In this chapter we gather together for reference and review those parts of elementary mathematics that are necessary for the study of calculus. We assume that you are familiar with most of this material and that you don’t require detailed explanations. But first a few words about the nature of calculus and a brief outline of the history of the subject.

1.1 WHAT IS CALCULUS?

To a Roman in the days of the empire, a "calculus" was a pebble used in counting and gambling. Centuries later, "calculare" came to mean "to calculate," "to compute," "to figure out." For our purposes, calculus is elementary mathematics (algebra, geometry, trigonometry) enhanced by the limit process.

Calculus takes ideas from elementary mathematics and extends them to a more general situation. Some examples are on pages 2 and 3. On the left-hand side you will find an idea from elementary mathematics; on the right, this same idea as extended by calculus.

It is fitting to say something about the history of calculus. The origins can be traced back to ancient Greece. The ancient Greeks raised many questions (often paradoxical) about tangents, motion, area, the infinitely small, the infinitely large—questions that today are clarified and answered by calculus. Here and there the Greeks themselves provided answers (some very elegant), but mostly they provided only questions.

<table>
<thead>
<tr>
<th>Elementary Mathematics</th>
<th>Calculus</th>
</tr>
</thead>
<tbody>
<tr>
<td>slope of a line $y = mx + b$</td>
<td>slope of a curve $y = f(x)$</td>
</tr>
</tbody>
</table>

(Table continues)
CHAPTER 1 PRECALCULUS REVIEW

- Tangent line to a circle
- Tangent line to a more general curve
- Area of a region bounded by line segments
- Area of a region bounded by curves
- Length of a line segment
- Length of a curve
- Volume of a rectangular solid
- Volume of a solid with a curved boundary
- Motion along a straight line with constant velocity
- Motion along a curved path with varying velocity
- Work done by a constant force
- Work done by a varying force
After the Greeks, progress was slow. Communication was limited, and each scholar
was obliged to start almost from scratch. Over the centuries, some ingenious solutions
to calculus-type problems were devised, but no general techniques were put forth.
Progress was impeded by the lack of a convenient notation. Algebra, founded in the
nineteenth century by Arab scholars, was not fully systematized until the sixteenth century.
Then, in the seventeenth century, Descartes established analytic geometry, and the stage
was set.

The actual invention of calculus is credited to Sir Isaac Newton (1642–1727),
an Englishman, and to Gottfried Wilhelm Leibniz (1646–1716), a German. Newton's
invention is one of the few good turns that the great plague did mankind. The plague
forced the closing of Cambridge University in 1665, and young Isaac Newton of Trinity
College returned to his home in Lincolnshire for eighteen months of meditation, out
of which grew his method of fluxions, his theory of gravitation, and his theory of light.
The method of fluxions is what concerns us here. A treatise with this title was written
by Newton in 1672, but it remained unpublished until 1736, nine years after his death.
The new method (calculus to us) was first announced in 1687, but in vague general
terms without symbolism, formulas, or applications. Newton himself seemed reluctant
to publish anything tangible about his new method, and it is not surprising that its
development on the Continent, in spite of a late start, soon overtook Newton and went
beyond him.

Leibniz started his work in 1673, eight years after Newton. In 1675 he initiated the
basic modern notation: \(dx\) and \(\int\). His first publications appeared in 1684 and 1686. These
made little stir in Germany, but the two brothers Bernoulli of Basel (Switzerland) took
up the ideas and added profusely to them. From 1690 onward, calculus grew rapidly and
reached roughly its present state in about a hundred years. Certain theoretical subtleties
were not fully resolved until the twentieth century.

### 1.2 REVIEW OF ELEMENTARY MATHEMATICS

In this section we review the terminology, notation, and formulas of elementary math-
ematics.

#### Sets

A set is a collection of distinct objects. The objects in a set are called the elements or
members of the set. We will denote sets by capital letters \(A, B, C, \ldots\) and use lowercase
letters \(a, b, c, \ldots\) to denote the elements.
CHAPTER 1  PRECALCULUS REVIEW

For a collection of objects to be a set it must be well-defined; that is, given any object \( x \), it must be possible to determine with certainty whether or not \( x \) is an element of the set. Thus the collection of all even numbers, the collection of all lines parallel to a given line \( l \), the solutions of the equation \( x^2 = 9 \) are all sets. The collection of all intelligent adults is not a set. It’s not clear who should be included.

Notions and Notation
- the object \( x \) is in the set \( A \) \( x \in A \)
- the object \( x \) is not in the set \( A \) \( x \notin A \)
- the set of all \( x \) which satisfy property \( P \) \( \{ x : P \} \)
- \( A \subseteq B \) A is a subset of \( B \)
- \( B \supseteq A \) \( B \) contains \( A \)
- \( A \cup B \) the union of \( A \) and \( B \)
- \( A \cap B \) the intersection of \( A \) and \( B \)
- \( \emptyset \) the empty set

These are the only notions from set theory that you will need at this point.

Real Numbers

Classification
- positive integers\(^1\) 1, 2, 3, . . .
- integers 0, 1, −1, 2, −2, 3, −3, . . .
- rational numbers \( p/q \), with \( p, q \) integers, \( q \neq 0 \);
  for example, \( 5/2, -19/7, -4/1 = -4 \)
- irrational numbers real numbers that are not rational;
  for example \( \sqrt{2}, \sqrt{7}, \pi \)

Decimal Representation

Each real number can be expressed as a decimal. To express a rational number \( p/q \) as a decimal, we divide the denominator \( q \) into the numerator \( p \). The resulting decimal either terminates or repeats:

\[
\frac{3}{5} = 0.6, \quad \frac{27}{20} = 1.35, \quad \frac{43}{8} = 5.375
\]

are terminating decimals;

\[
\frac{2}{3} = 0.666 \ldots = 0.\overline{6}, \quad \frac{15}{11} = 1.363636 \ldots = 1.\overline{36}, \quad \text{and}
\]

\[
\frac{116}{37} = 3.135135 \ldots = 3.\overline{135}
\]

are repeating decimals. (The bar over the sequence of digits indicates that the sequence repeats indefinitely.) The converse is also true; namely, every terminating or repeating decimal represents a rational number.

\(^1\)Also called natural numbers.
1.2 REVIEW OF ELEMENTARY MATHEMATICS

The decimal expansion of an irrational number can neither terminate nor repeat. The expansions
\[ \sqrt{2} = 1.414213562 \ldots \quad \text{and} \quad \pi = 3.141592653 \ldots \]
do not terminate and do not develop any repeating pattern.

If we stop the decimal expansion of a given number at a certain decimal place, then the result is a rational number that approximates the given number. For instance, 1.414 = 1414/1000 is a rational number approximation to \( \sqrt{2} \) and 3.14 = 314/100 is a rational number approximation to \( \pi \). More accurate approximations can be obtained by using more decimal places from the expansions.

The Number Line (Coordinate Line, Real Line)

On a horizontal line we choose a point \( O \). We call this point the origin and assign to it coordinate 0. Now we choose a point \( U \) to the right of \( O \) and assign to it coordinate 1. See Figure 1.2.1. The distance between \( O \) and \( U \) determines a scale (a unit length). We go on as follows: the point \( a \) units to the right of \( O \) is assigned coordinate \( a \); the point \( a \) units to the left of \( O \) is assigned coordinate \( -a \).

In this manner we establish a one-to-one correspondence between the points of a line and the numbers of the real number system. Figure 1.2.2 shows some real numbers represented as points on the number line. Positive numbers appear to the right of 0, negative numbers to the left of 0.

Order Properties

(i) Either \( a < b \), \( b < a \), or \( a = b \). (trichotomy)
(ii) If \( a < b \) and \( b < c \), then \( a < c \).
(iii) If \( a < b \), then \( a + c < b + c \) for all real numbers \( c \).
(iv) If \( a < b \) and \( c > 0 \), then \( ac < bc \).
(v) If \( a < b \) and \( c < 0 \), then \( ac > bc \).

(Techniques for solving inequalities are reviewed in Section 1.3.)

Density

Between any two real numbers there are infinitely many rational numbers and infinitely many irrational numbers. In particular, there is no smallest positive real number.

Absolute Value

\[ |a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases} \]

other characterizations \[ |a| = \max(a, -a); |a| = \sqrt{a^2}. \]
geometric interpretation \[ |a| = \text{distance between } a \text{ and } 0; \]
\[ |a - c| = \text{distance between } a \text{ and } c. \]
CHAPTER 1 PRECALCULUS REVIEW

properties
(i) \(|a| = 0 \text{ iff } a = 0.\)†
(ii) \(|-a| = |a|.
(iii) \(|ab| = |a||b|.
(iv) \(|a + b| \leq |a| + |b|.
(v) \(|a| - |b| \leq |a - b|.
(vi) |a|^2 = |a|^2.

Techniques for solving inequalities that feature absolute value are reviewed in Section 1.3.

Intervals
Suppose that \(a < b\). The open interval \((a, b)\) is the set of all numbers between \(a\) and \(b\):
\[(a, b) = \{x : a < x < b\}.\]
The closed interval \([a, b]\) is the open interval \((a, b)\) together with the endpoints \(a\) and \(b\):
\[\[a, b\] = \{x : a \leq x \leq b\}.\]
There are seven other types of intervals:
\[\(a, b\] = \{x : a < x \leq b\},\]
\[[a, b) = \{x : a \leq x < b\},\]
\[(a, \infty) = \{x : a < x\},\]
\[[a, \infty) = \{x : a \leq x\},\]
\[(-\infty, b) = \{x : x < b\},\]
\[(-\infty, b] = \{x : x \leq b\},\]
\[(-\infty, \infty) = \text{the set of real numbers}.\]

Interval notation is easy to remember: we use a square bracket to include an end-point and a parenthesis to exclude it. On a number line, inclusion is indicated by a solid dot, exclusion by an open dot. The symbols \(\infty\) and \(-\infty\), read “infinity” and “negative infinity” (or “minus infinity”), do not represent real numbers. In the intervals listed above, the symbol \(\infty\) is used to indicate that the interval extends indefinitely in the positive direction; the symbol \(-\infty\) is used to indicate that the interval extends indefinitely in the negative direction.

Open and Closed
Any interval that contains no endpoints is called open: \((a, b), (a, \infty), (-\infty, b), (-\infty, \infty)\) are open. Any interval that contains each of its endpoints (there may be one or two) is called closed: \([a, b], [a, \infty), (-\infty, b]\) are closed. The intervals \((a, b)\) and \([a, b]\) are called half-open (half-closed): \((a, b)\) is open on the left and closed on the right; \([a, b]\) is closed on the left and open on the right. Points of an interval that are not endpoints are called interior points of the interval.

† By “iff” we mean “if and only if.” This expression is used so often in mathematics that it’s convenient to have an abbreviation for it.
†† The absolute value of the sum of two numbers cannot exceed the sum of their absolute values. This is analogous to the fact that in a triangle the length of one side cannot exceed the sum of the lengths of the other two sides.
Boundedness

A set $S$ of real numbers is said to be:

(i) Bounded above if there exists a real number $M$ such that
$$x \leq M \quad \text{for all } x \in S;$$
such a number $M$ is called an upper bound for $S$.

(ii) Bounded below if there exists a real number $m$ such that
$$m \leq x \quad \text{for all } x \in S;$$
such a number $m$ is called a lower bound for $S$.

(iii) Bounded if it is bounded above and below.†

Note that if $M$ is an upper bound for $S$, then any number greater than $M$ is also an upper bound for $S$, and if $m$ is a lower bound for $S$, then any number less than $m$ is also a lower bound for $S$.

Examples. The intervals $(-\infty, 2]$ and $(-\infty, 2)$ are both bounded above by 2 (and by every number greater than 2), but these sets are not bounded below. The set of positive integers $\{1, 2, 3, \ldots\}$ is bounded below by 1 (and by every number less than 1), but the set is not bounded above; there being no number $M$ greater than or equal to all positive integers, the set has no upper bound. All finite sets of numbers are bounded—(bounded below by the least element and bounded above by the greatest). Finally, the set of all integers, $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, is unbounded in both directions; it is unbounded above and unbounded below. ❑

Factorials

Let $n$ be a positive integer. By $n$ factorial, denoted $n!$, we mean the product of the integers from $n$ down to 1:

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1.$$  

In particular

$$1! = 1, \quad 2! = 2 \cdot 1 = 2, \quad 3! = 3 \cdot 2 \cdot 1 = 6, \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24,$$
and so on.

For convenience we define $0! = 1$.

Algebra

Powers and Roots

a real, $p$ a positive integer

$$a^1 = a, \quad a^p = a \cdot a \cdots a,$$
a $\neq 0$ : $a^0 = 1, \quad a^{-p} = 1/a^p$

laws of exponents

$$a^{p+q} = a^p a^q, \quad a^{p-q} = a^p/a^q, \quad (a^p)^q = a^{pq}$$

a real, $q$ odd

$$a^{1/q}$$ called the $q$th root of $a$, is the number $b$ such that $b^q = a$

a nonnegative, $q$ even

$$a^{1/q}$$ is the nonnegative number $b$ such that $b^q = a$

notation

$a^{1/q}$ can be written $\sqrt[q]{a}$ ($a^{1/2}$ is written $\sqrt{a}$)

rational exponents

$$a^{p/q} = (a^{1/q})^p$$

†In defining bounded above, bounded below, and bounded we used the conditional “if,” not “iff.” We could have used “iff,” but that would have been unnecessary. Definitions are by their very nature “if” statements.
Examples
\[2^0 = 1,\quad 2^1 = 1,\quad 2^2 = 2 \cdot 2 = 4,\quad 2^3 = 2 \cdot 2 \cdot 2 = 8, \text{ and so on}\]
\[2^{3+5} = 2^3 \cdot 2^5 = 32 \cdot 32 = 1,024,\quad 2^{3-5} = 2^3 \div 2^5 = 1/32 = 1/4\]
\[(2^3)^2 = 2^6 = 64,\quad (2^3)^3 = 2^9 = 512\]
\[8^{1/3} = 2,\quad (-8)^{1/3} = -2,\quad 16^{1/2} = \sqrt{16} = 4,\quad 16^{1/4} = 2\]
\[8^{5/3} = (8^{1/3})^5 = 2^5 = 32,\quad 8^{-5/3} = (8^{1/3})^{-5} = 2^{-5} = 1/32\]

Basic Formulas
\[(a + b)^2 = a^2 + 2ab + b^2\]
\[(a - b)^2 = a^2 - 2ab + b^2\]
\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]
\[(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\]
\[a^2 - b^2 = (a - b)(a + b)\]
\[a^3 - b^3 = (a - b)(a^2 + ab + b^2)\]
\[a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)\]

More generally:
\[a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})\]

Quadratic Equations
The roots of a quadratic equation
\[ax^2 + bx + c = 0\quad \text{with} \quad a \neq 0\]
are given by the general quadratic formula
\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]
If \(b^2 - 4ac > 0\), the equation has two real roots; if \(b^2 - 4ac = 0\), the equation has one real root; if \(b^2 - 4ac < 0\), the equation has no real roots, but it has two complex roots.

