{k-1}_q \binom{n-r}{m-k}_q \right) = \sum_{n=m}^{\infty} \binom{n}{m}_q t^n,
\]

for \(|t| < 1\). Therefore,

\[
\sum_{r=k}^{k+n-m} q^{k(n-r)} \frac{r-1}{k-1}_q \binom{n-r}{m-k}_q = \binom{n}{m}_q.
\]
1.5 GENERAL $q$-BINOMIAL FORMULA AND $q$-EXPONENTIAL FUNCTIONS

A general $q$-binomial formula and, alternatively, a general negative $q$-binomial formula are given in the following theorem. The adopted proof is due to Heine (1847, 1878).

**Theorem 1.8.** Let $x$, $t$, and $q$ be real numbers, with $|t| < 1$ and $|q| < 1$. Then,

$$\prod_{i=1}^{\infty} \frac{1 + tq^{i-1}}{1 + tq^{i}} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} k+1 \\ k \end{array} \right]_{q} x^{k} t^{k}. \quad (1.21)$$

Alternatively,

$$\prod_{i=1}^{\infty} \frac{1 - t^{q^{i-1}}}{1 - t^{q^{i}}} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} x + k - 1 \\ k \end{array} \right]_{q} x^{k}. \quad (1.22)$$

**Proof.** Let us consider the convergent series

$$h(x; t) = \sum_{k=0}^{\infty} q^{\left( \frac{k}{2} \right)} \left[ \begin{array}{c} x \\ k \end{array} \right]_{q} t^{k}, \quad |t| < 1, \quad |q| < 1$$

and let us compute the difference

$$h(x; t) - h(x - 1; t) = \sum_{k=0}^{\infty} q^{\left( \frac{k}{2} \right)} \left\{ \left[ \begin{array}{c} x \\ k \end{array} \right]_{q} - \left[ \begin{array}{c} x - 1 \\ k \end{array} \right]_{q} \right\} t^{k}, \quad |t| < 1, \quad |q| < 1.$$

Using the recurrence relation (1.1), it follows that

$$h(x; t) - h(x - 1; t) = \sum_{k=1}^{\infty} q^{\left( \frac{k-1}{2} \right)} \left[ \begin{array}{c} x - 1 \\ k - 1 \end{array} \right]_{q} t^{k-1}$$

$$= tq^{x-1} \sum_{k=1}^{\infty} q^{\left( \frac{k-1}{2} \right)} \left[ \begin{array}{c} x - 1 \\ k - 1 \end{array} \right]_{q} t^{k-1}$$

$$= tq^{x-1} h(x - 1; t).$$

Also

$$h(x; t) - h(x; tq) = \sum_{k=0}^{\infty} q^{\left( \frac{k}{2} \right)} \left[ \begin{array}{c} x \\ k \end{array} \right]_{q} (1 - q^{k}) t^{k}$$

$$= t(1 - q^{x}) \sum_{k=1}^{\infty} q^{\left( \frac{k-1}{2} \right)} \left[ \begin{array}{c} x - 1 \\ k - 1 \end{array} \right]_{q} (tq)^{k-1}$$

$$= t(1 - q^{x}) h(x - 1; tq).$$
Eliminating $h(x - 1; tq)$ from the expressions

$$h(x; tq) = (1 + tq^x)h(x - 1; tq)$$

and

$$h(x; t) - h(x; tq) = t(1 - q^x)h(x - 1; tq),$$

we get the relation

$$h(x; t) = \frac{1 + t}{1 + tq^x}h(x; tq).$$

Iterating this relation $n - 1$ times, we find

$$h(x; t) = \prod_{i=1}^{n} \frac{1 + tq^{i-1}}{1 + tq^{x+i-1}}h(x; tq^n).$$

Letting $n \to \infty$, and since $\lim_{n \to \infty} h(x; tq^n) = h(x; 0) = 1$, for $|q| < 1$, we deduce the expression

$$h(x; t) = \prod_{i=1}^{\infty} \frac{1 + tq^{i-1}}{1 + tq^{x+i-1}},$$

which shows (1.21). The alternative expression (1.22) is deduced from (1.21), by replacing successively $x$ by $-x$ and $t$ by $-tq^x$ and finally using the relation

$$\begin{bmatrix} -x \\ k \end{bmatrix}_q = (-1)^k q^{-xk - \binom{k}{2}} \begin{bmatrix} x + k - 1 \\ k \end{bmatrix}_q.$$

Hence, the proof of the theorem is completed. \(\square\)

A $q$-analogue of the exponential function can be obtained from (1.21) by replacing $t$ by $(1 - q)t$ and then taking the limit as $x \to \infty$. Since, for $|q| < 1$,

$$\lim_{x \to \infty} (1 - q)[x - j]_q = \lim_{x \to \infty} (1 - q^{x-j}) = 1, \quad \lim_{x \to \infty} (1 - q)^k[x]_{k,q} = 1,$$

a $q$-exponential function is deduced as

$$E_q(t) = \prod_{i=1}^{\infty} (1 + t(1 - q)q^{i-1}) = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{t^k}{[k]_q!}, \quad -\infty < t < \infty. \quad (1.23)$$

Another $q$-exponential function can be similarly obtained from (1.22) as

$$e_q(t) = \prod_{i=1}^{\infty} (1 - t(1 - q)q^{i-1})^{-1} = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!}, \quad |t| < 1/(1 - q). \quad (1.24)$$

The limiting expressions of the $q$-exponential functions, as $q \to 1$,

$$\lim_{q \to 1} E_q(t) = e^t, \quad \lim_{q \to 1} e_q(t) = e^t,$$
justify their name. Clearly, these functions satisfy the relation

\[ E_q(t) e_q(-t) = 1, \quad |t| < 1/(1 - q), \quad |q| < 1. \]

Also, since \([k]_{q-1}! = q^{-\binom{k}{2}} [k]_q!\), the following relation is readily obtained:

\[ E_{q-1}(t) = e_q(t), \quad |t| < 1/(1 - q), \quad |q| < 1. \]

A \(q\)-analogue of the logarithmic function can be obtained from (1.22) by subtracting 1 from both sides, dividing the resulting expression by \([x]_q\) and taking the limit as \(x \to 0\). Since

\[
\lim_{x \to 0} \frac{1}{[x]_q} \left( x + k - 1 \right)_q = \frac{1}{[k]_q} \cdot \frac{\lim_{x \to 0} [x + 1]_q [x + 2]_q \cdots [x + k - 1]_q}{[k - 1]_q!} = \frac{1}{[k]_q},
\]

a \(q\)-logarithmic function is deduced as

\[ -l_q(1 - t) = \lim_{x \to 0} \frac{1}{[x]_q} \left( \prod_{i=1}^{\infty} \frac{1 - t q^{i-1}}{1 - t q^{i-1}} - 1 \right) = \sum_{k=1}^{\infty} \frac{t^k}{[k]_q}, \quad |t| < 1. \quad (1.25) \]

The limiting expression of the \(q\)-logarithmic function, as \(q \to 1\),

\[ \lim_{q \to 1} l_q(1 - t) = \log(1 - t), \quad |t| < 1, \]

justifies its name.

### 1.6 \(q\)-STIRLING NUMBERS

Consider the noncentral \(q\)-factorial of \(t\) of order \(n\) and noncentrality parameter \(r\),

\[ [t - r]_{n,q} = [t - r]_q [t - r - 1]_q \cdots [t - r - n + 1]_q, \]

with \(t, q,\) and \(r\) real numbers, \(q \neq 1\), and \(n\) a positive integer. Using the relation

\[ [t - r - j]_q = q^{-r-j} ([t]_q - [r + j]_q), \quad j = 0, 1, \ldots, \]

it may be expressed as

\[ [t - r]_{n,q} = q^{-\binom{n}{2} - rn} ([t]_q - [r]_q) ([t]_q - [r + 1]_q) \cdots ([t]_q - [r + n - 1]_q). \]

This is a polynomial of the \(q\)-number \([t]_q\) of degree \(n\). Executing the multiplications and arranging the terms in ascending order of powers of \([t]_q\), we get

\[ [t - r]_{n,q} = q^{-\binom{n}{2} - rn} \sum_{k=0}^{n} s_q(n, k; r) [t]_q^k, \quad n = 0, 1, \ldots. \quad (1.26) \]
Inversely, the $n$th power of the $q$-number $[t]_q$ may be expressed in the form of a polynomial of noncentral $q$-factorials of $t$. Specifically,

$$[t]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2} + r k} S_q(n, k; r)[t - r]_{k,q}, \quad n = 0, 1, \ldots, \quad (1.27)$$

or equivalently

$$[t + r]^n_q = \sum_{k=0}^{n} q^{\binom{k}{2} + r k} S_q(n, k; r)[t]_{k,q}, \quad n = 0, 1, \ldots. \quad (1.28)$$

The coefficients $s_q(n, k; r)$ and $S_q(n, k; r)$ are called noncentral $q$-Stirling numbers of the first and second kind, respectively. Note that for $r = 0$ the noncentral $q$-Stirling numbers of the first and second kind reduce to $s_q(n, k; 0) = s_q(n, k)$ and $S_q(n, k; 0) = S_q(n, k)$, the usual (central) $q$-Stirling numbers of the first and second kind, respectively.

Furthermore, consider the expansion of the noncentral ascending $q$-factorial of $t$ of order $n$ and noncentrality parameter $r$,

$$[t + r + n - 1]_{n,q} = [t + r]_q[t + r + 1]_q \cdots [t + r + n - 1]_q$$

$$= q^{\binom{n}{2} + r n}([t]_q + q^{-1}[r]_{q^{-1}}) \cdots ([t]_q + q^{-1}[r + n - 1]_{q^{-1}}),$$

into a polynomial of $[t]_q$,

$$[t + r + n - 1]_{n,q} = q^{\binom{n}{2} + r n} \sum_{k=0}^{n} |s_{q^{-1}}(n, k; r)| [t]_q^k, \quad n = 0, 1, \ldots, \quad (1.29)$$

for $q \neq 0$. Since

$$[t + r + n - 1]_{n,q} = [-1]^n_q[-t - r]_{n,q^{-1}}, \quad [-t]_q = [-1]_q[t]_{q^{-1}},$$

where $[-1]_q = -q^{-1}$, on using (1.26), with $-t$ instead of $t$ and $q^{-1}$ instead of $q$, it follows that

$$|s_{q^{-1}}(n, k; r)| = [-1]_{q^{-1}}^{n-k} s_{q^{-1}}(n, k; r),$$

or, equivalently, that

$$|s_q(n, k; r)| = [-1]_{q}^{-(n-k)} s_q(n, k; r). \quad (1.30)$$

The coefficient $|s_q(n, k; r)|$, which for $0 < q < 1$ or $1 < q < \infty$ and $r \geq 0$ is positive, is called signless (or absolute) noncentral $q$-Stirling number of the first kind.

The noncentral $q$-Stirling numbers of the first and second kind constitute a pair of orthogonal bivariate sequences. This is shown in the next theorem.
The noncentral q-Stirling numbers of the first kind satisfy the orthogonality relations
\[
\sum_{m=k}^{n} s_q(n, m; r) S_q(m, k; r) = \delta_{n,k}, \quad \sum_{m=k}^{n} S_q(n, m; r) s_q(m, k; r) = \delta_{n,k},
\]
(1.31)
where \( \delta_{n,k} = 1 \), if \( k = n \) and \( \delta_{n,k} = 0 \), if \( k \neq n \), is the Kronecker delta.

**Proof.** Expanding the noncentral \( q \)-factorial of \( t \) of order \( n \) and noncentrality parameter \( r \), \([t - r]_{n,q}\), into powers of \([r]_q\), by using (1.26), and in the resulting expression expanding the powers of \([r]_q\) into noncentral \( q \)-factorials \([t - r]_{k,q}\), for \( k = 0, 1, \ldots, n \), by using (1.27), we get
\[
[t - r]_{n,q} = q^{-\binom{n}{2}} \sum_{m=0}^{n} s_q(n, m; r) [t]_q^m
= q^{-\binom{n}{2}} \sum_{m=0}^{n} s_q(n, m; r) \sum_{k=0}^{m} q^{\binom{k}{2} + rk} S_q(m, k; r) [t - r]_{k,q}.
\]
Furthermore, interchanging the order of summation, we deduce the relation
\[
[t - r]_{n,q} = \sum_{k=0}^{n} q^{-\binom{n}{2} + \binom{k}{2}} - (n-k) \sum_{m=k}^{n} s_q(n, m; r) S_q(m, k; r) [t - r]_{k,q},
\]
which implies the first of (1.31). Similarly, expanding the \( n \)th power of the \( q \)-number \([t]_q\), \([t]^n\), into noncentral \( q \)-factorials \([t - r]_{m,q}\), \( m = 0, 1, \ldots, n \), and in the resulting expression expanding the noncentral \( q \)-factorials \([t - r]_{m,q}\), \( m = 0, 1, \ldots, n \), into powers of \([r]_q\), we deduce the second of (1.31). \( \square \)

Triangular recurrence relations for the noncentral \( q \)-Stirling numbers of the first and second kind are derived in the following theorem.

