Preliminaries on Deterministic and Random Signals

Signals are the object of communication systems and as such the vehicle for communications. In particular, it is only through random processes, which are unpredictable to some extent, that we can carry information from source to destination. In this chapter we review some basic results about deterministic and random processes that the reader should be familiar with. The abridgement of such results in this chapter is also the occasion to establish the notation we will use throughout the book. We let \( j = \sqrt{-1} \). In Table 1.1 we collect useful trigonometric identities.

1.1 TIME AND FREQUENCY DOMAIN REPRESENTATION

1.1.1 CONTINUOUS TIME SIGNALS

**Definition 1.1** In our context a continuous time signal is a complex-valued function of a real variable

\[
x : \mathbb{R} \mapsto \mathbb{C}.
\]  

(1.1)
Table 1.1 Trigonometric identities.

<table>
<thead>
<tr>
<th>Trigonometric identities</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) )</td>
<td></td>
</tr>
<tr>
<td>( \cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) )</td>
<td></td>
</tr>
<tr>
<td>( \sin(\alpha) \sin(\beta) = \frac{1}{2} \left[ \cos(\alpha - \beta) - \cos(\alpha + \beta) \right] )</td>
<td></td>
</tr>
<tr>
<td>( \cos(\alpha) \cos(\beta) = \frac{1}{2} \left[ \cos(\alpha + \beta) + \cos(\alpha - \beta) \right] )</td>
<td></td>
</tr>
<tr>
<td>( \sin(\alpha) \cos(\beta) = \frac{1}{2} \left[ \sin(\alpha + \beta) + \sin(\alpha - \beta) \right] )</td>
<td></td>
</tr>
<tr>
<td>( \sin(2\alpha) = 2\sin(\alpha) \cos(\alpha) )</td>
<td></td>
</tr>
<tr>
<td>( \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) )</td>
<td></td>
</tr>
<tr>
<td>( \sin^2(\alpha) = \frac{1}{2} [1 - \cos(2\alpha)] )</td>
<td></td>
</tr>
<tr>
<td>( \cos^2(\alpha) = \frac{1}{2} [1 + \cos(2\alpha)] )</td>
<td></td>
</tr>
</tbody>
</table>

Euler identities

\[
\begin{align*}
\cos(\alpha) &= \frac{e^{j\alpha} + e^{-j\alpha}}{2} \\
\sin(\alpha) &= \frac{e^{j\alpha} - e^{-j\alpha}}{2j} \\
e^{j\alpha} &= \cos(\alpha) + j\sin(\alpha)
\end{align*}
\]

The independent real variable often has the meaning of time and is thus denoted by the letter \( t \) and measured in seconds (s) or their multiples or submultiples. On the contrary the function \( x(t) \) can represent any quantity that possibly evolves with time, such as the voltage across the terminals of an electrical device, the current in a mesh, the temperature at some point in space, the optical power emitted by a laser source, and so on; its value will then accordingly be expressed in the proper measurement units: volt (V), ampere (A), kelvin (K), watt (W), etc. Most of these quantities are intrinsically real, and the signals that represent them will only take real values. Such signals are called real-valued or simply real signals. The graphical representation of a real signal \( x \) is the plot of \( x(t) \) versus \( t \), as illustrated in Figure 1.1.

![Graphical representation of a real-valued signal representing a voltage.](image)
Example 1.1 A  The following are a few real-valued functions that are very useful in the definition of signals.

- The Heaviside step function
  \[ 1(x) = \begin{cases} 
  1, & x > 0 \\
  1/2, & x = 0 \\
  0, & x < 0.
  \end{cases} \]

- We also make use of
  \[ 1_0(x) = \begin{cases} 
  1, & x \geq 0 \\
  0, & x < 0.
  \end{cases} \]

- The sign function
  \[ \text{sgn}(x) = \begin{cases} 
  -1, & x < 0 \\
  0, & x = 0 \\
  1, & x > 0.
  \end{cases} \]

- The rectangle function
  \[ \text{rect}(x) = \begin{cases} 
  1, & |x| < 1/2 \\
  1/2, & |x| = 1/2 \\
  0, & |x| > 1/2.
  \end{cases} \]

- The triangle function
  \[ \text{triang}(x) = \begin{cases} 
  1 - |x|, & |x| < 1 \\
  0, & |x| \geq 1.
  \end{cases} \]

- The Woodward sine cardinal function
  \[ \text{sinc}(x) = \begin{cases} 
  \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\
  1, & x = 0.
  \end{cases} \]

- The raised cosine function
  \[ \text{rcos}(x, \rho) = \begin{cases} 
  1, & 0 \leq |x| \leq (1 - \rho)/2 \\
  \cos^2 \left( \frac{\pi}{2} \frac{|x| - (1 - \rho)/2}{\rho} \right), & (1 - \rho)/2 < |x| \leq (1 + \rho)/2 \\
  0, & |x| > (1 + \rho)/2.
  \end{cases} \]
PRELIMINARIES ON DETERMINISTIC AND RANDOM SIGNALS