Geometry
Elementary Figures

<table>
<thead>
<tr>
<th>Triangle</th>
<th>Equilateral Triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Triangle Image" /></td>
<td><img src="image2" alt="Equilateral Triangle Image" /></td>
</tr>
<tr>
<td>area = (\frac{1}{2}bh)</td>
<td>area = (\frac{1}{2}\sqrt{3} s^2)</td>
</tr>
</tbody>
</table>
### 1.2 REVIEW OF ELEMENTARY MATHEMATICS

<table>
<thead>
<tr>
<th><strong>Rectangle</strong></th>
<th><strong>Rectangular Solid</strong></th>
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<tbody>
<tr>
<td><img src="Rectangle.png" alt="Rectangle" /></td>
<td><img src="RectangularSolid.png" alt="Rectangular Solid" /></td>
</tr>
<tr>
<td>area = $lw$</td>
<td>volume = $lwh$</td>
</tr>
<tr>
<td>perimeter = $2l + 2w$</td>
<td>surface area = $2lw + 2lh + 2wh$</td>
</tr>
<tr>
<td>diagonal = $\sqrt{l^2 + w^2}$</td>
<td>diagonal = $\sqrt{l^2 + w^2}$</td>
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<thead>
<tr>
<th><strong>Square</strong></th>
<th><strong>Cube</strong></th>
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<tbody>
<tr>
<td><img src="Square.png" alt="Square" /></td>
<td><img src="Cube.png" alt="Cube" /></td>
</tr>
<tr>
<td>area = $x^2$</td>
<td>volume = $x^3$</td>
</tr>
<tr>
<td>perimeter = $4x$</td>
<td>surface area = $6x^2$</td>
</tr>
<tr>
<td>diagonal = $x\sqrt{2}$</td>
<td>diagonal = $x\sqrt{2}$</td>
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</tbody>
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<table>
<thead>
<tr>
<th><strong>Circle</strong></th>
<th><strong>Sphere</strong></th>
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<tbody>
<tr>
<td><img src="Circle.png" alt="Circle" /></td>
<td><img src="Sphere.png" alt="Sphere" /></td>
</tr>
<tr>
<td>area = $\pi r^2$</td>
<td>volume = $\frac{4}{3}\pi r^3$</td>
</tr>
<tr>
<td>circumference = $2\pi r$</td>
<td>surface area = $4\pi r^2$</td>
</tr>
</tbody>
</table>

Sector of a Circle: radius $r$, central angle $\theta$ measured in radians (see Section 1.6).

| ![Sector of a Circle](Sector.png) | ![Sector of a Circle](Sector.png) |
| arc length = $r\theta$ | area = $\frac{1}{2}r^2\theta$ |

(Table continues)
Find the real roots of the equation.

\[ x = \frac{\pi}{6}, 2 \pi \]

Indicate on a number line the numbers

\[ n \in \mathbb{Z} \]

Sketch the set on a number line.

\[ n \cap [0, 1) \]

Exercises 41–47. State whether the set is bounded above, bounded below, bounded. If a set is bounded above, give an upper bound; if it is bounded below, give a lower bound; if it is bounded, give an upper bound and a lower bound.

Exercises 48–50.

48. Order the following numbers and place them on a number line: \( \sqrt{7}, \sqrt{8}, \sqrt{9}, \sqrt{10} \).

49. Let \( x_0 = 2 \) and define \( x_n = 17 + 2x_{n-1} \) for \( n = 1, 2, 3, \ldots \). Find at least five values for \( x_n \). Is the set \( S = \{x_0, x_1, x_2, \ldots, x_n, \ldots \} \) bounded above, bounded below, bounded? If so, give a lower bound and/or an upper bound for \( S \). If \( n \) is a large positive integer, what is the approximate value of \( x_n \)?

50. Reword Exercise 49 with \( x_0 = 3 \) and \( x_n = 231 + 4x_{n-1} \).

Exercises 51–56. Write the expression in factored form.

51. \( x^2 - 10x + 25 \)

52. \( 9x^2 - 4 \)

53. \( 8x^4 + 64 \)

54. \( 27x^3 - 8 \)

55. \( 4x^2 + 12x + 9 \)

56. \( 4x^4 + 4x^2 + 1 \)

Exercises 57–64. Find the real roots of the equation.

57. \( x^2 - x - 2 = 0 \)

58. \( x^2 - 9 = 0 \)

59. \( x^2 - 6x + 9 = 0 \)

60. \( 2x^2 - 5x - 3 = 0 \)

61. \( x^2 - 2x + 2 = 0 \)

62. \( x^2 + 8x + 16 = 0 \)

63. \( x^2 + 4x + 13 = 0 \)

64. \( x^2 - 2x + 5 = 0 \)

Exercises 65–69. Evaluate.

65. \( 5! \)

66. \( 6! \)

67. \( \frac{8!}{2!} \)

68. \( \frac{9!}{3!} \)

69. \( \frac{7!}{2!} \)
1.3 REVIEW OF INEQUALITIES

All our work with inequalities is based on the order properties of the real numbers given in Section 1.2. In this section we work with the type of inequalities that arise frequently in calculus, inequalities that involve a variable.

To solve an inequality in \( x \) is to find the numbers \( x \) that satisfy the inequality. These numbers constitute a set, called the **solution set** of the inequality.

We solve inequalities much as we solve an equation, but there is one important difference. We can maintain an inequality by adding the same number to both sides, or by subtracting the same number from both sides, or by multiplying or dividing both sides by the same positive number. But if we multiply or divide by a negative number, then the inequality is reversed:

\[
\begin{align*}
-3(4 - x) &\leq 12. \\
\text{SOLUTION} & \quad \text{Multiplying both sides of the inequality by } -\frac{1}{3}, \text{ we have} \\
4 - x &\geq -4. \quad \text{(the inequality has been reversed)} \\text{Subtracting 4, we get} \\
-x &\geq -8.
\end{align*}
\]
To isolate $x$, we multiply by $-1$. This gives

$$x \leq 8.$$  

(The inequality has been reversed again)

The solution set is the interval $(-\infty, 8]$. 

There are generally several ways to solve a given inequality. For example, the last inequality could have been solved as follows:

$$-3(4 - x) \leq 12,$$

$$-12 + 3x \leq 12,$$

$$3x \leq 24,$$  

(we added 12)

$$x \leq 8.$$  

(we divided by 3)

To solve a quadratic inequality, we try to factor the quadratic. Failing that, we can complete the square and go on from there. This second method always works.

**Example 2** Solve the inequality

$$x^2 - 4x + 3 > 0.$$

**SOLUTION** Factoring the quadratic, we obtain

$$(x - 1)(x - 3) > 0.$$  

The product $(x - 1)(x - 3)$ is zero at 1 and 3. Mark these points on a number line (Figure 1.3.1). The points 1 and 3 separate three intervals:

$$(-\infty, 1), \ (1, 3), \ (3, \infty).$$

On each of these intervals the product $(x - 1)(x - 3)$ keeps a constant sign:

- on $(-\infty, 1)$ [to the left of 1] sign of $(x - 1)(x - 3) = (-)(-) = +$;
- on $(1, 3)$ [between 1 and 3] sign of $(x - 1)(x - 3) = (+)(-) = -$;
- on $(3, \infty)$ [to the right of 3] sign of $(x - 1)(x - 3) = (+)(+) = +$.

The product $(x - 1)(x - 3)$ is positive on the open intervals $(-\infty, 1)$ and $(3, \infty)$. The solution set is the union $(-\infty, 1) \cup (3, \infty)$. 

**Example 3** Solve the inequality

$$x^2 - 2x + 5 \leq 0.$$

**SOLUTION** Not seeing immediately how to factor the quadratic, we use the method that always works: completing the square. Note that

$$x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x - 1)^2 + 4.$$  

This tells us that

$$x^2 - 2x + 5 \geq 4$$  

for all real $x$, and thus there are no numbers that satisfy the inequality we are trying to solve. To put it in terms of sets, the solution set is the empty set $\emptyset$. 

---

**Figure 1.3.1**
In practice we frequently come to expressions of the form
\[(x - a_1)^{k_1}(x - a_2)^{k_2} \cdots (x - a_n)^{k_n}\]
where $k_1, k_2, \ldots, k_n$ are positive integers, $a_1 < a_2 < \cdots < a_n$. Such an expression is zero at $a_1, a_2, \ldots, a_n$. It is positive on those intervals where the number of negative factors is even and negative on those intervals where the number of negative factors is odd.

Take, for instance,
\[(x + 2)(x - 1)(x - 3)\].
This product is zero at $-2, 1, 3$. It is negative on $(-\infty, -2)$, positive on $(-2, 1)$, negative on $(1, 3)$, and positive on $(3, \infty)$.

Example 4 Solve the inequality
\[(x + 3)^2(x - 1)(x - 4)^2 < 0.\]
\[\text{SOLUTION} \quad \text{We view } (x + 3)^2(x - 1)(x - 4)^2 \text{ as the product of three factors: } (x + 3)^2, (x - 1), (x - 4)^2. \text{ The product is zero at } -3, 1, 4. \text{ These points separate the intervals } (-\infty, -3), (-3, 1), (1, 4), (4, \infty). \text{ On each of these intervals the product keeps a constant sign:}
\begin{align*}
\text{positive on } & (-\infty, -3), \\
\text{negative on } & (-3, 1), \\
\text{positive on } & (1, 4), \\
\text{positive on } & (4, \infty).
\end{align*}
\text{The solution set is the open interval } (-3, 1).\]

1.3 REVIEW OF INEQUALITIES

Inequalities and Absolute Value

Now we take up inequalities that involve absolute values. With an eye toward developing the concept of limits (Chapter 2), we introduce two Greek letters: $\delta$ (delta) and $\epsilon$ (epsilon).
CHAPTER 1  PRECALCULUS REVIEW

As you know, for each real number \( a \)
\[
|a| = \begin{cases} 
   a & \text{if } a \geq 0, \\
   -a & \text{if } a < 0.
\end{cases}
|a| = \max\{a, -a\},  \quad |a| = \sqrt{a^2}.
\]

We begin with the inequality
\[ |x| < \delta \]
where \( \delta \) is some positive number. To say that \( |x| < \delta \) is to say that \( x \) lies within \( \delta \) units of 0 or, equivalently, that \( x \) lies between \( -\delta \) and \( \delta \). Thus
\[
|x| < \delta \iff -\delta < x < \delta.
\]

The solution set is the open interval \((-\delta, \delta)\).

Somewhat more delicate is the inequality
\[ 0 < |x - c| < \delta. \]
Here we have \( |x - c| < \delta \) with the additional requirement that \( x \neq c \). Consequently,
\[
0 < |x - c| < \delta \iff c - \delta < x < c + \delta.
\]

The solution set is the union of two open intervals: \((c - \delta, c + \delta)\).

The following results are an immediate consequence of what we just showed.
\[
|\frac{1}{2} x | < \frac{1}{2} \iff -\frac{1}{2} < x < \frac{1}{2};  \\
|x - 5| < 1 \iff 4 < x < 6;  \\
0 < |x - 5| < 1 \iff 4 < x < 5 \quad \text{or} \quad 5 < x < 6;  \\
|x + 2| < 3.
\]

Example 5  Solve the inequality \( |x + 2| < 3 \).

SOLUTION  Once we recognize that \( |x + 2| = |x - (-2)| \), we are in familiar territory:
\[ |x - (-2)| < 3 \iff -2 - 3 < x < -2 + 3 \iff -5 < x < 1. \]
1.3 REVIEW OF INEQUALITIES

The solution set is the open interval (−5, 1).

Example 6  Solve the inequality $|3x - 4| < 2$.

SOLUTION  Since $|3x - 4| = |3(x - \frac{4}{3})| = |3|x - \frac{4}{3}| = 3|x - \frac{4}{3}|$, the inequality can be written

$3|x - \frac{4}{3}| < 2$.

This gives

$|x - \frac{4}{3}| < \frac{2}{3}, \quad \frac{4}{3} - \frac{2}{3} < x < \frac{4}{3} + \frac{2}{3}, \quad \frac{2}{3} < x < \frac{4}{3}$.

The solution set is the open interval ($\frac{2}{3}, \frac{4}{3}$).

ALTERNATIVE SOLUTION  There is usually more than one way to solve an inequality. In this case, for example, we can write $|3x - 4| < 2$ as

$-2 < 3x - 4 < 2$

and proceed from there. Adding 4 to the inequality, we get

$2 < 3x < 6$.

Division by 3 gives the result we had before:

$\frac{2}{3} < x < 2$.

Let $\epsilon > 0$. If you think of $|a|$ as the distance between $a$ and 0, then

$|a| > \epsilon$ iff $a > \epsilon$ or $a < -\epsilon$.

Example 7  Solve the inequality $|2x + 3| > 5$.

SOLUTION  In general

$|a| > \epsilon$ iff $a > \epsilon$ or $a < -\epsilon$.

So here

$2x + 3 > 5$ or $2x + 3 < -5$.

The first possibility gives $2x > 2$ and thus

$x > 1$.
The second possibility gives $2x < -8$ and thus $x < -4$.

The total solution is therefore the union $(-\infty, -4) \cup (1, \infty)$.

We come now to one of the fundamental inequalities of calculus: for all real numbers $a$ and $b$,

$$|a + b| \leq |a| + |b|.$$  \hspace{1cm} (1.3.6)

This is called the triangle inequality in analogy with the geometric observation that “in any triangle the length of each side is less than or equal to the sum of the lengths of the other two sides.”

**Proof of the Triangle Inequality**  The key here is to think of $|x|$ as $\sqrt{x^2}$.

Note first that

$$(a + b)^2 = a^2 + 2ab + b^2 \leq |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2.$$

Comparing the extremes of the inequality and taking square roots, we have

$$\sqrt{(a + b)^2} \leq |a| + |b|.$$

(Exercise 51)

The result follows from observing that

$$\sqrt{(a + b)^2} = |a + b|.$$  \hspace{1cm} (Exercise 51)

Here is a variant of the triangle inequality that also comes up in calculus: for all real numbers $a$ and $b$,

$$||a| - |b|| \leq |a - b|.$$  \hspace{1cm} (1.3.7)

The proof is left to you as an exercise.