**Theorem 1.10.** The noncentral \( q \)-Stirling numbers of the first kind \( s_q(n, k; r) \), \( k = 0, 1, \ldots, n \), \( n = 0, 1, \ldots \), satisfy the triangular recurrence relation
\[
s_q(n+1, k; r) = s_q(n, k-1; r) - [n + r]_q s_q(n, k; r),
\]
(1.32)
for \( k = 1, 2, \ldots, n + 1 \) and \( n = 0, 1, \ldots \), with initial conditions
\[
s_q(0, 0; r) = 1, s_q(0, k; r) = 0, k > 0, s_q(n, 0; r) = q^{\binom{n}{2} + rn} [-r]_{n,q}, n > 0.
\]
Also, the noncentral \( q \)-Stirling numbers of the second kind \( S_q(n, k; r) \), \( k = 0, 1, \ldots, n \), \( n = 0, 1, \ldots \), satisfy the triangular recurrence relation
\[
S_q(n+1, k; r) = S_q(n, k-1; r) + [k + r]_q S_q(n, k; r),
\]
(1.33)
for $k = 1, 2, \ldots, n + 1$ and $n = 0, 1, \ldots$, with initial conditions

$$S_q(0, 0; r) = 1, \quad S_q(0, k; r) = 0, k > 0, \quad S_q(n, 0; r) = [r]^n_q, n > 0.$$ 

Proof. Expanding both members of the recurrence relation

$$[t - r]_{n+1, q} = [t - (n + r)]_q [t - r]_{n, q} = q^{-(n-r)} [t]_q - [n + r]_q [t - r]_{n, q}$$

into powers of $[t]_q$, according to (1.26), we get the relation

$$q^{-\binom{n+1}{2} - r(n+1)} \sum_{k=0}^{n+1} s_q(n + 1, k; r)[t]^k_q = q^{-\binom{n}{2} - r(n-r)} \left\{ \sum_{j=0}^{n} s_q(n, j; r)[t]^j_q \right\} - \sum_{k=0}^{n} [n + r]_q s_q(n, k; r)[t]^k_q,$$

or equivalently, the relation

$$\sum_{k=0}^{n+1} s_q(n + 1, k; r)[t]^k_q = \sum_{k=0}^{n+1} s_q(n, k - 1; r)[t]^k_q - \sum_{k=0}^{n} [n + r]_q s_q(n, k; r)[t]^k_q,$$

which implies (1.32). The initial conditions follow directly from (1.26).

Also, expanding both members of the recurrence relation

$$[t + r]_{n+1, q} = [t + r]_q [t + r]^{n}_q$$

into factorials of the $q$-number $[t]_q$, according to (1.28), we have

$$\sum_{k=0}^{n+1} q^{\binom{k}{2} + rk} S_q(n + 1, k; r)[t]_{k,q} = \sum_{j=0}^{n} q^{\binom{j}{2} + rj} S_q(n, j; r)[t + r]_{j,q},$$

Since

$$[t + r]_{q}[t]_{j,q} = [(t - j) + (j + r)]_q [t]_{j,q} = (q^{j+r}[t - j]_q + [j + r]_q [t]_{j,q},$$

whence

$$[t + r]_{q}[t]_{j,q} = q^{j+r}[t]_{j+1,q} + [j + r]_q [t]_{j,q},$$

we deduce the relation

$$\sum_{k=0}^{n+1} q^{\binom{k}{2} + rk} S_q(n + 1, k; r)[t]_{k,q}$$

$$= \sum_{j=0}^{n} q^{\binom{j+1}{2} + r(j+1)} S_q(n, j; r)[t]_{j+1,q} + \sum_{j=0}^{n} q^{\binom{j}{2} + rj} [j + r]_q S_q(n, j; r)[t]_{j,q}$$

$$= \sum_{k=1}^{n+1} q^{\binom{k}{2} + rk} S_q(n, k - 1; r)[t]_{k,q} + \sum_{k=0}^{n} q^{\binom{k}{2} + rk} [k + r]_q S_q(n, k; r)[t]_{k,q},$$

which implies (1.33). The initial conditions follow directly from (1.28).
The triangular recurrence relation (1.33) can be used for the determination of the (power) generating function of the sequence of the noncentral \( q \)-Stirling numbers of the second kind \( S_q(n, k; r) \), \( n = k, k + 1, \ldots \), for fixed \( k \). This generating function transformed yields the expansion of the reciprocal noncentral \( q \)-factorials into reciprocal \( q \)-powers, which inverted provides the expansion of the reciprocal \( q \)-powers into reciprocal noncentral \( q \)-factorials. Specifically, we have the following theorem.

**Theorem 1.11.** The generating function of the sequence of the noncentral \( q \)-Stirling numbers of the second kind \( S_q(n, k; r) \), \( n = k, k + 1, \ldots \), for fixed \( k \), is given by

\[
\varphi_k(u; q, r) = \sum_{n=k}^{\infty} S_q(n, k; r)u^n = u^k \prod_{j=0}^{k-1} (1 - [r + j]_q u)^{-1},
\]

(1.34)

for \(|u| < 1/[r + k]_q\) and \( k = 0, 1, \ldots \).

**Proof.** Note first that the series in (1.34) is convergent, since

\[
\lim_{n \to \infty} S_q(n, k; r)\left([r + k]_q u\right)^n = \frac{1}{[k]_q!} \lim_{n \to \infty} ([r + k]_q u)^n = 0,
\]

for \(|u| < 1/[r + k]_q\). Furthermore, multiplying the triangular recurrence relation (1.33) by \( u^{n+1} \) and summing the resulting expression for \( n = k - 1, k, \ldots \), and since \( S_q(k - 1, k; r) = 0 \), we get

\[
\sum_{n=k-1}^{\infty} S_q(n + 1, k; r)u^{n+1} = u \sum_{n=k-1}^{\infty} S_q(n, k - 1; r)u^n + [k + r]_q u \sum_{n=k}^{\infty} S_q(n, k; r)u^n,
\]

for \( k = 1, 2, \ldots \). Consequently

\[
\varphi_k(u; q, r) = u\varphi_{k-1}(u; q, r) + [k + r]_q u \varphi_k(u; q, r), \quad k = 1, 2, \ldots
\]

and so

\[
\varphi_k(u; q, r) = u(1 - [k + r]_q u)^{-1} \varphi_{k-1}(u; q, r), \quad k = 1, 2, \ldots .
\]

Applying this recurrence relation repeatedly and since

\[
\varphi_0(u; q, r) = \sum_{n=0}^{\infty} S_q(n, 0; r)u^n = \sum_{n=0}^{\infty} ([r]_q u)^n = (1 - [r]_q u)^{-1},
\]

we deduce (1.34). \( \square \)

**Corollary 1.3.** The reciprocal noncentral \( q \)-factorial \( 1/[t - r]_{k+1,q} \) is expanded into reciprocal \( q \)-powers \( 1/[t]_{q+1}^{n+1} \), \( n = k, k + 1, \ldots \), \( k + 1, \ldots \), as

\[
\frac{1}{[t - r]_{k+1,q}} = q^{\binom{k+1}{2} + r(k+1)} \sum_{n=k}^{\infty} S_q(n, k; r) \frac{1}{[t]_{q}^{n+1}}, \quad t > k + r.
\]

(1.35)
Inversely, the reciprocal $q$-power $1/[t]_{q}^{k+1}$ is expanded into reciprocal noncentral $q$-factorials $1/[t-r]_{n+1,q}$, $n = k, k + 1, \ldots, k = 0, 1, \ldots$, as

$$
\frac{1}{[t]_{q}^{k+1}} = \sum_{n=k}^{\infty} q^{\left(\frac{n+1}{2}\right)-r(n+1)} s_{q}(n, k; r) \frac{1}{[t-r]_{n+1,q}}, \quad t > k + r. \quad (1.36)
$$

Proof. Setting in (1.34) $u = 1/[t]_{q}$ and since

$$([t]_{q} - [r]_{q})([t]_{q} - [r + 1]_{q}) \cdots ([t]_{q} - [r + k]_{q}) = q^{\left(\frac{k+1}{2}\right)+r(k+1)}[t-r]_{k+1,q},$$

we conclude (1.35). In the expansion (1.35), replacing the bound variable $n$ by $m$ and the fixed number $k$ by $n$ and then multiplying the resulting expression by

$$q^{-\left(\frac{n+1}{2}\right)-r(n+1)} s_{q}(n, k; r)$$

and summing for $n = k, k + 1, \ldots$, we find

$$
\sum_{n=k}^{\infty} q^{-\left(\frac{n+1}{2}\right)-r(n+1)} s_{q}(n, k; r) \frac{1}{[t-r]_{n+1,q}} = \sum_{n=k}^{\infty} \sum_{m=n}^{\infty} S_{q}(m, n; r)s_{q}(n, k; r) \frac{1}{[t]_{q}^{m+1}}
$$

$$
= \sum_{m=k}^{\infty} \left\{ \sum_{n=k}^{m} S_{q}(m, n; r)s_{q}(n, k; r) \right\} \frac{1}{[t]_{q}^{m+1}}.
$$

Therefore, by the second of the orthogonality relations (1.31), we deduce the relation

$$
\sum_{n=k}^{\infty} q^{-\left(\frac{n+1}{2}\right)-r(n+1)} s_{q}(n, k; r) \frac{1}{[t]_{n+1,q}} = \sum_{m=k}^{\infty} \delta_{m,k} \frac{1}{[t]_{q}^{m+1}},
$$

which implies (1.36). \hfill \Box

Expressions of the noncentral $q$-Stirling numbers in terms of multiple sums of products of $q$-numbers, over combinations of positive integers are provided in the next theorem.

**Theorem 1.12.** The noncentral $q$-Stirling number of the first kind $s_{q}(n, k; r)$, $k = 1, 2, \ldots, n$, $n = 1, 2, \ldots$, is given by the multiple sum

$$
s_{q}(n, k; r) = (-1)^{n-k} \sum [r + i_1]_{q}[r + i_2]_{q} \cdots [r + i_{n-k}]_{q}, \quad (1.37)
$$

where the summation is extended over all $(n-k)$-combinations $\{i_1, i_2, \ldots, i_{n-k}\}$ of the $n$ nonnegative integers $\{0, 1, \ldots, n - 1\}$.
Also, the noncentral \( q \)-Stirling number of the second kind \( S_q(n,k;r) \), \( k = 1, 2, \ldots, n, \) is given by the multiple sum
\[
S_q(n,k;r) = \sum [r + i_1]_q[r + i_2]_q \cdots [r + i_{n-k}]_q,
\]
(1.38)
where the summation is extended over all \( (n - k) \)-combinations \( \{i_1, i_2, \ldots, i_{n-k}\} \), with repetition, of the \( k + 1 \) nonnegative integers \( \{0, 1, \ldots, k\} \).

**Proof.** According to the definition (1.26), and since
\[
[t - r]_{n,q} = q^{(q - 1)n - r} ([t]_q - [r]_q) ([t]_q - [r + 1]_q) \cdots ([t]_q - [r + n - 1]_q),
\]
we have
\[
([t]_q - [r]_q) ([t]_q - [r + 1]_q) \cdots ([t]_q - [r + n - 1]_q) = \sum_{k=1}^{n} s_q(n,k;r)(t)_q^k,
\]
for \( n = 2, 3, \ldots \) The \( i \)th factor of the product of the left-hand side, \( f_i(t; q) = [t]_q - [r + i]_q, \) \( i = 0, 1, \ldots, n - 1 \), is a monomial with constant term \(-[r + i]_q\). Executing the multiplications, the \( k \)th-order power of \([t]_q\) is formed by multiplying the constant terms of any \( n - k \) factors \( \{i_1, i_2, \ldots, i_{n-k}\} \), out of the \( n \) factors \( \{0, 1, \ldots, n - 1\} \), together with the first-order terms \([t]_q\) of the remaining \( k \) factors. Since the coefficient of the first-order term \([t]_q\), in any factor, equals one, by the multiplication principle, (1.37) is deduced.

Also, expanding each factor in (1.34) by using the geometric series, we get
\[
\phi_k(u; q, r) = \sum_{n=k}^{\infty} S_q(n,k;r) u^n = u^k \prod_{j=0}^{\infty} \left( \sum_{m_j=0}^{\infty} \frac{[r + j]_q^{m_j} u^{m_j}}{[r]_q} \right),
\]
and so
\[
S_q(n,k;r) = \sum_{m_0}^{\infty} [r]_q^{m_0} [r + 1]_q^{m_1} \cdots [r + k]_q^{m_k},
\]
where the summation is extended over all integers \( m_j \geq 0, j = 0, 1, \ldots, k \), with \( m_0 + m_1 + \cdots + m_k = n - k \). This expression is equivalent to (1.38).

