CHAPTER 1

Introduction

1.1 Theory, Modelling, and Implementation

This book aims to give a balanced representation of the theoretical foundations of mathematical finance, especially derivative pricing, state-of-the-art models, which are actually used in practice, and their implementation.

In practice, none of the three aspects—theory, modeling, and implementation—can be considered alone. Knowledge of the theory is worthless if it isn’t applied. Theory provides the tools for consistent modeling. A model without implementation is essentially useless. Good implementation requires a deep understanding of the model and the underlying theory.

With this in mind, the book tries to build a bridge between academia and practice and from theory to object-oriented implementation.

1.2 Interest Rate Models and Interest Rate Derivatives

The text concentrates on the modeling of interest rates as stochastic (uncertain) quantities and the valuation of interest rate derivatives under such models. However, this is not a specialized text. Although the mathematical modeling of stock prices was the historical starting point and interest rates were assumed to be constant, some important theoretical aspects are significant only for stochastic interest rates (e.g., the change of numeraire techniques). For this reason, it is meaningful to start with interest rate models. Another reason to start with interest rate models is that interest rate models are the foundation of hybrid models. Since this manuscript, the reference asset is most likely an interest rate-related product, a usual stochastic interest
rates implies the need to build upon an interest rate model; see Figures 1.1 and 1.2. We will do so in Chapter 29. Nevertheless, the first model studied will be, of course, the Black-Scholes model for a single stock, after which we will move to stochastic interest rates.

![Figure 1.1: Hybrid Models](image)

The numéraire, the reference asset in the modeling of price processes, is most likely an interest rate product. This choice is not mathematically necessary but convenient for almost all models. Interest rate processes are the natural starting point for the modeling of price processes.

### Black-Scholes Model

<table>
<thead>
<tr>
<th>Equity Model</th>
<th>dS = μ S dt + σ S dW(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest Rate Model</td>
<td>dB = r B dt</td>
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</table>

### Equity Hybrid LIBOR Market Model

<table>
<thead>
<tr>
<th>Equity Model</th>
<th>dS = μ S dt + σ S dW(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest Rate Model</td>
<td>dL_1 = μ_1 L_1 dt + σ_1 L_1 dW_1(t)</td>
</tr>
</tbody>
</table>

![Figure 1.2: The Black-Scholes model may be interpreted as a hybrid model with deterministic interest rates. The solution of dV(t) = rV(t)dt + σV(t)dW(t), i.e., it is deterministic and given in closed form. Thus, the interest rate component is trivial. Within a LIBOR market model, the interest rate is a stochastic quantity, which also changes properties of the stock process.](image)
1.3 About This Book

1.3.1 How to Read This Book

The text may be read in a modular way, i.e., the chapters have been kept as free-standing as possible. Chapters 2 and 3 provide the foundations in the order of their dependencies. The reader familiar with the concepts of stochastic processes and martingales may skip the chapter and use it as reference only. To get a feeling for the mathematical concepts, one should read the special sections Interpretation and Motivation. Readers familiar with programming and implementation may prefer Chapter 13 as an illustration of the basic concepts.

The appendix gives a selection of the results and techniques from diverse areas (linear algebra, calculus, optimization), which are used in the text and in the implementation, but which are less important for understanding the essential concepts.

1.3.2 Abridged Version

For a crash course focusing on particular aspects some chapters may be skipped. What follows are a few suggestions in this direction.

1.3.2.1 Abridged version "Binomial Pricing"

Foundations (Chapter 2) → Replication (Chapter 3)
- Black-Scholes Model (Chapter 4)
- Discretization / Monte Carlo Simulation (Chapter 13)

1.3.2.2 Abridged version "LHCR Market Model"

Foundations (Chapter 2) → Replication (Chapter 3)
- Libor Market Structures (Chapter 8) → Black Model (Chapter 10)
- Heston Model (Chapter 19)
- Instantaneous and Terminal Correlation (Chapter 21)
- Shape of the Interest Rate Curve (Chapter 23)

1.3.2.3 Abridged version "Heston-Hull-White Model"

Foundations (Chapter 2) → Replication (Chapter 3)
- Interest Rate Structures (Chapter 8) → Black Model (Chapter 10)
- The Density of the Underlying of a European Option (Chapter 5)
- Heston-Hull-White Models (Chapter 27)
1.3.3 Special Sections

The text contains special sections giving notes on interpretation, motivation, and practical aspects. These are marked by the following symbols:

- **Interpretation**: Provides an interpretation of the preceding topic.
- **Motivation**: Provides motivation for the following topic. Sometimes notes deficiencies in the previous results.
- **Further Reading**: Suggested literature and associated topics.
- **Experiment**: Guide for a software experiment where aspects of the preceding topic can be explored.
- **Tip**: Hints for practical use and software implementation of the preceding topic.

1.3.4 Notation

We will model the time evolution of stocks or interest rates with random variables parametrized through a time parameter $t$. Such stochastic processes may depend on other parameters like morality or interest rate period. We will separate these into different kinds of parameters by a semicolon, see Figure 1.3.
1.3.5 Exercises

Please help us improve this book. Please send error reports and suggestions to:

Christian Hites <email@christian-hites.de>

Thank you.

1.3.6 Resources

In connection with this book, the following resources are available:

- Interactive experiments and exercises:
  https://www.christian-hites.de/Community/applets

- Java™ source code:
  https://www.ChristianHites.de/

- Figures (in Color): The figures in this book are reproduced in black and white. The original color figures may be obtained from:
  https://www.christian-hites.de/Community/book

- Updates: For updates and errata connections see:
  https://www.christian-hites.de/Community/book/errata