Chapter 1

MAGNETIC AND MAGNETICALLY COUPLED CIRCUITS

1.1 INTRODUCTION

Before diving into the analysis of electromechanical motion devices, it is helpful to review briefly some of our previous work in physics and in basic electric circuit analysis. In particular, the analysis of magnetic circuits, the basic properties of magnetic materials, and the derivation of equivalent circuits of stationary, magnetically coupled devices are topics presented in this chapter. Much of this material will be a review for most, since it is covered either in a sophomore physics course for engineers or in introductory electrical engineering courses in circuit theory. Nevertheless, reviewing this material and establishing concepts and terms for later use sets the appropriate stage for our study of electromechanical motion devices.

Perhaps the most important new concept presented in this chapter is the fact that in all electromechanical devices, mechanical motion must occur, either translational or rotational, and this motion is reflected into the electric system either as a change of flux linkages in the case of an electromagnetic system or as a change of charge in the case of an electrostatic system. We will deal primarily with electromagnetic systems. If the magnetic system is linear, then the change in flux linkages results, owing to a change in the inductance. In other words, we will find that the inductances of the electric
circuits associated with electromechanical motion devices are functions of the mechanical motion. In this chapter, we shall learn to express the self- and mutual inductances for simple translational and rotational electromechanical devices, and to handle these changing inductances in the voltage equations describing the electric circuits associated with the electromechanical system.

Throughout this text, we will give short problems (SPs) with answers following most sections. If we have done our job, each short problem should take less than ten minutes to solve. Also, it may be appropriate to skip or deemphasize some material in this chapter depending upon the background of the students. For example, those familiar with the concept of phasors may opt to skip all or most of the following section. At the close of each chapter, we shall take a moment to look back over some of the important aspects of the material that we have just covered and mention what is coming next and how we plan to fit things together as we go along.

## 1.2 PHASOR ANALYSIS

Phasors are used to analyze steady-state performance of ac circuits and devices. This concept can be readily established by expressing a steady-state sinusoidal variable as

\[ F_a = F_p \cos \theta_{ef} \quad (1.2-1) \]

where capital letters are used to denote steady-state quantities and \( F_p \) is the peak value of the sinusoidal variation, which is generally voltage or current but could be any electrical or mechanical sinusoidal variable. For steady-state conditions, \( \theta_{ef} \) may be written as

\[ \theta_{ef} = \omega_c t + \theta_{ef}(0) \quad (1.2-2) \]

where \( \omega_c \) is the electrical angular velocity and \( \theta_{ef}(0) \) is the time-zero position of the electrical variable. Substituting (1.2-2) into (1.2-1) yields

\[ F_a = F_p \cos[\omega_c t + \theta_{ef}(0)] \quad (1.2-3) \]

Since

\[ e^{j\alpha} = \cos \alpha + j \sin \alpha \quad (1.2-4) \]

equation (1.2-3) may also be written as

\[ F_a = \text{Re} \left\{ F_p e^{j[\omega_c t + \theta_{ef}(0)]} \right\} \quad (1.2-5) \]
where \( \text{Re} \) is shorthand for the "real part of." Equations (1.2-3) and (1.2-5) are equivalent. Let us rewrite (1.2-5) as

\[
F_a = \text{Re} \left\{ F_p e^{j\theta_{ef}^{(0)}} e^{j\omega_ft} \right\} \tag{1.2-6}
\]

We need to take a moment to define what is referred to as the root-mean-square (rms) of a sinusoidal variation. In particular, the rms value is defined as

\[
F = \left( \frac{1}{T} \int_0^T F_a^2(t) dt \right)^{\frac{1}{2}} \tag{1.2-7}
\]

where \( F \) is the rms value of \( F_a(t) \) and \( T \) is the period of the sinusoidal variation. It is left to the reader to show that the rms value of (1.2-3) is \( F_p/\sqrt{2} \). Therefore, we can express (1.2-6) as

\[
F_a = \text{Re} \left[ \sqrt{2} F e^{j\theta_{ef}^{(0)}} e^{j\omega_ft} \right] \tag{1.2-8}
\]

By definition, the phasor representing \( F_a \), which is denoted with a raised tilde, is

\[
\tilde{F}_a = F e^{j\theta_{ef}^{(0)}} \tag{1.2-9}
\]

which is a complex number. The reason for using the rms value as the magnitude of the phasor will be addressed later in this section. Equation (1.2-6) may now be written as

\[
F_a = \text{Re} \left[ \sqrt{2} \tilde{F}_a e^{j\omega_ft} \right] \tag{1.2-10}
\]

A shorthand notation for (1.2-9) is

\[
\tilde{F}_a = F / \theta_{ef}^{(0)} \tag{1.2-11}
\]

Equation (1.2-11) is commonly referred to as the polar form of the phasor. The cartesian form is

\[
\tilde{F}_a = F \cos \theta_{ef}^{(0)} + jF \sin \theta_{ef}^{(0)} \tag{1.2-12}
\]

When using phasors to calculate steady-state voltages and currents, we think of the phasors as being stationary at \( t = 0 \). On the other hand, a phasor is related to the instantaneous value of the sinusoidal quantity it represents. Let us take a moment to consider this aspect of the phasor and, thereby, give some physical meaning to it. From (1.2-4), we realize that \( e^{j\omega_ft} \) is a constant-amplitude line of unity length rotating counterclockwise at an angular velocity of \( \omega_e \). Therefore,
\[ \sqrt{2} F e^{j\omega_e t} = \sqrt{2} F \left\{ \cos[\omega_e t + \theta_{ef}(0)] + j \sin[\omega_e t + \theta_{ef}(0)] \right\} \]  
(1.2-13)

is a constant-amplitude line \( \sqrt{2} F \) in length rotating counterclockwise at an angular velocity of \( \omega_e \) with a time-zero displacement from the positive real axis of \( \theta_{ef}(0) \). Since \( \sqrt{2} F \) is the peak value of the sinusoidal variation, the instantaneous value of \( F_a \) is the real part of (1.2-13). In other words, the real projection of the phasor \( \tilde{F}_a \) is the instantaneous value of \( F_a/\sqrt{2} \) at time zero. As time progresses, \( \tilde{F}_a e^{j\omega_e t} \) rotates at \( \omega_e \) in the counterclockwise direction, and its real projection, in accordance with (1.2-10), is the instantaneous value of \( F_a/\sqrt{2} \). Thus, for

\[ F_a = \sqrt{2} F \cos \omega_e t \]  
(1.2-14)

the phasor representing \( F_a \) is

\[ \tilde{F}_a = F e^{j0} = F/0^\circ = F + j0 \]  
(1.2-15)

For

\[ F_a = \sqrt{2} F \sin \omega_e t \]  
(1.2-16)

the phasor is

\[ \tilde{F}_a = F e^{-j\pi/2} = F/-90^\circ = 0 - jF \]  
(1.2-17)

Although there are several ways to arrive at (1.2-17) from (1.2-16), it is helpful to ask yourself where must the rotating phasor be positioned at time zero so that, when it rotates counterclockwise at \( \omega_e \), its real projection is \( (1/\sqrt{2})F_p \sin \omega_e t \)? Is it clear that a phasor of amplitude \( F \) positioned at \( \pi/2 \) represents \( -\sqrt{2} F \sin \omega_e t \)?

In order to show the facility of the phasor in the analysis of steady-state performance of ac circuits and devices, it is useful to consider a series circuit consisting of a resistance, an inductance, and a capacitance. Thus,

\[ v_a = R i_a + L \frac{di_a}{dt} + \frac{1}{C} \int i_a dt \]  
(1.2-18)

For steady-state operation, let

\[ V_a = \sqrt{2} V \cos[\omega_e t + \theta_{ev}(0)] \]  
(1.2-19)

\[ I_a = \sqrt{2} I \cos[\omega_e t + \theta_{av}(0)] \]  
(1.2-20)

where the subscript \( a \) is used to distinguish the instantaneous value from the
rms value of the steady-state variable. The steady-state voltage equation may be obtained by substituting (1.2-19) and (1.2-20) into (1.2-18), whereupon we can write

\[
\sqrt{2}V \cos[\omega_c t + \theta_{ei}(0)] = R\sqrt{2}I \cos[\omega_c t + \theta_{ei}(0)] \\
+ \omega_c L \sqrt{2}I \cos[\omega_c t + \frac{1}{2}\pi + \theta_{ei}(0)] \\
+ \frac{1}{\omega_c C} \sqrt{2}I \cos[\omega_c t - \frac{1}{2}\pi + \theta_{ei}(0)]
\]  

(1.2-21)

The second term in the right-hand side of (1.2-21), which is \(L \frac{dI_a}{dt}\), can be written

\[
\omega_c L \sqrt{2}I \cos [\omega_c t + \frac{1}{2}\pi + \theta_{ei}(0)] = \omega_c L Re[\sqrt{2}Ie^{j\frac{1}{2}\pi}e^{j\theta_{ei}(0)}e^{j\omega_c t}]
\]  

(1.2-22)

Since \(\tilde{I}_a = Ie^{j\theta_{ei}(0)}\), we can write

\[
L \frac{d\tilde{I}_a}{dt} = \omega_c Le^{j\frac{1}{2}\pi} \tilde{I}_a
\]  

(1.2-23)

Since \(e^{j\frac{1}{2}\pi} = j\), (1.2-23) may be written

\[
L \frac{d\tilde{I}_a}{dt} = j\omega_c L \tilde{I}_a
\]  

(1.2-24)

If we follow a similar procedure, we can show that

\[
\frac{1}{C} \int I_a dt = -j \frac{1}{\omega_c C} \tilde{I}_a
\]  

(1.2-25)

It is interesting that differentiation of a steady-state sinusoidal variable rotates the phasor counterclockwise by \(\frac{1}{2}\pi\), whereas integration rotates the phasor clockwise by \(\frac{1}{2}\pi\).

The steady-state voltage equation given by (1.2-21) can be written in phasor form as

\[
\tilde{V}_a = \left[ R + j(\omega_c L - \frac{1}{\omega_c C}) \right] \tilde{I}_a
\]  

(1.2-26)

We can express (1.2-26) compactly as

\[
\tilde{V}_a = Z \tilde{I}_a
\]  

(1.2-27)

where \(Z\), the impedance, is a complex number; it is not a phasor. It is often expressed as
\[ Z = R + j(X_L - X_C) \]  
\hspace{1cm} (1.2-28)

where \( X_L = \omega_L L \) is the inductive reactance and \( X_C = \frac{1}{\omega_C C} \) is the capacitive reactance.