- The inverse raised cosine function
  \[ \text{ircos}(x, \rho) = \frac{\sin(x) \cos(\pi \rho x)}{1 - (2\rho x)^2}. \]  \hspace{1cm} (1.3)

- The square root raised cosine function
  \[ \text{rrcos}(x, \rho) = \sqrt{\text{rcos}(x, \rho)}. \]  \hspace{1cm} (1.4)

- The inverse square root raised cosine function
  \[ \text{irrcos}(x, \rho) = \frac{\sin[\pi(1 - \rho)x] + 4\rho x \cos[\pi(1 + \rho)x]}{\pi[1 - (4\rho x)^2]x}. \]  \hspace{1cm} (1.5)

Many signals can be obtained from the above basic functions with operations of time shifting, time scaling, and amplitude scaling. For example we can define the step signal starting at \( t_1 \) with amplitude \( A_1 \) shown in Figure 1.2(a) as \( x_1(t) = A_1 (t - t_1) \) and the rectangular pulse with amplitude \( A_2 \) and width \( T_2 \) shown in Figure 1.2(b) as \( x_2(t) = A_2 \text{rect}(t/T_2) \).

![Figure 1.2](image)

**Figure 1.2** Plots of: (a) the signal \( x_1(t) = A_1(t - t_1) \); (b) the signal \( x_2(t) = A_2 \text{rect}(t/T_2) \).

**Example 1.1 B** The Dirac delta function or impulse \( \delta(t) \) is also very useful in defining signals, although it is not a ‘function’ strictly speaking. It is characterized by the following properties:

- it vanishes outside the origin
  \[ \delta(t) = 0, \quad t \neq 0; \]

- its integral over any interval including the origin is unity
  \[ \int_a^b \delta(t) \, dt = \begin{cases} 1, & 0 \in (a, b) \\ 0, & 0 \not\in (a, b); \end{cases} \]

- the sifting property
  \[ \int_{-\infty}^{+\infty} x(t) \delta(t - t_0) \, dt = x(t_0). \]

It is easy to see that \( \delta(0) \) cannot have a finite value, since the above integrals would vanish in all cases: thus, it is commonly said that the Dirac impulse is infinite at the origin.

It is also easy to verify that the Dirac impulse and the step function are related by

\[ \int_{-\infty}^{t} \delta(\tau) \, d\tau = 1(t), \quad \delta(t) = \frac{d}{dt} 1(t). \]
A very common graphical representation of the Dirac impulse is given by an arrow pointing upward, as in Figure 1.3.

![Figure 1.3](image)

Figure 1.3  Plots of: (a) the Dirac impulse; (b) the same impulse shifted in time.

As we show in Figure 1.4, for a complex signal $x$ two different representations are commonly used, each combining two plots: the real and imaginary parts representation, combining plots of the real part $\Re[x(t)]$ and imaginary part $\Im[x(t)]$ versus $t$ as in Figure 1.4(b); and the amplitude and phase representation, combining plots of the amplitude

$$|x(t)| = \sqrt{(\Re[x(t)])^2 + (\Im[x(t)])^2}$$

(1.6)

and phase

$$\angle x(t) = \arg x(t) = \arctan \left( \frac{\Im[x(t)]}{\Re[x(t)]} \right)$$

(1.7)

versus $t$ as in Figure 1.4(c). Hence

$$x(t) = \Re[x(t)] + j \Im[x(t)]$$

(1.8)

or

$$x(t) = |x(t)| e^{j \angle x(t)}.$$  

(1.9)

It is far less common to use the three-dimensional representation of Figure 1.4(a).

A continuous time signal can also be identified through its Fourier transform (Ftf) that is itself a complex function of a real variable, defined as

$$\mathcal{X} : \mathbb{R} \mapsto \mathbb{C}, \quad \mathcal{X}(f) = \mathcal{F}[x][f] = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} \, dt. \quad (1.10)$$

The variable denoted with $f$ has the meaning of frequency, the inverse of time, and is thus measured in hertz (Hz), whereas the dimension of $\mathcal{X}$ is the product of the dimensions of $x$ and $t$. For example, if the signal $x$ represents a voltage evolving with time, its Fourier transform will be measured in volts times seconds (V s) or volts per Hz (V/Hz). Moreover the transform is usually complex, even for a real signal, and it is quite common to choose the amplitude/phase plots for its graphical representation.

1We assume that the range of the arctan is $(-\pi, \pi)$ by inspection of the sign of $\Re[x(t)]$. 
The Fourier transformation (1.10), yielding $X$ from $x$, can be inverted allowing one to unambiguously recover the signal $x$ from its transform $X$ by the relation

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} \, df. \quad (1.11)$$

Since knowledge of $X$ is equivalent to knowledge of $x$, $X$ can be seen as an alternative way to represent the signal $x$ and thus it is usually referred to as the *frequency domain* representation of $x$. In this context, the function $x(t)$ is called the *time domain* representation of the signal.

The reader should already be familiar with the remarkable concept of Fourier transform. Here we limit ourselves to recall some of its properties in Table 1.2, and collect a few signal/transform pairs in Table 1.3.
Some general properties of the Fourier transform.

