Goals of this Book and Global Overview

0.1 WHAT IS THIS BOOK?

The goal of this book is to develop robust, accurate and efficient numerical methods to price a number of derivative products in quantitative finance. We focus on one-factor and multi-factor models for a wide range of derivative products such as options, fixed income products, interest rate products and ‘real’ options. Due to the complexity of these products it is very difficult to find exact or closed solutions for the pricing functions. Even if a closed solution can be found it may be very difficult to compute. For this and other reasons we need to resort to approximate methods. Our interest in this book lies in the application of the finite difference method (FDM) to these problems.

This book is a thorough introduction to FDM and how to use it to approximate the various kinds of partial differential equations for contingent claims such as:

- One-factor European and American options
- One-factor and two-factor barrier options with continuous and discrete monitoring
- Multi-asset options
- Asian options, continuous and discrete monitoring
- One-factor and two-factor bond options
- Interest rate models
- The Heston model and stochastic volatility
- Merton jump models and extensions to the Black–Scholes model.

Finite difference theory has a long history and has been applied for more than 200 years to approximate the solutions of partial differential equations in the physical sciences and engineering.

What is the relationship between FDM and financial engineering? To answer this question we note that the behaviour of a stock (or some other underlying) can be described by a stochastic differential equation. Then, a contingent claim that depends on the underlying is modelled by a partial differential equation in combination with some initial and boundary conditions. Solving this problem means that we have found the value for the contingent claim.

Furthermore, we discuss finite difference and variational schemes that model free and moving boundaries. This is the style for exercising American options, and we employ a number of new modelling techniques to locate the position of the free boundary.

Finally, we introduce and elaborate the theory of partial integro-differential equations (PIDEs), their applications to financial engineering and their approximations by FDM. In particular, we show how the basic Black–Scholes partial differential equation is augmented by an integral term in order to model jumps (the Merton model). Finally, we provide worked-out C++ code on the CD that accompanies this book.
0.2 WHY HAS THIS BOOK BEEN WRITTEN?

There are a number of reasons why this book has been written. First, the author wanted to produce a text that showed how to apply numerical methods (in this case, finite difference schemes) to quantitative finance. Furthermore, it is important to justify the applicability of the schemes rather than just rely on numerical recipes that are sometimes difficult to apply to real problems. The second desire was to construct robust finite difference schemes for use in financial engineering, creating algorithms that describe how to solve the discrete set of equations that result from such schemes and then to map them to C++ code.

0.3 FOR WHOM IS THIS BOOK INTENDED?

This book is for quantitative analysts, financial engineers and others who are involved in defining and implementing models for various kinds of derivatives products. No previous knowledge of partial differential equations (PDEs) or of finite difference theory is assumed. It is, however, assumed that you have some knowledge of financial engineering basics, such as stochastic differential equations, Ito calculus, the Black–Scholes equation and derivative pricing in general. This book will be of value to those financial engineers who use the binomial and trinomial methods to price options, as these two methods are special cases of explicit finite difference schemes. This book will also hopefully be employed by those engineers who use simulation methods (for example, the Monte Carlo method) to price derivatives, and it is hoped that the book will help to bridge the gap between the stochastics and PDE approaches. Finally, this book could be interesting for mathematicians, physicists and engineers who wish to see how a well-known branch of numerical analysis is applied to financial engineering. The information in the book may even improve your job prospects!

0.4 WHY SHOULD I READ THIS BOOK?

In the author’s opinion, this is one of the first self-contained introductions to the finite difference method and its applications to derivatives pricing. The book introduces the theory of PDE and FDM and their applications to quantitative finance, and can be used as a self-contained guide to learning and discovering the most important finite difference schemes for derivative pricing problems. Some of the advantages of the approach and the resulting added value of the book are:

- A defined process starting from the financial models through PDEs, FDM and algorithms
- An application of robust, accurate and efficient finite difference schemes for derivatives pricing applications.