The instantaneous power is

\[ P = V_a I_a \]
\[ = \sqrt{2}V \cos[\omega_e t + \theta_{ev}(0)] \sqrt{2}I \cos[\omega_e t + \theta_{ei}(0)] \]  
\hspace{1cm} (1.2-29)

After some manipulation, we can write (1.2-29) as

\[ P = V I \cos[\theta_{ev}(0) - \theta_{ei}(0)] + V I \cos[2\omega_e t + \theta_{ev}(0) + \theta_{ei}(0)] \]  
\hspace{1cm} (1.2-30)

Therefore, the average power \( P_{\text{ave}} \) may be written

\[ P_{\text{ave}} = |\tilde{V}||\tilde{I}| \cos[\theta_{ev}(0) - \theta_{ei}(0)] \]  
\hspace{1cm} (1.2-31)

where \( |\tilde{V}| \) and \( |\tilde{I}| \) are the magnitude of the phasors (rms value), \( \theta_{ev}(0) - \theta_{ei}(0) \) is the power factor angle \( \varphi_{pf} \), and \( \cos[\theta_{ev}(0) - \theta_{ei}(0)] \) is referred to as the power factor. If current is positive in the direction of voltage drop then (1.2-31) is positive if power is consumed and negative if power is generated. It is interesting to point out that in going from (1.2-29) to (1.2-30), the coefficient of the two right-hand terms is \( \frac{1}{2}(\sqrt{2}V\sqrt{2}I) \) or one-half the product of the peak values of the sinusoidal variables. Therefore, it was considered more convenient to use the rms values for the phasors, whereupon average power could be calculated by the product of the magnitude of the voltage and current phasors as given by (1.2-31).

We see from (1.2-30) that the instantaneous power of a single-phase ac circuit oscillates at \( 2\omega_e t \) about an average value. Let us take a moment to calculate the steady-state power of a two-phase ac system. Balanced, steady-state, two-phase variables (\( a \) and \( b \) phase) may be expressed as

\[ V_a = \sqrt{2}V \cos[\omega_e t + \theta_{ev}(0)] \]  
\hspace{1cm} (1.2-32)

\[ I_a = \sqrt{2}I \cos[\omega_e t + \theta_{ei}(0)] \]  
\hspace{1cm} (1.2-33)

\[ V_b = \sqrt{2}V \cos[\omega_e t - \frac{1}{2}\pi + \theta_{ev}(0)] \]  
\hspace{1cm} (1.2-34)

\[ I_b = \sqrt{2}I \cos[\omega_e t - \frac{1}{2}\pi + \theta_{ei}(0)] \]  
\hspace{1cm} (1.2-35)

The total instantaneous power is
\[ P = V_a I_a + V_b I_b \quad (1.2-36) \]

Substituting (1.2-32) through (1.2-35) into (1.2-36) and after some trigonometric manipulation, the total power for a balanced two-phase system becomes

\[ P = 2 |\tilde{V}_a| |\tilde{I}_a| \cos \varphi_{pf} \quad (1.2-37) \]

It is important to note that the \(2\omega_c t\) oscillation is not present. In other words, the total instantaneous steady-state power is constant. In the case of a three-phase balanced system, the phasors of the three voltages or currents are displaced 120° and the instantaneous steady-state power is also constant and three times the average power of one phase. In other words the 2 in (1.2-37) becomes 3 when considering a three-phase system.

**Example 1A.** It is often instructive to construct a phasor diagram. For example, let us consider a voltage equation of the form

\[ \tilde{V} = Z \tilde{I} + \tilde{E} \quad (1A-1) \]

where \(Z\) is given by (1.2-28). Let us assume that \(\tilde{V}\) and \(\tilde{I}\) are known and that we are to calculate \(\tilde{E}\). The phasor diagram may be used as a rough check on these calculations. Let us construct this phasor diagram by assuming that \(|X_L| > |X_C|\) and \(\tilde{V}\) and \(\tilde{I}\) are known as shown in Fig. 1A-1. Solving (1A-1) for \(\tilde{E}\) yields

\[ \tilde{E} = \tilde{V} - [R + j(X_L - X_C)]\tilde{I} \quad (1A-2) \]

To perform this graphically, start at the origin in Fig. 1A-1 and walk to the terminus of \(\tilde{V}\). Now, we want to subtract \(R\tilde{I}\). To achieve the proper orientation to do this, stand at the terminus of \(\tilde{V}\), turn, and look in the \(\tilde{I}\) direction which is at the angle \(\theta_{el}(0)\). But we must subtract \(R\tilde{I}\); hence, \(-\tilde{I}\) is 180° from \(\tilde{I}\), so do an about-face and now we are headed in the \(-\tilde{I}\) direction, which is \(\theta_{el}(0) - 180°\). Start walking in the direction of \(-\tilde{I}\) for the distance \(R|\tilde{I}|\) and then stop. While still facing in the \(-\tilde{I}\) direction, let us consider the next term. Now since we have assumed that \(|X_L| > |X_C|\), we must subtract \(j(X_L - X_C)\tilde{I}\), so let us face in the direction of \(-j\tilde{I}\). We are still looking in the \(-\tilde{I}\) direction, so we need only to j ourselves. Thus, we must rotate 90° in the counterclockwise direction, whereupon we are standing at the end of \(\tilde{V} - R\tilde{I}\) looking in the direction of \(\theta_{el}(0) - 180° + 90°\). Start walking in this direction for the distance of \((X_L - X_C)|\tilde{I}|\), whereupon we are at
the terminus of $\vec{V} - [R + j(X_L - X_C)]\vec{I}$. According to (1A-2), $\vec{E}$ is the phasor drawn from the origin of the phasor diagram to where we are.

The average steady-state power for a single-phase circuit may be calculated using (1.2-31). We will mention in passing that the reactive power is defined as

$$Q = |\vec{V}| |\vec{I}| \sin[\theta_{ev}(0) - \theta_{ei}(0)] \quad (1A-3)$$

The units of $Q$ are var (voltampere reactive). An inductance is said to absorb reactive power and thus, by definition, $Q$ is positive for an inductor and negative for a capacitor. Actually, $Q$ is a measure of the interchange of energy stored in the electric (capacitor) and magnetic (inductance) fields.

**SP1.2-1** If $\vec{V} = 1/0^\circ$ and $\vec{I} = 1/180^\circ$ in the direction of the voltage drop, calculate $Z$ and $P_{ave}$. Is power generated or consumed? [(-1 + j0) ohms, 1 watt, generated]

**SP1.2-2** For SP1.2-1, express instantaneous voltage, current, and power if the frequency is 60 Hz. [$V = \sqrt{2}\cos 377t$, $I = \sqrt{2}\cos(377t + \pi)$, $P = -1 + 1\cos(754t + \pi)$]

**SP1.2-3** $A = \sqrt{2}/0^\circ$, $B = \sqrt{2}/90^\circ$. Calculate $A + B$ and $A \times B$. [2/45°, 2/90°]

**SP1.2-4** In Example 1A, $X_L > X_C$ and yet $\vec{I}$ was given as leading $\vec{V}$. How can this be? [EB]
1.3 MAGNETIC CIRCUITS

An elementary magnetic circuit is shown in Fig. 1.3-1. This system consists of an electric conductor wound \( N \) times about the magnetic member, which is generally some type of ferromagnetic material. In this example system, the magnetic member contains an air gap of uniform length between points \( a \) and \( b \). We will assume that the magnetic system (circuit) consists only of the magnetic member and the air gap. Recall that Ampere’s law states that the line integral of the field intensity \( \mathbf{H} \) about a closed path is equal to the net current enclosed within this closed path of integration. That is,

\[
\oint \mathbf{H} \cdot d\mathbf{L} = i_n \tag{1.3-1}
\]

where \( i_n \) is the net current enclosed. Let us apply Ampere’s law to the closed path depicted as a dashed line in Fig. 1.3-1. In particular,

\[
\int_a^b H_i dL + \int_b^a H_g dL = Ni \tag{1.3-2}
\]

![Diagram of a magnetic circuit](image)

Figure 1.3-1: Elementary magnetic circuit.

where the path of integration is assumed to be in the clockwise direction. This equation requires some explanation. First, we are assuming that the field intensity exists only in the direction of the given path, hence we have dropped the vector notation. The subscript \( i \) denotes the field intensity \( (H_i) \) in the ferromagnetic material (iron or steel) and \( g \) denotes the field intensity \( (H_g) \) in the air gap. The path of integration is taken as the mean
length about the magnetic member, for purposes we shall explain later. The right-hand side of (1.3-2) represents the net current enclosed. In particular, we have enclosed the current \( i \), \( N \) times. This has the units of amperes but is commonly referred to as ampere-turns (At) or magnetomotive force (mmf). We will find that the mmf in magnetic circuits is analogous to the electromotive force (emf) in electric circuits. Note that the current enclosed is positive in (1.3-2) if the current \( i \) is positive. The sign of the right-hand side of (1.3-2) may be determined by the so-called "corkscrew" rule. That is, the current enclosed is positive if its assumed positive direction is in the same direction as the advance of a right-hand screw if it were turned in the direction of the path of integration, which in Fig. 1.3-1 is clockwise. Before continuing, it should be mentioned that we refer to \( \mathbf{H} \) as the field intensity; however, some authors prefer to call \( \mathbf{H} \) the field strength.

If we carry out the line integration, (1.3-2) can be written

\[
H_i l_i + H_g l_g = N i \tag{1.3-3}
\]

where \( l_i \) is the mean length of the magnetic material and \( l_g \) is the length across the air gap. Now, we have some explaining to do. We have assumed that the magnetic circuit consists only of the ferromagnetic material and the air gap, and that the magnetic field intensity is always in the direction of the path of integration or, in other words, perpendicular to a cross section of the magnetic material taken in the same sense as the air gap is cut through the material. The assumed direction of the magnetic field intensity is valid except in the vicinity of the corners. The direction of the field intensity changes gradually rather than abruptly at the corners. Nevertheless, the "mean length approximation" is widely used as an adequate means of analyzing this type of magnetic circuit.