**Remark 1.5.** The sign of the \( q \)-Stirling numbers. The sign of the noncentral \( q \)-Stirling numbers of the first and second kind in a base \( q \), with \( 0 < q < 1 \) or \( 1 < q < \infty \), according to expressions (1.37) and (1.38), depends on the noncentrality parameter \( r \) as follows:

The noncentral \( q \)-Stirling number of the first kind \( s_q(n,k;r) \) has the sign of \((-1)^{n-k}\) for \( r \geq 0 \), whereas it is a nonnegative \( q \)-number for \( r + n - 1 < 0 \). Also,
the noncentral \( q \)-Stirling number of the second kind \( S_q(n, k; r) \) is a nonnegative \( q \)-number for \( r \geq 0 \), whereas it has sign of \((-1)^{n-k}\) for \( r + k < 0 \).

In particular, for \( r = 0 \), the \( q \)-Stirling number of the first kind \( s_q(n, k) \) has the sign of \((-1)^{n-k}\). Also, the \( q \)-Stirling number of the second kind \( S_q(n, k) \) is a nonnegative \( q \)-number.

Expressions of the noncentral \( q \)-Stirling numbers in the form of single summations of elementary terms are given in the following theorem.

**Theorem 1.13.** The noncentral \( q \)-Stirling number of the first kind \( s_q(n, k; r) \), \( k = 1, 2, \ldots, n \), \( n = 1, 2, \ldots \), is given by

\[
s_q(n, k; r) = \frac{1}{(1 - q)^{n-k}} \sum_{j=k}^{n} (-1)^{j-k} q^{(n-j)/2} j! \begin{bmatrix} n \\ j \\ k \end{bmatrix}_q, \tag{1.39}
\]

Also, the noncentral \( q \)-Stirling number of the second kind \( S_q(n, k; r) \), \( k = 1, 2, \ldots, n \), \( n = 1, 2, \ldots \), is given by

\[
S_q(n, k; r) = \frac{1}{(1 - q)^{n-k}} \sum_{j=k}^{n} (-1)^{j-k} q^{r(j-k)} \begin{bmatrix} n \\ j \\ k \end{bmatrix}_q. \tag{1.40}
\]

In addition,

\[
S_q(n, k; r) = \frac{1}{[k]_q!} \sum_{j=0}^{k} (-1)^{k-j} q^{(j+1)/2} j! \begin{bmatrix} k \\ j \\ r \end{bmatrix}_q j^n. \tag{1.41}
\]

**Proof.** The \( q \)-binomial formula (1.14), by replacing successively \( q \) by \( q^{-1} \), \( t \) by \( -q^{i-1} \), the dummy variable \( k \) by \( j \) and \( n \) by \( k \), it becomes

\[
\prod_{i=1}^{k} (1 - q^{-r} q^{-(i-1)}) = \sum_{j=0}^{k} (-1)^{j} q^{(j+1)/2} j! \begin{bmatrix} k \\ j \end{bmatrix}_q q^{j-r}.
\]

Furthermore, using the relations

\[
\prod_{i=1}^{n} (1 - q^{-r} q^{-(i-1)}) = (1 - q)^n [t - r]_{n,q}, \quad \begin{bmatrix} n \\ j \\ q^{-1} \end{bmatrix}_q = q^{-j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q,
\]

it is expressed as

\[
[t - r]_{n,q} = \frac{1}{(1 - q)^n} \sum_{j=0}^{n} (-1)^{j} q^{(j+1)/2} j! \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j-r}. \tag{1.42}
\]
Multiplying both members of (1.42) by \(q(\frac{n}{2})^{+r n}\), reducing the exponent of \(q\) by using the relation \(\binom{n-j}{2} = \binom{n}{2} - \binom{j}{2} - j(n-j)\), and expanding the \(q\)-function \(q^i = (1 - (1-q)[t]_q)^j\) into powers of the \(q\)-number \([t]_q\), we get

\[
q(\frac{n}{2})^{+r n}[t-r]_{n,q} = \sum_{j=0}^{n} (-1)^j q^{(\frac{n-j}{2})^{+r(n-j)}} \left[ \begin{array}{c} n \\vdash \end{array} \right] (\frac{n}{k}) \sum_{k=0}^{j} (-1)^k \binom{k}{j} \frac{(1-q)^k}{(1-q)^n}[t]_q^k.
\]

Furthermore, interchanging the order of summation, we find the expansion

\[
q(\frac{n}{2})^{+r n}[t-r]_{n,q} = \sum_{k=0}^{n} \left\{ \frac{1}{(1-q)^{n-k}} \sum_{j=k}^{n} (-1)^{j-k} q^{(\frac{n-j}{2})^{+r(n-j)}} \left[ \begin{array}{c} n \\vdash \end{array} \right] (\frac{n}{j}) \binom{j}{k} \right\} [t]_q^k.
\]

Comparing this expansion to (1.26), we readily deduce (1.39).

Also, expanding the \(n\)th power of the \(q\)-number of \(t+r\), into powers of \(q'\), we have

\[
[t+r]_q^n = \frac{(1-q'^{+r})^n}{(1-q)^n} = \frac{1}{(1-q)^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} q'^r q^j.
\]

Then, expanding the powers of \(q'\) into factorials, by using (1.18), we get

\[
[t+r]_q^n = \frac{1}{(1-q)^n} \sum_{j=0}^{n} (-1)^j q'^r \binom{n}{j} \sum_{j=0}^{n} (-1)^{j-k} (1-q)^k q^{(\frac{k}{2})} \left[ \begin{array}{c} k \\vdash \end{array} \right] [t]_{k,q}^k
\]

\[
= \sum_{k=0}^{n} q^{(\frac{k}{2})} \left\{ \frac{1}{(1-q)^{n-k}} \sum_{j=k}^{n} (-1)^{j-k} q'^r \binom{n}{j} \left[ \begin{array}{c} j \\vdash \end{array} \right] [t]_{j,q}^j \right\}
\]

Comparing this expansion to (1.28), we readily deduce (1.40). The last expression can be transformed into (1.41) by using the expression

\[
\left[ \begin{array}{c} j \\vdash \end{array} \right] = \frac{1}{[k]_q!(1-q)^k} \sum_{i=0}^{k} (-1)^i q^{(\frac{i+1}{2})^{+i} - ik} \left[ \begin{array}{c} k \\vdash \end{array} \right] [i]_q^i,
\]

which is deduced from (1.42) by replacing \(n\) by \(k\), the bound variable \(j\) by \(i\), setting \(t = j, r = 0\) and dividing both members of the resulting expression by \([k]_q!\). Specifically, we get

\[
S_q(n, k; r) = \frac{1}{[k]_q!} \sum_{j=0}^{n} (-1)^{j-k} \binom{n}{j} \sum_{i=0}^{k} (-1)^i q^{(\frac{i+1}{2})^{+i} - k(r+i)} \left[ \begin{array}{c} k \\vdash \end{array} \right] \frac{q^{(r+i)j}}{(1-q)^n}
\]

\[
= \frac{1}{[k]_q!} \sum_{i=0}^{k} (-1)^{k-i} q^{(\frac{i+1}{2})^{+i} - k(r+i)} \left[ \begin{array}{c} k \\vdash \end{array} \right] \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(q^{r+i}j)}{(1-q)^n}.
\]
Since, by the classical binomial formula,
\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(q^{r+i})^j}{(1-q)^n} = \frac{(1-q^{r+i})^n}{(1-q)^n} = [r+i]_q^n, \]
the last expression implies (1.41).

The explicit expressions of the $q$-Stirling numbers of the first and second kind, (1.39) and (1.40), with $r = 0$, can be inverted to express the binomial coefficients in terms of the $q$-binomial coefficients and vice versa. These inversions are obtained in the following corollary of Theorem 1.13.

**Corollary 1.4.** Let $k$ and $j$ be positive integers and let $q$ be a real number, with $q \neq 1$. Then,
\[ \binom{k}{j} = \sum_{m=j}^{k} (-1)^{m-j}(1-q)^{m-j}s_q(m,j)\left[ \frac{k}{m} \right]_q \]  
(1.43)
and
\[ \left[ \frac{k}{j} \right]_q = \sum_{m=j}^{k} (-1)^{m-j}(1-q)^{m-j}S_q(m,j)\left( \frac{k}{m} \right). \]  
(1.44)

**Proof.** Expression (1.39), with $r = 0$, may be inverted to obtain the binomial coefficient $\binom{k}{j}$ in terms of $q$-binomial coefficients as follows. Specifically, in (1.39), with $r = 0$, we replace $n$ by $m$, the bound (dummy) variable $j$ by $i$ and multiply the resulting expression by $(-1)^{m-j}(1-q)^{m-j}\left[ \frac{k}{m} \right]_q$.

Then, summing for $m = j, j+1, \ldots, k$, we get
\[
\sum_{m=j}^{k} (-1)^{m-j}(1-q)^{m-j}s_q(m,j)\left[ \frac{k}{m} \right]_q = \sum_{m=j}^{k} \left[ \frac{k}{m} \right]_q \sum_{i=j}^{m} (-1)^{m-i}q^{\frac{m-i}{2}}\left[ \frac{m}{i} \right]_q \left( \frac{k}{j} \right) 
\]
\[ = \sum_{i=j}^{k} \left( \frac{i}{j} \right) \left\{ \sum_{m=i}^{k} (-1)^{m-i}q^{\frac{m-i}{2}}\left[ \frac{m}{i} \right]_q \left[ \frac{k}{m} \right]_q \right\} 
\]
and using the orthogonality relation (1.16), we deduce expression (1.43).

Similarly, in (1.40), with $r = 0$, we replace $n$ by $m$, the bound variable $j$ by $i$ and multiply the resulting expression by $(-1)^{m-j}(1-q)^{m-j}\left( \frac{k}{m} \right)$. 


Then, summing for \( m = j, j + 1, \ldots, k \), we get
\[
\sum_{m=j}^{k} (-1)^{m-j}(1 - q)^{m-j}S_q(m, j) \binom{k}{m} = \sum_{m=j}^{k} \binom{k}{m} \sum_{i=j}^{m} (-1)^{m-i} \binom{m}{i} [j]_q
\]
\[
= \sum_{i=j}^{k} \binom{i}{j} \sum_{m=i}^{k} (-1)^{m-i} \binom{m}{i} \binom{k}{m}
\]
and using the orthogonality relation
\[
\sum_{m=i}^{k} (-1)^{m-i} \binom{m}{i} \binom{k}{m} = \delta_{i,k},
\]
we deduce relation (1.44).

1.7 GENERALIZED q-Factorial Coefficients

Let \( t, q, s \), and \( r \) be real numbers, with \( q \neq 1 \). The noncentral generalized q-factorial of \( t \) of order \( n \), scale parameter \( s \) and noncentrality parameter \( r \),
\[
[st + r]_{n,q} = [st + r]_q [st + r - 1]_q \cdots [st + r - n + 1]_q
\]
\[
= q^{\binom{n}{2} + nr} ([s]_q [r]_q - [-r]_q) \cdots ([s]_q [r]_q - [n - r - 1]_q),
\]
may be expressed as a polynomial of \( q^s \)-factorials of \( t \) as
\[
[st + r]_{n,q} = q^{\binom{n}{2} + nr} \sum_{k=0}^{n} q^{\binom{k}{2}} C_q(n, k; s, r)[t]_{k, q^s}, \quad n = 0, 1, \ldots .
\]  
(1.45)
The coefficient \( C_q(n, k; s, r) \) is called noncentral generalized q-factorial coefficient.

Furthermore, the expansion of the generalized ascending q-factorial of \( t \) of order \( n \), scale parameter \( s \) and noncentrality parameter \( r \),
\[
[st + r + n - 1]_{n,q} = [st + r]_q [st + r + 1]_q \cdots [st + r + n - 1]_q
\]
\[
= [-1]_q^{n}[st - r]_{n,q-1},
\]
into a polynomial of the \( q^s \)-factorials of \( t \) may be deduced from (1.45) as
\[
[st + r + n - 1]_{n,q} = q^{\binom{n}{2} + nr} \sum_{k=0}^{n} q^{\binom{k}{2}} |C_q(n, k; -s, -r)|[t]_{k, q^s},
\]  
(1.46)
for \( n = 0, 1, \ldots \), where
\[
|C_q^{-1}(n, k; -s, -r)| = [-1]_q^{n}C_q^{-1}(n, k; -s, -r).
\]
The coefficient \(|C_q^{-1}(n, k; -s, -r)|\), which for \( 0 < q < 1 \) or \( 1 < q < \infty \) and \( s \) and \( r \) positive numbers, is nonnegative, is called absolute noncentral generalized q-factorial coefficient.
The noncentral generalized $q$-factorial coefficients as the scale parameter tends to zero or infinity converge to the noncentral $q$-Stirling numbers of the first and second kind, respectively. Specifically, expression (1.45) may be written as

$$[t + r]_{n,q} = q^{-\left(\frac{n}{2}\right) + r n} \sum_{k=0}^{n} q^{\frac{k}{2}} \{ [s]^{-k} C_{q}(n, k; s, r) \} \{ [s]^{k} [t/s]_{k,q} \}.$$  

Since $\lim_{s \to 0} [s]^{k} [t/s]_{k,q} = [t]^{k}_{q}$, it follows that

$$\lim_{s \to 0} [s]^{-k} C_{q}(n, k; s, r) = s_{q}(n, k; -r). \quad (1.47)$$

Similarly, writing expression (1.45) in the form

$$[s]^{-n} \{ s(t + r) \}_{n,q}^{1/s} = q^{-\left(\frac{n}{2}\right)} \sum_{k=0}^{n} \left( \frac{k}{2} \right)^{k} q^{(n-k) \{ s \}_{n,q}^{1/s} C_{q}(n, k; s, rs)} \{ t \}_{k,q}$$

and since

$$\lim_{s \to \infty} [s]^{-n} \{ s(t + r) \}_{n,q}^{1/s} = [t + r]_{q}^{n},$$

it follows, by virtue of (1.28), that

$$\lim_{s \to \infty} q^{r(n-k)} \{ s \}_{n,q}^{1/s} C_{q}(n, k; s, rs) = S_{q}(n, k; r). \quad (1.48)$$

A triangular recurrence relation for the generalized $q$-factorial coefficients are derived in the following theorem.