**Exercises 1.3**

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Inequality</th>
<th>Solution Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$2 + 3x &lt; 5$</td>
<td>$x &lt; \frac{3}{2}$</td>
</tr>
<tr>
<td>2.</td>
<td>$\frac{1}{2}(2x + 3) &lt; 6$</td>
<td>$x &lt; \frac{9}{2}$</td>
</tr>
<tr>
<td>3.</td>
<td>$16x + 64 \leq 16$</td>
<td>$x \leq -3$</td>
</tr>
<tr>
<td>4.</td>
<td>$3x + 5 &gt; \frac{1}{3}(x - 2)$</td>
<td>$x &gt; -\frac{13}{8}$</td>
</tr>
<tr>
<td>5.</td>
<td>$\frac{1}{2}(1 + x) &lt; \frac{1}{2}(1 - x)$</td>
<td>$x &lt; 0$</td>
</tr>
<tr>
<td>6.</td>
<td>$3x - 2 \leq 1 + 6x$</td>
<td>$x \geq -\frac{3}{5}$</td>
</tr>
<tr>
<td>7.</td>
<td>$x^2 - 1 &lt; 0$</td>
<td>$-1 &lt; x &lt; 1$</td>
</tr>
<tr>
<td>8.</td>
<td>$x^2 + 9x + 20 &lt; 0$</td>
<td>$-5 &lt; x &lt; -4$</td>
</tr>
<tr>
<td>9.</td>
<td>$x^2 - x - 6 \geq 0$</td>
<td>$x \leq -2$ or $x \geq 3$</td>
</tr>
<tr>
<td>10.</td>
<td>$x^2 - 4x + 4 &gt; 0$</td>
<td>$x \neq 2$</td>
</tr>
<tr>
<td>11.</td>
<td>$2x^2 + x - 1 \leq 0$</td>
<td>$x \leq -\frac{1}{2}$ or $x \geq 1$</td>
</tr>
<tr>
<td>12.</td>
<td>$3x^2 + 4x - 4 \geq 0$</td>
<td>$x \leq -\frac{2}{3}$ or $x \geq 1$</td>
</tr>
<tr>
<td>13.</td>
<td>$x(x - 1)(x - 2) &gt; 0$</td>
<td>$x &lt; 0$ or $1 &lt; x &lt; 2$</td>
</tr>
<tr>
<td>14.</td>
<td>$x(2x - 1)(3x - 5) \leq 0$</td>
<td>$x \leq 0$ or $\frac{1}{2} \leq x \leq \frac{5}{3}$</td>
</tr>
<tr>
<td>15.</td>
<td>$x^3 - 2x^2 + x \geq 0$</td>
<td>$x \leq -1$ or $0 \leq x \leq 2$</td>
</tr>
<tr>
<td>16.</td>
<td>$x^2 - 4x + 4 \leq 0$</td>
<td>$x = 2$</td>
</tr>
<tr>
<td>17.</td>
<td>$x^2(x - 2)(x + 3)^2 &lt; 0$</td>
<td>$x &lt; -3$ or $x = 2$ or $x &gt; 3$</td>
</tr>
<tr>
<td>18.</td>
<td>$x^2(x - 3)(x + 4)^2 &gt; 0$</td>
<td>$x &lt; -4$ or $x = -3$ or $x &gt; 4$</td>
</tr>
<tr>
<td>19.</td>
<td>$x^3 - 2x(x + 6) &gt; 0$</td>
<td>$x &gt; 6$</td>
</tr>
<tr>
<td>20.</td>
<td>$7x(x - 4)^2 &lt; 0$</td>
<td>$x &lt; 4$</td>
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<thead>
<tr>
<th>Exercise</th>
<th>Inequality</th>
<th>Solution Set</th>
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<tbody>
<tr>
<td>21.</td>
<td>$</td>
<td>x</td>
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<td>22.</td>
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<td>x</td>
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<tr>
<td>23.</td>
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<td>x</td>
</tr>
<tr>
<td>24.</td>
<td>$</td>
<td>x - 1</td>
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<tr>
<td>25.</td>
<td>$</td>
<td>x - 2</td>
</tr>
<tr>
<td>26.</td>
<td>$</td>
<td>x - \frac{1}{2}</td>
</tr>
<tr>
<td>27.</td>
<td>$0 &lt;</td>
<td>x</td>
</tr>
<tr>
<td>28.</td>
<td>$0 &lt;</td>
<td>x - 2</td>
</tr>
<tr>
<td>29.</td>
<td>$0 &lt;</td>
<td>x - 2</td>
</tr>
<tr>
<td>30.</td>
<td>$0 &lt;</td>
<td>x - 2</td>
</tr>
<tr>
<td>31.</td>
<td>$0 &lt;</td>
<td>x - 3</td>
</tr>
<tr>
<td>32.</td>
<td>$0 &lt;</td>
<td>x - 3</td>
</tr>
<tr>
<td>33.</td>
<td>$</td>
<td>2x + 1</td>
</tr>
<tr>
<td>34.</td>
<td>$</td>
<td>5x - 3</td>
</tr>
<tr>
<td>35.</td>
<td>$</td>
<td>2x + 5</td>
</tr>
<tr>
<td>36.</td>
<td>$</td>
<td>3x + 1</td>
</tr>
</tbody>
</table>

**Exercises 21–36.** Solve the inequality and express the solution set as an interval or as the union of intervals.
37. $(-3, 3)$.
38. $(2, 2)$.
39. $(-3, 7)$.
40. $(0, 4)$.
41. $(-7, 3)$.
42. $(a, b)$.

Exercises 43–46. Determine all numbers $A > 0$ for which the statement is true.
43. If $|x| > 2$, then $2x < 4 < A$.
44. If $|x| < 2$, then $2x < 4 < A$.
45. If $|x| < A$, then $3|x| + 3 < 4$.
46. If $|x| > 2$, then $3|x| + 3 < A$.

47. Arrange the following in order: $1, x, \sqrt{x}, \frac{1}{x}, \frac{1}{\sqrt{x}}$. Given that: (a) $x > 1$; (b) $0 < x < 1$. 
48. Given that $x > 0$, compare $\sqrt{x + 1}$ and $\sqrt{\frac{x + 2}{x}}$.
49. Suppose that $ab > 0$. Show that if $a < b$, then $1/b < 1/a$.
50. Given that $a > 0$ and $b > 0$, show that if $a^2 \leq b^2$, then $a \leq b$.
51. Show that if $0 \leq a \leq b$, then $\sqrt{ab} \leq \sqrt{b}$.

1.4 COORDINATE PLANE; ANALYTIC GEOMETRY

Rectangular Coordinates

The one-to-one correspondence between real numbers and points on a line can be used to construct a coordinate system for the plane. In the plane, we draw two number lines that are mutually perpendicular and intersect at their origins. Let $O$ be the point of intersection. We set one of the lines horizontally with the positive numbers to the right of $O$ and the other vertically with the positive numbers above $O$. The point $O$ is called the origin, and the number lines are called the coordinate axes. The horizontal axis is usually labeled the $x$-axis and the vertical axis is usually labeled the $y$-axis. The coordinate axes separate four regions, which are called quadrants. The quadrants are numbered I, II, III, IV in the counterclockwise direction starting with the upper right quadrant. See Figure 1.4.1.

Rectangular coordinates are assigned to points of the plane as follows (see Figure 1.4.2). The point on the $x$-axis with line coordinate $a$ is assigned rectangular coordinates $(a, 0)$. The point on the $y$-axis with line coordinate $b$ is assigned rectangular coordinates $(0, b)$. Thus the origin is assigned coordinates $(0, 0)$. A point $P$ not on one of the coordinate axes is assigned coordinates $(a, b)$ provided that the line $l_1$ that passes through $P$ and is parallel to the $y$-axis intersects the $x$-axis at the point with coordinates $(a, 0)$, and the line $l_2$ that passes through $P$ and is parallel to the $x$-axis intersects the $y$-axis at the point with coordinates $(0, b)$.

This procedure assigns an ordered pair of real numbers to each point of the plane. Moreover, the procedure is reversible. Given any ordered pair $(a, b)$ of real numbers, there is a unique point $P$ in the plane with coordinates $(a, b)$.

To indicate $P$ with coordinates $(a, b)$ we write $P(a, b)$. The number $a$ is called the $x$-coordinate (the abscissa); the number $b$ is called the $y$-coordinate (the ordinate). The coordinate system that we have defined is called a rectangular coordinate system. It is often referred to as a Cartesian coordinate system after the French mathematician René Descartes (1596–1650).
CHAPTER 1 PRECALCULUS REVIEW

Distance and Midpoint Formulas

Let \( P_0(x_0, y_0) \) and \( P_1(x_1, y_1) \) be points in the plane. The formula for the distance \( d(P_0, P_1) \) between \( P_0 \) and \( P_1 \) follows from the Pythagorean theorem:

\[
d(P_0, P_1) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}. \quad \text{(Figure 1.4.3)}
\]

Let \( M(x, y) \) be the midpoint of the line segment \( P_0P_1 \). That

\[
x = \frac{x_0 + x_1}{2} \quad \text{and} \quad y = \frac{y_0 + y_1}{2},
\]

follows from the congruence of the triangles shown in Figure 1.4.4

![Figure 1.4.4](image)

Midpoint: \( M(x, y) = \left( \frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right) \)

Lines

(i) Slope Let \( l \) be the line determined by \( P_0(x_0, y_0) \) and \( P_1(x_1, y_1) \). If \( l \) is not vertical, then \( x_1 \neq x_0 \) and the slope of \( l \) is given by the formula

\[
m = \frac{y_1 - y_0}{x_1 - x_0}. \quad \text{(Figure 1.4.5)}
\]

With \( \theta \) (as indicated in the figure) measured counterclockwise from the x-axis,

\[
m = \tan \theta. \quad \text{(Figure 1.4.5)}
\]

The angle \( \theta \) is called the inclination of \( l \). If \( l \) is vertical, then \( \theta = \pi/2 \) and the slope of \( l \) is not defined.

(ii) Intercepts If a line intersects the x-axis, it does so at some point \((a, 0)\). We call \( a \) the x-intercept. If a line intersects the y-axis, it does so at some point \((0, b)\). We call \( b \) the y-intercept. Intercepts are shown in Figure 1.4.6.

†The trigonometric functions are reviewed in Section 1.6.
1.4 COORDINATE PLANE: ANALYTIC GEOMETRY

(iii) Equations

vertical line \( x = a \).

horizontal line \( y = b \).

point-slope form \( y - y_0 = m(x - x_0) \).

deslope-intercept form \( y = mx + b \).

two-intercept form \( \frac{x}{a} + \frac{y}{b} = 1 \).

general form \( Ax + By + C = 0 \).

(iv) Parallel and Perpendicular Nonvertical Lines

parallel \( m_1 = m_2 \).

perpendicular \( m_1m_2 = -1 \).

(v) The Angle Between Two Lines The angle between two lines that meet at right angles is \( \pi/2 \). Figure 1.4.7 shows two lines \( l_1, l_2 \) with inclinations \( \theta_1, \theta_2 \) that intersect but not at right angles. These lines form two angles, marked \( \alpha \) and \( \pi - \alpha \) in the figure. The smaller of these angles, the one between 0 and \( \pi/2 \), is called the angle between \( l_1 \) and \( l_2 \). This angle, marked \( \alpha \) in the figure, is readily obtained from \( \theta_1 \) and \( \theta_2 \). If neither \( l_1 \) nor \( l_2 \) is vertical, the angle \( \alpha \) between \( l_1 \) and \( l_2 \) can also be obtained from the slopes of the lines:

\[ \tan \alpha = \frac{m_1 - m_2}{1 + m_1m_2} \]

The derivation of this formula is outlined in Exercise 75 of Section 1.6.

Example 1 Find the slope and the \( y \)-intercept of each of the following lines:

\[ l_1 : 20x - 24y - 30 = 0, \quad l_2 : 2x - 3 = 0, \quad l_3 : 4y + 5 = 0. \]

SOLUTION The equation of \( l_1 \) can be written

\[ y = \frac{5}{6}x - \frac{5}{3}. \]
This is in the form \( y = mx + b \). The slope is \( \frac{5}{6} \), and the \( y \)-intercept is \( -\frac{5}{4} \).

The equation of \( l_2 \) can be written
\[ x = \frac{3}{2}. \]

The line is vertical and the slope is not defined. Since the line does not cross the \( y \)-axis, the line has no \( y \)-intercept.

The third equation can be written
\[ y = -\frac{5}{4}. \]

The line is horizontal. The slope is 0 and the \( y \)-intercept is \( -\frac{5}{4} \). The three lines are drawn in Figure 1.4.8.

\[ \text{Figure 1.4.8} \]

**Example 2** Write an equation for the line \( l_2 \) that is parallel to
\[ l_1 : 3x - 5y + 8 = 0 \]
and passes through the point \( P(-3, 2) \).

**SOLUTION** The equation for \( l_1 \) can be written
\[ y = \frac{3}{5}x + \frac{8}{5}. \]

The slope of \( l_1 \) is \( \frac{3}{5} \). The slope of \( l_2 \) must also be \( \frac{3}{5} \) (For nonvertical parallel lines, \( m_1 = m_2 \)).

Since \( l_2 \) passes through \((-3, 2)\) with slope \( \frac{3}{5} \), we can use the point-slope formula and write the equation as
\[ y - 2 = \frac{3}{5}(x + 3). \]

**Example 3** Write an equation for the line that is perpendicular to
\[ l_1 : x - 4y + 8 = 0 \]
and passes through the point \( P(2, -4) \).

**SOLUTION** The equation for \( l_1 \) can be written
\[ y = \frac{1}{4}x + 2. \]
The slope of \( l_1 \) is \( \frac{1}{4} \). The slope of \( l_2 \) is therefore \( -\frac{4}{1} \). Since \( l_2 \) passes through \((2, -4)\) with slope \( -4 \), we can use the point-slope formula and write the equation as
\[
y + 4 = -4(x - 2).
\]

**Example 4** Show that the lines
\[
l_1 : 3x - 4y + 8 = 0 \quad \text{and} \quad l_2 : 12x - 5y - 12 = 0
\]
intersect and find their point of intersection.

**SOLUTION** The slope of \( l_1 \) is \( \frac{3}{4} \) and the slope of \( l_2 \) is \( \frac{12}{5} \). Since \( l_1 \) and \( l_2 \) have different slopes, they intersect at a point.

To find the point of intersection, we solve the two equations simultaneously:
\[
\begin{align*}
3x - 4y + 8 &= 0 \\
12x - 5y - 12 &= 0
\end{align*}
\]
Multiplying the first equation by \(-4\) and adding it to the second equation, we obtain
\[
11y - 44 = 0
\]
\[
y = 4.
\]
Substituting \( y = 4 \) into either of the two given equations, we find that \( x = \frac{2}{3} \). The lines intersect at the point \( \left( \frac{2}{3}, 4 \right) \).

**Circle, Ellipse, Parabola, Hyperbola**

These curves and their remarkable properties are thoroughly discussed in Section 10.1. The information we give here suffices for our present purposes.
CHAPTER 1 PRECALCULUS REVIEW

Ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b > a
\]

Figure 1.4.10

Parabola

\[
y = ax^2, \quad a > 0
\]

\[
y = ax^2, \quad a < 0
\]

\[
y = ax^2 + bx + c, \quad a > 0
\]

\[
y = ax^2 + bx + c, \quad a < 0
\]

Figure 1.4.11

Hyperbola

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

\[
\frac{y^2}{p^2} - \frac{x^2}{p^2} = 1
\]

Figure 1.4.12
Remark  The circle, the ellipse, the parabola, and the hyperbola are known as the conic sections because each of these configurations can be obtained by slicing a “double right circular cone” by a suitably inclined plane. (See Figure 1.4.13.)

**EXERCISES 1.4**

**Exercises 1–4.** Find the distance between the points.

1. \(P_0(0, 5), \ P_1(6, -3)\)
2. \(P_0(2, 2), \ P_1(5, 5)\)
3. \(P_0(5, -2), \ P_1(-3, 2)\)
4. \(P_0(7, 3), \ P_1(-4, 7)\)

**Exercises 5–8.** Find the midpoint of the line segment \(P_0P_1\).

5. \(P_0(2, 4), \ P_1(6, 8)\)
6. \(P_0(3, -1), \ P_1(-1, 5)\)
7. \(P_0(2, -3), \ P_1(7, -3)\)
8. \(P_0(a, 3), \ P_1(3, a)\)

**Exercises 9–14.** Find the slope of the line through the points.

9. \(P_0(-2, 5), \ P_1(4, 1)\)
10. \(P_0(4, -3), \ P_1(-2, -7)\)
11. \(P_0(a, b), \ Q(b, a)\)
12. \(P_0(-4, -1), \ Q(-3, -1)\)
13. \(P_0(x_0, 0), \ Q(0, y_0)\)
14. \(P_0(0, 0), \ Q(x_0, y_0)\)

**Exercises 15–20.** Find the slope and \(y\)-intercept.

15. \(y = 2x - 4\)
16. \(6 - 5x = 0\)
17. \(3y = x + 6\)
18. \(8y = 3x + 8 = 0\)
19. \(7x = 3y + 4 = 0\)
20. \(y = 3\)

**Exercises 21–24.** Write an equation for the line with

21. slope 5 and \(y\)-intercept 2.
22. slope 5 and \(y\)-intercept –2.
23. slope –5 and \(y\)-intercept 2.
24. slope –5 and \(y\)-intercept –2.

**Exercises 25–26.** Write an equation for the horizontal line 3 units above the \(x\)-axis.

26. below the \(x\)-axis.