This book is more than just a cookbook: it motivates why a method does or does not work and you can learn from this knowledge in a meaningful way. This book is also a good companion to my other book, *Financial Instrument Pricing in C++* (Duffy, 2004). The algorithms in the present book can be mapped to C++, the de-facto object-oriented language for financial engineering applications. In short, it is hoped that this book will help you to master all the details needed for a good understanding of FDM in your daily work.
0.5 THE STRUCTURE OF THIS BOOK

The book has been partitioned into seven parts, each of which deals with one specific topic in detail. Furthermore, each part contains material that is required by its successor. In general, we interleave the parts by first discussing the theory (for example, basic finite difference schemes) in a given part and then applying this theory to a problem in financial engineering. This ‘separation of concerns’ approach promotes understandability of the material, and the parts in the book discuss the following topics:

I. The Continuous Theory of Partial Differential Equations
II. Finite Difference Methods: the Fundamentals
III. Applying FDM to One-Factor Instrument Pricing
IV. FDM for Multidimensional Problems
V. Applying FDM to Multi-Factor Instrument Pricing
VI. Free and Moving Boundary Value Problems
VII. Design and Implementation in C++

Part I presents an introduction to partial differential equations (PDE). This theory may be new for some readers and for this reason these equations are discussed in some detail. The relevance of PDE to instrument pricing is that a contingent claim or derivative can be modelled as an initial boundary value problem for a second-order parabolic partial differential equation. The partial differential equation has one time variable and one or more space variables. The focus in Part I is to develop enough mathematical theory to provide a basis for work on finite differences.

Part II is an introduction to the finite difference method for a number of partial differential equations that appear in instrument pricing problems. We learn FDM in the following way:
(1) We introduce the model PDEs for the heat, convection and convection–diffusion equations and propose several important finite difference schemes to approximate them. In particular, we discuss a number of schemes that are used in the financial engineering literature and we also introduce some special schemes that work under a range of parameter values. In this part, focus is on the practical application of FDM to parabolic partial differential equations in one space variable.

Part III examines the partial differential equations that describe one-factor instrument models and their approximation by the finite difference schemes. In particular, we concentrate on European options, barrier options and options with jumps, and propose several finite difference schemes for such options. An important class of problems discussed in this part is the class of barrier options with continuous or discrete monitoring and robust methods are proposed for each case. Finally, we model the partial integro-differential equations (PIDEs) that describe options with jumps, and we show how to approximate them by finite difference schemes.

Part IV discusses how to define and use finite difference schemes for initial boundary value problems in several space variables. First, we discuss ‘direct’ scheme where we discretise the time and space dimensions simultaneously. This approach works well with problems in two space dimensions but for problems in higher dimensions we may need to solve the problem as a series of simpler problems. There are two main contenders: first, alternating direction implicit (ADI) methods are popular in the financial engineering literature; second, we discuss operator splitting methods (or the method of fractional steps) that have their origins in the former Soviet Union. Finally, we discuss some modern developments in this area.
Part V applies the results and schemes from Part IV to approximating some multi-factor problems. In particular, we examine the Heston PDE with stochastic volatility, Asian options, rainbow options and two-factor bond models and how to apply ADI and operator splitting methods to them.

Part VI deals with instrument pricing problems with the so-called early exercise feature. Mathematically, these problems fall under the umbrella of free and moving boundary value problems. We concentrate on the theory of such problems and the application to one-factor American options. We also discuss ADI method in conjunction with free boundaries.

Part VII contains a number of chapters that support the work in the previous parts of the book. Here we address issues that are relevant to the design and implementation of the FDM algorithms in the book. We provide hints, guidelines and C++ sources to help the reader to make the transition to production code.

0.6 WHAT THIS BOOK DOES NOT COVER

This book is concerned with the application of the finite difference method to instrument pricing. This viewpoint implies that we concentrate on a number of issues while neglecting others. Thus, this book is not:

- an introduction to numerical analysis
- a guide to the theoretical foundations of the theory of finite differences
- an introduction to instrument pricing
- a full ‘production’ C++ course.

These problems are considered in detail in other books and will be discussed elsewhere.

0.7 CONTACT, FEEDBACK AND MORE INFORMATION

The author welcomes your feedback, comments and suggestions for improvement. As far as I am aware, all typos and errors have been removed from the text, but some may have slipped past unnoticed. Nevertheless, all errors are my responsibility.

I am a trainer and developer and my main professional interests are in quantitative finance, computational finance and object-oriented programming. In my free time I enjoy judo and studying foreign (natural) languages.

If you have any questions on this book, please do not hesitate to contact me at dduffy@datasim.nl.