**Theorem 1.14.** The noncentral generalized $q$-factorial coefficients $C_{q}(n, k; s, r)$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots$, satisfy the triangular recurrence relation

$$C_{q}(n + 1, k; s, r) = [s] C_{q}(n, k - 1; s, r) + ([sk] - [n - r]_{q}) C_{q}(n, k; s, r), \quad (1.49)$$

for $k = 1, 2, \ldots, n + 1$ and $n = 0, 1, \ldots$, with initial conditions

$$C_{q}(0, 0; s, r) = 1, \quad C_{q}(0, k; s, r) = 0, \quad k > 0,$$

and

$$C_{q}(n, 0; s, r) = q^{\left(\frac{n}{2}\right) - r n} [r]_{n,q}, \quad n > 0.$$  

**Proof.** Expanding both members of the recurrence relation

$$[st + r]_{n+1,q} = [st + r - n]_{q} \{ st + r \}_{n,q} = q^{-n+r} \{ [s]_{q} [r]_{q} - [n - r]_{q} \} \{ [st]_{n,q}$$

into $q$-factorials of $t$, using (1.45), we find

$$\sum_{k=0}^{n+1} q^{\frac{k}{2}} C_{q}(n + 1, k; s, r) [t]_{k,q} = \sum_{k=0}^{n} q^{\frac{k}{2}} C_{q}(n, k; s, r) [s]_{q} [t]_{q^r} [t]_{k,q}$$

$$+ \sum_{k=0}^{n} q^{\frac{k}{2}} [n - r]_{q} C_{q}(n, k; s, r) [t]_{k,q}.$$
Furthermore, using the expressions
\[ [t]_q [t]_{k,q^r} = q^k [t]_{k+1,q^r} + [k]_q [t]_{k,q^r}, \quad [s]_q [k]_{q^r} = [sk]_q, \]
we get the relation
\[
\sum_{k=0}^{n+1} q^{s \left( \frac{k}{2} \right)} C_q(n + 1, k; s, r)[t]_{k,q^r} = \sum_{k=0}^{n} q^{s \left( \frac{k+1}{2} \right)} [s]_q C_q(n, k; s, r)[t]_{k+1,q^r}
\]
\[+ \sum_{k=0}^{n} q^{s \left( \frac{k}{2} \right)} ([sk]_q - [n - r]_q) C_q(n, k; s, r)[t]_{k,q^r}.\]

Equating the coefficients of \([t]_{k,q^r}\) in both sides of the last relation, we get (1.49). The initial conditions follow directly from (1.45).

\[\square\]

Remark 1.6. The absolute noncentral generalized \(q\)-factorial coefficients
\[|C_{q^{-1}}(n, k; -s, -r)| = [-1]^n q^{r} C_{q^{-1}}(n, k; -s, -r),\]
for \(k = 0, 1, \ldots, n\) and \(n = 0, 1, \ldots\), with \(s\) and \(r\) positive numbers, according to (1.49), satisfy the triangular recurrence relation
\[|C_{q^{-1}}(n + 1, k; -s, -r)| = [s]_q |C_{q^{-1}}(n, k - 1; -s, -r)|
\[+ ([sk]_q + q^{-(n+r)}[n + r]_q)|C_{q^{-1}}(n, k; -s, -r)|,\]
for \(k = 1, 2, \ldots, n + 1\) and \(n = 0, 1, \ldots\), with initial conditions
\[|C_{q^{-1}}(0, 0; -s, -r)| = 1, \quad |C_{q^{-1}}(0, k; -s, -r)| = 0, k > 0,\]
and
\[|C_{q^{-1}}(n, 0; -s, -r)| = q^{-s \left( \frac{n}{2} \right) + rn} [r + n - 1]_{n,q}, n > 0.\]

The noncentral generalized \(q\)-factorial coefficient \(C_q(n, k; s, r)\) is a polynomial in \([s]_q\) of degree \(n\). Specifically, they have the following theorem.

Theorem 1.15. The noncentral generalized \(q\)-factorial coefficients are connected with the noncentral \(q\)-Stirling numbers of the first and second kind by
\[C_q(n, k; s, \rho - r) = q^{-s \rho (n-k)} \sum_{m=k}^{n} s_q(n, m; r) S_{q^r}(m, k; \rho)[s]_q^m. \quad (1.50)\]

Proof. Expanding the noncentral generalized \(q\)-factorial
\[ [s(t + \rho) - r]_{n,q} = [st + (s \rho - r)]_{n,q} \]
into powers of \([s(t + \rho)]_q = [s]_q [t + \rho]_{q^r}\) by using (1.26) and, in the resulting expression, expanding the powers of \([t + \rho]_{q^r}\) into \(q^k\)-factorials of \(t\), by using (1.28), we
deduce the expression

\[
[s(t + \rho) - r]_{n,q} = q^{-\binom{n}{2} - r n} \sum_{m=0}^{n} s_q(n, m; r)[s]_q^m [s(t + \rho)]_q^m
\]

\[
= q^{-\binom{n}{2} - r n} \sum_{m=0}^{n} s(n, m; r)[s]_q^m \sum_{k=0}^{m} q^{\binom{k}{2}} + s \rho k S_q(m, k; \rho)[t]_{k,q}^s
\]

\[
= q^{-\binom{n}{2} - r n} \sum_{k=0}^{n} q^{\binom{k}{2}} + s \rho k \left\{ \sum_{m=k}^{n} s_q(n, m; r)S_q(m, k; \rho)[s]_q^m \right\} [t]_{k,q}^s,
\]

and since, by (1.45),

\[
[st + (s \rho - r)]_{n,q} = q^{-\binom{n}{2} + (s \rho - r)n} \sum_{k=0}^{n} q^{\binom{k}{2}} C_q(n, k; s, s \rho - r)[t]_{k,q}^s,
\]

we deduce (1.50).

\[\square\]

**Remark 1.7.** The sign of the generalized q-factorial coefficients. Expression (1.50) in the particular case \( \rho = 0 \) may be written as

\[
C_q(n, k; s, r) = \sum_{m=k}^{n} s_q(n, m; -r)S_q^r(m, k)[s]_q^m.
\]

Thus, according to this expression and Remark 1.4, the noncentral generalized q-factorial coefficients \( C_q(n, k; s, r), k = 0, 1, \ldots, n, n = 0, 1, \ldots, \) in a base \( q \), with \( 0 < q < 1 \) or \( 1 < q < \infty \), for \( s \) and \( r \) positive numbers and \( n < r + 1 \), are nonnegative q-numbers.

Also, expression (1.50) for the numbers \( |C_q(n, k; -s, -r)|, k = 0, 1, \ldots, n, n = 0, 1, \ldots, \) may be written as

\[
|C_q(n, k; -s, -r)| = \sum_{m=k}^{n} |s_q(n, m; r)|S_q^r(m, k)[s]_q^m.
\]

Thus, according to this expression and Remark 1.4, the absolute noncentral generalized q-factorial coefficients \( |C_q(n, k; -s, -r)|, k = 0, 1, \ldots, n, n = 0, 1, \ldots, \) in a base \( q \), with \( 0 < q < 1 \) or \( 1 < q < \infty \), for \( s \) and \( r \) positive numbers, are nonnegative q-numbers.

**Theorem 1.16.** The reciprocal q-factorial \( 1/[t]_{k+1,q}^s \) is expanded into reciprocal noncentral generalized q-factorials \( 1/[st + r]_{n+1,q}^s, n = k, k+1, \ldots, k = 0, 1, \ldots, \) as

\[
\frac{1}{[t]_{k+1,q}^s} = q^{\binom{k+1}{2}} \sum_{n=k}^{\infty} q^{-\binom{n+1}{2} + r(n+1)} [s]_q C_q(n, k; s, r) \frac{1}{[st + r]_{n+1,q}^s}. \tag{1.51}
\]

for \( t > k \).
Proof. Let us consider the series

\[ C_k(t; q) = \sum_{n=k}^{\infty} q^{-\frac{(n+1)}{2} + rn} C_q(n, k; s, r) \frac{1}{[st + r]_{n+1,q}}, \quad t > k, \]

for \( k = 0, 1, \ldots \). Multiplying both sides of the triangular recurrence relation (1.49) by

\[ \frac{q^{-\frac{(n+2)}{2} + r(n+1)}}{[st + r]_{n+2,q}} = \frac{q^{-\frac{(n+2)}{2} + r(n+2)}}{[st + r]_{n+2,q}} ([st]_q - [n - r + 1]_q), \]

we find the relation

\[ ([st]_q C_q(n + 1, k; s, r) - [n - r + 1]_q C_q(n + 1, k; s, r)) \cdot q^{-\frac{(n+2)}{2} + r(n+2)} = ([sk]_q C_q(n, k; s, r) - [n - r]_q C_q(n, k; s, r) + [s]_q C_q(n, k - 1; s, r)) \cdot q^{-\frac{(n+2)}{2} + r(n+1)} \]

Summing it for \( n = k - 1, k, \ldots \) and since \( C_q(k - 1, k; s) = 0 \), we obtain for \( C_k(t; q) \) the relation

\[ [st]_q C_k(t; q) = [sk]_q C_k(t; q) + [s]_q C_{k-1}(t; q), \quad k = 1, 2, \ldots . \]

Using the expressions

\[ [st]_q = [s]_q [t]_q, \quad [sk]_q = [s]_q [k]_q, \quad [t]_q, [k]_q = q^k [t - k]_q, \]

we deduce the recurrence relation

\[ C_k(t; q) = \frac{q^{-sk}}{[t - k]_q} C_{k-1}(t; q), \quad k = 1, 2, \ldots , \]

and applying it repeatedly, we find

\[ C_k(t; q) = \frac{q^{-sk} (k+1)}{[t-1]_{k,q}} C_0(t; q). \]

Since \( C_q(n, 0; s, r) = q \left( \binom{n}{2} - n \right) [r]_{n,q} \), for \( n > 0 \), we get the initial value

\[ [st]_q C_0(t; q) = \sum_{n=0}^{\infty} q^{-n+r} \frac{[st]_q [r]_{n,q}}{[st + r]_{n+1,q}} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{[r]_{n,q}}{[st + r]_{n,q}} - \frac{[r]_{n+1,q}}{[st + r]_{n+1,q}} \right) = 1, \]
and so \( C_0(t; q) = 1/[st]_q = 1/([s]_q[t]_q) \). Therefore,

\[
C_k(t; q) = \frac{q^{-s\binom{k+1}{2}}}{[s]_q[t]_{k+1,q}}
\]

and (1.51) is established.

Explicit expressions for the noncentral generalized \( q \)-factorial coefficients are derived in the following theorem.

