CONTENTS

Preface xiii

PART I INTRODUCTION

1 Modeling 3
   1.1 The model-based approach 3
      1.1.1 The modeling process 4
      1.1.2 The modeling advantage 5
   1.2 Organization of this book 5

2 Random variables 7
   2.1 Introduction 7
   2.2 Key functions and four models 9
      2.2.1 Exercises 17

3 Basic distributional quantities 19
   3.1 Moments 19
      3.1.1 Exercises 26
   3.2 Percentiles 27
      3.2.1 Exercises 28
   3.3 Generating functions and sums of random variables 29
### CONTENTS

3.3.1 Exercises .............................................. 30

3.4 Tails of distributions .................................. 31
  3.4.1 Classification based on moments ............... 31
  3.4.2 Comparison based on limiting tail behavior .... 32
  3.4.3 Classification based on the hazard rate function 33
  3.4.4 Classification based on the mean excess loss function 34
  3.4.5 Equilibrium distributions and tail behavior .... 36
  3.4.6 Exercises ............................................ 37

3.5 Measures of Risk ........................................ 38
  3.5.1 Introduction ........................................ 38
  3.5.2 Risk measures and coherence .................... 39
  3.5.3 Value-at-Risk ...................................... 40
  3.5.4 Tail-Value-at-Risk .................................. 42
  3.5.5 Exercises ............................................ 46

### PART II ACTUARIAL MODELS

4 Characteristics of Actuarial Models ................. 49
  4.1 Introduction .......................................... 49
  4.2 The role of parameters ................................ 49
    4.2.1 Parametric and scale distributions .......... 50
    4.2.2 Parametric distribution families ............ 52
    4.2.3 Finite mixture distributions ................. 52
    4.2.4 Data-dependent distributions ................. 55
    4.2.5 Exercises ........................................ 56

5 Continuous models ........................................ 59
  5.1 Introduction .......................................... 59
  5.2 Creating new distributions ......................... 59
    5.2.1 Multiplication by a constant ................. 60
    5.2.2 Raising to a power ................................ 60
    5.2.3 Exponentiation .................................... 62
    5.2.4 Mixing ............................................ 62
    5.2.5 Frailty models .................................... 65
    5.2.6 Splicing ........................................... 66
    5.2.7 Exercises ........................................ 68
  5.3 Selected distributions and their relationships .... 72
    5.3.1 Introduction ...................................... 72
    5.3.2 Two parametric families ....................... 72
    5.3.3 Limiting distributions ............................ 72
    5.3.4 Two heavy-tailed distributions ............... 74
    5.3.5 Exercises ........................................ 75
5.4  The linear exponential family 75
      5.4.1  Exercises 78

6  **Discrete distributions** 79
6.1  Introduction 79
      6.1.1  Exercise 80
6.2  The Poisson distribution 80
6.3  The negative binomial distribution 83
6.4  The binomial distribution 85
6.5  The \((a, b, 0)\) class 86
      6.5.1  Exercises 89
6.6  Truncation and modification at zero 89
      6.6.1  Exercises 94

7  **Advanced discrete distributions** 95
7.1  Compound frequency distributions 95
      7.1.1  Exercises 101
7.2  Further properties of the compound Poisson class 101
      7.2.1  Exercises 107
7.3  Mixed frequency distributions 107
      7.3.1  General mixed frequency distribution 107
      7.3.2  Mixed Poisson distributions 109
      7.3.3  Exercises 113
7.4  Effect of exposure on frequency 114

**Appendix: An inventory of discrete distributions** 114
      A.0.1  Exercises 115

8  **Frequency and severity with coverage modifications** 117
8.1  Introduction 117
8.2  Deductibles 117
      8.2.1  Exercises 122
8.3  The loss elimination ratio and the effect of inflation for ordinary
      deductibles 122
      8.3.1  Exercises 124
8.4  Policy limits 125
      8.4.1  Exercises 127
8.5  Coinsurance, deductibles, and limits 127
      8.5.1  Exercises 129
8.6  The impact of deductibles on claim frequency 131
      8.6.1  Exercises 134
CONTENTS

9 Aggregate loss models 137
  9.1 Introduction 137
  9.1.1 Exercises 140
  9.2 Model choices 140
  9.2.1 Exercises 141
  9.3 The compound model for aggregate claims 141
  9.3.1 Exercises 148
  9.4 Analytic results 155
  9.4.1 Exercises 158
  9.5 Computing the aggregate claims distribution 159
  9.6 The recursive method 161
    9.6.1 Applications to compound frequency models 163
    9.6.2 Underflow/overflow problems 165
    9.6.3 Numerical stability 165
    9.6.4 Continuous severity 166
    9.6.5 Constructing arithmetic distributions 166
    9.6.6 Exercises 169
  9.7 The impact of individual policy modifications on aggregate payments 173
    9.7.1 Exercises 176
  9.8 The individual risk model 176
    9.8.1 The model 176
    9.8.2 Parametric approximation 178
    9.8.3 Compound Poisson approximation 180
    9.8.4 Exercises 182

PART III CONSTRUCTION OF EMPIRICAL MODELS

10 Review of mathematical statistics 187
  10.1 Introduction 187
  10.2 Point estimation 188
    10.2.1 Introduction 188
    10.2.2 Measures of quality 189
    10.2.3 Exercises 195
  10.3 Interval estimation 196
    10.3.1 Exercises 198
  10.4 Tests of hypotheses 198
    10.4.1 Exercise 202

11 Estimation for complete data 203
  11.1 Introduction 203
  11.2 The empirical distribution for complete, individual data 207
11.2.1 Exercises 211
11.3 Empirical distributions for grouped data 211
11.3.1 Exercises 214