**Exercises 27–28.** Write an equation for the vertical line 3 units to the left of the \(y\)-axis.
29. to the right of the \(y\)-axis.
30. parallel to the \(y\)-axis.

**Exercises 29–34.** Find an equation for the line that passes through the point \(P(2, 7)\) and is

29. parallel to the \(x\)-axis.
30. parallel to the \(y\)-axis.
31. parallel to the line \(3y = 2x + 6 = 0\).
32. perpendicular to the line \(y = 2x + 5 = 0\).
33. perpendicular to the line \(3y = 2x + 6 = 0\).
34. parallel to the line \(y = 2x + 5 = 0\).

**Exercises 35–38.** Determine the point(s) where the line intersects the circle.

35. \(y = x, \quad x^2 + y^2 = 1\)
36. \(y = mx, \quad x^2 + y^2 = 4\)
37. \(4x + 3y = 24, \quad x^2 + y^2 = 25\)
38. \(y = mx + b, \quad x^2 + y^2 = b^2\)

**Exercises 39–42.** Find the point where the lines intersect.

39. \(l_1: 4x - y = 3 = 0, \quad l_2: 3x - 4y + 1 = 0\)
40. \(l_1: 3x + y = 5 = 0, \quad l_2: 7x - 10y + 27 = 0\)
41. \(l_1: 4x - y + 2 = 0, \quad l_2: 19x + y = 0\)
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42. \( a_1 : 5x - 6y + 1 = 0, \quad a_2 : 8x + 5y + 2 = 0 \).

43. Find the area of the triangle with vertices (1, -2), (-1, 3), (2, 4).

44. Find the area of the triangle with vertices (-1, 1), (3, \( \sqrt{2} \)), (2, -1).

45. Determine the slope of the line that intersects the circle \( x^2 + y^2 = 169 \) only at the point (5, 12).

46. Find an equation for the line which is tangent to the circle \( x^2 + y^2 - 2x + 6y - 15 = 0 \) at the point (4, 1). HINT: A line is tangent to a circle at a point if it is perpendicular to the radius at that point.

47. The point P(-1, -3) is on a circle centered at C(-1, 3). Find an equation for the line tangent to the circle at P.

Exercises 48-51. Estimate the point(s) of intersection.

48. \( a_1 : 3x - 4y = 7, \quad a_2 : -5x + 2y = 11 \).

49. \( a_1 : 2.41x + 3.29y = 5, \quad a_2 : 5.13x - 4.27y = 13 \).

50. \( a_1 : 2x - 3y = 5, \quad a_2 : x^2 + y^2 = 4 \).

51. circle : \( x^2 + y^2 = 9, \) parabola : \( y = x^2 - 4x + 5 \).

Exercises 52-53. The perpendicular bisector of the line segment \( \overline{PQ} \) is the line which is perpendicular to \( \overline{PQ} \) and passes through the midpoint of \( \overline{PQ} \). Find an equation for the perpendicular bisector of the line segment that joins the two points.

52. \( P(-1, 3), \quad Q(3, -4) \).

53. \( P(-1, -4), \quad Q(4, 9) \).

Exercises 54-56. The points are the vertices of a triangle. State whether the triangle is isosceles (two sides of equal length), a right triangle, both of these, or neither of these.

54. \( P(-4, -3), \quad P(-4, -1), \quad P(2, 1) \).

55. \( P(-2, 5), \quad P(1, 3), \quad P(-1, 0) \).

56. \( P(-1, 2), \quad P(1, 3), \quad P(4, 1) \).

57. Show that the distance from the origin to the line \( Ax + By + C = 0 \) is given by the formula
\[
d(0, 1) = \frac{|C|}{\sqrt{A^2 + B^2}}.
\]

58. An equilateral triangle is a triangle the three sides of which have the same length. Given two of the vertices of an equilateral triangle are (0, 0) and (4, 3), find all possible locations for a third vertex. How many such triangles are there?

59. Show that the midpoint M of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.

60. A median of a triangle is a line segment from a vertex to the midpoint of the opposite side. Find the lengths of the medians of the triangle with vertices (-1, -2), (2, 1), (4, -3).

61. The vertices of a triangle are (1, 0), (3, 4), (-1, 6). Find the point(s) where the medians of this triangle intersect.

62. Show that the medians of a triangle intersect in a single point (called the centroid of the triangle). HINT: Introduce a coordinate system such that one vertex is at the origin and one side is on the positive x-axis; see the figure.

63. Prove that each diagonal of a parallelogram bisects the other. HINT: Introduce a coordinate system with one vertex at the origin and one side on the positive x-axis.

64. \( P(x_1, y_1), \quad P(x_2, y_2), \quad P(x_3, y_3), \quad P(x_4, y_4) \) are the vertices of a quadrilateral. Show that the quadrilateral formed by joining the midpoints of adjacent sides is a parallelogram.

65. Except in scientific work, temperature is usually measured in degrees Fahrenheit (F) or in degrees Celsius (C). The relation between F and C is linear. (In the equation that relates F to C, both F and C appear to the first degree.) The freezing point of water in the Fahrenheit scale is 32°F, in the Celsius scale it is 0°C. The boiling point of water in the Fahrenheit scale is 212°F, in the Celsius scale it is 100°C. Find an equation that gives the Fahrenheit temperature F in terms of the Celsius temperature C. Is there a temperature at which the Fahrenheit and Celsius readings are equal? If so, find it.

66. In scientific work, temperature is measured on an absolute scale, called the Kelvin scale (after Lord Kelvin, who initiated this mode of temperature measurement). The relation between Fahrenheit temperature F and absolute temperature K is linear. Given that K = 273 when F = 32°F, and K = 373° when F = 212°F, express K in terms of F. Then use your result in Exercise 65 to determine the connection between Celsius temperature and absolute temperature.

1.5  FUNCTIONS

The fundamental processes of calculus (called differentiation and integration) are processes applied to functions. To understand these processes and to be able to carry them out, you have to be comfortable working with functions. Here we review some of the basic ideas and the nomenclature. We assume that you are familiar with all of this.
Functions can be applied in a very general setting. At this stage, and throughout the first thirteen chapters of this text, we will be working with what are called real-valued functions of a real variable, functions that assign real numbers to real numbers.

Domain and Range
Let’s suppose that \( D \) is some set of real numbers and that \( f \) is a function defined on \( D \). Then \( f \) assigns a unique number \( f(x) \) to each number \( x \) in \( D \). The number \( f(x) \) is called the value of \( f \) at \( x \), or the image of \( x \) under \( f \). The set \( D \), the set on which the function is defined, is called the domain of \( f \), and the set of values taken on by \( f \) is called the range of \( f \). In set notation

\[
\text{dom}(f) = D, \quad \text{range}(f) = \{f(x) : x \in D\}.
\]

We can specify the function \( f \) by indicating exactly what \( f(x) \) is for each \( x \) in \( D \).

Some examples. We begin with the squaring function \( f(x) = x^2 \), for all real numbers \( x \).

The domain of \( f \) is explicitly given as the set of real numbers. Particular values taken on by \( f \) can be found by assigning particular values to \( x \). In this case, for example,

\[
\begin{align*}
f(4) &= 4^2 = 16, \\
f(-3) &= (-3)^2 = 9, \\
f(0) &= 0^2 = 0.
\end{align*}
\]

As \( x \) runs through the real numbers, \( x^2 \) runs through all the nonnegative numbers. Thus the range of \( f \) is \([0, \infty)\). In abbreviated form, we can write

\[
\text{dom}(f) = (-\infty, \infty), \quad \text{range}(f) = [0, \infty)
\]

and we can say that \( f \) maps \((-\infty, \infty)\) onto \([0, \infty)\).

Now let’s look at the function \( g \) defined by

\[
g(x) = \sqrt{2x + 4}, \quad x \in [0, 6].
\]

The domain of \( g \) is given as the closed interval \([0, 6]\). At \( x = 0 \), \( g \) takes on the value 2:

\[
g(0) = \sqrt{2 \cdot 0 + 4} = \sqrt{4} = 2;
\]

at \( x = 6 \), \( g \) has the value 4:

\[
g(6) = \sqrt{2 \cdot 6 + 4} = \sqrt{16} = 4.
\]

As \( x \) runs through the numbers in \([0, 6]\), \( g(x) \) runs through the numbers from 2 to 4. Therefore, the range of \( g \) is the closed interval \([2, 4]\). The function \( g \) maps \([0, 6]\) onto \([2, 4]\).

Some functions are defined piecewise. As an example, take the function \( h \), defined by setting

\[
h(x) = \begin{cases} 2x + 1, & \text{if } x < 0 \\ x^2, & \text{if } x \geq 0. \end{cases}
\]

As explicitly stated, the domain of \( h \) is the set of real numbers. As you can verify, the range of \( h \) is also the set of real numbers. Thus the function \( h \) maps \((-\infty, \infty)\) onto \((-\infty, \infty)\). A more familiar example is the absolute value function \( f(x) = |x| \). Here

\[
f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}
\]

The domain of this function is \((-\infty, \infty)\) and the range is \([0, \infty)\).

Remark. Functions are often given by equations of the form \( y = f(x) \) with \( x \) restricted to some set \( D \). In this setup \( x \) is called the independent variable (or the argument of the function) and \( y \), clearly dependent on \( x \), is called the dependent variable.
CHAPTER 1

The Graph of a Function

If $f$ is a function with domain $D$, then the graph of $f$ is the set of all points $P(x, f(x))$ with $x \in D$. Thus the graph of $f$ is the graph of the equation $y = f(x)$ with $x$ restricted to $D$, namely

$$\text{the graph of } f = \{(x, y) : x \in D, y = f(x)\}.$$ 

The most elementary way to sketch the graph of a function is to plot points. We plot enough points so that we can “see” what the graph may look like and then connect the points with a “curve.” Of course, if we can identify the curve in advance (for example, if we know that the graph is a straight line, a parabola, or some other familiar curve), then it is much easier to draw the graph.

The graph of the squaring function $f(x) = x^2$, $x \in (-\infty, \infty)$ is the parabola shown in Figure 1.5.1. The points that we plotted are indicated in the table and marked on the graph. The graph of the function $g(x) = \sqrt{2x + 4}$, $x \in [0, 6]$ is the arc shown in Figure 1.5.2

$$\begin{array}{c|c|c}
 x & x^2 & \sqrt{2x + 4} \\
 \hline
 -2 & 4 & 2.8 \\
 0 & 0 & 0 \\
 1 & 1 & 2.4 \\
 2 & 4 & 2.8 \\
 4 & 16 & 3.5 \\
 6 & 36 & 3.5 \\
 \end{array}$$

Figure 1.5.1

The graph of the function

$$h(x) = \begin{cases} 
2x + 1, & \text{if } x < 0 \\
x^2, & \text{if } x \geq 0 
\end{cases}$$

and the graph of the absolute value function are shown in Figures 1.5.3 and 1.5.4.

Although the graph of a function is a “curve” in the plane, not every curve in the plane is the graph of a function. This raises a question: How can we tell whether a curve is the graph of a function?

A curve $C$ which intersects each vertical line at most once is the graph of a function: for each $P(x, y) \in C$, define $f(x) = y$. A curve $C$ which intersects some vertical line more than once is not the graph of a function: $P(x, y_1)$ and $P(x, y_2)$ are both on $C$, then how can we decide what $f(x)$ is? Is it $y_1$; or is it $y_2$?

These observations lead to what is called the vertical line test: a curve $C$ in the plane is the graph of a function if no vertical line intersects $C$ at more than one point. Thus circles, ellipses, hyperbolas are not the graphs of functions. The curve shown in Figure 1.5.5 is the graph of a function, but the curve shown in Figure 1.5.6 is not the graph of a function.
Graphing calculators and computer algebra systems (CAS) are valuable aids to graphing, but, used mindlessly, they can detract from the understanding necessary for more advanced work. We will not attempt to teach the use of graphing calculators or the ins and outs of computer software, but technology-oriented exercises will appear throughout the text.

**Even Functions, Odd Functions; Symmetry**

For even integers \( n \), \((-x)^n = x^n\); for odd integers \( n \), \((-x)^n = -x^n\). These simple observations prompt the following definitions:

A function \( f \) is said to be even if

\[
f(-x) = f(x) \quad \text{for all } x \in \text{dom}(f),
\]

a function \( f \) is said to be odd if

\[
f(-x) = -f(x) \quad \text{for all } x \in \text{dom}(f).
\]

The graph of an even function is symmetric about the y-axis, and the graph of an odd function is symmetric about the origin. (Figures 1.5.7 and 1.5.8.)

The absolute value function is even:

\[
f(-x) = | -x | = |x| = f(x).
\]
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Figure 1.5.7

Even function

Figure 1.5.8

Odd function

Its graph is symmetric about the y-axis. (See Figure 1.5.4.) The function \( f(x) = 4x - x^3 \) is odd:

\[
(−x, f(−x)) \quad \text{and} \quad (x, f(x))
\]

The graph, shown in Figure 1.5.9, is symmetric about the origin.

Convention on Domains

If the domain of a function \( f \) is not explicitly given, then by convention we take as domain the maximal set of real numbers \( x \) for which \( f(x) \) is a real number. For the function \( f(x) = x^3 + 1 \), we take as domain the set of real numbers. For \( g(x) = \sqrt{x} \), we take as domain the set of nonnegative numbers. For \( h(x) = \frac{1}{x - 2} \) we take as domain the set of all real numbers \( x \neq 2 \). In interval notation,

\[
\text{dom}(f) = (−\infty, \infty), \quad \text{dom}(g) = [0, \infty), \quad \text{and} \quad \text{dom}(h) = (−\infty, 2) \cup (2, \infty).
\]

The graphs of the three functions are shown in Figure 1.5.10.

Example 1

Give the domain of each function:

(a) \( f(x) = \frac{x + 1}{x^2 + x - 6} \)

(b) \( g(x) = \sqrt{4 - x^2} \)

SOLUTION

(a) You can see that \( f(x) \) is a real number iff \( x^2 + x - 6 \neq 0 \). Since

\[
x^2 + x - 6 = (x + 3)(x - 2),
\]

(b) \( g(x) \) is defined for all \( x \) such that \( 4 - x^2 \geq 0 \). This is equivalent to \( x^2 \leq 4 \), or \( -2 \leq x \leq 2 \).

Figure 1.5.10
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(a) The domain of \( f \) is the set of real numbers other than \(-3\) and \(2\). This set can be expressed as \((-\infty, -3) \cup (-3, 2) \cup (2, \infty)\).

(b) For \( g(x) \) to be a real number, we need \(4 - x^2 \geq 0\) and \(x \neq 1\).

Since \(4 - x^2 \geq 0 \iff x^2 \leq 4 \iff -2 \leq x \leq 2\), the domain of \( g \) is the set of all numbers \( x \) in the closed interval \([-2, 2]\) other than \( x = 1 \). This set can be expressed as the union of two half-open intervals:

\([-2, 1) \cup (1, 2]\).

Example 2

Give the domain and range of the function:

\[ f(x) = \sqrt{2} - x + 5. \]

**Solution**

First we look for the domain. Since \(\sqrt{2} - x \) is a real number \( \iff 2 - x \geq 0 \), we need \( x \leq 2 \). But at \( x = 2 \), \(\sqrt{2} - x = 0 \) and its reciprocal is not defined. We must therefore restrict \( x \) to \( x < 2 \). The domain is \((-\infty, 2)\).

Now we look for the range. As \( x \) runs through \((-\infty, 2)\), \(\sqrt{2} - x \) takes on all positive values and so does its reciprocal. The range of \( f \) is therefore \((5, \infty)\). The function \( f \) maps \((-\infty, 2)\) onto \((5, \infty)\).