**Theorem 1.17.** The noncentral generalized \( q \)-factorial coefficient \( C_q(n, k; s, r) \), for \( k = 1, 2, \ldots, n \), \( n = 1, 2, \ldots \), is given by

\[
C_q(n, k; s, r) = \frac{[s]^k_q}{(1 - q)^{n-k}} \sum_{j=k}^{n} (-1)^{j-k} q^{\binom{n-j}{j}} \sum_{j=0}^{r(n-j)} \frac{\binom{n}{j}}{\binom{j}{k}} [j]_{q^s}. \tag{1.52}
\]

Also

\[
C_q(n, k; s, r) = \frac{q^{\binom{n}{2} - r n}}{[k]_q^r} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{j+1}{2} - sk} \binom{k}{j} [sj + r]_{n,q}. \tag{1.53}
\]

**Proof.** Let us consider (1.42), with \( t \) replaced by \( st \), and multiply both its sides by \( q^{\binom{n}{2}} \). Then, using the relation \( \binom{n-j}{j} = \binom{n}{j} - j(n-j) \), we deduce the expression

\[
q^{\binom{n}{2}} [st]_{n,q} = \frac{1}{(1 - q)^n} \sum_{j=0}^{n} (-1)^{j} q^{\binom{n-j}{j}} \binom{n}{j} (q^s)^j.
\]

Expanding the \( q^s \)-function \( q^{(i,j)} \) into \( q^s \)-factorials of \( t \), using (1.18), we get

\[
q^{\binom{n}{2}} [st]_{n,q} = \frac{1}{(1 - q)^n} \sum_{j=0}^{n} (-1)^{j} q^{\binom{n-j}{j}} \binom{n}{j} \sum_{k=0}^{j} (-1)^{k} (1 - q)^k q^{\binom{k}{2}} [j]_{q^s} [t]_q^k
\]

\[
= \sum_{k=0}^{n} q^{\binom{k}{2}} \left\{ \frac{[s]^k_q}{(1 - q)^{n-k}} \sum_{j=k}^{n} (-1)^{j-k} q^{\binom{n-j}{j}} \binom{n}{j} \binom{j}{k} [j]_{q^s} [t]_q^k \right\}. \tag{1.53}
\]

Comparing this expansion to (1.45), we readily deduce (1.52). The last expression can be transformed into (1.53) by using the expression

\[
\binom{j}{k} [k]_{q^s} = \frac{1}{[k]_q!(1 - q)^k[s]^k_q} \sum_{i=0}^{k} (-1)^{i} q^{\binom{i+1}{2} - sk} [i]_{q^s} [s]_{q^s}^{i},
\]

which is deduced from (1.42) by replacing \( n \) by \( k \), \( q \) by \( q^s \), the bound variable \( j \) by \( i \), setting \( t = j \) and \( r = 0 \), dividing both members of the resulting expression by \( [k]_q! \).
and using the relation \((1 - q^r) = (1 - q)[s]_q\). Specifically, we get

\[
C_q(n, k; s, r) = \frac{q^{-rn}}{[k]_{q^r}} \sum_{j=0}^{n} (-1)^{j-k} q^{\frac{(n-j)}{2}} \sum_{i=0}^{k} (-1)^{i} q^{i+1} \left[ \begin{array}{c} i \\ j \end{array} \right] \frac{q^{(s+r)i}}{(1 - q)^n}.
\]

Since, by the \(q\)-binomial formula (1.14),

\[
\sum_{j=0}^{n} (-1)^{j} q^{\frac{(n-j)}{2}} \left[ \begin{array}{c} n \\ j \end{array} \right] \frac{q^{(s+r)i}}{(1 - q)^n} = q^{\frac{n}{2}} \sum_{j=0}^{n} (-1)^{j} q^{\frac{(i)}{2}} \left[ \begin{array}{c} i \\ j \end{array} \right] \frac{q^{(s+r)i}}{(1 - q)^n} = q^{\frac{n}{2}} \prod_{m=1}^{n} \frac{(1 - q^{si+r-m+1})}{(1 - q)^n} = q^{\frac{n}{2}} [s| + r]_{n, q},
\]

the last expression implies (1.53).

\[
\square
\]

### 1.8 \(q\)-FACTORIAL AND \(q\)-BINOMIAL MOMENTS

The calculation of the mean and variance and generally the calculation of the moments of a discrete \(q\)-distribution is quite difficult. Several techniques have been used for the calculation of the mean and variance of particular \(q\)-distributions. In this section, we introduce the \(q\)-factorial and \(q\)-binomial moments of a discrete \(q\)-distribution, the calculation of which is as easy as that of the usual factorial and binomial moments of the classical discrete distributions. These moments, apart from the interest in their own, are used as an intermediate step in the calculation of the usual factorial and binomial moments of the \(q\)-distributions.

**Definition 1.1.** Let \(X\) be a nonnegative integer-valued random variable, with probability function \(f(x) = P(X = x)\), \(x = 0, 1, \ldots\). The expected values

\[
E([X]_{m, q}) = \sum_{x=m}^{\infty} [x]_{m, q} f(x), \quad m = 1, 2, \ldots \tag{1.54}
\]

and

\[
E \left( \left[ \begin{array}{c} X \\ m \end{array} \right] \right) = \sum_{x=m}^{\infty} \left[ \begin{array}{c} x \\ m \end{array} \right] f(x), \quad m = 1, 2, \ldots \tag{1.55}
\]

provided they exist, are called the \(m\)th-order \(q\)-factorial and the \(m\)th-order \(q\)-binomial moments, respectively, of the random variable \(X\).
Note that, the $q$-factorial and the $q$-binomial moments are closely connected by

$$E\left(\left[\frac{X}{m}\right]_q\right) = \frac{E(\left[\frac{X}{m}\right]_{q,m})}{[m]_q!}, \quad E(\left[\frac{X}{m}\right]_{q}) = [m]_q! E\left(\left[\frac{X}{m}\right]_q\right).$$

Also in particular, for $m = 1$, the $q$-expected value or the $q$-mean of $X$, denoted by $E(\left[\frac{X}{q}\right])$ or by $\mu_q$, is defined by

$$\mu_q = E(\left[\frac{X}{q}\right]) = \sum_{x=1}^{\infty} [x]_q f(x),$$

provided the series is convergent. Furthermore, the $q$-variance of $X$, denoted by $V(\left[\frac{X}{q}\right])$ or by $\sigma_q^2$, is defined by

$$\sigma_q^2 = V(\left[\frac{X}{q}\right]) = E(\left[\frac{X}{q}\right]^2) = \sum_{x=1}^{\infty} (\left[\frac{x}{q}\right] - \mu_q)^2 f(x),$$

provided the series is convergent. Clearly

$$V(\left[\frac{X}{q}\right]) = E(\left[\frac{X}{q}\right]^2) - [E(\left[\frac{X}{q}\right])]^2.$$  \hspace{1cm} (1.58)

Also, since $q[X - 1]_q = [X]_q - 1$, it follows that

$$q[X]_{2,q} = q[X]_q[X - 1]_q = [X]_q([X]_q - 1) = [X^2]_q - [X]_q$$

and so the $q$-variance may be expressed in terms of the $q$-factorial moments as

$$V(\left[\frac{X}{q}\right]) = qE(\left[\frac{X}{q}^2\right]) + E(\left[\frac{X}{q}\right]) - [E(\left[\frac{X}{q}\right])]^2.$$ \hspace{1cm} (1.59)

**Remark 1.8.** $q$-Deformed distributions in Quantum Physics. Consider a non-negative integer-valued random variable $X$ with probability mass function $f_X(x) = P(X = x)$, $x = 0, 1, \ldots$. Furthermore, consider the $q$-number transformation $Y = [X]_q$, which in the language of quantum physics is known as a $q$-deformation. The distribution of the random variable $Y$, with probability function

$$f_Y([x]_q) = P(Y = [x]_q) = P(X = x) = f_X(x), \quad x = 0, 1, \ldots,$$

is called $q$-deformed distribution. The mean and the variance of the $q$-deformed distribution of $Y$ are the $q$-mean and the $q$-variance of the distribution of $X$.

The usual binomial and factorial moments are expressed in terms of the $q$-binomial and the $q$-factorial moments, respectively, through the $q$-Stirling numbers of the first kind, in the following theorem.

**Theorem 1.18.** The usual binomial moments are expressed in terms of the $q$-binomial moments by

$$E\left(\left[\frac{X}{j}\right]\right) = \sum_{m=j}^{\infty} (-1)^{m-j}(1 - q)^{m-j} s_q(m,j) E\left(\left[\frac{X}{m}\right]_q\right),$$ \hspace{1cm} (1.60)
for \( j = 1, 2, \ldots \), and equivalently, the usual factorial moments are expressed in terms of the \( q \)-factorial moments by

\[
E[(X)_j] = j! \sum_{m=j}^{\infty} (-1)^{m-j}(1-q)^{m-j} s_q(m, j) \frac{E[(X)_{m,q}]}{[m]_q!},
\]

(1.61)

for \( j = 1, 2, \ldots \), where \( s_q(m, j) \) is the \( q \)-Stirling number of the first kind.

**Proof.** Multiplying both sides of expression (1.43),

\[
\binom{x}{j} = j! \sum_{m=j}^{x} (-1)^{m-j}(1-q)^{m-j} s_q(m, j) \frac{x^m}{[m]_q!}, \quad j = 1, 2, \ldots,
\]

by the probability function \( f(x) \) of the random variable \( X \) and summing for all \( x = 0, 1, \ldots \), we deduce, according to (1.55), expression (1.60). Furthermore, since

\[
E[(X)_j] = E[(X)^j], \quad E\left( \binom{X}{m}_q \right) = \frac{E[(X)_{m,q}]}{[m]_q!},
\]

expression (1.61) is readily deduced from (1.60). \( \square \)

The probability function of a nonnegative integer-valued random variable \( X \) may be expressed in terms of its \( q \)-binomial (or \( q \)-factorial) moments, by inverting expression (1.55) (or (1.54)). Such an expression is derived in the following theorem.

**Theorem 1.19.** The probability function \( f(x) = P(X = x) \), \( x = 0, 1, \ldots \), of a nonnegative integer-valued random variable \( X \), is expressed in terms of its \( q \)-binomial moments \( E\left( \binom{X}{m}_q \right), \ m = 0, 1, \ldots \), by

\[
f(x) = \sum_{m=x}^{\infty} (-1)^{m-x} q^{\frac{m-x}{2}} \left[ \frac{m}{x}_q \right] E\left( \binom{X}{m}_q \right), \quad x = 0, 1, \ldots,
\]

(1.62)

provided that the series is absolutely convergent.

**Proof.** Writing expression (1.55) with \( x \) replaced by \( k \) and then multiplying it by

\[
(-1)^{m-x} q^{\frac{m-x}{2}} \left[ \frac{m}{x}_q \right],
\]

and summing the resulting expression for \( m = x, x+1, \ldots \), we get

\[
\sum_{m=x}^{\infty} (-1)^{m-x} q^{\frac{m-x}{2}} \left[ \frac{m}{x}_q \right] E\left( \binom{X}{m}_q \right) = \sum_{m=x}^{\infty} (-1)^{m-x} q^{\frac{m-x}{2}} \left[ \frac{m}{x}_q \right] \sum_{k=m}^{\infty} \left[ \frac{k}{m}_q \right] f(k)
\]

\[
= \sum_{k=x}^{\infty} \left\{ \sum_{m=x}^{k} (-1)^{m-x} q^{\frac{m-x}{2}} \left[ \frac{m}{x}_q \right] \left[ \frac{k}{m}_q \right] \right\} f(k).
\]
Therefore, using the second of the orthogonality relations (1.16),
\[ \sum_{m=x}^{k} (-1)^{m-x} q^{\binom{m-x}{2}} \binom{m}{x}_q \binom{k}{m}_q = \delta_{k,x}, \]
we deduce (1.62). □

1.9 REFERENCE NOTES

The introduction of the \( q \)-number and its notation stems from Jackson (1910a), who published important and influential papers on the subject. A list of his publications is included in the obituary note by Chaudry (1962).

Gauss (1863) introduced the \( q \)-binomial coefficients (or Gaussian polynomials) and presented their triangular recurrence relations, derived in Theorem 1.1, and the vertical recurrence relations, which are given as Exercise 1.2. Also, the summation formula given as Exercise 1.4 is from Gauss (1863). The distribution of the number theoretic random variable examined in Example 1.1 was discussed by Rawlings (1994a). The combinatorial interpretation of the \( q \)-binomial coefficient as the number of subspaces of a vector space, presented in Example 1.2, was given in Goldman and Rota (1970). Also, its appearance as generating function of the number of partitions of an integer into parts of restricted size, with variable (indeterminate) \( q \), presented in Example 1.3, was noted by Sylvester (1882).

The \( q \)-Vandermonde’s (\( q \)-factorial convolution) formulae, and equivalently the \( q \)-Cauchy’s (\( q \)-binomial convolution) formulae, together with the general \( q \)-binomial formulae were derived by Cauchy (1843), Jacobi (1846), and Heine (1847, 1878). It is worth noticing that the origin of the general \( q \)-binomial formulae is quite uncertain; Hardy (1940) attributed these formulae to Euler. The derivation of the power series expressions of the two \( q \)-exponential functions (1.23) and (1.24) are, indeed, from Euler (1748). The limit formulas for the \( q \)-exponential functions, which are given in Exercise 1.11, are from Rawlings (1994b). Several other interesting \( q \)-series expansions are presented in the classical book of Andrews (1976); Exercises 1.12 and 1.13, on the univariate and multivariate Rogers–Szegö polynomials, are taken from this book. A motivated introduction and a clear presentation of the \( q \)-gamma function and the \( q \)-beta integral can be found in the excellent book of Andrews et al. (1999).

An authoritative and comprehensive account of the basic \( q \)-hypergeometric series is given by Gasper and Rahman (2004).