Let us now take a cross section of the magnetic material as shown in Fig. 1.3-2. From our study of physics, we know that for linear, isotropic magnetic materials the flux density \( \mathbf{B} \) is related to the field intensity as

\[
\mathbf{B} = \mu \mathbf{H} \tag{1.3-4}
\]

Where \( \mu \) is the permeability of the medium. Hence, we can write (1.3-3) in terms of flux density as

\[
\frac{B_i}{\mu_i} l_i + \frac{B_g}{\mu_g} l_g = N i \tag{1.3-5}
\]

The surface integral of the flux density is equal to the flux \( \Phi \), thus
Figure 1.3-2: Cross section of magnetic material.

\[ \Phi = \int_A \mathbf{B} \cdot d\mathbf{S} \]  
(1.3-6)

If we assume that the flux density is uniform over the cross-sectional area, then

\[ \Phi_i = B_i A_i \]  
(1.3-7)

where \( \Phi_i \) is the total flux in the magnetic material and \( A_i \) is the associated cross-sectional area. In the air gap,

\[ \Phi_g = B_g A_g \]  
(1.3-8)

where \( A_g \) is the cross-sectional area of the gap. From physics, it is known that the streamlines of flux density \( \mathbf{B} \) are closed; hence, the flux in the air gap is equal to the flux in the core. That is, \( \Phi_i = \Phi_g \); and, if the air gap is small, \( A_i \approx A_g \), and, therefore, \( B_i \approx B_g \). However, the effective area of the air gap is larger than that of the magnetic material, since the flux will tend to balloon or spread out (fringing effect), covering a maximum area midway across the air gap. Generally, this is taken into account by assuming that \( A_g = kA_i \), where \( k \), which is greater than unity, is determined primarily by the length of the air gap. Although we shall keep this in mind, it is sufficient for our purposes to assume \( A_g = A_i \). If we let \( \Phi_i = \Phi_g = \Phi \) and substitute (1.3-7) and (1.3-8) into (1.3-5), we obtain

\[ \frac{l_i}{\mu_i A_i} \Phi + \frac{l_g}{\mu_g A_g} \Phi = N i \]  
(1.3-9)

The analogy to Ohm's law is at hand. \( N i \) (m.m.f) is analogous to the voltage (emf), and the flux \( \Phi \) is analogous to the current. We can complete this analogy if we recall that the resistance of a conductor is proportional to its
length and inversely proportional to its conductivity and cross-sectional area. Similarly, \( l_i/\mu_i A_i \) and \( l_g/\mu_g A_g \) are the reluctances of the magnetic material and air gap, respectively. Generally, the permeability is expressed in terms of relative permeability as

\[
\mu_i = \mu_{ri}\mu_0 \tag{1.3-10}
\]

\[
\mu_g = \mu_{rg}\mu_0 \tag{1.3-11}
\]

where \( \mu_0 \) is the permeability of free space \((4\pi \times 10^{-7} \text{ Wb/A} \cdot \text{m})\) and \( \mu_{ri} \) and \( \mu_{rg} \) are the relative permeability of the magnetic material and the air gap, respectively. For all practical purposes, \( \mu_{rg} = 1 \); however, \( \mu_{ri} \) may be as large as 500 to 4000 depending upon the type of ferromagnetic material. We will use \( \mathcal{R} \) to denote reluctance so as to distinguish reluctance from resistance, which will be denoted by \( r \) or \( R \). We can now write (1.3-9) as

\[
(\mathcal{R}_i + \mathcal{R}_g)\Phi = Ni \tag{1.3-12}
\]

where \( \mathcal{R}_i \) and \( \mathcal{R}_g \) are the reluctance of the iron and air gap, respectively.

**Example 1B.** A magnetic system is shown in Fig. 1B-1. The total number of turns is 100, the relative permeability of the iron is 1000, and the current is 10 A. Calculate the total flux in the center leg.

Let us draw the electric circuit analog of this magnetic system for which we will need to calculate the reluctance of the various paths:

\[
\mathcal{R}_{ab} = \frac{L_{ab}}{\mu_{ri}\mu_0 A_i} = \frac{0.22}{1000(4\pi \times 10^{-7})(0.04)^2} = 109,419 \text{ H}^{-1} \tag{1B-1}
\]

Similarly,

\[
\mathcal{R}_{bcd} = \frac{0.25 + 0.22 + 0.25}{(1000)(4\pi \times 10^{-7})(0.04)^2} = 358,099 \text{ H}^{-1} \tag{1B-2}
\]

Neglecting the air gap length,

\[
\mathcal{R}_{bef} = \mathcal{R}_{gha} = \frac{1}{4} \mathcal{R}_{bcd} = 179,049 \text{ H}^{-1} \tag{1B-3}
\]
The reluctance of the air gap is

$$R_{fg} = \frac{0.002}{(4\pi \times 10^{-7})(0.04)^2} = 994,718 \text{ H}^{-1} \quad (1B-4)$$

The electric circuit analog is given in Fig. 1B-2. The polarity of the mmf is determined by the right-hand rule. That is, if we grasp one of the turns of the winding with our right hand with the thumb pointed in the direction of positive current, then our fingers will point in the direction of positive flux which flows in the direction of an mmf rise. Or if we grasp the winding (center leg) with the fingers of our right hand in the direction of positive current, then our thumb will be in the direction of positive flux and in the direction of a rise in mmf.

We can now apply dc circuit theory to solve for the total flux, $\Phi_1 + \Phi_2$, flowing in the center leg. For example, we can use loop equations or, as we will do here, reduce the series-parallel circuit to an equivalent reluctance. The equivalent reluctance of the parallel combination is
\[
R_{eq} = \frac{(R_{beda})(R_{bef} + R_{fg} + R_{gha})}{R_{beda} + R_{bef} + R_{fg} + R_{gha}} \\
= \frac{(358,099)(179,049 + 994,718 + 179,049)}{358,099 + 179,049 + 994,718 + 179,049} \\
= \frac{(358,099)(1,352,816)}{1,710,915} = 283,148 \text{ H}^{-1} \quad (1B-5)
\]

\[
\Phi_1 + \Phi_2 = \frac{N_i}{R_{ob} + R_{eq}} \\
= \frac{(100)(10)}{109,419 + 283,148} = 2.547 \times 10^{-3} \text{ Wb} \quad (1B-6)
\]

**Example 1C.** Consider the magnetic system shown in Fig. 1C-1. The windings are supplied from ac sources and, in the steady state, \( I_1 = \sqrt{2} \cos \omega t \) and \( I_2 = \sqrt{2} 0.3 \cos(\omega t + 45^\circ) \), where capital letters are used to denote steady-state conditions. \( N_1 = 150 \) turns, \( N_2 = 90 \) turns, and \( \mu_r = 3000 \). Calculate the flux in the center leg.

The electric circuit analog is given in Fig. 1C-2. The reluctance \( R_x \) is the reluctance of the center leg and \( R_y \) is the reluctance of one of the two parallel paths from the top of the center leg through an outside leg to the bottom of the center leg. In particular,

![Figure 1C-1: A two-winding magnetic system with dimensions in centimeters.](image)

\( N_1 = 150 \)  
\( N_2 = 90 \)  
\( \mu_r = 3000 \)
Figure 1C-2: Electric-circuit analog of Fig. 1C-1.

\[
\mathcal{R}_y = \frac{2(0.03 + 0.06 + 0.02) + 0.12}{3000(4\pi \times 10^{-7})(0.06)(0.04)} = 37,578 \text{ H}^{-1} \quad (1C-1)
\]

\[
\mathcal{R}_x = \frac{0.12}{3000(4\pi \times 10^{-7})0.06^2} = 8,842 \text{ H}^{-1} \quad (1C-2)
\]

Since the currents are sinusoidal, the mmf's will be sinusoidal. Thus, it is convenient to use phasors to solve for \( \Phi_1 \) and \( \Phi_2 \). The loop equations are

\[
\begin{align*}
\tilde{\text{mmf}}_1 &= \mathcal{R}_y \tilde{\Phi}_1 + \mathcal{R}_x (\tilde{\Phi}_1 - \tilde{\Phi}_2) \\
\tilde{\text{mmf}}_2 &= \mathcal{R}_x (\tilde{\Phi}_2 - \tilde{\Phi}_1) + \mathcal{R}_y \tilde{\Phi}_2
\end{align*} \quad (1C-3) \quad (1C-4)
\]

which may be written in matrix form as

\[
\begin{bmatrix}
\tilde{\text{mmf}}_1 \\
\tilde{\text{mmf}}_2
\end{bmatrix} =
\begin{bmatrix}
\mathcal{R}_x + \mathcal{R}_y & -\mathcal{R}_x \\
-\mathcal{R}_x & \mathcal{R}_x + \mathcal{R}_y
\end{bmatrix}
\begin{bmatrix}
\tilde{\Phi}_1 \\
\tilde{\Phi}_2
\end{bmatrix} \quad (1C-5)
\]

A review of matrix algebra is given in Appendix B. Now,

\[
\tilde{\text{mmf}}_1 = N_1 \tilde{I}_1 = (150)(1/0^\circ) = 150/0^\circ \text{ At} \quad (1C-6)
\]

\[
\tilde{\text{mmf}}_2 = N_2 \tilde{I}_2 = (90)(0.3/45^\circ) = 27/45^\circ \text{ At} \quad (1C-7)
\]

Solving (1C-5) yields

\[
\tilde{\Phi}_1 = (3.434 + j0.081) \times 10^{-3} \text{ Wb} \quad (1C-8)
\]

\[
\tilde{\Phi}_2 = (1.065 + j0.427) \times 10^{-3} \text{ Wb} \quad (1C-9)
\]
The flux flowing down through the center leg is

\[
\Phi_1 - \Phi_2 = (2.369 - j0.346) \times 10^{-3} \\
= 2.39 \times 10^{-3} / -8.3^\circ \text{ Wb}
\]  

(1C-10)

**SP1.3-1** Calculate \(\Phi_1\) in Example 1B. [\(\Phi_1 = 2.014 \times 10^{-3} \text{ Wb}\)]

**SP1.3-2** Calculate \(\Phi_1 + \Phi_2\) in Example 1B when \(I = \sqrt{2} \cdot 10 \cos(\omega t - 30^\circ)\). 
[\(\Phi_1 + \Phi_2 = 2.547 \times 10^{-3} / -30^\circ \text{ Wb, rms}\)]

**SP1.3-3** Remove the center leg of the magnetic system shown in Fig. 1C-1. Calculate the total flux when \(I_1 = 9\) and \(I_2 = -15\) A. [Zero]

**SP1.3-4** Express the sinusoidal variation represented by \(\Phi_2\) given by (1C-9). 
[\(\sqrt{2}(1.147 \times 10^{-3})\cos(\omega t + 21.8^\circ)\)]

### 1.4 PROPERTIES OF MAGNETIC MATERIALS

We may be aware from our study of physics that, when ferromagnetic materials such as iron, nickel, cobalt, or alloys of these elements, such as various types of steels, are placed in a magnetic field, the flux produced is markedly larger (500 to 4000 times, for example) than that which would be produced when a nonmagnetic material is subjected to the same magnetic field. We must take some time to review briefly the basic properties of ferromagnetic materials and to establish terminology for later use.

Let us begin by considering the relationship between \(B\) and \(H\) shown in Fig. 1.4-1 which is typical of silicon steel used in transformers. We will assume that the ferromagnetic core is initially completely demagnetized (both \(B\) and \(H\) are zero). As we apply an external \(H\) field by increasing the current in a winding wound around the core, the flux density \(B\) also increases, but nonlinearly, as shown in Fig. 1.4-1. After \(H\) reaches a value of approximately 150 A/m, the flux density rises more slowly and the material begins to saturate when \(H\) is several hundred Ampere-turns per meter.

In ferromagnetic materials, the combination of the magnetic moments produced by the electrons orbiting the nucleus of an atom and the electron itself spinning on its axis produce a net magnetic moment of the atom that is not canceled by an opposing magnetic moment of a neighboring atom. Ferromagnetic materials have been found to be divided into magnetic domains wherein all magnetic moments (dipoles) are aligned. Although the
magnetic moments are all aligned within a magnetic domain, the direction of this alignment will differ from one domain to another.