Functions are used in applications to show how variable quantities are related. The domain of a function that appears in an application is dictated by the requirements of the application.

Example 3

U.S. Postal Service regulations require that the length plus the girth (the perimeter of a cross section) of a package for mailing cannot exceed 108 inches. A rectangular box with a square end is designed to meet the regulation exactly (see Figure 1.5.11). Express the volume \( V \) of the box as a function of the edge length of the square end and give the domain of the function.

**Solution**

Let \( x \) denote the edge length of the square end and let \( h \) denote the length of the box. The girth is the perimeter of the square, or \(4x\). Since the box meets the regulations exactly,

\[ 4x + h = 108 \quad \text{and therefore} \quad h = 108 - 4x. \]

The volume of the box is given by \( V = x^2 h \) and so it follows that

\[ V(x) = x^2 (108 - 4x) = 108x^2 - 4x^3. \]

Since neither the edge length of the square end nor the length of the box can be negative, we have

\[ x \geq 0 \quad \text{and} \quad h = 108 - 4x \geq 0. \]

The second condition requires \( x \leq 27 \). The full requirement on \( x \), \(0 \leq x \leq 27\), gives \( \text{dom}(V) = [0, 27] \).

Example 4

A soft-drink manufacturer wants to fabricate cylindrical cans. (See Figure 1.5.12.) The can is to have a volume of 12 fluid ounces, which we take to be approximately 22 cubic inches. Express the total surface area \( S \) of the can as a function of the radius and give the domain of the function.
1. takes on

Give the domain of the function and sketch

\( f(x) = x^2 + 3x + 2 \)

2. \( f(x) = \frac{2x - 1}{x^2 + 4} \)

3. \( f(x) = \sqrt{x^2 + 2x} \)

4. \( f(x) = |x + 3| - 5x \)

5. \( f(x) = \frac{2x}{|x + 2| + x^2} \)

6. \( f(x) = 1 - \frac{1}{(x + 1)^2} \)

Solutions 7–10: Calculate (a) \( f(-1) \), (b) \( f(1) \), (c) \( f(a + b) \).

7. \( f(x) = x^2 - 2x \)

8. \( f(x) = \frac{x}{x^2 + 1} \)

9. \( f(x) = \sqrt{1 + x^2} \)

10. \( f(x) = \frac{x}{\sqrt{x^2 - 1}} \)

11. Exercises 11 and 12: Calculate \( f(a + h) \) and \( f(a + h) - f(a)/h \) for \( h \neq 0 \).

12. \( f(x) = 2x^2 - 3x \)

13. \( f(x) = 2x - x \)

14. \( f(x) = \sqrt{1 + x} \)

15. \( f(x) = x^2 + 4x + 5 \)

16. \( f(x) = 4x + 10 + h^2 - x^2 \)

17. \( f(x) = \frac{2}{\sqrt{x^2 + 5}} \)

18. \( f(x) = \frac{x}{|x|} \)

19. Exercises 19–30: Give the domain and range of the function.

19. \( f(x) = |x| \)

20. \( g(x) = x^2 - 1 \)

21. \( f(x) = 2x + 3 \)

22. \( g(x) = \sqrt{x} + 5 \)

23. \( f(x) = 1 \)

24. \( g(x) = \sqrt{\frac{x}{3}} \)

25. \( f(x) = \sqrt[3]{x} \)

26. \( g(x) = \sqrt{x - 4} \)

27. \( f(x) = \sqrt{7 - x} \)

28. \( g(x) = \sqrt{x - 1} - 1 \)

29. \( f(x) = \frac{1}{\sqrt{x}} \)

30. \( g(x) = \frac{1}{\sqrt{4-h}} \)

31. Exercises 31–40: Give the domain of the function and sketch the graph.

31. \( f(x) = 1 \)

32. \( f(x) = -1 \)

33. \( f(x) = 2x \)

34. \( f(x) = 2x + 1 \)

35. \( f(x) = \frac{1}{3}x + 2 \)

36. \( f(x) = -\frac{1}{2}x - 3 \)

37. \( f(x) = \sqrt{x^2} \)

38. \( f(x) = \sqrt{9 - x^2} \)

39. \( f(x) = x^2 - x - 6 \)

40. \( f(x) = |x - 1| \)

41. Exercises 41–44: Sketch the graph and give the domain and range of the function.

41. \( f(x) = \begin{cases} -x, & x < 0 \\ 1, & x > 0 \end{cases} \)

42. \( f(x) = \begin{cases} x^2, & x \leq 0 \\ 1 - x, & x > 0 \end{cases} \)

43. \( f(x) = \begin{cases} 1 + x, & 0 \leq x \leq 1 \\ x, & 1 < x < 2 \\ \frac{1}{x} + 1, & 2 \leq x \end{cases} \)

44. \( f(x) = \begin{cases} x^2, & x < 0 \\ -1, & 0 \leq x < 2 \\ x, & 2 \leq x \end{cases} \)

45. Exercises 45–48: State whether the curve is the graph of a function. If it is, give the domain and the range.
46. The graph of \( f(x) = x^3 \) looks something like this:

47. The graph of \( f(x) = x^2 + 1 \) looks something like this:

48. The graph of \( f(x) = x(x-1) \) looks something like this:

Exercises 49–54. State whether the function is odd, even, or neither.

49. \( f(x) = x^3 \)

50. \( f(x) = x^2 + 1 \)

51. \( g(x) = x(x-1) \)

52. \( g(x) = x^2 + 1 \)

53. \( f(x) = \frac{x^2}{1 - |x|} \)

54. \( f(x) = x + 1 \)

55. \( f(x) = \frac{x}{x^2 - 9} \)

56. \( f(x) = \sqrt{x^2 - 8} \)

57. The graph of \( f(x) = \frac{1}{4}x^2 + \frac{1}{2}x^2 - 12x - 6 \) looks something like this:

(a) Use a graphing utility to sketch an accurate graph of \( f \).
(b) Find the zero(s) of \( f \) if any. Use three decimal place accuracy.
(c) Find the coordinates of the points marked \( A \) and \( B \), accurate to three decimal places.

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(b) Find the zero(s) of \( f \) such that \( f(x) = 0 \) accurate to three decimal places.
(c) Find the coordinates of the points marked \( A \) and \( B \), accurate to three decimal places.

58. The graph of \( f(x) = -x^4 + 8x^2 + x - 1 \) looks something like this:

(a) Use a graphing utility to sketch an accurate graph of \( f \).
(b) Find the zero(s) of \( f \), if any. Use three decimal place accuracy.
(c) Find the coordinates of the points marked \( A \) and \( B \), accurate to three decimal places.

Exercises 59 and 60. Use a graphing utility to draw several views of the graph of the function. Select the one that most accurately shows the important features of the graph. Give the domain and range of the function.

59. \( f(x) = |x^2 - 3x^2 - 24x + 4| \)

60. \( f(x) = \sqrt{x^2 - 8} \)

61. Determine the range of \( y = x^2 - 4x - 5 \)
(a) by writing \( y \) in the form \( x = a^2 + b \).
(b) by first solving the equation for \( x \).

62. Determine the range of \( y = 2x^2 - x \)
(a) by writing \( y \) in the form \( x = a^2 + b \).
(b) by first solving the equation for \( x \).

63. Express the area of a circle as a function of the circumference.

64. Express the volume of a sphere as a function of the surface area.

65. Express the volume of a cube as a function of the area of one of the faces.

66. Express the volume of a cube as a function of the total surface area.

67. Express the surface area of a cube as a function of the length of the diagonal of a face.

68. Express the volume of a cube as a function of one of the diagonals.

69. Express the area of an equilateral triangle as a function of the length of a side.

70. A right triangle with hypotenuse \( c \) is revolved about one of its legs to form a cone. (See the figure.) Given that \( x \) is the length of the other leg, express the volume of the cone as a function of \( x \).
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71. A Norman window is a window in the shape of a rectangle surmounted by a semicircle. (See the figure.) Given that the perimeter of the window is 15 feet, express the area as a function of the width \( x \).

72. A window has the shape of a rectangle surmounted by an equilateral triangle. Given that the perimeter of the window is 15 feet, express the area as a function of the length of one side of the equilateral triangle.

73. Express the area of the rectangle shown in the accompanying figure as a function of the \( x \)-coordinate of the point \( P \).

74. A right triangle is formed by the coordinate axes and a line through the point \((2, 5)\). (See the figure.) Express the area of the triangle as a function of the \( y \)-intercept.

75. A string 28 inches long is to be cut into two pieces, one piece to form a square and the other to form a circle. Express the total area enclosed by the square and circle as a function of the perimeter of the square.

76. A tank in the shape of an inverted cone is being filled with water. (See the figure.) Express the volume of water in the tank as a function of the depth \( h \).

77. Suppose that a cylindrical mailing container exactly meets the U.S. Postal Service regulations given in Example 3. (See the figure.) Express the volume of the container as a function of the radius of an end.

1.6  THE ELEMENTARY FUNCTIONS

The functions that figure most prominently in single-variable calculus are the polynomials, the rational functions, the trigonometric functions, the exponential functions, and the logarithm functions. These functions are generally known as the elementary functions. Here we review polynomials, rational functions, and trigonometric functions. Exponential and logarithm functions are introduced in Chapter 7.

Polynomials

We begin with a nonnegative integer \( n \). A function of the form

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

for all real \( x \),
where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and $a_n \neq 0$ is called a (real) polynomial of degree $n$.

If $n = 0$, the polynomial is simply a constant function:

$$P(x) = a_0$$

for all real $x$.

Nonzero constant functions are polynomials of degree 0. The function $P(x) = 0$ for all real $x$ is also a polynomial, but we assign no degree to it.

Polynomials satisfy a condition known as the factor theorem: if $P$ is a polynomial and $r$ is a real number, then

$$P(r) = 0 \text{ iff } (x - r) \text{ is a factor of } P(x).$$

The real numbers $r$ at which $P(x) = 0$ are called the zeros of the polynomial.

The linear functions

$$P(x) = ax + b, \quad a \neq 0$$

are the polynomials of degree 1. Such a polynomial has only one zero: $r = -b/a$. The graph is the straight line $y = ax + b$.

The quadratic functions

$$P(x) = ax^2 + bx + c, \quad a \neq 0$$

are the polynomials of degree 2. The graph of such a polynomial is the parabola $y = ax^2 + bx + c$. If $a > 0$, the vertex is the lowest point on the curve; the curve opens up. If $a < 0$, the vertex is the highest point on the curve. (See Figure 1.6.1.)

The zeros of the quadratic function $P(x) = ax^2 + bx + c$ are the roots of the quadratic equation

$$ax^2 + bx + c = 0.$$ 

The three possibilities are depicted in Figure 1.6.2. Here we are taking $a > 0$.

Polynomials of degree 3 have the form $P(x) = ax^3 + bx^2 + cx + d, a \neq 0$. These functions are called cubics. In general, the graph of a cubic has one of the two following shapes, again determined by the sign of $a$ (Figure 1.6.3). Note that we have not tried to
locate these graphs with respect to the coordinate axes. Our purpose here is simply to
indicate the two typical shapes. You can see, however, that for a cubic there are three
possibilities: three real roots, two real roots, one real root. (Each cubic has at least one
real root.)

Cubics

Figure 1.6.3

Polynomials become more complicated as the degree increases. In Chapter 4 we
use calculus to analyze polynomials of higher degree.

Rational Functions

A rational function is a function of the form

\[ R(x) = \frac{P(x)}{Q(x)} \]

where \( P \) and \( Q \) are polynomials. Note that every polynomial \( P \) is a rational function:

\[ P(x) = \frac{P(x)}{1} \]

is the quotient of two polynomials. Since division by 0 is meaningless, a rational function \( R = P/Q \) is not defined at those points \( x \) (if any) where \( Q(x) = 0 \); \( R \) is defined at all other points. Thus, \( \text{dom}(R) = \{ x : Q(x) \neq 0 \} \).

Rational functions \( R = P/Q \) are more difficult to analyze than polynomials and
more difficult to graph. In particular, we have to examine the behavior of \( R \) near the
zeros of the denominator and the behavior of \( R \) for large values of \( x \), both positive and
negative. If, for example, the denominator \( Q \) is zero at \( x = a \) but the numerator \( P \) is
not zero at \( x = a \), then the graph of \( R \) tends to the vertical as \( x \) tends to \( a \) and the
line \( x = a \) is called a vertical asymptote. If \( a \) becomes very large positive or very
large negative the values of \( R \) tend to some number \( b \), then the line \( y = b \) is called a
horizontal asymptote. Vertical and horizontal asymptotes are mentioned here only in
passing. They will be studied in detail in Chapter 4. Below are two simple examples.

(i) The graph of

\[ R(x) = \frac{1}{x^2 - 4x + 4} = \frac{1}{(x - 2)^2} \]

is shown in Figure 1.6.4. The line \( x = 2 \) is a vertical asymptote; the line \( y = 0 \) (the
\( x \)-axis) is a horizontal asymptote.

(ii) The graph of

\[ R(x) = \frac{x^2}{x^2 - 1} = \frac{x^2}{(x - 1)(x + 1)} \]

is shown in Figure 1.6.5. The lines \( x = 1 \) and \( x = -1 \) are vertical asymptotes; the
line \( y = 1 \) is a horizontal asymptote.
The Trigonometric Functions

Radian Measure  Degree measure, traditionally used to measure angles, has a serious drawback. It is artificial; there is no intrinsic connection between a degree and the geometry of a rotation. Why choose 360° for one complete revolution? Why not 100° or 400°?

There is another way of measuring angles that is more natural and lends itself better to the methods of calculus: measuring angles in radians.

Angles arise from rotations. We will measure angles by measuring rotations. Suppose that the points of the plane are rotated about some point 0. The point 0 remains fixed, but all other points P trace out circular arcs on circles centered at 0. The farther P is from 0, the longer the circular arc (Figure 1.6.6). The magnitude of a rotation about 0 is by definition the length of the arc generated by the rotation as measured on a circle at a unit distance from 0.

Now let θ be any real number. The rotation of radian measure θ (we shall simply call it the rotation θ) is by definition the rotation of magnitude |θ| in the counterclockwise direction if θ > 0, in the clockwise direction if θ < 0. If θ = 0, there is no movement; every point remains in place.

In degree measure a full turn is effected over the course of 360°. In radian measure, a full turn is effected during the course of 2π radians. (The circumference of a circle of radius 1 is 2π.) Thus

\[ 2\pi \text{ radians} = 360 \text{ degrees} \]

one radian = 360/2π degrees \( \approx 57.30° \)

one degree = 2π/360 radians \( \approx 0.0175 \text{ radians} \).

The following table gives some common angles (rotations) measured both in degrees and in radians.

<table>
<thead>
<tr>
<th>degrees</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
<th>180°</th>
<th>270°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>radians</td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\pi}{2} )</td>
<td>( \frac{2\pi}{3} )</td>
<td>( \frac{3\pi}{4} )</td>
<td>( \frac{5\pi}{6} )</td>
<td>( \pi )</td>
<td>( \frac{3\pi}{2} )</td>
<td>( 2\pi )</td>
</tr>
</tbody>
</table>

Cosine and Sine  In Figure 1.6.7 you can see a circle of radius 1 centered at the origin of a coordinate plane. We call this the unit circle. On the circle we have marked the point A(1, 0).

Now let θ be any real number. The rotation θ takes A(1, 0) to some point P, also on the unit circle. The coordinates of P are completely determined by θ and have names
related to $\theta$. The second coordinate of $P$ is called the sine of $\theta$ (we write $\sin \theta$) and the first coordinate of $P$ is called the cosine of $\theta$ (we write $\cos \theta$). Figure 1.6.8 illustrates the idea. To simplify the diagram, we have taken $\theta$ from 0 to $2\pi$.