The \( q \)-Stirling numbers of the second kind were introduced by Carlitz (1933) in connection with an enumeration problem in abelian groups. In a second paper, Carlitz (1948) found it convenient to generalize these numbers to what are nowadays called noncentral \( q \)-Stirling numbers of the second kind. Furthermore, Gould (1961) studied the \( q \)-Stirling numbers of the first and second kind, which were defined as sums of all \( k \)-factor products that are formed from the first \( n \) \( q \)-natural numbers, without and with repeated factors, respectively. The \( q \)-Lah numbers appeared in Hahn (1949). Also, Garsia and Remmel (1980) discussed these numbers, as \( q \)-Laguerre numbers.
The central and noncentral generalized $q$-factorial coefficients were discussed in Charalambides (1996, 2002, 2004, 2005b). The $q$-factorial moments and their connection to the usual factorial moments were discussed in Charalambides and Papadatos (2005) and Charalambides (2005a). Jackson (1910a, 1910b, 1951) extensively studied $q$-derivatives and $q$-integrals. The generalized Stirling and Lah numbers were introduced by Tauber (1962, 1965) and further studied by Comtet (1972) and Platonov (1976).

1.10 Exercises

1.1 Let $m$ and $k$ be positive integers, with $k \leq m$.
   
   (a) Show that
   $$\left[ \begin{array}{c} m \\ k \end{array} \right]_q = \left[ \begin{array}{c} m \\ m-k \end{array} \right]_q .$$
   
   (b) Furthermore, let $x$ be a real number. Show that
   $$\left[ \begin{array}{c} x \\ m \end{array} \right]_q \left[ \begin{array}{c} m \\ k \end{array} \right]_q = \left[ \begin{array}{c} x \\ m-k \end{array} \right]_q \left[ \begin{array}{c} x-m+k \\ k \end{array} \right]_q$$
   and
   $$\left[ \begin{array}{c} x \\ m \end{array} \right]_q \left[ \begin{array}{c} x-m \\ k \end{array} \right]_q = \left[ \begin{array}{c} x \\ m-k \end{array} \right]_q \left[ \begin{array}{c} x \\ m \end{array} \right]_q \left[ \begin{array}{c} m+k \\ k \end{array} \right]_q .$$

1.2 Vertical recurrence relations for the $q$-binomial coefficients. For $n$, $k$, and $m$ positive integers, show that
   $$\sum_{r=m}^{n} q^{r-k} \left[ \begin{array}{c} r-1 \\ k-1 \end{array} \right]_q = \left[ \begin{array}{c} n \\ k \end{array} \right]_q - \left[ \begin{array}{c} m-1 \\ k \end{array} \right]_q$$
   and, alternatively, that
   $$\sum_{r=m}^{n} q^{(n-r)k} \left[ \begin{array}{c} r-1 \\ k-1 \end{array} \right]_q = \left[ \begin{array}{c} n \\ k \end{array} \right]_q - q^{(n-m+1)k} \left[ \begin{array}{c} m-1 \\ k \end{array} \right]_q .$$

   In particular, conclude that
   $$\sum_{r=k}^{n} q^{r-k} \left[ \begin{array}{c} r-1 \\ k-1 \end{array} \right]_q = \left[ \begin{array}{c} n \\ k \end{array} \right]_q$$
   and, alternatively, that
   $$\sum_{r=k}^{n} q^{(n-r)k} \left[ \begin{array}{c} r-1 \\ k-1 \end{array} \right]_q = \left[ \begin{array}{c} n \\ k \end{array} \right]_q .$$

1.3 A horizontal recurrence relation for the $q$-binomial coefficients. For $n$, $k$, and $m$ positive integers, show that
   $$\sum_{r=k}^{m} (-1)^{r-k} q^{r+1 \choose 2} \left[ \begin{array}{c} n+1 \\ r+1 \end{array} \right]_q = q^{k+1 \choose 2} \left[ \begin{array}{c} n \\ k \end{array} \right]_q + (-1)^{m-k} q^{m+2 \choose 2} \left[ \begin{array}{c} n \\ m+1 \end{array} \right]_q .$$
and conclude that
\[ \sum_{r=k}^{n} (-1)^{r-k} q^{\left(\frac{r+1}{2}\right)} \binom{n+1}{r+1} q^{\left(\frac{k+1}{2}\right) \binom{n}{k} q}. \]

1.4 *A Gauss summation formula.* Show that
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} q = \begin{cases} \frac{(1-q)}{[1+q]} m, & n = 2m, \\ 0, & n = 2m+1, \end{cases} \]
for \( m \) a nonnegative integer.

1.5 Let \( x, y, \) and \( q \) be real numbers, with \( q \neq 1 \), and let \( n \) be a positive integer. Using \( q \)-Vandermonde’s formula, show that
\[ \sum_{k=0}^{n} (-1)^k q^{k\left(\frac{k+1}{2}\right)} \binom{n}{k} q^{x+n} \binom{x}{k} q^{x+n} = 1/\binom{x+n}{n} q, \]
for \( x + y \neq 0, 1, \ldots, n - 1 \), and conclude that
\[ \sum_{k=0}^{n} (-1)^k q^{k\left(\frac{k+1}{2}\right)} \binom{n}{k} q^{x+n} \binom{x}{k} q^{x+n} = \sum_{k=0}^{n} (-1)^k q^{k\left(\frac{k+1}{2}\right)} \binom{n}{k} q^{x+n} \binom{x}{k} q^{x+n}, \]

1.6 Show that the sequence of sums
\[ S_n = \sum_{k=1}^{n} (-1)^k q^{k\left(\frac{k+1}{2}\right)} \frac{1}{[k] q^{k}}, \quad n = 1, 2, \ldots, \]
satisfies the recurrence relation
\[ S_n = S_{n-1} + \frac{q^n}{[n] q}, \quad n = 2, 3, \ldots, \]
with \( S_1 = q \), and conclude that
\[ S_n = \sum_{k=1}^{n} \frac{q^k}{[k] q}, \quad n = 2, 3, \ldots. \]

1.7 Show that the sequence of sums
\[ S_n(x) = \sum_{k=0}^{n} (-1)^k q^{k\left(\frac{k+1}{2}\right)-n(x+k)} \binom{n}{k} q^{x+k} \binom{x}{k} q, \quad n = 0, 1, \ldots, \]
for \( x \neq -1, -2, \ldots, -n \), satisfies the recurrence relation
\[ S_n(x) = \frac{[n] q}{[x+n] q} S_{n-1}(x), \quad n = 1, 2, \ldots, \]
with $S_0(x) = 1$, and conclude that

$$S_n(x) = 1/\binom{x + n}{n}_q.$$ 

1.8 Let $n$ be a positive integer and let $t, u, w$, and $q$ be real numbers, with $q \neq 1$. Show that

$$\sum_{k=0}^{n} \binom{n}{k}_q u^k \prod_{i=1}^{n-k} (w + tq^{i-1}) = \sum_{r=0}^{n-r} \binom{n}{r}_q \prod_{i=1}^{n-r} (w + tq^{i-1})$$

and conclude that

$$\sum_{k=0}^{n} \binom{n}{k}_q u^k \prod_{i=1}^{n-k} (1 - tq^{i-1}) = \sum_{r=0}^{n-r} \binom{n}{r}_q \prod_{i=1}^{n-r} (w - tq^{i-1})$$

and

$$\sum_{k=0}^{n} \binom{n}{k}_q t^k \prod_{i=1}^{n-k} (1 - tq^{i-1}) = 1.$$ 

1.9 Additional negative $q$-binomial formulae. For $n$ a positive integer and $t$ and $q$ real numbers, with $0 < t < \infty$ and $0 < q < 1$ or $1 < q < \infty$, show that

$$\sum_{k=0}^{\infty} \binom{n + k - 1}{k}_q \frac{t^k q^{\binom{k}{2}}}{\prod_{i=1}^{k} (1 + tq^{n+i-1})} = \prod_{i=1}^{n} (1 + tq^{i-1}),$$

or, equivalently, that

$$\sum_{k=0}^{\infty} \binom{n + k - 1}{k}_q \frac{q^k}{\prod_{i=1}^{k} (1 + tq^{n+i-1})} = \prod_{i=1}^{n} (1 + tq^{i-1}) t^n q^{\binom{n}{2}}.$$ 

1.10 A $q$-geometric series. Consider the $q$-geometric progression

$$g_k = \prod_{j=1}^{k} (1 - tq^{j-1})q^k, \quad k = 0, 1, \ldots,$$

for $0 < t < \infty$ and $0 < q < 1$ or $1 < q < \infty$, with $g_0 = 1$. 
(a) Show that the sum of its first $n$ terms is given by

$$\sum_{k=0}^{n-1} \prod_{j=1}^{k} (1 - tq^{j-1})q^k = \frac{1 - \prod_{j=1}^{n} (1 - tq^{j-1})}{t}$$

and (b) deduce the limit of the $q$-geometric series as

$$\sum_{k=0}^{\infty} \prod_{j=1}^{k} (1 - tq^{j-1})q^k = \frac{1 - E_q (-t/(1-q))}{t},$$

for $0 < t < \infty$ and $0 < q < 1$, where $E_q(t) = \prod_{i=1}^{\infty} (1 + t(1-q)q^{i-1})$ is a $q$-exponential function.
1.11 Limit formulas for the $q$-exponential functions. Show that the $q$-exponential functions $E_q(t) = \prod_{i=1}^{\infty} (1 + t(1 - q)q^{i-1})$ and $e_q(t) = \prod_{i=1}^{\infty} (1 - t(1 - q)q^{i-1})^{-1}$, for $|q| < 1$ and $|t| < 1/(1 - q)$, may be obtained as

$$E_q(t) = \lim_{n \to \infty} \prod_{i=1}^{n} (1 + tq^{i-1}/[n]_q) \quad \text{and} \quad e_q(t) = \lim_{n \to \infty} \prod_{i=1}^{n} (1 - tq^{i-1}/[n]_q)^{-1},$$

respectively.

1.12 Rogers–Szegö polynomial. The polynomial

$$H_n(t; q) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_q t^k, \quad -\infty < t < \infty, \quad 0 < q < 1, \quad n = 0, 1, \ldots,$$

is called Rogers–Szegö polynomial.

(a) Derive its $q$-exponential generating function as

$$\sum_{n=0}^{\infty} H_n(t; q) \frac{u^n}{[n]_q!} = e_q(u)e_q(ut),$$

where $e_q(u) = \prod_{i=1}^{\infty} (1 - u(1 - q)q^{i-1})^{-1}$ is a $q$-exponential function.

(b) Show that

$$H_{n+1}(t; q) = (1 + t)H_n(t; q) - t(1 - q)[n]_q H_{n-1}(t; q), \quad n = 1, 2, \ldots,$$

with $H_0(t; q) = 1$ and $H_1(t; q) = t$.

1.13 $q$-Multinomial coefficients. The $q$-number

$$\left[ \frac{n}{k_1, k_2, \ldots, k_{r-1}} \right]_q = \frac{[n]_q!}{[k_1]_q![k_2]_q! \cdots [k_{r-1}]_q!},$$

where $k_r = n - k_1 - k_2 - \cdots - k_{r-1}$, for $k_i = 0, 1, \ldots, n$, $i = 1, 2, \ldots, r$ and $n = 0, 1, \ldots$, is called $q$-multinomial coefficient. The multivariate analogue of the Rogers–Szegö polynomial may be defined as

$$H_n(t_1, t_2, \ldots, t_{r-1}; q) = \sum \left[ \frac{n}{k_1, k_2, \ldots, k_{r-1}} \right]_q t_1^{k_1} t_2^{k_2} \cdots t_{r-1}^{k_{r-1}},$$

where the summation is extended over all $k_i = 0, 1, \ldots, n$, $i = 1, 2, \ldots, r$, such that $k_1 + k_2 + \cdots + k_{r-1} + k_r = n$. Derive its $q$-exponential generating function as

$$\sum_{n=0}^{\infty} H_n(t_1, t_2, \ldots, t_{r-1}; q) \frac{u^n}{[n]_q!} = e_q(u)e_q(ut_1)e_q(ut_2) \cdots e_q(ut_{r-1}),$$

where $e_q(u) = \prod_{i=1}^{\infty} (1 - u(1 - q)q^{i-1})^{-1}$ is a $q$-exponential function.
1.14 Noncentral $q$-Stirling numbers of the first kind. Show that the noncentral $q$-Stirling numbers of the first kind are connected with the usual $q$-Stirling numbers of the first kind by

$$s_q(n, k; r) = \sum_{j=k}^{n} (-1)^{j-k} q^{r(n-j)} \binom{j}{k} [r]_q^{j-k} s_q(n, j)$$

and

$$s_q(n, k; r) = \sum_{j=k}^{n} q^{(n-j)+r(n-j)} \binom{n}{j} [-r]_{n-j,q} s_q(j, k).$$

1.15 (Continuation). Show that

$$|s_q(n, j; r + \theta)| = q^n \sum_{k=j}^{n} \binom{k}{j} [s_q(n, k; r)|q^{(n-k)(\theta - 1)}|\theta]_q^{k-j},$$

where $|s_q(n, k; r)|$ is the noncentral signless $q$-Stirling number of the first kind.