When a ferromagnetic material is subjected to an external magnetic field, those domains, which originally tend to be aligned with the applied magnetic field, grow at the expense of those domains with magnetic moments that are less aligned. Thereby, the flux is increased from that which would occur with a nonmagnetic material. This is known as domain-wall motion [1]. As the strength of the magnetic field increases, the aligned domains continue to grow in nearly a linear fashion. Thus, a nearly linear $B$-$H$ curve results ($B \propto \mu_r \mu_0 H$) until the ability of the aligned domains to take from the unaligned domains starts to slow. This gives rise to the knee of the $B$-$H$ curve and the beginning of saturation. At this point, the displacements of the domain walls are complete. That is, there are no longer unaligned domains from which to take. However, the remaining domains may still not be in perfect alignment with the external $H$ field. A further increase in $H$ will cause a rotation of the atomic dipole moments within the remaining domains toward a more perfect alignment. However, the marginal increase in $B$ due to rotation is less than the original increase in $B$ due to domain-wall motion, resulting in a decrease in slope of the $B$-$H$ curve. The magnetic material is said to be completely saturated when the remaining domains are perfectly aligned. In this case, the slope of the $B$-$H$ curve becomes $\mu_0$ [1]. If it is assumed that
the magnetic flux is uniform through most of the magnetic material, then $B$ is proportional to $\Phi$ and $H$ is proportional to mmf. Hence, a plot of flux versus current is of the same shape as the $B$-$H$ curve.

A transformer is generally designed so that some saturation occurs during normal operation. Electric machines are also designed similarly in that a machine generally operates slightly in the saturated region during normal, rated operating conditions. Since saturation causes the coefficients of the differential equations describing the behavior of an electromagnetic device to be functions of the winding currents, a transient analysis is difficult without the aid of a computer. However, it is not our purpose to set forth methods of analyzing nonlinear magnetic systems.

In the previous discussion, we have assumed that the ferromagnetic material is initially demagnetized and that the applied field intensity is gradually increased from zero. However, if a ferromagnetic material is subjected to an alternating field intensity, the resulting $B$-$H$ curve exhibits hysteresis. For example, let us assume that a ferromagnetic material is subjected to an alternating field intensity (alternating current flowing in the winding) and initially the flux density and field intensity are both zero. As $H$ increases from zero, $B$ increases along the initial $B$-$H$ curve, as shown in Fig. 1.4-2. However the field intensity varies sinusoidally and, when $H$ decreases from a maximum, $B$ does not follow back down the original $B$-$H$ curve. After several cycles, the magnetic system will reach a steady-state condition and the plot of $B$ versus $H$ will form a hysteresis loop or a double-valued function, as shown in Fig. 1.4-2. What is happening is very complex. In simple terms, the growth of aligned domains for an incremental change in $H$ in one direction is not equal to the growth of oppositely aligned domains if this change in $H$ were suddenly reversed. We could become quite involved by discussing minor hysteresis loops which would occur if, during the sinusoidal variation of $H$, it were suddenly stopped at some nonzero value then reversed, stopped, and reversed again [1]. We shall only mention this phenomenon in passing.

A family of hysteresis loops is shown in Fig. 1.4-3. In each case, the applied $H$ is sinusoidal; however, the amplitude of the $H$ field is varied to give the family of loops shown in Fig. 1.4-3. A magnetization or $B$-$H$ curve for a given material is obtained by connecting the tips of the hysteresis loops, as shown by the dashed line in Fig. 1.4-3. The locus of the tips of the hysteresis curves is about the same as the original $B$-$H$ curve in Fig. 1.4-1, which corresponds to a gradual increase of $H$ in an initially demagnetized material. If $H$ were suddenly stopped at zero, the flux density remaining
Figure 1.4-2: Hysteresis loop.

Figure 1.4-3: Family of steady-state hysteresis loops.
in the ferromagnetic material is called the residual flux density \((B_r)\). The negative field intensity necessary to bring this residual flux density to zero is called the coercive force \((H_c)\). These two quantities are indicated in Fig. 1.4-3 for the largest hysteresis loop shown.

Energy is required to increase the size of the magnetic domains of the ferromagnetic material. It can be shown that the energy necessary to align alternately the magnetic domains is equal to the area enclosed by the hysteresis loop. This energy causes a rise in the temperature of the magnetic material, and the power associated with this energy loss is called the hysteresis loss.

When a solid block of magnetic material such as that shown in Fig. 1.3-1 is subjected to an alternating field intensity, the resulting alternating flux induces current in the solid magnetic material, which will circulate in a loop perpendicular to the flux density \((B)\) inducing it. These so-called eddy currents have two undesirable effects. First, the mmf established by these circulating currents opposes the mmf produced by the winding, and this opposition is greatest at the center of the material because that tends to be also the center of the current loops. Thus, the flux would tend not to flow through the center of the solid magnetic member, thereby not utilizing the full benefits of the ferromagnetic material. Second, there is an \(i^2r\) loss associated with these eddy currents, called eddy current loss, which is dissipated as heat. These two adverse effects can be minimized in several ways, but the most common is to build the ferromagnetic core of laminations (thin strips) insulated from each other and oriented in the direction of the magnetic field \((B\) or \(H)\). These thin strips offer a much smaller area in which the eddy currents can flow; hence, smaller currents and smaller losses result.

The core losses associated with ferromagnetic materials are the combination of the hysteresis and eddy current losses. Electromagnetic devices are designed to minimize these losses; however, they are always present and are often taken into account in a linear system analysis by assuming that their effects on the electric system can be represented by a resistance.

**SP1.4-1** The magnetic circuit of Fig. 1.3-1 is constructed by using silicon sheet steel. Its magnetization curve is given by Fig. 1.4-1. The gap length \(l_g\) is 1 mm, the mean core length \(l_i\) is 100 cm, \(N = 500\), and \(A_i = A_g = 25\text{ cm}^2\). Determine the current needed to produce a flux \(\Phi\) of \(2.5 \times 10^{-3}\text{ Wb}\). [\textit{Hint: First establish }H_i, H_g, \text{ and use (1.3-3).}] [\textit{I = 1.99 A}]
1.5 STATIONARY MAGNETICALLY COUPLED CIRCUITS

Magnetically coupled electric circuits are central to the operation of transformers and electromechanical motion devices. In the case of transformers, stationary circuits are magnetically coupled for the purpose of changing the ac voltage and current levels. In the case of electromechanical devices, circuits in relative motion are magnetically coupled for the purpose of transferring energy between the mechanical and electric systems. Since magnetically coupled circuits play such an important role in energy conversion, it is important to establish the equations that describe their behavior and to express these equations in a form convenient for analysis. Many of these goals may be achieved by considering two stationary electric circuits that are magnetically coupled, as shown in Fig. 1.5-1. The two windings consist of turns $N_1$ and $N_2$, and they are wound on a common core, which is a ferromagnetic material with a permeability large relative to that of air. The magnetic core is not illustrated in three dimensions.

![Diagram of magnetically coupled circuits](image)

Figure 1.5-1: Magnetically coupled circuits.

Before proceeding, a comment or two is in order. Generally, the concept of an ideal transformer is introduced in a basic circuits course. In the ideal case, $v_2$ in Fig. 1.5-1 is $(N_2/N_1)v_1$ and $i_2$ is $-(N_1/N_2)i_1$. Only the turns-ratio of the transformer is considered. However, this treatment is often not sufficient for a detailed analysis of transformers, and it is seldom appropriate in the
analysis of electromechanical motion devices, since an air gap is necessary for motion to occur; hence, the windings are not as tightly coupled as in the case of transformers and the leakage flux must be taken into account.

In general, the flux produced by each winding can be separated into two components: a leakage component denoted with the subscript \( l \) and a magnetizing component denoted by the subscript \( m \). Each of these components is depicted by a single streamline with the positive direction determined by applying the right-hand rule to the directions of current flow in the winding. (The right-hand rule was reviewed in Example 1B.) The leakage flux associated with a given winding links only that winding, whereas the magnetizing flux, whether it is due to current in winding 1 or winding 2, links both windings. In some cases, \( i_2 \) is selected positive out of the top of winding 2 and a dot is placed at that terminal. Although the "dot notation" is convenient for transformers, it is seldom used in the case of electromechanical devices.

The flux linking each winding may be expressed as

\[
\Phi_1 = \Phi_{l1} + \Phi_{m1} + \Phi_{m2} \tag{1.5-1}
\]

\[
\Phi_2 = \Phi_{l2} + \Phi_{m2} + \Phi_{m1} \tag{1.5-2}
\]

The leakage flux \( \Phi_{l1} \) is produced by current flowing in winding 1 and it links only the turns of winding 1. Likewise, the leakage flux \( \Phi_{l2} \) is produced by current flowing in winding 2 and it links only the turns of winding 2. The flux \( \Phi_{m1} \) is produced by current flowing in winding 1 and it links all turns of windings 1 and 2. Similarly, the magnetizing flux \( \Phi_{m2} \) is produced by current flowing in winding 2 and it also links all turns of windings 1 and 2. Both \( \Phi_{m1} \) and \( \Phi_{m2} \) are called magnetizing fluxes. With the selected positive directions of current flow and the manner in which the windings are wound, magnetizing flux produced by positive current flowing in one winding adds to the magnetizing flux produced by positive current flowing in the other winding. For this case, we will find that the mutual inductance is positive.

It is appropriate to point out that this is an idealization of the actual magnetic system. It seems logical that all of the leakage flux will not link all the turns of the winding producing it; hence, \( \Phi_{l1} \) and \( \Phi_{l2} \) are "equivalent" leakage fluxes. Similarly, all of the magnetizing flux of one winding may not link all of the turns of the other winding. To acknowledge this practical aspect of the magnetic system, \( N_1 \) and \( N_2 \) are often considered to be the equivalent number of turns rather than the actual number.

The voltage equations may be expressed as
\[ v_1 = r_1 i_1 + \frac{d\lambda_1}{dt} \quad (1.5-3) \]
\[ v_2 = r_2 i_2 + \frac{d\lambda_2}{dt} \quad (1.5-4) \]

In matrix form,
\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad (1.5-5)
\]

A review of matrix algebra is given in Appendix B. The resistances \( r_1 \) and \( r_2 \) and the flux linkages \( \lambda_1 \) and \( \lambda_2 \) are related to windings 1 and 2, respectively. Since it is assumed that \( \Phi_1 \) links the equivalent turns of winding 1 \( (N_1) \) and \( \Phi_2 \) links the equivalent turns of winding 2 \( (N_2) \), the flux linkages may be written as

\[ \lambda_1 = N_1 \Phi_1 \quad (1.5-6) \]
\[ \lambda_2 = N_2 \Phi_2 \quad (1.5-7) \]

where \( \Phi_1 \) and \( \Phi_2 \) are given by (1.5-1) and (1.5-2), respectively.