For each real $\theta$, the rotation $\theta$ and the rotation $\theta + 2\pi$ take the point $A$ to exactly the same point $P$. It follows that for each $\theta$,

$$\sin(\theta + 2\pi) = \sin \theta, \quad \cos(\theta + 2\pi) = \cos \theta.$$ 

In Figure 1.6.9 we consider two rotations: a positive rotation $\theta$ and its negative counterpart $-\theta$. From the figure, you can see that

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta.$$ 

The sine function is an odd function and the cosine function is an even function.

In Figure 1.6.10 we have marked the effect of consecutive rotations of $\frac{1}{2}\pi$ radians:

$$(a, b) \rightarrow (-b, a) \rightarrow (-a, -b) \rightarrow (b, -a).$$

In each case, $(x, y) \rightarrow (-y, x)$. Thus,

$$\sin(\theta + \frac{1}{2}\pi) = \cos \theta, \quad \cos(\theta + \frac{1}{2}\pi) = -\sin \theta.$$ 

A rotation of $\pi$ radians takes each point to the point antipodal to it: $(x, y) \rightarrow (-x, -y)$. Thus

$$\sin(\theta + \pi) = -\sin \theta, \quad \cos(\theta + \pi) = -\cos \theta.$$ 

Tangent, Cotangent, Secant, Cosecant

There are four other trigonometric functions: the tangent, the cotangent, the secant, the cosecant. These are obtained as follows:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}.$$ 

The most important of these functions is the tangent. Note that the tangent function is an odd function

$$\tan(-\theta) = \frac{-\sin(-\theta)}{-\cos(-\theta)} = -\tan \theta, \quad \tan(\theta + \pi) = \frac{-\sin(\theta + \pi)}{-\cos(\theta + \pi)} = \tan \theta.$$
1.6 THE ELEMENTARY FUNCTIONS

Particular Values  The values of the sine, cosine, and tangent at angles (rotations) frequently encountered are given in the following table.

<table>
<thead>
<tr>
<th>$0$</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{2\pi}{3}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{5\pi}{6}$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$1$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>$1$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>$0$</td>
<td>$\sqrt{3}$</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{3}$</td>
<td>$\infty$</td>
<td>$-\sqrt{2}$</td>
<td>$-\sqrt{3}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

The (approximate) values of the trigonometric functions for any angle $\theta$ can be obtained with a hand calculator or from a table of values.

Identities  Below we list the basic trigonometric identities. Some are obvious; some have just been verified; the rest are derived in the exercises.

(i) unit circle

$\sin^2 \theta + \cos^2 \theta = 1$,  \hspace{1em} $\tan^2 \theta + 1 = \sec^2 \theta$,  \hspace{1em} $1 + \cot^2 \theta = \csc^2 \theta$.

(The first identity is obvious; the other two follow from the first)

(ii) periodicity

$\sin(\theta + 2\pi) = \sin \theta$,  \hspace{1em} $\cos(\theta + 2\pi) = \cos \theta$,  \hspace{1em} $\tan(\theta + \pi) = \tan \theta$.

(iii) odd and even

$\sin(-\theta) = -\sin \theta$,  \hspace{1em} $\cos(-\theta) = \cos \theta$,  \hspace{1em} $\tan(-\theta) = -\tan \theta$.

(The sine and tangent are odd functions; the cosine is even)

(iv) sines and cosines

$\sin(\theta + \pi) = -\sin \theta$,  \hspace{1em} $\cos(\theta + \pi) = -\cos \theta$,  \hspace{1em} $\sin(\theta + \frac{\pi}{2}) = \cos \theta$,  \hspace{1em} $\cos(\theta + \frac{\pi}{2}) = -\sin \theta$,  \hspace{1em} $\sin(\frac{\pi}{2} - \theta) = \cos \theta$,  \hspace{1em} $\cos(\frac{\pi}{2} - \theta) = \sin \theta$.

(Only the third pair of identities still has to be verified)

(v) addition formulas

$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$,  \hspace{1em} $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$,  \hspace{1em} $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$,  \hspace{1em} $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$.

(Taken up in the exercises)

(vi) double-angle formulas

$\sin 2\theta = 2\sin \theta \cos \theta$,  \hspace{1em} $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$.

(Follow from the addition formulas)

1 A function $f$ with an unbounded domain is said to be periodic if there exists a number $p > 0$ such that, if $\theta$ is in the domain of $f$, then $\theta + p$ is in the domain and $f(\theta + p) = f(\theta)$. The least number $p$ with this property (if there is a least one) is called the period of the function. The sine and cosine have period $2\pi$. Their reciprocals, the cosecant and secant, also have period $2\pi$. The tangent and cotangent have period $\pi$. 


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(vii) half-angle formulas

\[
\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta), \quad \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta)
\]

(follow from the double-angle formulas)

In Terms of a Right Triangle For angles \(\theta\) between 0 and \(\pi/2\), the trigonometric functions can also be defined as ratios of the sides of a right triangle. (See Figure 1.6.11.)

\[
\begin{align*}
\sin \theta &= \frac{\text{opposite side}}{\text{hypotenuse}} \\
csc \theta &= \frac{\text{hypotenuse}}{\text{opposite side} } \\
\cos \theta &= \frac{\text{adjacent side}}{\text{hypotenuse}} \\
sec \theta &= \frac{\text{hypotenuse}}{\text{adjacent side} } \\
\tan \theta &= \frac{\text{opposite side}}{\text{adjacent side}} \\
cot \theta &= \frac{\text{adjacent side}}{\text{opposite side} }
\end{align*}
\]

Exercise 81

Arbitrary Triangles Let \(a, b, c\) be the sides of a triangle and let \(A, B, C\) be the opposite angles. (See Figure 1.6.12.)

area \(\frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A\).

law of sines \(\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\) (taken up in the exercises)

law of cosines \(a^2 = b^2 + c^2 - 2bc \cos A,\)

\(b^2 = a^2 + c^2 - 2ac \cos B,\)

\(c^2 = a^2 + b^2 - 2ab \cos C\).

Graphs Usually we work with functions \(y = f(x)\) and graph them in the \(xy\)-plane. To bring the graphs of the trigonometric functions into harmony with this convention, we replace \(\theta\) by \(x\) and write \(y = \sin x, y = \cos x, y = \tan x\). (These are the only functions that we are going to graph here.) The functions have not changed, only the symbols: \(x\) is the rotation that takes \((1, 0)\) to the point \(P(\cos x, \sin x)\). The graphs of the sine, cosine, and tangent appear in Figure 1.6.13.

The graphs of sine and cosine are waves that repeat themselves on every interval of length \(2\pi\). These waves appear to chase each other. They do chase each other. In the chase the cosine wave remains \(\frac{1}{2}\pi\) units behind the sine wave:

\(\cos x = \sin(x + \frac{1}{2}\pi)\).

Changing perspective, we see that the sine wave remains \(\frac{1}{4}\pi\) units behind the cosine wave:

\(\sin x = \cos(x + \frac{1}{4}\pi)\).

All these waves crest at \(y = 1\), drop down to \(y = -1\), and then head up again.

The graph of the tangent function consists of identical pieces separated every \(\pi\) units by asymptotes that mark the points \(x\) where \(\cos x = 0\).
1.6 THE ELEMENTARY FUNCTIONS

1.6 EXERCISES

Exercises 1–10. State whether the function is a polynomial, a rational function (but not a polynomial), or neither a polynomial nor a rational function. If the function is a polynomial, give the degree.

1. \( f(x) = 3 \).
2. \( f(x) = 1 + \frac{1}{x} \).
3. \( g(x) = \frac{x}{x^2 + 1} \).
4. \( h(x) = x^2 - 4 \).
5. \( F(x) = \frac{x^3 + 3x^2 - 2x}{x^2 - 1} \).
6. \( f(x) = 5x^4 - \pi x^2 + \frac{3}{2} \).
7. \( f(x) = \sqrt{x} \).
8. \( g(x) = \sqrt{x^2 + 8x + 2} \).
9. \( f(x) = \frac{x^2 - 1}{x^2 - 1} \).
10. \( h(x) = \frac{(\sqrt{x} + 2)(\sqrt{x} - 2)}{x^2 + 4} \).

Exercises 11–16. Determine the domain of the function and sketch the graph.

11. \( f(x) = 3x + \frac{1}{x} \).
12. \( f(x) = \frac{1}{x + 1} \).
13. \( g(x) = x^2 - 3x - 6 \).
14. \( F(x) = x^3 - x \).
15. \( f(x) = \frac{1}{x^2} \).
16. \( g(x) = x + \frac{1}{x^2} \).

Exercises 17–22. Convert the degree measure into radian measure.

17. 172.5°.
19. 300°.
20. 450°.
21. 15°.
22. 3°.

Exercises 23–28. Convert the radian measure into degree measure.

23. \(-\frac{3\pi}{2} \).
24. \(5\pi/4 \).

Figure 1.6.13
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25. \( 5\pi /3 \).
26. \(-11\pi /6 \).

27. \( 3\pi /2 \).
28. \(-\pi /7 \).

29. Show that in a circle of radius \( a \) a central angle of \( \theta \) radians subtends an arc of length \( r\theta \).

30. Show that in a circular disk of radius \( a \), a sector with a central angle of \( \theta \) radians has area \( \frac{1}{2} a^2 \theta \). Take \( \theta \) between 0 and \( 2\pi \).

HINT: The area of the circle is \( \pi a^2 \).

Exercises 31–38. Find the number(s) \( x \) in the interval \([0, 2\pi]\) which satisfy the equation.

31. \( \sin x = 1/2 \).
32. \( \cos x = -1/2 \).
33. \( \tan x = 2 \).
34. \( \sqrt{\sin x} = 1 \).
35. \( \cos x = \sqrt{7}/2 \).
36. \( \sin 2x = -\sqrt{7}/2 \).
37. \( \cos 2x = 0 \).
38. \( \tan x = -\sqrt{7} \).

Exercises 39–44. Evaluate to four decimal place accuracy.

39. \( \sin 51^\circ \).
40. \( \cos 17^\circ \).
41. \( \sin(2.352) \).
42. \( \cos(-13.461) \).
43. \( \tan 72.4^\circ \).
44. \( \cot(7.311) \).

Exercises 45–52. Find the solutions \( x \) in the interval \([0, 2\pi]\) that satisfy the equation.

45. \( \sin x = 0.5231 \).
46. \( \cos x = 0.8243 \).
47. \( \tan x = 6.7192 \).
48. \( \cot x = -3.0649 \).
49. \( \sec \theta = -4.4073 \).
50. \( \csc x = 10.260 \).

Exercises 51–52. Solve the equation \( f(x) = y_0 \) for \( x \) in \([0, 2\pi]\) by using a graphing utility. Display the graph of \( f \) and the line \( y = y_0 \) in one figure, then use the trace function to find the point(s) of intersection.

51. \( f(x) = \sin 3x \); \( y_0 = -1/\sqrt{2} \).
52. \( f(x) = \cos 2x \); \( y_0 = 1 \).

Exercises 53–58. Give the domain and range of the function.

53. \( f(x) = | \sin x | \).
54. \( g(x) = \sin^2 x + \cos^2 x \).
55. \( f(x) = 2 \cos 3x \).
56. \( F(x) = 1 + \sin x \).
57. \( f(x) = 1 + \tan^2 x \).
58. \( h(x) = \sqrt{\tan^2 x} \).

Exercises 59–62. Determine the period. (The least positive number \( p \) for which \( f(x + p) = f(x) \) for all \( x \).)

59. \( f(x) = \sin \pi x \).
60. \( f(x) = \cos 2x \).
61. \( f(x) = \cos 4x \).
62. \( f(x) = \sin \frac{x}{2} \).

Exercises 63–68. Sketch the graph of the function.

63. \( f(x) = 3 \sin 2x \).
64. \( f(x) = 1 + \sin x \).
65. \( g(x) = 1 - \cos x \).
66. \( F(x) = \tan \frac{x}{4} \).
67. \( f(x) = \sqrt{\sin^2 x} \).
68. \( g(x) = -2 \cos x \).

Exercises 69–74. State whether the function is odd, even, or neither.

69. \( f(x) = \sin 3x \).
70. \( g(x) = \tan x \).
71. \( f(x) = 1 + \cos 2x \).
72. \( g(x) = \sec x \).

73. \( f(x) = x^2 + \sin x \).
74. \( h(x) = \frac{\cos x}{x^2 + 1} \).

75. Suppose that \( \theta_1 \) and \( \theta_2 \) are two nonvertical lines. If \( w_1 w_2 = -1 \), then \( \theta_1 \) and \( \theta_2 \) intersect at right angles. Show that if \( \theta_1 \) and \( \theta_2 \) do not intersect at right angles, then the angle \( u \) between \( \theta_1 \) and \( \theta_2 \) (see Section 1.4) is given by the formula

\[
\tan u = \frac{w_1 - w_2}{1 + w_1 w_2}.
\]

HINT: Derive the identity

\[
\tan(\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}
\]

by expressing the right side in terms of sines and cosines.

Exercises 76–79. Find the point where the lines intersect and determine the angle between the lines.

76. \( \theta_1: 4x - y - 3 = 0 \); \( \theta_2: 3x - 4y + 1 = 0 \).
77. \( \theta_1: 3x + y - 5 = 0 \); \( \theta_2: 7x - 10y + 27 = 0 \).
78. \( \theta_1: 4x - y + 2 = 0 \); \( \theta_2: 19x + y = 0 \).
79. \( \theta_1: 5x - 6y + 1 = 0 \); \( \theta_2: 8x + 5y + 2 = 0 \).

80. Show that the function \( f(x) = 1/x \) is periodic but has no period.

81. Verify that, for angles \( \theta \) between 0 and \( \pi /2 \), the definition of the trigonometric functions in terms of the unit circle and the definitions in terms of a right triangle are in agreement. HINT: Set the triangle as in the figure.

The setting for Exercises 82, 83, 84 is a triangle with sides \( a, b, c \) and opposite angles \( A, B, C \).

82. Show that the area of the triangle is given by the formula

\[
\text{Area} = \frac{1}{2}ab \sin C.
\]

HINT: Drop a perpendicular from one vertex to the opposite side and use the two right triangles formed.

83. Confirm the law of sines:

\[
\sin A = \sin B = \sin C.
\]

HINT: Drop a perpendicular from one vertex to the opposite side and use the two right triangles formed.

84. Confirm the law of cosines:

\[
a^2 = b^2 + c^2 - 2bc \cos A.
\]

HINT: Drop a perpendicular from angle \( B \) to side \( b \) and use the two right triangles formed.
85. Verify the identity
\[ \cos(a - b) = \cos a \cos b + \sin a \sin b. \]
HINT: With \( P \) and \( Q \) as in the accompanying figure, calculate the length of \( PQ \) by applying the law of cosines.

86. Use Exercise 85 to show that
\[ \cos(a + \beta) = \cos a \cos \beta - \sin a \sin \beta. \]

87. Verify the following identities:
\[ \sin(a + \beta) = \sin a \cos \beta + \cos a \sin \beta. \]
HINT: \( \sin(a + \beta) = \cos\left(\frac{\pi}{2} - (a + \beta)\right) \)

88. Verify that
\[ \sin(a - \beta) = \sin a \cos \beta - \cos a \sin \beta. \]
HINT: \( \sin(a - \beta) = \cos\left(\frac{\pi}{2} - (a - \beta)\right) \)

89. Use Exercise 88 to show that
\[ \sin(a - \beta) = \sin a \cos \beta - \cos a \sin \beta. \]

90. It has been said that “all of trigonometry lies in the undulations of the sine wave.” Explain.

91. (a) Use a graphing utility to graph the polynomials
\[ f(x) = x^3 + 2x^2 - 5x^2 - 3x + 1, \]
\[ g(x) = -x^4 + x^3 + 4x^2 - 3x + 2. \]

(b) Based on your graphs in part (a), make a conjecture about the general shape of the graphs of polynomials of degree 4.