1.16 (Continuation). Show that

$$s_q(n, 1) = (-1)^{n-1} [n - 1]_q! \quad \text{and} \quad s_q(n, 2) = (-1)^{n-2} [n - 1]_q! \zeta_{n-1,q},$$

where $\zeta_{n,q} = \sum_{j=1}^{n} 1/[j]_q$.

1.17 Noncentral $q$-Stirling numbers of the second kind. Show that the noncentral $q$-Stirling numbers of the second kind are connected with the usual $q$-Stirling numbers of the second kind by

$$S_q(n, k; r) = \sum_{j=k}^{n} q^{(j-k)} \binom{j}{k} [r]_{j-k,q} S_q(n, j)$$

and

$$S_q(n, k; r) = \sum_{j=k}^{n} q^{r(j-k)} \binom{n}{j} [r]_q^{n-j} S_q(j, k).$$

1.18 Bivariate generating functions of the noncentral $q$-Stirling numbers. Show that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} s_q(n, k; r)t^k \frac{u^n}{[n]_q!} = \prod_{i=1}^{\infty} \frac{1 + uq^{r+i-1}}{1 + (1 - (1 - q)t)uq^{i-1}}$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} q^{\binom{k}{2}+r} s_q(n, k; r)t^k \frac{u^n}{[n]_q!} = E_q(-t) \sum_{j=0}^{\infty} e^{(j+r)u} \frac{t^j}{[j]_q!},$$

where $E_q(t) = \prod_{i=1}^{\infty} (1 + t(1 - q)q^{i-1})$ is a $q$-exponential function.
1.19 (Continuation). Show that

\[
\sum_{k=j}^{n} \binom{k}{j} (1 - q)^{n-k} s_q(n, k; r) = q\binom{n-j}{2} + r(n-j) \binom{n}{j}_{q}
\]

and

\[
\sum_{k=j}^{n} \binom{k}{j} q^{(k-j)/2} + r(n-j) (1 - q)^{n-k} S_q(n, k; r) = \binom{n}{j}.
\]

1.20 (Continuation). Show that

\[
\sum_{n=k}^{\infty} \binom{n+j}{j} (1 - q)^{n-k} S_q(n, k; r) = q^{-(k+1)/2} - r(k+1-j)+k+r+k \binom{k+j}{j}_{q}.
\]

1.21 Noncentral generalized q-factorial coefficients. Show that the noncentral generalized q-factorial coefficients are connected with the usual generalized q-factorial coefficients by

\[
C_q(n, k; s, r) = q^{-r(n-k)} \sum_{j=k}^{n} q^{s\binom{j-k}{2}} \binom{j}{k}_{q} [r/s]_{q}^{-j} q_{k,q} C_q(n,j;s)
\]

and

\[
C_q(n, k; s, r) = \sum_{j=k}^{n} q^{-(n-j)} \binom{n-j}{j}_{q} [r]_{n-j=1} q_{k,q} C_q(i, k; s).
\]

1.22 Noncentral q-Lah numbers. Consider the expansion

\[
[-(t-r)]_{n,q} = q^{\binom{n}{2}} - r \sum_{k=0}^{n} q^{\binom{k}{2}} L_q(n, k; r)[r]_{k,q}.
\]

Since \([-(t-r)]_{n,q} = [t-r + n - 1]_{n,q} / [-1]_{q}^{n}\) and setting

\[
|L_q(n, k; r)| = [-1]_{q}^{n} L_q(n, k; r),
\]

it can be written as

\[
[t-r + n - 1]_{n,q} = q^{\binom{n}{2}} - r \sum_{k=0}^{n} q^{\binom{k}{2}} |L_q(n, k; r)| [r]_{k,q}.
\]

The coefficients \(L_q(n, k; r)\) and \(|L_q(n, k; r)|\) are called noncentral q-Lah numbers and noncentral signless q-Lah number, respectively. Note that for \(r = 0\) the noncentral q-Lah number and the noncentral signless q-Lah number reduce to \(L_q(n, k; 0) = L_q(n, k)\) and \(|L_q(n, k; 0)| = |L_q(n, k)|\), the usual (central) q-Lah number and the signless q-Lah number, respectively. Show that

\[
|L_q(n, k; r)| = q^{-\binom{n}{2} + \binom{k}{2}} + r(n-k) \binom{n}{k}_{q}^{-1} [n-r-1]_{q}! [k-r-1]_{q}.
\]
1.23 $q$-Eulerian numbers. Consider the expansion of the $n$th power of a $q$-number into $q$-binomial coefficients of order $n$

$$[t]^n_q = \sum_{k=0}^{n} q^{\binom{k}{2}} A_q(n,k) \left[ \begin{array}{c} t + n - k \\ n \end{array} \right]_q, \quad n = 0, 1, \ldots$$

The coefficient $A_q(n,k)$ is called $q$-Eulerian number. (a) Show that

$$A_q(n,k) = A_q(n,n-k+1), \quad k = 0, 1, \ldots, n, \quad n = 0, 1, \ldots,$$

and (b) derive the explicit expression

$$A_q(n,k) = q^{-\binom{k}{2}} \sum_{r=0}^{n} (-1)^r q^{\binom{r}{2}} \left[ \begin{array}{c} n + 1 \\ r \end{array} \right]_q [k-r]^n_q,$$

for $k = 0, 1, \ldots, n$ and $n = 0, 1, \ldots$

1.24 (Continuation). Show that the $q$-Eulerian numbers $A_q(n,k), \quad k = 0, 1, \ldots, n, \quad n = 0, 1, \ldots,$ satisfy the triangular recurrence relation

$$A_q(n+1,k) = [k]_q A_q(n,k) + [n-k+2]_q A_q(n,k-1),$$

for $k = 1, 2, \ldots, n+1, n = 0, 1, \ldots,$ with initial conditions

$$A_q(0,0) = 1, \quad A_q(n,0) = 0, n > 0, \quad A_q(n,k) = 0, k > n.$$

1.25 Generalized Stirling numbers of the first kind. Consider the expansion

$$\prod_{i=1}^{n} (t - a_i) = \sum_{k=0}^{n} s(n,k;a)t^k,$$

or equivalently, the expansion

$$\prod_{i=1}^{n} (t + a_i) = \sum_{k=0}^{n} \left| s(n,k;a) \right| t^k,$$

where $|s(n,k;a)| = (-1)^{n-k}s(n,k;a)$, with $a = (a_1, a_2, \ldots, a_n)$. The coefficient $s(n,k;a)$ is called generalized Stirling number of the first kind and the coefficient $|s(n,k;a)|$, which for $a_i \geq 0, i = 1, 2, \ldots, n$, is nonnegative, is called generalized signless Stirling number of the first kind.

(a) Show that

$$|s(n,k;a)| = \sum a_{i_1}a_{i_2}\cdots a_{i_{n-k}},$$

where the summation is extended over all $(n-k)$-combinations, $\{i_1, i_2, \ldots, i_{n-k}\}$, of the $n$ indices $\{1, 2, \ldots, n\}$. Note that $|s(n,n-k;a)|$
is the elementary symmetric function (with respect to the \( n \) variables \( a_1, a_2, \ldots, a_n \)). Alternatively,

\[
|s(n, k; a)| = \left( \prod_{i=1}^{n} a_i \right) \sum_{i=1}^{n} \frac{1}{a_1 a_2 \cdots a_i},
\]

where the summation is extended over all \( k \)-combinations, \( \{j_1, j_2, \ldots, j_k\} \), of the \( n \) indices \( \{1, 2, \ldots, n\} \).

(b) Derive the triangular recurrence relation

\[
|s(n, k; a)| = |s(n - 1, k - 1; a)| + a_n |s(n - 1, k; a)|,
\]

for \( k = 1, 2, \ldots, n \) and \( n = 1, 2, \ldots \), with initial conditions

\[
|s(0, 0; a)| = 1, \quad |s(n, 0; a)| = a_n a_{n-1} \cdots a_1, n > 0, \quad |s(n, k; a)| = 0, k > n.
\]

(c) Show that

\[
|s(n, k; a)| = \theta^{-(n-k)} q^{-\left(\frac{n}{2}\right) + \left(\frac{k}{2}\right)} \left(\frac{n}{q}\right)_{k}, \quad \text{for } a_i = 1/\theta q^{i-1}, \quad i = 1, 2, \ldots \]

and

\[
|s(n, k; a)| = (\theta/q)^{n-k} |s_q(n, k; r)|, \quad \text{for } a_i = \theta [r + i - 1], \quad i = 1, 2, \ldots
\]

1.26 Generalized Stirling numbers of the second kind. Consider the expansion

\[
i^n = \sum_{k=0}^{n} S(n, k; a) \prod_{i=1}^{k} (t - a_i).
\]

The coefficient \( S(n, k; a) \) is called generalized Stirling number of the second kind.

(a) Derive the triangular recurrence relation

\[
S(n, k; a) = S(n - 1, k - 1; a) + a_{k+1} S(n - 1, k; a),
\]

for \( k = 1, 2, \ldots, n \) and \( n = 1, 2, \ldots \), with initial conditions

\[
S(0, 0; a) = 1, \quad S(n, 0; a) = a_n^n, n > 0, \quad S(n, k; a) = 0, k > n.
\]

(b) Show that

\[
\prod_{i=1}^{k} (1 - a_i \mu)^{-1} = \sum_{n=k}^{\infty} S(n - 1, k - 1; a) u^{n-k}
\]

and conclude that

\[
S(n + k - 1, n - 1; a) = \sum_{i=1}^{k} r_1 a_i r_2 a_i \cdots a_i r_n a_i,
\]

where the summation is extended over all \( r_i = 0, 1, \ldots, k \), \( i = 1, 2, \ldots, n \), such that \( r_1 + r_2 + \cdots + r_n = k \). Note that \( S(n + k - 1, n - 1; a) \) is the homogeneous product sum symmetric function.
(c) Also, show that
\[ S(n, k; a) = \theta^{n-k} \left[ \begin{array}{c} n \\ k \end{array} \right]_q, \quad \text{for} \ a_i = \theta q^{i-1}, \quad i = 1, 2, \ldots \]
and
\[ S(n, k; a) = \theta^{n-k} S_q(n, k; r), \quad \text{for} \ a_i = \theta [r + i - 1]_q, \quad i = 1, 2, \ldots \]

1.27 (Continuation).
(a) Show that the generalized Stirling numbers of the first and second kind satisfy the following orthogonality relations
\[ \sum_{j=k}^{n} s(n, j; a) s(j, k; a) = \delta_{n,k}, \quad \sum_{j=k}^{n} S(n, j; a) S(j, k; a) = \delta_{n,k}, \]
where \( \delta_{n,k} = 1, \) if \( k = n \) and \( \delta_{n,k} = 0, \) if \( k \neq n, \) is the Kronecker delta.
(b) Also, show that
\[ 1 \prod_{k=1}^{n} \left( t - a_i \right) = \sum_{n=k}^{\infty} S(n-1, k-1; a) \frac{1}{t^n} \]
and
\[ \frac{1}{t^k} = \sum_{n=k}^{\infty} \left| s(n-1, k-1; a) \left| \sum_{n=k}^{\infty} \frac{1}{\prod_{i=1}^{n} (t + a_i)} \right. \right. \]

1.28 Generalized Lah numbers. Consider the expansion
\[ \prod_{i=1}^{n} (t - a_i) = \sum_{k=0}^{n} C(n, k; a, b) \prod_{j=1}^{k} (t - b_j), \]
with \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_k). \) The coefficient \( C(n, k; a, b) \) is called generalized Lah number.
(a) Show that
\[ C(n, k; a, b) = \sum_{m=k}^{n} s(n, m; a) S(m, k; b) \]
and
\[ C(n, k; a, b) = [a]^n_q [b]^{-k} C_q \left( n, k; s, r \right), \quad s = b/a, \quad r = c/a, \]
for \( a_i = [a(i - 1) - c]_q, \ b_i = [b(i - 1)]_q, \ i = 1, 2, \ldots, \) where \( C_q \left( n, k; s, r \right) \) is the noncentral generalized \( q \)-factorial coefficient.
(b) Derive the triangular recurrence relation
\[ C(n, k; a, b) = C(n-1, k-1; a, b) + (b_{k+1} - a_n) C(n-1, k; a, b), \]
for \( k = 1, 2, \ldots, n, \ n = 1, 2, \ldots, \) with initial conditions
\[
C(0, 0; \mathbf{a}, \mathbf{b}) = 1, \ C(n, 0; \mathbf{a}, \mathbf{b}) = \prod_{i=1}^{n} (b_1 - a_i), \ n > 0, \ C(n, k; \mathbf{a}, \mathbf{b}) = 0, \ k > n.
\]

(c) Also, show that
\[
\frac{1}{\prod_{j=1}^{k+1} (t - b_j)} = \sum_{n=k}^{\infty} C(n, k; \mathbf{a}, \mathbf{b}) \frac{1}{\prod_{i=1}^{n+1} (t - a_i)}.
\]

1.29 $q$-Derivative operator and $q$-exponential functions. The $q$-derivative operator, denoted by $D_q = \frac{d_q}{d_q t}$, is defined by
\[
D_q f(t) = \frac{d_q f(t)}{d_q t} = \frac{f(t) - f(qt)}{(1-q)t},
\]
so that $D_q 1 = 0$. The higher-order $q$-derivatives are defined recursively by
\[
D_q^k f(t) = D_q(D_q^{k-1} f(t)), \quad k = 2, 3, \ldots.
\]

(a) Show that
\[
D_q t^m = [m]_q t^{m-1}, \quad D_q^{-1} t^m = [m]_q t^{m-1} = q^{-(m-1)}[m]_q t^{m-1}, \quad m \neq 0
\]
and
\[
D_q^k t^m = [m]_{k_q} t^{m-k}, \quad D_q^{k-1} t^m = q^{-mk + \left(\frac{k+1}{2}\right)}[m]_{k_q} t^{m-k}, \quad m \neq 0.
\]

(b) Also, show that the $q$-exponential functions
\[
e_q(t) = \prod_{i=1}^{\infty} (1 - t(1-q)q^{i-1})^{-1} = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q}, \quad |t| < 1/(1-q)
\]
and
\[
E_q(t) = \prod_{i=1}^{\infty} (1 + t(1-q)q^{i-1}) = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \frac{t^k}{[k]_q}, \quad -\infty < t < \infty.
\]
satisfy the $q$-differential equations
\[
D_q e_q(t) = e_q(t), \quad D_q^{-1} E_q(t) = E_q(t), \quad D_q E_q(t) = E_q(qt).
\]

1.30 (Continuation).