If we assume that the magnetic system is linear, we may apply Ohm’s law for magnetic circuits to express the fluxes. Thus, the fluxes may be written as

\[ \Phi_{11} = \frac{N_1 i_1}{R_{t1}} \quad (1.5-8) \]
\[ \Phi_{m1} = \frac{N_1 i_1}{R_m} \quad (1.5-9) \]
\[ \Phi_{t2} = \frac{N_2 i_2}{R_{t2}} \quad (1.5-10) \]
\[ \Phi_{m2} = \frac{N_2 i_2}{R_m} \quad (1.5-11) \]

where \( R_{t1} \) and \( R_{t2} \) are the reluctances of the leakage paths, and \( R_m \) is the reluctance of the path of magnetizing fluxes. Typically, the reluctances associated with leakage paths are much larger than the reluctance of the magnetizing path. The reluctance associated with an individual leakage path is difficult to determine exactly, and it is usually approximated from test data.
or by using the computer to solve the field equations numerically. On the other hand, the reluctance of the magnetizing path of the core shown in Fig. 1.5-1 may be computed with sufficient accuracy as in Example 1B.

Substituting (1.5-8) through (1.5-11) into (1.5-1) and (1.5-2) yields

\[
\Phi_1 = \frac{N_1 i_1}{R_{i1}} + \frac{N_1 i_1}{R_m} + \frac{N_2 i_2}{R_m} \tag{1.5-12}
\]

\[
\Phi_2 = \frac{N_2 i_2}{R_{i2}} + \frac{N_2 i_2}{R_m} + \frac{N_1 i_1}{R_m} \tag{1.5-13}
\]

Substituting (1.5-12) and (1.5-13) into (1.5-6) and (1.5-7) yields

\[
\lambda_1 = \frac{N_1^2}{R_{i1}} i_1 + \frac{N_1 N_2}{R_m} i_1 + \frac{N_1 N_2}{R_m} i_2 \tag{1.5-14}
\]

\[
\lambda_2 = \frac{N_2^2}{R_{i2}} i_2 + \frac{N_2 N_1}{R_m} i_2 + \frac{N_2 N_1}{R_m} i_1 \tag{1.5-15}
\]

When the magnetic system is linear, the flux linkages are generally expressed in terms of inductances and the currents. We see that the coefficients of the first two terms on the right-hand side of (1.5-14) depend upon \( N_1 \) and the reluctance of the magnetic system, independent of the existence of winding 2. An analogous statement may be made regarding (1.5-15) with the roles of winding 1 and winding 2 reversed. Hence, the self-inductances are defined as

\[
L_{i1} = \frac{N_1^2}{R_{i1}} + \frac{N_1^2}{R_m} = L_{i1} + L_{m1} \tag{1.5-16}
\]

\[
L_{i2} = \frac{N_2^2}{R_{i2}} + \frac{N_2^2}{R_m} = L_{i2} + L_{m2} \tag{1.5-17}
\]

where \( L_{i1} \) and \( L_{i2} \) are the leakage inductances and \( L_{m1} \) and \( L_{m2} \) are the magnetizing inductances of windings 1 and 2, respectively. From (1.5-16) and (1.5-17), it follows that the magnetizing inductances may be related as

\[
\frac{L_{m2}}{N_2^2} = \frac{L_{m1}}{N_1^2} \tag{1.5-18}
\]

The mutual inductances are defined as the coefficient of the third term on the right-hand side of (1.5-14) and (1.5-15). In particular,
\[ L_{12} = \frac{N_1 N_2}{R_m} \]  
(1.5-19)
\[ L_{21} = \frac{N_2 N_1}{R_m} \]  
(1.5-20)

We see that \( L_{12} = L_{21} \) and, with the assumed positive direction of current flow and the manner in which the windings are wound, the mutual inductances are positive. If, however, the assumed positive directions of current were such that \( \Phi_{m1} \) opposed \( \Phi_{m2} \), then the mutual inductances would be negative.

The mutual inductances may be related to the magnetizing inductances. Comparing (1.5-16) and (1.5-17) with (1.5-19) and (1.5-20), we see that
\[ L_{12} = \frac{N_2}{N_1} L_{m1} = \frac{N_1}{N_2} L_{m2} \]  
(1.5-21)

The flux linkages may now be written as
\[ \lambda_1 = L_{11}i_1 + L_{12}i_2 \]  
(1.5-22)
\[ \lambda_2 = L_{21}i_1 + L_{22}i_2 \]  
(1.5-23)

where \( L_{11} \) and \( L_{22} \) are defined by (1.5-16) and (1.5-17), respectively, and \( L_{12} \) and \( L_{21} \), by (1.5-21). The self-inductances \( L_{11} \) and \( L_{22} \) are always positive; however, the mutual inductances \( L_{12}(L_{21}) \) may be positive or negative, as previously mentioned.

Although the voltage equations given by (1.5-3) and (1.5-4) may be used for purposes of analysis, it is customary to perform a change of variables which yields the well-known equivalent \( T \) circuit of two windings coupled by a linear magnetic circuit. To set the stage for this derivation, let us express the flux linkages from (1.5-22) and (1.5-23) as
\[ \lambda_1 = L_{11}i_1 + L_{m1} \left( i_1 + \frac{N_2}{N_1}i_2 \right) \]  
(1.5-24)
\[ \lambda_2 = L_{22}i_2 + L_{m2} \left( \frac{N_1}{N_2}i_1 + i_2 \right) \]  
(1.5-25)

With \( \lambda_1 \) in terms of \( L_{m1} \) and \( \lambda_2 \) in terms of \( L_{m2} \), we see two logical candidates for substitute variables, in particular, \((N_2/N_1)i_2\) or \((N_1/N_2)i_1\). If we let
\[ i'_2 = \frac{N_2}{N_1} i_2 \]  \hspace{1cm} (1.5-26)

then we are using the substitute variable \( i'_2 \), which, when flowing through winding 1, produces the same mmf as the actual \( i_2 \) flowing through winding 2; \( N_1 i'_2 = N_2 i_2 \). This is said to be referring the current in winding 2 to winding 1 or to a winding with \( N_1 \) turns, whereupon winding 1 becomes the reference winding. On the other hand, if we let

\[ i'_1 = \frac{N_1}{N_2} i_1 \]  \hspace{1cm} (1.5-27)

then \( i'_1 \) is the substitute variable that produces the same mmf when flowing through winding 2 as \( i_1 \) does when flowing in winding 1; \( N_2 i'_1 = N_1 i_1 \). This change of variables is said to refer the current of winding 1 to winding 2 or to a winding with \( N_2 \) turns, whereupon winding 2 becomes the reference winding.

We will demonstrate the derivation of the equivalent \( T \) circuit by referring the current of winding 2 to a winding with \( N_1 \) turns; thus \( i'_2 \) is expressed by (1.5-26). We want the instantaneous power to be unchanged by this substitution of variables. Therefore,

\[ v'_2 i'_2 = v_2 i_2 \]  \hspace{1cm} (1.5-28)

Hence,

\[ v'_2 = \frac{N_1}{N_2} v_2 \]  \hspace{1cm} (1.5-29)

Flux linkages, which have the units of \( V \cdot s \), are related to the substitute flux linkages in the same way as voltages. In particular,

\[ \lambda'_2 = \frac{N_1}{N_2} \lambda_2 \]  \hspace{1cm} (1.5-30)

Now, replace \( (N_2/N_1)i_2 \) with \( i'_2 \) in the expression for \( \lambda_1 \), given by (1.5-24). Next, solve (1.5-26) for \( i_2 \) and substitute it into \( \lambda_2 \) given by (1.5-25). Now, multiply this result by \( N_1/N_2 \) to obtain \( \lambda'_2 \) and then substitute \( (N_2/N_1)^2 Lm1 \) for \( Lm2 \) in \( \lambda'_2 \). If we do all this, we will obtain

\[ \lambda_1 = L_{11} i_1 + L_{m1} (i_1 + i'_2) \]  \hspace{1cm} (1.5-31)

\[ \lambda'_2 = L_{12} i'_2 + L_{m1} (i_1 + i'_2) \]  \hspace{1cm} (1.5-32)
where

\[ L'_{22} = \left( \frac{N_1}{N_2} \right)^2 L_{22} \]  \hspace{1cm} (1.5-33)

The flux linkage equations given by (1.5-31) and (1.5-32) may also be written as

\[ \lambda_1 = L_{11} i_1 + L_{m1} i_2' \]  \hspace{1cm} (1.5-34)

\[ \lambda_2' = L_{m1} i_1 + L'_{22} i_2' \]  \hspace{1cm} (1.5-35)

where

\[ L'_{22} = \left( \frac{N_1}{N_2} \right)^2 L_{22} = L'_{12} + L_{m1} \]  \hspace{1cm} (1.5-36)

and \( L_{22} \) is defined by (1.5-17).

If we multiply (1.5-4) by \( N_1/N_2 \) to obtain \( v_2' \), the voltage equations become

\[
\begin{bmatrix}
  v_1 \\
  v_2'
\end{bmatrix} =
\begin{bmatrix}
  r_1 & 0 \\
  0 & r_2'
\end{bmatrix}
\begin{bmatrix}
  i_1 \\
  i_2'
\end{bmatrix} +
\frac{d}{dt}
\begin{bmatrix}
  \lambda_1 \\
  \lambda_2'
\end{bmatrix}
\]  \hspace{1cm} (1.5-37)

where

\[ r_2' = \left( \frac{N_1}{N_2} \right)^2 r_2 \]  \hspace{1cm} (1.5-38)

The previous voltage equations, (1.5-37), together with the flux linkage equations, (1.5-34) through (1.5-35), suggest the equivalent \( T \) circuit shown in Fig. 1.5-2. This method may be extended to include any number of windings wound on the same core.

---

**Figure 1.5-2:** Equivalent \( T \) circuit with winding 1 selected as reference winding.
Earlier in this section, we mentioned that in the case of an ideal transformer only the turns ratio is considered, that is, \( v_2 = (N_2/N_1)v_1 \) and \( i_2 = -(N_1/N_2)i_1 \). We can now more fully appreciate the assumptions that are made in this type of analysis. In particular, the resistances \( r_1 \) and \( r_2 \) and the leakage inductances \( L_{\Omega 1} \) and \( L_{\Omega 2} \) are neglected, and it is assumed that the magnetizing inductance is large so that the magnetizing current \( i_1 + i_2 \) is negligibly small.

The information presented in this section forms the basis of the equivalent circuits for many types of electric machines. Using a turns ratio to refer the voltages and currents of rotor circuits of electric machines to a winding with the same number of turns as the stator windings is common practice. In fact, the equivalent circuits for many ac machines are of the same form as shown in Fig. 1.5-2, with the addition of a voltage source referred to as a speed voltage. We shall talk much more about this speed voltage later—where it comes from and how it fits into the equivalent circuit.

**Example 1D.** It is instructive to illustrate the method of deriving an equivalent \( T \) circuit from open- and short-circuit measurements. When winding 2 of the two-winding transformer shown in Fig. 1.5-2 is open circuited and a voltage of 110 V (rms) at 60 Hz is applied to winding 1, the average power supplied to winding 1 is 6.66 W. The measured current in winding 1 is 1.05 A (rms). Next, with winding 2 short-circuited, the current flowing in winding 1 is 2 A when the applied voltage is 30 V at 60 Hz. The average input power is 44 W. If we assume \( L_{\Omega 1} = L'_{\Omega 2} \), an approximate equivalent \( T \) circuit can be determined from these measurements with winding 1 selected as the reference winding.