(c) Test your conjecture by graphing
\[ f(x) = x^4 - 4x^3 + 4x^2 + 2 \quad \text{and} \quad g(x) = -x^4. \]

1.7 COMBINATIONS OF FUNCTIONS

In this section we review the elementary ways of combining functions.

Algebraic Combinations of Functions

Here we discuss with some precision ideas that were used earlier without comment.

On the intersection of their domains, functions can be added and subtracted:
\[ (f + g)(x) = f(x) + g(x), \quad (f - g)(x) = f(x) - g(x); \]
they can be multiplied:
\[ (fg)(x) = f(x)g(x); \]
and, at the points where \( g(x) \neq 0 \), we can form the quotient:

\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}
\]

a special case of which is the reciprocal:

\[
\left( \frac{1}{g} \right)(x) = \frac{1}{g(x)}.
\]

**Example 1**

Let

\[
\begin{align*}
  f(x) &= \sqrt{x+3} \\
  g(x) &= \sqrt{5-x-2}
\end{align*}
\]

(a) Give the domain of \( f \) and of \( g \).

(b) Determine the domain of \( f + g \) and specify \((f + g)(x)\).

(c) Determine the domain of \( f/g \) and specify \((f/g)(x)\).

**SOLUTION**

(a) We can form \( \sqrt{x+3} \) iff \( x + 3 \geq 0 \), which holds iff \( x \geq -3 \). Thus \( \text{dom}(f) = [-3, \infty) \). We can form \( \sqrt{5-x-2} \) iff \( 5 - x \geq 0 \), which holds iff \( x \leq 5 \). Thus \( \text{dom}(g) = (-\infty, 5] \).

\[
\begin{align*}
  \text{dom}(f + g) &= \text{dom}(f) \cap \text{dom}(g) = [-3, \infty) \cap (-\infty, 5] = [-3, 5] \\
  (f + g)(x) &= f(x) + g(x) = \sqrt{x+3} + \sqrt{5-x-2}.
\end{align*}
\]

(b) To obtain the domain of the quotient, we must exclude from \([-3, 5]\) the numbers \( x \) at which \( g(x) = 0 \). There is only one such number: \( x = 1 \). Therefore

\[
\begin{align*}
  \text{dom}\left( \frac{f}{g} \right) &= \{ x \in [-3, 5] : x \neq 1 \} = [-3, 1) \cup (1, 5], \\
  \left( \frac{f}{g} \right)(x) &= \frac{f(x)}{g(x)} = \frac{\sqrt{x+3}}{\sqrt{5-x-2}}.
\end{align*}
\]

We can multiply functions \( f \) by real numbers \( \alpha \) and form what are called scalar multiples of \( f \):

\[
(\alpha f)(x) = \alpha f(x).
\]

With functions \( f \) and \( g \) and real numbers \( \alpha \) and \( \beta \), we can form linear combinations:

\[
(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x).
\]

These are just specific instances of the products and sums that we defined at the beginning of the section.

You have seen all these algebraic operations many times before:

(i) The polynomials are simply finite linear combinations of powers \( x^n \), each of which is a finite product of identity functions \( f(x) = x \). (Here we are taking the point of view that \( x^0 = 1 \).)

(ii) The rational functions are quotients of polynomials.

(iii) The secant and cosecant are reciprocals of the cosine and the sine.

(iv) The tangent and cotangent are quotients of sine and cosine.

**Vertical Translations (Vertical Shifts)**

Adding a positive constant \( c \) to a function raises the graph by \( c \) units. Subtracting a positive constant \( c \) from a function lowers the graph by \( c \) units. (Figure 1.7.1.)
Chapter 1.7: Combinations of Functions

You have seen how to combine functions algebraically. There is another (probably less familiar) way to combine functions, called composition. To describe it, we begin with two functions, \( f \) and \( g \), and a number \( x \) in the domain of \( g \). By applying \( g \) to \( x \), we get the number \( g(x) \). If \( g(x) \) is in the domain of \( f \), then we can apply \( f \) to \( g(x) \) and thereby obtain the number \( f(g(x)) \).

What is \( f(g(x)) \)? It is the result of first applying \( g \) to \( x \) and then applying \( f \) to \( g(x) \). The idea is illustrated in Figure 1.7.2. This new function—also in the domain of \( g \)—is called the composition of \( f \) with \( g \) and is denoted by \( f \circ g \). (See Figure 1.7.3.) The symbol \( f \circ g \) is read “\( f \) circle \( g \).”

**DEFINITION 1.7.1 COMPOSITION**

Let \( f \) and \( g \) be functions. For those \( x \) in the domain of \( g \) for which \( g(x) \) is in the domain of \( f \), we define the composition of \( f \) with \( g \), denoted \( f \circ g \), by setting

\[
(f \circ g)(x) = f(g(x)).
\]

In set notation,

\[
\text{dom}(f \circ g) = \{ x \in \text{dom}(g) : g(x) \in \text{dom}(f) \}.
\]

**Example 2** Suppose that

\[
g(x) = x^2
\]

(the squaring function)
and 

\[ f(x) = x + 3. \]

Then 

\[ (f \circ g)(x) = f(g(x)) = g(x) + 3 = x^2 + 3. \]

Thus, \( f \circ g \) is the function that first squares and then adds 3.

On the other hand, the composition of \( g \) with \( f \) gives 

\[ (g \circ f)(x) = g(f(x)) = (x + 3)^2. \]

Thus, \( g \circ f \) is the function that first adds 3 and then squares.

Since \( f \) and \( g \) are everywhere defined, both \( f \circ g \) and \( g \circ f \) are also everywhere defined. Note that \( g \circ f \) is not the same as \( f \circ g \). ❏

**Example 3** Let 

\[ f(x) = x^2 - 1 \]

and 

\[ g(x) = \sqrt{3 - x}. \]

The domain of \( g \) is \((-\infty, 3]\). Since \( f \) is everywhere defined, the domain of \( f \circ g \) is also \((-\infty, 3] \). On that interval 

\[ (f \circ g)(x) = f(g(x)) = (\sqrt{3 - x})^2 - 1 = (3 - x) - 1 = 2 - x. \]

Since \( g(f(x)) = \sqrt{1 - f(x)} \), we can form \( g(f(x)) \) only for those \( x \) in the domain of \( f \) for which \( f(x) \leq 3 \). As you can verify, this is the set \([-2, 2] \). On \([-2, 2] \), 

\[ (g \circ f)(x) = g(f(x)) = \sqrt{3 - (x^2 - 1)} = \sqrt{4 - x^2}. \]

**Horizontal Translations (Horizontal Shifts)**

Adding a positive constant \( c \) to the argument of a function shifts the graph \( c \) units left: the function \( g(x) = f(x + c) \) takes on at \( x \) the value that \( f \) takes on at \( x + c \). Subtracting a positive constant \( c \) from the argument of a function shifts the graph \( c \) units to the right: the function \( h(x) = f(x - c) \) takes on at \( x \) the value that \( f \) takes on at \( x - c \). (See Figure 1.7.4.)

[Figure 1.7.4]

![Figure 1.7.4](image)

We can form the composition of more than two functions. For example, the triple composition \( f \circ g \circ h \) consists of first \( h \), then \( g \), and then \( f \): 

\[ (f \circ g \circ h)(x) = f(g(h(x))). \]
We can go on in this manner with as many functions as we like.

**Example 4**

If \( f(x) = \frac{1}{x} \), \( g(x) = x^2 + 1 \), \( h(x) = \cos x \),

then

\[
(f \circ g \circ h)(x) = f(g(h(x))) = \frac{1}{g(h(x))} = \frac{1}{\cos^2 x + 1}
\]

**Example 5**

Find functions \( f \) and \( g \) such that \( f \circ g = F \) given that \( F(x) = (x + 1)^5 \).

**A Solution**

The function consists of first adding 1 and then taking the fifth power.

We can therefore set

\[ g(x) = x + 1 \]  

and

\[ f(x) = x^5. \]

As you can see,

\[
(f \circ g)(x) = f(g(x)) = [g(x)]^5 = (x + 1)^5.
\]

**Example 6**

Find three functions \( f \), \( g \), \( h \) such that \( f \circ g \circ h = F \) given that \( F(x) = |x| + 3 \).

**A Solution**

\( F \) takes the absolute value, adds 3, and then inverts. Let \( h \) take the absolute value:

\[ h(x) = |x|. \]

Let \( g \) add 3:

\[ g(x) = x + 3. \]

Let \( f \) do the inverting:

\[ f(x) = \frac{1}{x} \]

With this choice of \( f \), \( g \), \( h \), we have

\[
(f \circ g \circ h)(x) = f(g(h(x))) = \frac{1}{g(h(x))} = \frac{1}{h(x) + 3} = \frac{1}{|x| + 3}.
\]

**Exercises 1.7**

**Exercises 1–8.** Set \( f(x) = 2x^2 - 3x + 1 \) and \( g(x) = x^2 + 1/x \).

Calculate the indicated value.

1. \((f + g)(2)\).
2. \((f - g)(-1)\).
3. \((f \cdot g)(-2)\).
4. \((\frac{f}{g})(1)\).
5. \((2f - 3g)(\frac{1}{2})\).
6. \((\frac{f + 2g}{f})(-1)\).
7. \((f \circ g)(1)\).
8. \((g \circ f)(1)\).

**Exercises 9–12.** Determine \( f + g \), \( f - g \), \( f \cdot g \), \( f/g \), and give the domain of each.

9. \( f(x) = 2x - 3 \), \( g(x) = 2 - x \).
10. \( f(x) = x^2 - 1 \), \( g(x) = x + 1/x \).
11. \( f(x) = \sqrt{x} - 1 \), \( g(x) = x - \sqrt{x} + 3 \).
12. \( f(x) = \sin^2 x \), \( g(x) = \cos 2x \).
CHAPTER 1 PRECALCULUS REVIEW

13. Given that \( f(x) = x + 1/\sqrt{x} \) and \( g(x) = \sqrt{x} - 2/\sqrt{x} \), find (a) \( f + g \), (b) \( f - g \), (c) \( f/g \).

14. Given that 
\[
f(x) = \begin{cases} 1 - x, & x \leq 1 \\ x > 1 \end{cases}
\]
and \( g(x) = \begin{cases} 0, & x < 2 \\ -1, & x \geq 2 \end{cases} \), find \( f + g \), \( f - g \), \( f \cdot g \), \( f/g \).

Exercises 15–22. Sketch the graph with \( f \) and \( g \) as shown in the figure.

Exercises 23–30. Form the composition \( f \circ g \) and give the domain.
23. \( f(x) = 2x + 5 \), \( g(x) = x^2 \).
24. \( f(x) = x^2 \), \( g(x) = 2x + 5 \).
25. \( f(x) = \sqrt{x} \), \( g(x) = x^2 + 5 \).
26. \( f(x) = x^2 + x \), \( g(x) = \sqrt{x} \).
27. \( f(x) = 1/x \), \( g(x) = (x - 2)/x \).
28. \( f(x) = 1/x + 1 \), \( g(x) = x^2 \).
29. \( f(x) = \sqrt{x} \), \( g(x) = \cos 2x \).
30. \( f(x) = \sqrt{\pi} \), \( g(x) = 2 \cos x \) for \( x \in [0, 2\pi] \).

Exercises 31–34. Form the composition \( f \circ g \circ h \) and give the domain.
31. \( f(x) = 4x \), \( g(x) = x - 1 \), \( h(x) = x^2 \).
32. \( f(x) = x - 1 \), \( g(x) = 4x \), \( h(x) = x^2 \).
33. \( f(x) = x + 1 \), \( g(x) = \frac{1}{x} \), \( h(x) = x^2 \).
34. \( f(x) = \sqrt{x} \), \( g(x) = \frac{1}{x + 1} \), \( h(x) = x^2 \).

Exercises 35–38. Find \( f \) such that \( f \circ g = F \) given that
35. \( g(x) = \frac{1 + x^2}{x^2} \), \( F(x) = \frac{1 + x^4}{x^2} \).
36. \( g(x) = x^2 \), \( F(x) = ax^2 + b \).
37. \( g(x) = 3x \), \( F(x) = 2 \sin x \).
38. \( g(x) = -x^2 \), \( F(x) = \sqrt{\pi} + x^2 \).

Exercises 39–42. Find \( f \) such that \( f \circ g = F \) given that
39. \( f(x) = x^3 \), \( F(x) = (1 - 1/x^4)^2 \).
40. \( f(x) = x + 1 \), \( F(x) = x^2 + 1/\sqrt{x} \).
41. \( f(x) = x^2 + 1 \), \( F(x) = (2x^2 - 1)^2 + 1 \).
42. \( f(x) = \sin x \), \( F(x) = \sin 3x \).
Exercises 43–46. Find \( f \circ g \) given that \( f(x) = x^2 \) and \( g(x) = x \).
43. \( f(x) = x^2 + 1 \), \( g(x) = x^2 \).
44. \( f(x) = 3x + 1 \), \( g(x) = x^2 \).
45. \( f(x) = x^2 - 1 \), \( g(x) = \sin x \).
46. \( f(x) = x^2 + 1 \), \( g(x) = \sqrt{\pi} - x \).

47. Find \( g \) given that \( f \circ g = f(x) + c \).
48. Find \( f \) given that \( f \circ g = g(x) + c \).
49. Find \( g \) given that \( f \circ g = 1/f(x) \).
50. Find \( f \) given that \( f \circ g(x) = g(x) \).
51. Take \( f \) as a function on \([0, 2]\) with range \([0, 9]\) and take \( g \) as defined below.

Exercises 52–55. Suppose that \( f \) and \( g \) are odd functions. What can you conclude about \( f \circ g \)?
52. Suppose that \( f \) and \( g \) are even functions. What can you conclude about \( f \circ g \)?
53. Suppose that \( f \) is an even function and \( g \) is an odd function. What can you conclude about \( f \circ g \)?
54. Suppose that \( f \) is an even function and \( g \) is an odd function. What can you conclude about \( f \circ g \)?

55. Given that \( f(x) = x^2 \), \( x \geq 0 \), \( f \) is defined as follows:

\[ f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases} \]

How is \( f \) defined for \( x > 0 \) if \( (a) f \) is even? \( (b) f \) is odd?
56. For \( x > 0 \), \( f(x) = x^2 + 1 \). How is \( f \) defined for \( x < 0 \) if \( (a) f \) is even? \( (b) f \) is odd?
57. Given that \( f \) is defined for all real numbers, show that the function \( g(x) = f(x) + f(-x) \) is an even function.
58. Given that \( f \) is defined for all real numbers, show that the function \( h(x) = f(x) - f(-x) \) is an odd function.
59. Show that every function defined for all real numbers can be written as the sum of an even function and an odd function.
60. For \( x \neq 0, 1 \), define

\[ f(x) = 2, \quad f(x) = x. \]

This family of functions is closed under composition, that is, the composition of any two of these functions is again one of these functions. Tabulate the results of composing these functions one with the other by filling in the table shown in the figure. To indicate that \( f \circ f = f \), write “f” in the \( 2^\text{nd} \) row, \( 2^\text{nd} \) column. We have already made two entries in the table. Check out these two entries and then fill in the rest of the table.
1.8 A NOTE ON MATHEMATICAL PROOF; MATHEMATICAL INDUCTION

Mathematical Proof

The notion of proof goes back to Euclid’s *Elements*, and the rules of proof have changed little since they were formulated by Aristotle. We work in a deductive system where truth is argued on the basis of assumptions, definitions, and previously proved results. We cannot claim that such and such is true without clearly stating the basis on which we make that claim. A theorem is an implication; it consists of a hypothesis and a conclusion:

$$\text{if (hypothesis)} \ldots, \text{then (conclusion)} \ldots$$

Here is an example:

If $a$ and $b$ are positive numbers, then $ab$ is positive.