(a) Show that the noncentral $q$-Stirling numbers of the first kind may be written as
\[
s_q(n, k; r) = q^{\binom{n}{2} + nr} \left[ \frac{1}{[k]_q} D_q^k [t - r]_{n,q}\right]_{t=0}, \quad k = 0, 1, \ldots, n, \quad n = 0, 1, \ldots
\]
(b) Also, derive the following \(q\)-Leibnitz formula:

\[
D_q^n(f(t)g(t)) = \sum_{k=0}^{\infty} \left[ n \atop k \right] \frac{D_q^{n-k} f(q^k t)}{D_q^k g(t)}.
\]

**1.31 q-Integral and q-logarithmic function.** The \(q\)-integral is defined by

\[
\int_0^x f(t)d_q t = x(1 - q) \sum_{k=0}^{\infty} f(xq^k)q^k,
\]

provided that the series converges, and

\[
\int_a^b f(t)d_q t = \int_0^b f(t)d_q t - \int_0^a f(t)d_q t.
\]

Note that, for a function \(f(t)\) that is continuous on \([a, b]\), it holds

\[
\int_a^b D_q f(t)d_q t = f(b) - f(a).
\]

Show that

\[
\int_0^x t^n d_q t = \frac{x^{n+1}}{[n+1]_q}, \quad n \neq -1
\]

and

\[
\int_1^x \frac{d_q t}{t} = l_q(x),
\]

with \(l_q(x)\) the \(q\)-logarithmic function, for which

\[
-l_q(1 - x) = \sum_{k=1}^{\infty} \frac{x^k}{[k]_q}, \quad |x| < 1.
\]

**1.32 A q-gamma function.** Consider the \(q\)-integral

\[
I_{n,q} = \int_0^\infty t^n E_q(-qt)d_q t, \quad n = 0, 1, \ldots, \quad 0 < q < 1,
\]

where \(E_q(t) = \prod_{i=1}^{\infty}(1 + t(1 - q)q^{i-1})\), \(-\infty < t < \infty\) and \(|q| < 1\), is a \(q\)-exponential function.

(a) Applying a \(q\)-integration by parts,

\[
\int_a^b g(t)d_q f(t) = [f(t)g(t)]_a^b - \int_a^b f(qt)d_q g(t),
\]

derive the first-order recurrence relation

\[
I_{n,q} = [n]_q I_{n-1,q}, \quad n = 1, 2, \ldots, \quad 0 < q < 1,
\]

with initial condition \(I_{0,q} = 1\), and conclude that

\[
I_{n,q} = \int_0^\infty t^n E_q(-qt)d_q t = [n]_q !, \quad n = 0, 1, \ldots, \quad 0 < q < 1.
\]
The expression of \( q \)-factorial of \( n \), for \( n \) a positive integer and \( 0 < q < 1 \),
\[
[n]_q! = \frac{\prod_{i=1}^{n}(1-q^i)}{(1-q)^n} \frac{\prod_{i=n+1}^{\infty}(1-q^i)}{\prod_{i=1}^{\infty}(1-q^{n+i})},
\]
may be extended to a real number \( x \) as
\[
\Gamma_q(x) = \frac{\prod_{i=1}^{\infty}(1-q^i)}{(1-q)^x} \frac{\prod_{i=1}^{\infty}(1-q^{x+i-1})}{\prod_{i=1}^{\infty}(1-q^{x+i})}, \quad |q| < 1.
\]
The \( q \)-function \( \Gamma_q(x) \) is called \( q \)-gamma function. Show that
\[
\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1
\]
and \( \lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x) \), where \( \Gamma(x) \) denotes the usual gamma function.

**1.33 Another \( q \)-gamma function.** Consider the \( q \)-integral
\[
J_{n,q} = \int_{0}^{\infty} t^n e_q(-t)dt, \quad n = 0, 1, \ldots, \quad 0 < q < 1,
\]
where \( e_q(t) = \prod_{n=1}^{\infty} (1-t(1-q)q^{-1})^{-1} \), \( |t| < 1/(1-q) \) and \( |q| < 1 \) is a \( q \)-exponential function.

(a) Applying a \( q \)-integration by parts, derive the first-order recurrence relation
\[
J_{n,q} = q^{-n}[n]_q! J_{n-1,q}, \quad n = 1, 2, \ldots, \quad 0 < q < 1,
\]
with initial condition \( J_{0,q} = 1 \), and conclude that
\[
J_{n,q} = q^{-\frac{n(n+1)}{2}} [n]_q!\]

(b) The relation connecting the \( q \)-factorials of \( n \), with bases (parameters) inverse to each other, \([n]_{q^{-1}}! = q^{-\frac{n}{2}} [n]_q!\), suggests the definition of a second \( q \)-gamma function as
\[
\gamma_q(x) = \frac{q^{-\frac{(x-1)}{2}}} {(1-q)^{x-1}} \frac{\prod_{i=1}^{\infty}(1-q^i)}{\prod_{i=1}^{\infty}(1-q^{x+i-1})}, \quad |q| < 1.
\]
It should be noted that the definition of the first \( q \)-gamma function \( \Gamma_q(x) \), which was given in Exercise 1.32 for \( |q| < 1 \), is extended for \( |q| > 1 \) through the second \( q \)-gamma function \( \gamma_q(x) \), by the relation
\[
\Gamma_q(x) \equiv \gamma_{q^{-1}}(x) = \frac{q^{\frac{x}{2}}} {(q-1)^{x-1}} \frac{\prod_{i=1}^{\infty}(1-q^{-i})}{\prod_{i=1}^{\infty}(1-q^{-(x+i-1)})}, \quad |q| > 1.
\]
Show that \( \lim_{q \to 1^+} \Gamma_q(x) = \Gamma(x) \), where \( \Gamma(x) \) denotes the gamma function.
1.34 A q-Beta function. Consider the q-integral

\[ B_q(x, y) = \int_0^1 \prod_{i=1}^\infty \frac{1 - t q^i}{1 - t q^{i-1}} t^{x-1} d_q t = \int_0^1 t^{x-1} \prod_{i=1}^\infty \frac{1 - t q^i}{1 - t q^{i-1}} d_q t, \]

for \( x > 0, \ y > 0 \) and \( 0 < q < 1 \).

(a) Show that

\[ B_q(x, y) = \frac{\Gamma_q(x) \Gamma_q(y)}{\Gamma_q(x+y)}, \]

where

\[ \Gamma_q(x) = \prod_{i=1}^\infty \frac{(1 - q^i)}{(1 - q)^{x-1} \prod_{i=1}^\infty (1 - q^{x+i-1})}, \quad |q| < 1. \]

is a q-gamma function, and conclude that

\[ \lim_{q \to 1} B_q(x, y) = B(x, y), \]

where \( B(x, y) \) is the beta function. The q-function \( B_q(x, y) \) is called q-beta function.

(b) In particular, for \( x = r + 1 \) and \( y = n - r \) positive integers, deduce that

\[ B_q(r+1, n-r) = \int_0^1 \prod_{i=1}^{n-r-1} (1 - t q^i) t^{r-1} d_q t = \frac{[r]_q ![n-r-1]_q!}{[n]_q!}. \]

1.35 The q-operator \( \Theta_q = t D_q \). The operator \( \Theta_q = t D_q \) is the q-analogue of the well-known operator \( \Theta = t D \), to which it reduces for \( q = 1 \).

(a) Show that

\[ \Theta_q = [\Theta]_q = \frac{1 - q^\Theta}{1 - q}. \]

(b) Express the operator \( \Theta_q \) in terms of the operator \( D_q \) as

\[ \Theta_q^n = \sum_{k=0}^n q^{\binom{n}{2}} S_q(n, k) t^k D_q^k, \]

and, inversely, express the operator \( D_q \) in terms of the operator \( \Theta_q \) as

\[ D_q^n = q^{-\binom{n}{2}} t^{-n} \sum_{k=0}^n s_q(n, k) \Theta_q^k, \]

where \( s_q(n, k) \) and \( S_q(n, k) \) are the q-Stirling numbers of the first and second kind, respectively.
1.36 *q*-Difference operator. The *q*-difference operator, denoted by \( \Delta_q \), is defined by

\[ \Delta_q f(t) = f(t + 1) - f(t). \]

The higher-order *q*-differences are defined recursively by

\[ \Delta_q^k f(t) = \Delta_q^{k-1} f(t + 1) - q^{k-1} \Delta_q^{k-1} f(t), \quad k = 2, 3, \ldots \]

Clearly, the *k*th-order *q*-difference operator is expressed in terms of the usual shift operator \( E \) by

\[ \Delta_q^k = \prod_{i=1}^{k} (E - q^{i-1}), \quad k = 1, 2, \ldots . \]

(a) Show that

\[ \Delta_q^k [t]_{m,q} = [m]_q [t]_{m-1,q} q^{t-m+1}, \quad m \neq 0 \]

and

\[ \Delta_q^k [t]_{m,q} = [m]_{k,q} [t]_{m-k,q} q^{k(t-m+k)}, \quad m \neq 0. \]

(b) Also, show that the noncentral *q*-Stirling numbers of the second kind and the noncentral generalized *q*-factorial coefficients may be written as

\[ S_q(n, k; r) = q^{-\binom{k}{2}} r^n \left[ \frac{1}{[k]_q}, \Delta_q^k [t+r]_{n,q} \right]_{t=0}, \]

for \( k = 0, 1, \ldots, n, n = 0, 1, \ldots, \) and

\[ C_q(n, k; s, r) = q^{\binom{n}{2}} s^{\binom{k}{2}} r^n \left[ \frac{1}{[k]_q}, \Delta_q^k [s+t+r]_{n,q} \right]_{t=0}, \]

for \( k = 0, 1, \ldots, n, n = 0, 1, \ldots, \) respectively.

1.37 *q*-Factorial moments as *q*-derivatives of probability generating functions. Let \( X \) be a nonnegative integer-valued random variable with probability generating function

\[ P_X(t) = \sum_{x=0}^{\infty} f(x) t^x, \quad |t| \leq 1, \]

where \( f(x) = P(X = x), x = 0, 1, \ldots, \) is the probability function. Show that (a)

\[ \left[ \frac{d^m P_X(t)}{dt^m} \right]_{t=1} = E[(X)_m], \quad m = 1, 2, \ldots, \]

where \( E[(X)_m] = \sum_{x=m}^{\infty} (x)_m f(x) \) is the *m*th factorial moment and (b)

\[ \left[ \frac{d_q^m P_X(t)}{d_q t^m} \right]_{t=1} = E([(X]_{m,q}), \quad m = 1, 2, \ldots, \]

where \( E([(X]_{m,q}) = \sum_{x=m}^{\infty} [x]_{m,q} f(x) \) is the *m*th *q*-factorial moment.
1.38 \textit{q-Factorial moments of a discrete q-uniform distribution.} Let $X_n$ be a discrete $q$-uniform random variable, with probability function

$$f_{X_n}(x) = P(X_n = x) = \frac{q^x}{[n]_q}, \quad x = 0, 1, \ldots, n - 1.$$  

(a) Find the $m$th $q$-factorial moment $E([X_n]_{m,q})$, $m = 1, 2, \ldots$, and deduce the $q$-expected value $E([X_n]_q)$ and the $q$-variance $V([X_n]_q)$.

(b) Derive the probability generating function $P_{X_n}(t) = \sum_{x=0}^{n-1} f_{X_n}(x)t^x$. 