The average power supplied to winding 1 may be expressed as

\[
P_1 = |\tilde{V}_1| \cdot |\tilde{I}_1| \cos \varphi_{pf} \tag{1D-1}
\]

where

\[
\varphi_{pf} = \theta_{ev}(0) - \theta_{ei}(0) \tag{1D-2}
\]

Here, \( \tilde{V}_1 \) and \( \tilde{I}_1 \) are phasors with the positive direction of \( \tilde{I}_1 \) taken in the direction of the voltage drop, and \( \theta_{ev}(0) \) and \( \theta_{ei}(0) \) are the phase angles of \( \tilde{V}_1 \) and \( \tilde{I}_1 \), respectively. Solving for \( \varphi_{pf} \) during the open-circuit test, we have
\[ \varphi_{pf} = \cos^{-1} \frac{P_1}{|V_1||I_1|} = \cos^{-1} \frac{6.66}{(110)(1.05)} = 86.7^\circ \]  

(1D-3)

Although \( \varphi_{pf} = -86.7^\circ \) is also a legitimate solution of (1D-3), the positive solution is taken since \( \bar{V}_1 \) leads \( \bar{I}_1 \) in an inductive circuit. With winding 2 open-circuited, the input impedance of winding 1 is

\[ Z = \frac{\bar{V}_1}{\bar{I}_1} = r_1 + j(X_{l1} + X_{m1}) \]  

(1D-4)

With \( \bar{V}_1 \) as the reference phasor, \( \bar{V}_1 = 110/0^\circ \), \( \bar{I}_1 = 1.05/-86.7^\circ \). Thus,

\[ r_1 + j(X_{l1} + X_{m1}) = \frac{110/0^\circ}{1.05/-86.7^\circ} = 6 + j104.6 \, \Omega \]  

(1D-5)

If we neglect core losses, then, from (1D-5), \( r_1 = 6 \, \Omega \). We also see from (1D-5) that \( X_{l1} + X_{m1} = 104.6 \, \Omega \).

For the short-circuit test, we will assume that \( \bar{I}_1 = -\bar{I}_2 \) since transformers are designed so that at rated frequency \( X_{m1} \gg |r_2' + jX_{l2}'| \). Hence, using (1D-1) again,

\[ \varphi_{pf} = \cos^{-1} \frac{44}{(30)(2)} = 42.8^\circ \]  

(1D-6)

In this case, the input impedance is \( Z = (r_1 + r_2') + j(X_{l1} + X_{l2}') \). This may be determined as

\[ Z = \frac{30/0^\circ}{2/-42.8^\circ} = 11 + j10.2 \, \Omega \]  

(1D-7)

Hence, \( r_2' = 11 - r_1 = 5 \, \Omega \) and, since it is assumed that \( X_{l1} = X_{l2}' \), both are \( 10.2/2 = 5.1 \, \Omega \). Therefore, \( X_{m1} = 104.6 - 5.1 = 99.5 \, \Omega \).

In summary, \( r_1 = 6 \, \Omega \), \( L_{l1} = 13.5 \, \text{mH} \), \( L_{m1} = 263.9 \, \text{mH} \), \( r_2' = 5 \, \Omega \), \( L_{l2} = 13.5 \, \text{mH} \). Make sure we converted from \( X \)'s to \( L \)'s correctly.

**SP1.5-1** Remove the center leg of the magnetic system shown in Fig. 1C-1. Calculate \( L_{l1}, L_{l2}, \) and \( L_{l2} \). Neglect the leakage inductances. \([L_{l1} = 299.4 \, \text{mH}, L_{l2} = 107.8 \, \text{mH}, L_{l2} = 179.5 \, \text{mH}]\)

**SP1.5-2** Consider the transformer and parameters calculated in Example 1D. Winding 2 is short-circuited and 12 V (dc) is applied to winding 1.
Calculate the steady-state values of $i_1$ and $i_2$. Repeat with winding 2 open-circuited. \([i_1 = 2 \text{ A and } i_2 = 0 \text{ in both cases}]\)

**SP1.5-3** Calculate the no-load (winding 2 open-circuited) current for the transformer given in Example 1D if $V_1 = \sqrt{2} 10 \cos 100t$. \([\vec{I}_1 = 0.352/-77.8^\circ \text{ A}]\)

### 1.6 OPEN- AND SHORT-CIRCUIT CHARACTERISTICS OF STATIONARY MAGNETICALLY COUPLED CIRCUITS

It is instructive to observe the open- and short-circuit characteristics of a transformer with two windings. For this purpose, a transformer with the parameters given in Example 1D was simulated on a computer. The open-circuit characteristics are shown in Figs. 1.6-1 and 1.6-2. The variables plotted are $\lambda$, $v_1$, $i_1$, $v'_2$, and $i'_2$. The variable $\lambda$ is equal to $L_{m1}(i_1 + i'_2)$, which is the last term on the right-hand side of (1.5-31) and (1.5-32). This is the flux linkage of winding 1 due to the flux in the transformer iron. It is often referred to as the magnetizing flux linkage(s) and denoted $\lambda_m$, $\lambda_{mag}$, or $\lambda_\varphi$, whereas $i_1 + i'_2$ is called the magnetizing current.

Initially, the windings are unexcited. At time zero \((t = 0)\), the voltage applied to winding 1 with winding 2 open-circuited is $v_1 = \sqrt{2} 110 \cos 377t$ in Fig. 1.6-1 and $v_1 = \sqrt{2} 110 \sin 377t$ in Fig. 1.6-2. The waveforms of the steady-state current $i_1$ are identical in Figs. 1.6-1 and 1.6-2; however, since the inductive reactance is large, applying a sine wave voltage for $v_1$ at time zero results in a much larger transient offset in $i_1$ than when $v_1 = \sqrt{2} 110 \cos 377t$. (You are asked to show this in a problem at the end of the chapter.) Since $v_1 = \sqrt{2} 110 \sin 377t$ causes a larger transient offset, it makes it easier for us to identify the transient period. Therefore, we shall continue with $v_1$ as a sine wave. Although it is difficult to determine the time constant for the offset of the current $i_1$ (or $\lambda$) to decay to one-third of its original value, it is on the order of 50 ms. The calculated value of the no-load time constant is $\tau_{nl} = (L_{q1} + L_{m1})/r_1 = 46.2 \text{ ms}$. Before leaving Figs. 1.6-1 and 1.6-2, note that, during steady-state conditions, $I_1$ lags $V_1$ by something close to 90° (86.7°, from Example 1D).

Let us now go to the short-circuit characteristics. The transient and steady-state response with $v_1 = \sqrt{2} 110 \sin 377t$ and with $v'_2 = 0$ are shown
Figure 1.6-1: Open-circuit conditions of a two-winding transformer with 
\[ v_1 = \sqrt{2} \ 110 \cos 377t. \]
Figure 1.6-2: Open-circuit conditions of a two-winding transformer with
\[ v_1 = \sqrt{2} \cdot 110 \sin 377t. \]
in Fig. 1.6-3. There are several things to note. From Fig. 1.6-3, it appears that the time constant associated with the decay of \( i_1 \) is small, less than 5 ms. Now let us look at the magnetizing flux linkage \( \lambda \). We see that it is smaller in amplitude than in the no-load case. We would expect this since during short-circuit conditions \( i_2 \equiv -i_2' \), whereby the mmf’s of the two windings oppose, and the resulting flux in the transformer iron is less than for the no-load condition where \( i_2' = 0 \). Looking at this in another way, we realize that \( i_1 \) and \(-i_2'\) will be essentially equal during short-circuit conditions whenever the impedance of the magnetizing branch \((j\omega_c L_{m1})\) is much larger (say 8 to 10 times larger) than \( r_2' + j\omega_c L'_{12} \). Here \( \omega_c = 377 \) rad/s.

It is interesting to note that the decay of the magnetizing flux linkage \( \lambda \) is much slower than the apparent decay of the currents. As we mentioned, the time constant associated with \( i_1 \) is small; however, there is indeed a small difference between \( i_1 \) and \(-i_2'\), and this small current (magnetizing current),

![Graphs of currents and voltages](image)

**Figure 1.6-3**: Short-circuit conditions of a two-winding transformer with 
\[ v_1 = \sqrt{2} \times 110 \sin 377t. \]
which is actually a small part of $i_1$, must flow in the large inductance $L_{m1}$. Hence, the magnetizing current is associated with a longer time constant than the much larger component of the current $i_1$ which circulates through the series $r_2'$ and $L_{i2}'$.

Let us take a brief look at the effects of saturation of the transformer iron. For this purpose we will assume that the $\lambda$ versus $(i_1 + i_2')$ plot of the core of the transformer is that shown in Fig. 1.6-4. The slope of the straight-line part of this plot is $L_{m1}$. The saturation characteristics shown in Fig. 1.6-4 were implemented on the computer following the method outlined in [2]. Since $\lambda$ is small during short-circuit conditions (Fig. 1.6-3), saturation does not occur. However, it is a different situation when we talk about the open-circuit conditions shown in Fig. 1.6-5, which is the same as Fig. 1.6-2 with saturation taken into account. Here, we see that during steady-state open-circuit conditions, the current $i_1$, which is the total magnetizing current since $i_2'$ is zero, is rich in third harmonic. What is happening? Well, we realize that, if there were no $r_1$ and $L_{i1}$, then the time rate of change of $\lambda$ must equal $v_1$, the applied voltage. In this case, the peak value of $\lambda$ would be $\sqrt{2} \times 110/377 = \sqrt{2} \times 0.29 \text{ V} \cdot \text{s}$. We see from Fig. 1.6-4 that saturation

![Figure 1.6-4: $\lambda$ versus $i_1 + i_2'$.](image)
must occur in order for the core to produce this peak value of $\lambda$. In the saturated region, a much larger increase in current per unit increase in $\lambda$ is required than when the transformer is not saturated. Hence, a peaking of the magnetizing current occurs. Now, in real life there is $r_1$ and $L_{i1}$ and, hence, a voltage drop will occur across each of these components. Thus, the magnitude of $d\lambda/dt$ is somewhat less than that of $v_1$; nevertheless, saturation must occur to produce the required steady-state peak value of $\lambda$, which is approximately $\sqrt{2} 0.26 \text{ V} \cdot \text{s}$ from Fig. 1.6-5.