12 Estimation for modified data 217
12.1 Point estimation 217
12.1.1 Exercises 224
12.2 Means, variances, and interval estimation 225
12.2.1 Exercises 234
12.3 Kernel density models 236
12.3.1 Exercises 239
12.4 Approximations for large data sets 240
12.4.1 Introduction 240
12.4.2 Using individual data points 242
12.4.3 Interval-based methods 245
12.4.4 Exercises 249

PART IV PARAMETRIC STATISTICAL METHODS

13 Frequentist estimation 253
13.1 Method of moments and percentile matching 253
13.1.1 Exercises 257
13.2 Maximum likelihood estimation 259
13.2.1 Introduction 259
13.2.2 Complete, individual data 261
13.2.3 Complete, grouped data 262
13.2.4 Truncated or censored data 263
13.2.5 Exercises 266
13.3 Variance and interval estimation 272
13.3.1 Exercises 278
13.4 Nonnormal confidence intervals 280
13.4.1 Exercise 282
13.5 Maximum likelihood estimation of decrement probabilities 282
13.5.1 Exercise 284

14 Frequentist Estimation for discrete distributions 285
14.1 Poisson 285
14.2 Negative binomial 289
14.3 Binomial 291
14.4 The \((a, b, 1)\) class 293
14.5 Compound models 297
14.6 Effect of exposure on maximum likelihood estimation 299
## Contents

14.7 Exercises 300

15 **Bayesian estimation** 305

15.1 Definitions and Bayes' Theorem 305
15.2 Inference and prediction 309
  15.2.1 Exercises 315
15.3 Conjugate prior distributions and the linear exponential family 320
  15.3.1 Exercises 321
15.4 Computational issues 322

16 **Model selection** 323

16.1 Introduction 323
16.2 Representations of the data and model 324
16.3 Graphical comparison of the density and distribution functions 325
  16.3.1 Exercises 330
16.4 Hypothesis tests 330
  16.4.1 Kolmogorov-Smirnov test 330
  16.4.2 Anderson-Darling test 332
  16.4.3 Chi-square goodness-of-fit test 333
  16.4.4 Likelihood ratio test 337
  16.4.5 Exercises 339
16.5 Selecting a model 342
  16.5.1 Introduction 342
  16.5.2 Judgment-based approaches 342
  16.5.3 Score-based approaches 343
  16.5.4 Exercises 350

## Part V Credibility

17 **Introduction and Limited Fluctuation Credibility** 357

17.1 Introduction 357
17.2 Limited fluctuation credibility theory 359
17.3 Full credibility 360
17.4 Partial credibility 363
17.5 Problems with the approach 366
17.6 Notes and References 367
17.7 Exercises 367

18 **Greatest accuracy credibility** 371

18.1 Introduction 371
18.2 Conditional distributions and expectation 373
18.3 The Bayesian methodology 377
18.4  The credibility premium 385
18.5  The Bühlmann model 388
18.6  The Bühlmann–Straub model 392
18.7  Exact credibility 397
18.8  Notes and References 401
18.9  Exercises 402

19  Empirical Bayes parameter estimation 415
19.1  Introduction 415
19.2  Nonparametric estimation 418
19.3  Semiparametric estimation 428
19.4  Notes and References 430
19.5  Exercises 430

PART VI  SIMULATION

20  Simulation 437
20.1  Basics of simulation 437
  20.1.1  The simulation approach 438
  20.1.2  Exercises 442
20.2  Simulation for specific distributions 442
  20.2.1  Discrete mixtures 442
  20.2.2  Time or age of death from a life table 443
  20.2.3  Simulating from the \((\alpha, b, 0)\) class 444
  20.2.4  Normal and lognormal distributions 446
  20.2.5  Exercises 447
20.3  Determining the sample size 448
  20.3.1  Exercises 449
20.4  Examples of simulation in actuarial modeling 450
  20.4.1  Aggregate loss calculations 450
  20.4.2  Examples of lack of independence 450
  20.4.3  Simulation analysis of the two examples 451
  20.4.4  Using simulation to determine risk measures 454
  20.4.5  Statistical analyses 454
  20.4.6  Exercises 456

A  An inventory of continuous distributions 459
A.1  Introduction 459
A.2  Transformed beta family 463
  A.2.1  Four-parameter distribution 463
  A.2.2  Three-parameter distributions 463
  A.2.3  Two-parameter distributions 465
CONTENTS

A.3 Transformed gamma family 467
  A.3.1 Three-parameter distributions 467
  A.3.2 Two-parameter distributions 468
  A.3.3 One-parameter distributions 469
A.4 Distributions for large losses 470
  A.4.1 Extreme value distributions 470
  A.4.2 Generalized Pareto distributions 471
A.5 Other distributions 471
A.6 Distributions with finite support 473

B An inventory of discrete distributions 475
  B.1 Introduction 475
  B.2 The \((a, b, 0)\) class 476
  B.3 The \((a, b, 1)\) class 477
    B.3.1 The zero-truncated subclass 477
    B.3.2 The zero-modified subclass 479
  B.4 The compound class 480
    B.4.1 Some compound distributions 480
  B.5 A hierarchy of discrete distributions 482

C Frequency and severity relationships 483

D The recursive formula 485

E Discretization of the severity distribution 487
  E.1 The method of rounding 487
  E.2 Mean preserving 488
  E.3 Undiscretization of a discretized distribution 488

F Numerical optimization and solution of systems of equations 491
  F.1 Maximization using Solver 491
  F.2 The simplex method 495
  F.3 Using Excel® to solve equations 496

References 501

Index 507