A common mistake is to ignore the hypothesis and persist with the conclusion: to insist, for example, that $ab > 0$ just because $a$ and $b$ are numbers.

Another common mistake is to confuse a theorem

$$\text{if } A \quad \text{then } B$$

with its converse

$$\text{if } B \quad \text{then } A.$$  

The fact that a theorem is true does not mean that its converse is true: While it is true that

$$\text{if } a \text{ and } b \text{ are positive numbers, then } ab \text{ is positive},$$
it is not true that

\[ (−2)(−3) \text{ is positive but } −2 \text{ and } −3 \text{ are not positive}. \]

A third, more subtle mistake is to assume that the hypothesis of a theorem represents the only condition under which the conclusion is true. There may well be other conditions under which the conclusion is true. Thus, for example, not only is it true that

\[ \text{if } a \text{ and } b \text{ are positive numbers, then } ab \text{ is positive} \]

but it is also true that

\[ \text{if } a \text{ and } b \text{ are negative numbers, then } ab \text{ is positive}. \]

In the event that a theorem

\[ \text{if } A, \text{ then } B \]

and its converse

\[ \text{if } B, \text{ then } A \]

are both true, then we can write

\[ A \text{ if and only if } B \]

or more briefly \( A \text{ iff } B \).

We know, for example, that

\[ \text{if } x \geq 0, \text{ then } |x| = x; \]

we also know that

\[ \text{if } |x| = x, \text{ then } x \geq 0. \]

We can summarize this by writing

\[ x \geq 0 \text{ iff } |x| = x. \]

**Remark** We’ll use “iff” frequently in this text but not in definitions. As stated earlier in a footnote, definitions are by their very nature iff statements. For example, we can say that “a number \( r \) is called a zero of \( P \) if \( P(r) = 0 \),” we don’t have to say “a number \( r \) is called a zero of \( P \) iff \( P(r) = 0 \).” In this situation, the “only if” part is taken for granted.

A final point. One way of proving

\[ \text{if } A, \text{ then } B \]

is to assume that

\[ A \text{ holds and } B \text{ does not hold} \]

and then arrive at a contradiction. The contradiction is taken to indicate that \( 1 \) is a false statement and therefore

\[ \text{if } A \text{ holds, then } B \text{ must hold}. \]

Some of the theorems of calculus are proved by this method.

Calculus provides procedures for solving a wide range of problems in the physical and social sciences. The fact that these procedures give us answers that seem to make sense is comforting, but it is only because we can prove our theorems that we can have confidence in the mathematics that is being applied. Accordingly, the study of calculus should include the study of some proofs.
1.8 Mathematical Induction

Mathematical induction is a method of proof which can be used to show that certain propositions are true for all positive integers $n$. The method is based on the following axiom:

1.8.1 AXIOM OF INDUCTION

Let $S$ be a set of positive integers. If

(A) $1 \in S,$ and

(B) $k \in S$ implies that $k + 1 \in S,$

then all the positive integers are in $S$.

You can think of the axiom of induction as a kind of “domino theory.” If the first domino falls (Figure 1.8.1), and if each domino that falls causes the next one to fall, then, according to the axiom of induction, each domino will fall.

While we cannot prove that this axiom is valid (axioms are by their very nature assumptions and therefore not subject to proof), we can argue that it is plausible.

Let’s assume that we have a set $S$ that satisfies conditions (A) and (B). Now let’s choose a positive integer $m$ and “argue” that $m \in S$.

From (A) we know that $1 \in S$. Since $1 \in S$, we know that $1 + 1 \in S$, and thus that $(1 + 1) + 1 \in S$, and so on. Since $m$ can be obtained from 1 by adding 1 successively $(m - 1)$ times, it seems clear that $m \in S$.

To prove that a given proposition is true for all positive integers $n$, we let $S$ be the set of positive integers for which the proposition is true. We prove first that $1 \in S$; that is, that the proposition is true for $n = 1$. Next we assume that the proposition is true for some positive integer $k$, and show that it is true for $k + 1$; that is, we show that $k \in S$ implies that $k + 1 \in S$. Then by the axiom of induction, we conclude that $S$ contains the set of positive integers and therefore the proposition is true for all positive integers.

Example 1 We’ll show that

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

for all positive integers $n$. 

![domino theory](image-url)
CHAPTER 1 PRECALCULUS REVIEW

**SOLUTION** Let $S$ be the set of positive integers $n$ for which

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.$$  

Then $1 \in S$ since

$$1 = \frac{1(1 + 1)}{2}.$$  

Next, we assume that $k \in S$; that is, we assume that

$$1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2}.$$  

Adding up the first $k + 1$ integers, we have

$$1 + 2 + 3 + \cdots + k + (k + 1) = \left[1 + 2 + 3 + \cdots + k\right] + (k + 1)$$

$$= \frac{k(k + 1)}{2} + (k + 1) \quad \text{(by the induction hypothesis)}$$

$$= \frac{k(k + 1) + 2(k + 1)}{2}$$

$$= \frac{(k + 1)(k + 2)}{2},$$

and so $k + 1 \in S$. Thus, by the axiom of induction, we can conclude that all positive integers are in $S$, that is, we can conclude that

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \quad \text{for all positive integers } n. \quad \square$$

**Example 2** We'll show that, if $x \geq -1$, then

$$(1 + x)^n \geq 1 + nx \quad \text{for all positive integers } n.$$  

**SOLUTION** We take $x \geq -1$ and let $S$ be the set of positive integers $n$ for which

$$(1 + x)^n \geq 1 + nx.$$  

Since

$$(1 + x)^1 = 1 + 1 \cdot x,$$

we have $1 \in S$.  

We now assume that $k \in S$. By the definition of $S,$

$$(1 + x)^k \geq 1 + kx.$$  

Since

$$(1 + x)^{k+1} = (1 + x)(1 + x)^k \geq (1 + kx)(1 + x) \quad \text{(explain)}$$

and

$$(1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x,$$

we can conclude that

$$(1 + x)^{k+1} \geq 1 + (k + 1)x$$

and thus that $k + 1 \in S$.  

We have shown that

$$1 \in S \quad \text{and that } \quad k \in S \implies k + 1 \in S.$$  

1.8 A NOTE ON MATHEMATICAL PROOF: MATHEMATICAL INDUCTION

By the axiom of induction, all positive integers are in S.

Remark An induction does not have to begin with the integer 1. If, for example, you want to show that some proposition is true for all integers \( n \geq 3 \), all you have to do is show that it is true for \( n = 3 \), and that, if it is true for \( n = k \), then it is true for \( n = k + 1 \). (Now you are starting the chain reaction by pushing on the third domino.)

EXERCISES 1.8

Exercises 1–10. Show that the statement holds for all positive integers \( n \).

1. \( 2n \leq 3n \).
2. \( 1 + 2n \leq 3n \).
3. \( 2^n + 2^n + 2^n + \cdots + 2^n = 2^n - 1 \).
4. \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \).
5. \( 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1) \).
6. \( 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 \).

HINT: Use Example 1.

7. \( 1^n + 2^n + \cdots + (n - 1)^n < n^n < 1^n + 2^n + \cdots + n^n \).
8. \( 1^n + 2^n + \cdots + (n + 1)^n < n^n < 1^n + 2^n + \cdots + n^n \).

9. \( \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < \sqrt{n} \).

10. \( \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} = \frac{n}{n + 1} \).

11. For what integers \( n \) is \( 3^n + 2^n \) divisible by 7? Prove that your answer is correct.

12. For what integers \( n \) is \( 9^n - 8n - 1 \) divisible by 64? Prove that your answer is correct.

13. Find a simplifying expression for the product

\[
\left( 1 - \frac{1}{2} \right) \left( 1 - \frac{1}{3} \right) \cdots \left( 1 - \frac{1}{n} \right)
\]

and verify its validity for all integers \( n \geq 2 \).

14. Find a simplifying expression for the product

\[
\left( 1 - \frac{1}{2} \right) \left( 1 - \frac{1}{3} \right) \cdots \left( 1 - \frac{1}{n} \right)
\]

and verify its validity for all integers \( n \geq 2 \).

15. Prove that an \( N \)-sided convex polygon has \( \frac{1}{2} N(N - 3) \) diagonals. Take \( N > 3 \).

16. Prove that the sum of the interior angles in an \( N \)-sided convex polygon is \( (N - 2)180^\circ \). Take \( N > 2 \).

17. Prove that all sets with \( n \) elements have \( 2^n \) subsets. Count the empty set \( \emptyset \) and the whole set as subsets.

18. Show that, given a unit length, for each positive integer \( n \), a line segment of length \( \sqrt{n} \) can be constructed by straight edge and compass.

19. Find the first integer \( n \) for which \( n^2 = n + 41 \) is a prime number.

CHAPTER 1. REVIEW EXERCISES

Exercises 1–4. Is the number rational or irrational?

1. \( \sqrt{25} \).
2. \( \sqrt{79} \).
3. \( 1.001001000 \ldots \).

Exercises 5–8. State whether the set is bounded above, bounded below, bounded. If the set is bounded above, give an upper bound; if it is bounded below, give a lower bound; if it is bounded, give an upper bound and a lower bound.

5. \( S = \{ 1, 3, 5, 7, \ldots \} \).
6. \( S = \{ x : x \leq 1 \} \).
7. \( S = \{ x : |x + 2| < 3 \} \).
8. \( S = \{ \sqrt{1/n^2} : n = 1, 2, 3, \ldots \} \).

Exercises 9–12. Find the real roots of the equation.

9. \( 2x^2 + x - 1 = 0 \).
10. \( x^2 + 2x + 5 = 0 \).
11. \( x^2 - 10x + 25 = 0 \).
12. \( 9x^3 - x = 0 \).

Exercises 13–22. Solve the inequality. Express the solution as an interval or as the union of intervals. Mark the solution on a number line.

13. \( 5x - 2 < 0 \).
14. \( 3x + 5 < \frac{1}{4}(4 - x) \).
15. \( x^2 - x - 6 \geq 0 \).
16. \( x(x - 3) - 2x + 1 \leq 0 \).
17. \( 2x + \frac{1}{x + 1} > 0 \).
18. \( x^2 - 2x - 3 \leq 0 \).
19. \( |x + 2| < 1 \).
20. \( |3x - 2| \geq 4 \).
21. \( \frac{2}{x + 4} > 2 \).
22. \( \frac{5}{x + 3} \leq 1 \).

Exercises 23–24. (a) Find the distance between the points \( P \) and \( Q \). (b) Find the midpoint of the line segment \( PQ \).

23. \( P(2, -3) \), \( Q(1, 4) \).
24. \( P(-3, -4) \), \( Q(-1, 6) \).
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Exercises 25–28. Find an equation for the line that passes through the point (2, –3) and is
25. parallel to the y-axis.
26. parallel to the line y = 1.
27. perpendicular to the line 2x – 3y = 6.
28. parallel to the line 3x + 4y = 12.

Exercises 29–30. Find the point where the lines intersect.
29. \( l_1 : x = 2y = 4 \), \( l_2 : 3x = 4y = 3 \).
30. \( l_1 : 3x = 4y = 2 \), \( l_2 : 3x + 2y = 0 \).

31. Find the point(s) where the line \( y = 8x – 6 \) intersects the parabola \( y = 2x^2 \).

32. Find an equation for the line tangent to the circle \( x^2 + y^2 + 2x – 6y = 3 = 0 \)
at the point (2, 1).

Exercises 33–38. Give the domain and range of the function.
33. \( f(x) = 4 – x^2 \).
34. \( f(x) = 3x – 2 \).
35. \( f(x) = \sqrt{x – 4} \).
36. \( f(x) = \sqrt{4 - x^2} \).
37. \( f(x) = \sqrt{4 - x^2} \).
38. \( f(x) = (x – 1)^2 \).

Exercises 39–40. Sketch the graph and give the domain and range of the function.
39. \( f(x) = \)
\[
\begin{align*}
& \begin{cases}
4 - 2x, & x \leq 2 \\
4 - x^2, & x > 2
\end{cases}
\end{align*}
\]
40. \( f(x) = \)
\[
\begin{align*}
& \begin{cases}
2 + x^2, & x < 0 \\
2 - x^2, & x > 0
\end{cases}
\end{align*}
\]

Exercises 41–44. Find the number(s) \( x \) in the interval \([0, 2\pi]\) which satisfy the equation.
41. \( \sin x = \frac{1}{2} \).
42. \( \cos 2x = \frac{1}{2} \).
43. \( \tan x = -1 \).
44. \( \sin 3x = 0 \).

Exercises 45–48. Sketch the graph of the function.
45. \( f(x) = \cos 2x \).
46. \( f(x) = -\cos 2x \).
47. \( f(x) = 3 \cos 2x \).
48. \( f(x) = \frac{1}{2} \cos 2x \).

Exercises 49–51. Form the combinations \( f \circ g, f - g, f' \cdot g, f / g \) and specify the domain of combination.
49. \( f(x) = 3x + 3 \), \( g(x) = x^2 - 1 \).
50. \( f(x) = x^2 - 4 \), \( g(x) = x + 1/x \).
51. \( f(x) = \cos^2 x \), \( g(x) = \sin 2x \) for \( x \in [0, 2\pi] \).

Exercises 52–54. Form the compositions \( f \circ g \) and \( g \circ f \) and specify the domain of each of these combinations.
52. \( f(x) = x^2 - 2x \), \( g(x) = x + 1 \).
53. \( f(x) = \sqrt{x + 1} \), \( g(x) = x^2 - 5 \).
54. \( f(x) = \sqrt{1 - x^2} \), \( g(x) = \sin 2x \).

55. (a) Write an equation in \( x \) and \( y \) for an arbitrary line \( l \) that passes through the origin.
(b) Verify that if \( P(a, b) \) lies on \( l \) and \( a \) is a real number, then the point \( Q(b, a) \) also lies on \( l \).
(c) What additional conclusion can you draw if \( a > 0 \) if \( a < 0 \)?

56. The roots of a quadratic equation. You can find the roots of a quadratic equation by resorting to the quadratic formula. The approach outlined below is more illuminating. Since division by the leading coefficient does not alter the roots of the equation, we can make the coefficient 1 and work with the equation
\[ x^2 + ax + b = 0. \]

(a) Show that the equation \( x^2 + ax + b = 0 \) can be written as
\[ (x - a)^2 - b^2 = 0, \] or
\[ (x - a)^2 = b^2. \]
(b) What are the roots of the equation \( x^2 + ax + b = 0 \)?
(c) What are the roots of the equation \( x^2 + ax + b = 0 \)?
(d) Show that the equation \( x^2 + ax + b = 0 \) has no real roots.

57. Knowing that
\[ |a + b| \leq |a| + |b| \] for all real \( a, b \) show that
\[ |a| - |b| \leq |a - b| \] for all real \( a, b \).

58. (a) Express the perimeter of a semicircle as a function of the diameter.
(b) Express the area of a semicircle as a function of the diameter.