There is one last item worthy of discussion. Recall that a relatively large transient offset in $\lambda$ occurs when we apply a sine wave voltage for $v_1$. This large transient offset drives the core into saturation. Note $\lambda$ during the first cycle in Fig. 1.6-5. Since the core is highly saturated, the magnetizing current necessary to produce the required $\lambda$ is very large. In Fig. 1.6-5 we see that this current, which occurs upon "energizing" the transformer, is nearly three times the normal steady-state magnetizing current. In some transformers, this may be as high as 50 to 100 times the normal magnetizing current, and
it may take several cycles before reaching steady-state conditions. For this reason, some transformers may "hum" loudly during energization as a result of forces created by the large inrush current. Also, note the waveform of \( v'_2 \) during the first cycle of energization. The effects of saturation are reflected into the open-circuit voltage of winding 2. Since during saturation the change of the flux linkages is small, the open-circuit voltage will be small, as depicted in Fig. 1.6-5. However, these changes would probably not be as distinct in the open-circuit voltage of an actual transformer.

**SP1.6-1** Use the plot of \( \lambda \) in Fig. 1.6-3 to approximate \( |\vec{I}_1 + \vec{I}_2'|. \) \( |\vec{I}_1 + \vec{I}_2'| \cong \frac{1}{2} \text{A} \]

**SP1.6-2** Calculate, using reasonable approximations, the phase angle between the steady-state current \( \vec{I}_1 \) and voltage \( \vec{V}_1 \) for the conditions of Fig. 1.6-3. Check your answer from the plots. \( [\vec{V}_1 \text{ leads } \vec{I}_1 \text{ by } 42.8^\circ] \)

**SP1.6-3** Consider the transformer given in Example 1D. Assume \( V_1 = \sqrt{2} 110 \cos 1000t \), and a load is connected across winding 2. The impedance of this load referred to winding 1 is \( 21 + j5 \Omega \). Calculate \( \vec{I}_2' \). Make valid approximations to reduce your work. \( [\vec{I}_2' \cong -2.4/ -45^\circ] \)

### 1.7 MAGNETIC SYSTEMS WITH MECHANICAL MOTION

In Chapter 2, relationships are derived for determining the electromagnetic force or torque established in electromechanical systems. Once this development is completed, three examples of elementary electromechanical systems are considered. It is convenient to introduce these three systems here for the purpose of establishing the voltage equations and expressions for the self- and mutual inductances, thereby setting the stage for the analysis to follow in Chapter 2. The first of these electromechanical systems is an elementary version of an electromagnet. It consists of a magnetic core, part of which is movable. The electric system exerts an electromagnetic force upon this movable member, thereby moving it relative to the stationary member. We shall analyze this device, and in Chapter 2 we shall observe its operating characteristics by computer traces. The second system is a rotational device commonly referred to as a reluctance machine. A large number of stepper motors operate on the reluctance-torque principle. The third device is also
a rotational device that has two windings: one on the stationary member and one on the rotational member. This device, although somewhat impracticable, illustrates the concept of windings or magnetic systems in relative motion.

**Elementary Electromagnet**

An elementary electromagnet that we will consider is shown in Fig. 1.7-1. This system consists of a stationary core with a winding of \( N \) turns and a block of magnetic material that is free to slide relative to the stationary member. It is shown in more detail in Chapter 2, wherein a spring, a damper, and an external force are associated with the movable member. We do not need to consider that level of detail here; instead, we will assume that the movable member is at a distance \( x \) from the stationary member, which may be a function of time, that is \( x = x(t) \).

The voltage equation that describes the electric system is

\[
v = ri + \frac{d\lambda}{dt} \tag{1.7-1}
\]

where the flux linkages are expressed as

\[
\lambda = N \Phi \tag{1.7-2}
\]

![Figure 1.7-1: Elementary electromagnet.](image-url)
The flux may be written as

\[ \Phi = \Phi_l + \Phi_m \] (1.7-3)

where \( \Phi_l \) is the leakage flux and \( \Phi_m \) is the magnetizing flux that is common to both the stationary and movable members. If the magnetic system is considered to be linear (saturation neglected), then, as in the case of the stationary coupled circuits, we can express the fluxes in terms of reluctances. That is,

\[ \Phi_l = \frac{Ni}{R_l} \] (1.7-4)

\[ \Phi_m = \frac{Ni}{R_m} \] (1.7-5)

where \( R_l \) and \( R_m \) are the reluctances of the leakage and magnetizing paths, respectively.

The flux linkages may now be written as

\[ \lambda = \left( \frac{N^2}{R_l} + \frac{N^2}{R_m} \right) i \] (1.7-6)

where the leakage inductance is

\[ L_l = \frac{N^2}{R_l} \] (1.7-7)

and the magnetizing inductance is

\[ L_m = \frac{N^2}{R_m} \] (1.7-8)

The reluctance of the magnetizing path is

\[ R_m = R_l + 2R_g \] (1.7-9)

where \( R_l \) is the total reluctance of the magnetic material of the stationary and movable members and \( R_g \) is the reluctance of one of the air gaps. If the cross-sectional area of the stationary and movable members is assumed to be equal and of the same material, the reluctances may be expressed as

\[ R_i = \frac{l_i}{\mu_0 \mu_i A_i} \] (1.7-10)
\[ R_g = \frac{x}{\mu_0 A_g} \]  

(1.7-11)

We will assume that \( A_g = A_i \). Even though, as we have mentioned previously, this may be somewhat of an oversimplification, it is sufficient for our purposes. Hence, \( R_m \) may be written as

\[ R_m = \frac{1}{\mu_0 A_i} \left( \frac{l_t}{\mu_{ri}} + 2x \right) \]  

(1.7-12)

The magnetizing inductance now becomes

\[ L_m = \frac{N^2}{(1/\mu_0 A_i)(l_t/\mu_{ri} + 2x)} \]  

(1.7-13)

In this analysis, the leakage inductance is assumed to be constant. The magnetizing inductance is clearly a function of displacement. That is, \( x = x(t) \) and \( L_m = L_m(x) \). Heretofore, when dealing with linear magnetic circuits wherein mechanical motion is not present as in the case of a transformer, the change of flux linkages with respect to time was simply \( L(di/dt) \). This is not the case here. When the inductance is a function of \( x(t) \)

\[ \lambda(i, x) = L(x)i = [L_t + L_m(x)]i \]  

(1.7-14)

and

\[ \frac{d\lambda(i, x)}{dt} = \frac{\partial \lambda}{\partial i} \frac{di}{dt} + \frac{\partial \lambda}{\partial x} \frac{dx}{dt} \]  

(1.7-15)

With (1.7-15) in mind, we see that the voltage equation (1.7-1) becomes

\[ v = ri + [L_t + L_m(x)] \frac{di}{dt} + i \frac{dL_m(x)}{dx} \frac{dx}{dt} \]  

(1.7-16)

Equation (1.7-16) is a nonlinear differential equation owing to the last two terms on the right-hand side.

Let us go back to the magnetizing inductance \( L_m \) as given by (1.7-13), for just a moment. In preparation for our work in Chapter 2, let us write (1.7-13) as

\[ L_m(x) = \frac{k}{k_0 + x} \]  

(1.7-17)

where

\[ k = \frac{N^2 \mu_0 A_i}{2} \]  

(1.7-18)
\[ k_0 = \frac{l_i}{2\mu_{ri}} \]  

(1.7-19)

When \( x = 0 \), \( L_m(x) \) is determined by the reluctance of the magnetic material. That is, for \( x = 0 \),

\[ L_m(0) = \frac{k}{k_0} = \frac{N^2\mu_0\mu_{ri}A_i}{l_i} \]  

(1.7-20)

Depending upon the parameters of the magnetic material, \( L_m(x) \) may be adequately predicted by

\[ L_m(x) = \frac{k}{x} \quad \text{for} \ x > 0 \]  

(1.7-21)

We will use this approximation in Chapter 2.

**Elementary Reluctance Machine**

An elementary reluctance machine is shown in Fig. 1.7-2. It consists of a stationary core with a winding of \( N \) turns and a movable member that rotates at an angular displacement and angular velocity of \( \theta_r \) and \( \omega_r \), respectively. The displacement is defined as

\[ \theta_r = \omega_r t + \theta_r(0) \]  

(1.7-22)

The voltage equation is of the form given by (1.7-1). Similarly, the flux may be divided into a leakage and magnetizing flux, as given by (1.7-3). It is convenient to express the flux linkages as

\[ \lambda = (L_l + L_m)i \]  

(1.7-23)

where \( L_l \) is the leakage inductance and \( L_m \) is the magnetizing inductance. The leakage inductance is essentially constant, independent of \( \theta_r \); however, the magnetizing inductance is a periodic function of \( \theta_r \). That is, \( L_m = L_m(\theta_r) \). In particular, with \( \theta_r = 0 \),

\[ L_m(0) = \frac{N^2}{R_m(0)} \]  

(1.7-24)

Here, the reluctance of the magnetizing path \( R_m \) is maximum due to the large air gap when the rotor is in the vertical (unaligned) position. Hence, \( L_m \) is a minimum in this position. Note that this same situation occurs not only at \( \theta_r = 0 \) but also when \( \theta_r = \pi, 2\pi \), and so on.

Now, with \( \theta_r = \frac{1}{2}\pi \)
Figure 1.7-2: Elementary reluctance machine.

\[ L_m \left( \frac{1}{2} \pi \right) = \frac{N^2}{R_m \left( \frac{1}{2} \pi \right)} \]  

(1.7-25)

Here, \( R_m \) is a minimum and, thus, \( L_m \) is a maximum. This same situation occurs at \( \theta_r = \frac{3}{2} \pi, \frac{5}{2} \pi \), and so on. Hence, the magnetizing inductance varies between maximum and minimum positive values twice per revolution of the rotating member (rotor). Let us make it easy for ourselves and assume that this variation may be adequately approximated by a sinusoidal function. In particular, let \( L_m(\theta_r) \) be expressed as

\[ L_m(\theta_r) = L_A - L_B \cos 2\theta_r \]  

(1.7-26)

whereupon

\[ L_m(0) = L_A - L_B \]  

(1.7-27)

\[ L_m \left( \frac{1}{2} \pi \right) = L_A + L_B \]  

(1.7-28)

and \( L_A > L_B \). The average value is \( L_A \) is illustrated in Fig. 1.7-3. The self-inductance may now be expressed as
Figure 1.7-3: Approximation of magnetizing inductance of an elementary reluctance machine.

\[
L(\theta_r) = L_l + L_m(\theta_r) \\
= L_l + L_A - L_B \cos 2\theta_r
\]  
(1.7-29)

The voltage equation is of the form given by (1.7-16) with \( x \) replaced by \( \theta_r \).

**Windings in Relative Motion**

The rotational device shown in Fig. 1.7-4 will be used to illustrate windings in relative motion. This device consists of two windings each containing several turns of a conductor. Winding 1 has \( N_1 \) turns and it is on the stationary member (stator); winding 2 has \( N_2 \) turns and it is on the rotating member (rotor). The \( \otimes \) indicates that the assumed direction of positive current flow in the conductors is into the paper, whereas \( \odot \) indicates positive current flow in the conductors is out of the paper. In a practical device, the turns of a winding are distributed over an arc (often 30 to 60°) of the stator and rotor. However, in this introductory consideration, it is sufficient to assume that the turns are concentrated in one position, as shown in Fig. 1.7-4. Also, the length of the air gap between the stator and rotor is shown exaggerated relative to the inside diameter of the stator.

The voltage equations may be written as

\[
v_1 = r_1 i_1 + \frac{d\lambda_1}{dt}
\]  
(1.7-30)
Figure 1.7-4: Elementary rotational electromechanical device. (a) end view; (b) cross-sectional view.

\[ v_2 = r_2i_2 + \frac{d\lambda_2}{dt} \]  

(1.7-31)

where \( r_1 \) and \( r_2 \) are the resistances of winding 1 and winding 2, respectively. The magnetic system is assumed linear; therefore, the flux linkages may be expressed as

\[ \lambda_1 = L_{11}i_1 + L_{12}i_2 \]  

(1.7-32)

\[ \lambda_2 = L_{21}i_1 + L_{22}i_2 \]  

(1.7-33)

The self-inductances \( L_{11} \) and \( L_{22} \) are constants and may be expressed in terms of leakage and magnetizing inductances as

\[ L_{11} = L_{i1} + L_{m1} \]

\[ = \frac{N_1^2}{\mathcal{R}_{i1}} + \frac{N_1^2}{\mathcal{R}_m} \]  

(1.7-34)

\[ L_{22} = L_{i2} + L_{m2} \]

\[ = \frac{N_2^2}{\mathcal{R}_{i2}} + \frac{N_2^2}{\mathcal{R}_m} \]  

(1.7-35)

where \( \mathcal{R}_m \) is the reluctance of the complete magnetic path of \( \Phi_{m1} \) and \( \Phi_{m2} \), which is through the rotor and stator iron and twice across the air gap. Clearly, it is the same for the magnetic system established by either winding 1 or winding 2.
Take a moment to note the designation of axis 1 and axis 2 in Fig. 1.7-4. These axes denote the positive direction of the respective magnetic systems with the assumed positive direction of current flow in the windings (right-hand rule). Now let us consider $L_{12}$. (Is it clear that $L_{12} = L_{21}$?) When $\theta_r$, which is defined by (1.7-22), is zero, then the coupling between windings 1 and 2 is maximum. In particular, with $\theta_r = 0$ the magnetic system of winding 1 aids that of winding 2 with positive currents assumed. Hence, the mutual inductance is positive and

$$L_{12}(0) = \frac{N_1 N_2}{\gamma_r m} \tag{1.7-36}$$

When $\theta_r = \frac{1}{2}\pi$, the windings are orthogonal. The mutual coupling is zero. Hence,

$$L_{12} \left( \frac{1}{2}\pi \right) = 0 \tag{1.7-37}$$

Again let us make it as simple as possible by assuming that the mutual inductance may be adequately predicted by

$$L_{12}(\theta_r) = L_{sr} \cos \theta_r \tag{1.7-38}$$

where $L_{sr}$ is the amplitude of the sinusoidal mutual inductance between the stator and rotor windings as given by (1.7-36).

In writing the voltage equations from (1.7-30) and (1.7-31) the total derivative of the flux linkages is required. This is accomplished by taking the partial derivative of both $\lambda_1$ and $\lambda_2$ with respect to $i_1$, $i_2$, and $\theta_r$. Writing these voltage equations is a problem at the end of the chapter.

**SP1.7-1** Let $L_m(x) = k/x$, $i = t$, and $x = t$. Express $d[L_m(x)i]/dt$. [Zero]

**SP1.7-2** Express $L(\theta_r)$ of the elementary reluctance machine if minimum reluctance occurs at $\theta_r = 0$. [$L(\theta_r) = L_l + L_A + L_0 \cos 2\theta_r$]

**SP1.7-3** Express $L_{11}$, $L_{22}$, and $L_{12}$ if positive $i_2$ is reversed from that shown in Fig. 1.7-4. [$L_{11}$ and $L_{22}$ are unchanged; $L_{12} = -L_{sr} \cos \theta_r$]

**SP1.7-4** Consider Fig. 1.7-4. $I_1 = 1$ A, $L_{sr} = 0.1$ H, $\omega_r = 100$ rad/s, $\theta_r(0) = 0$, and winding 2 is open-circuited. Express $V_2$. [$V_2 = -10 \sin 100t$]
1.8  RECAPPING

We will analyze electromechanical motion devices from the coupled-circuits viewpoint. Although the coupled windings of many electromechanical devices are in relative motion, the equivalent circuit of stationary coupled windings (the transformer) is the beginning of the equivalent circuits that we will develop for these devices in later chapters. We will find the concept of referring variables from one winding to the other very useful as we proceed.

The first step in the analysis of electromechanical motion devices of the electromagnetic type is to express the voltage and flux linkage equations in terms of self- and mutual inductances. We will not consider saturation in our analysis; instead we will restrict our work to linear magnetic systems and leave the analytical treatment of saturation to a more advanced study of these devices. In this chapter, we learned that electromagnetic, electromechanical motion devices are characterized by self- or mutual inductances that vary with displacement of the movable member.

In the next chapter, we will first develop an analytical means of determining the electromagnetic force or torque in electromechanical motion devices. Once we have accomplished this, we will be able to express the electromagnetic force in the elementary electromagnet and the electromagnetic torque in the elementary rotational devices that we have just considered.

1.9  REFERENCES


1.10  PROBLEMS

In all Problems sections, the more lengthy or involved problems are denoted by an asterisk.

1. Consider the magnetic system shown in Fig. 1.3-1. Let $\mu_r = 1500$, $N = 100$ turns, and $i = 2$ A. The cross section of the iron is square,
each side 4 cm in length. The air gap is 4 mm in length. The mean length of the iron is 200 times the air gap length. Neglect leakage flux and assume \( A_i = A_g \). Calculate the flux.

2. Repeat Example 1B with a second air gap of 2 mm in length cut midway between \( c \) and \( d \). Neglect leakage flux and assume \( A_i = A_g \).

3. An iron-core transformer that has two windings is shown in Fig. 1.10-1. \( N_1 = 50 \) turns, \( N_2 = 100 \) turns, and \( \mu_r = 4000 \). Calculate \( L_{12} \), \( L_{m1} \), and \( L_{m2} \).

![Figure 1.10-1: A two-winding iron-core transformer.]

4. An iron "doughnut" (toroid) with two coils is shown in Fig. 1.10-2. \( N_1 = 100 \) turns and \( N_2 = 200 \) turns, \( \mu_r = 10^4/4\pi \). Calculate \( L_{12} \).

5. An air gap is cut through the left leg of the magnetic system shown in Fig. 1C-1 so that the associated reluctance is 10 \( R_y \) rather than \( R_y \). Express \( L_{12} \) and \( L_{21} \) in terms of \( N_1 \), \( N_2 \), \( R_x \), and \( R_y \).

6. Two coupled coils have the following parameters: \( r_1 = 10 \Omega \), \( L_{11} = 0.1 L_{11} \), \( r_2 = 2.5 \Omega \), \( L_{12} = 0.1 L_{22} \), \( L_{11} = 100 \) mH, \( N_1 = 100 \) turns, \( L_{22} = 25 \) mH, \( N_2 = 50 \) turns. Develop an equivalent \( T \) circuit with (a) winding 1 as the reference winding and (b) winding 2 as reference winding.
7. Assume that the direction of positive current is reversed in winding 2 of Fig. 1.5-1. Express (a) $L_{12}$ in terms of $N_1$, $N_2$, and $R_m$; (b) $\lambda_1$ and $\lambda_2$ in the form of (1.5-22) and (1.5-23); (c) $\lambda_1$ and $\lambda_2'$ in the form of (1.5-31) and (1.5-32); and (d) $v_1$ and $v_2'$ in the form of (1.5-37).

8. The parameters of a transformer are: $r_1 = r_2' = 10 \Omega$, $L_{m1} = 300 \text{ mH}$, $L_{l1} = L_{l2}' = 30 \text{ mH}$. A 10-V peak-to-peak 30-Hz sinusoidal voltage is applied to winding 1. Winding 2 is short-circuited. Assume $i_1 = -\hat{i}_2'$. Calculate the phasor $\hat{I}_1$ with $\hat{V}_1$ at zero degrees.

9. A transformer with two windings has the following parameters: $r_1 = r_2 = 1 \Omega$, $L_{m1} = 1 \text{ H}$, $L_{l1} = L_{l2}' = 0.01 \text{ H}$, $N_1 = N_2$. A 2-$\Omega$ load resistance $R_L$ is connected across winding 2. $V_1 = 2 \cos 400t$. (a) Calculate $\hat{I}_1$. (b) Express $I_1$.

10. A transformer with two windings has the following parameters: $r_1 = 1 \Omega$, $L_{l1} = 0.01 \text{ H}$, $L_{m1} = 0.2 \text{ H}$, $N_2 = 2N_1$, $r_2 = 2 \Omega$, $L_{l2} = 0.04 \text{ H}$, $L_{m2} = 0.08 \text{ H}$. A 4-$\Omega$ resistance $R_L$ is connected across the terminals of winding 2 and a voltage $V_1 = \sqrt{2} 2 \cos 400t$ is applied to winding 1. Calculate and draw the phasor diagram showing $\hat{V}_1$, $\hat{I}_1$, $\hat{V}_2''$, and $\hat{I}_2'$. Neglect the magnetizing current.
*11. Consider the parameters of the transformer given in Example 1D. Calculate the input impedance measured from winding 1 with winding 2 short-circuited for (a) a dc source, (b) a 10-Hz source, and (c) a 400-Hz source. In each case, determine the input impedance first with the magnetizing current included and then with it neglected. The magnetizing current cannot be neglected as the frequency approaches zero. Why?

*12. Analytically obtain the expression for \(i_1\) in Figs. 1.6-1 and 1.6-2.

*13. If, in Figs. 1.6-1 and 1.6-2, the resistance \(r_1\) is zero, express \(i_1\) for \(t \geq 0\).

*14. Determine the phase of \(v_1\) in Fig. 1.6-3 in order to obtain the maximum offset \(i_1\). Neglect the magnetizing current.

15. For the elementary electromagnet shown in Fig. 1.7-1, assume that the cross-sectional area of the stationary and movable member is the same and \(A_i = A_g = 4\) cm\(^2\). Assume \(l_i = 20\) cm, \(N = 500\), and \(\mu_{ri} = 1000\). Express \(L_m(x)\) given by (1.7-17) and the approximation for \(x > 0\) given by (1.7-21). Determine the minimum value of \(x\) when this approximate expression for \(L_m(x)\) is less than 1.1 the value given by (1.7-17).

16. Express the voltage \(v\) of the elementary electromagnet given by (1.7-16) for \(L_m(x)\) given by (1.7-17), \(i = \sqrt{2}I_s\cos\omega_st\) and \(x = t\). Approximate \(v\) when \(t\) is large.

*17. Express the voltage equation for the elementary reluctance machine shown in Fig. 1.7-2. Use (1.7-29) for \(L(\theta_r)\).

18. Write the voltage equations for the coils in relative motion shown in Fig. 1.7-4. Use \(L_{11}, L_{22},\) and \(L_{12}\) as expressed by (1.7-38).