<table>
<thead>
<tr>
<th>Property</th>
<th>Signal</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>linearity</td>
<td>$x(t)$</td>
<td>$\mathcal{X}(f)$</td>
</tr>
<tr>
<td>duality</td>
<td>$ax(t) + by(t)$</td>
<td>$a\mathcal{X}(f) + b\mathcal{Y}(f)$</td>
</tr>
<tr>
<td>time inverse</td>
<td>$x(-t)$</td>
<td>$\mathcal{X}(-f)$</td>
</tr>
<tr>
<td>complex conjugate</td>
<td>$x^*(t)$</td>
<td>$\mathcal{X}^*(-f)$</td>
</tr>
<tr>
<td>real part</td>
<td>$\Re[x(t)] = \frac{1}{2}[x(t) + x^*(t)]$</td>
<td>$\frac{1}{2}[\mathcal{X}(f) + \mathcal{X}^*(-f)]$</td>
</tr>
<tr>
<td>imaginary part</td>
<td>$\Im[x(t)] = \frac{1}{2j}[x(t) - x^*(t)]$</td>
<td>$\frac{1}{2j}[\mathcal{X}(f) - \mathcal{X}^*(-f)]$</td>
</tr>
<tr>
<td>scaling</td>
<td>$x(at)$, $a \neq 0$</td>
<td>$\frac{1}{</td>
</tr>
<tr>
<td>time shift</td>
<td>$x(t - t_0)$</td>
<td>$e^{-j2\pi ft_0}\mathcal{X}(f)$</td>
</tr>
<tr>
<td>frequency shift</td>
<td>$x(t) e^{j2\pi f_0 t}$</td>
<td>$\mathcal{X}(f - f_0)$</td>
</tr>
<tr>
<td>modulation</td>
<td>$x(t) \cos(2\pi f_0 t + \varphi)$</td>
<td>$\frac{1}{2}[e^{j\varphi}\mathcal{X}(f - f_0) + e^{-j\varphi}\mathcal{X}(f + f_0)]$</td>
</tr>
<tr>
<td></td>
<td>$x(t) \sin(2\pi f_0 t + \varphi)$</td>
<td>$\frac{1}{2j}[e^{j\varphi}\mathcal{X}(f - f_0) - e^{-j\varphi}\mathcal{X}(f + f_0)]$</td>
</tr>
<tr>
<td>differentiation</td>
<td>$\frac{d}{dt}x(t)$</td>
<td>$j2\pi f \mathcal{X}(f)$</td>
</tr>
<tr>
<td>integration</td>
<td>$\int_{-\infty}^{t} x(\tau) d\tau = 1 * x(t)$</td>
<td>$\frac{1}{j2\pi f} \mathcal{X}(f) + \frac{1}{2} \mathcal{X}(0) \delta(f)$</td>
</tr>
<tr>
<td>convolution</td>
<td>$(x * y)(t)$</td>
<td>$\mathcal{X}(f)\mathcal{Y}(f)$</td>
</tr>
<tr>
<td>product</td>
<td>$x(t) y(t)$</td>
<td>$(\mathcal{X} * \mathcal{Y})(f)$</td>
</tr>
<tr>
<td>real signal</td>
<td>$x(t) = x^*(t)$</td>
<td>$\mathcal{X}(f) = \mathcal{X}^*(-f)$, $\mathcal{X}$ Hermitian, $\Re[\mathcal{X}(f)]$ even, $\Im[\mathcal{X}(f)]$ odd, $</td>
</tr>
<tr>
<td>imaginary signal</td>
<td>$x(t) = -x^*(t)$</td>
<td>$\mathcal{X}(f) = -\mathcal{X}^*(-f)$</td>
</tr>
<tr>
<td>even signal</td>
<td>$x(t) = x(-t)$</td>
<td>$\mathcal{X}(f) = \mathcal{X}(-f)$, $\mathcal{X}$ even</td>
</tr>
<tr>
<td>odd signal</td>
<td>$x(t) = -x(-t)$</td>
<td>$\mathcal{X}(f) = -\mathcal{X}(-f)$, $\mathcal{X}$ odd</td>
</tr>
</tbody>
</table>

Parseval’s theorem

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |\mathcal{X}(f)|^2 df = E_\mathcal{X}$$

Poisson sum formula

$$\sum_{k=-\infty}^{+\infty} x(kT_c) = \frac{1}{T_c} \sum_{\ell=-\infty}^{+\infty} \mathcal{X}(\ell/T_c)$$
Table 1.3 Examples of signal/Fourier transform pairs.

<table>
<thead>
<tr>
<th>Signal</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(t)$</td>
<td>$X(f)$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$\delta(f)$</td>
</tr>
<tr>
<td>1</td>
<td>$\delta(f)$</td>
</tr>
<tr>
<td>$e^{j2\pi f_0 t}$</td>
<td>$\delta(f - f_0)$</td>
</tr>
<tr>
<td>$\cos(2\pi f_0 t)$</td>
<td>$\frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)]$</td>
</tr>
<tr>
<td>$\sin(2\pi f_0 t)$</td>
<td>$\frac{1}{2j} [\delta(f - f_0) - \delta(f + f_0)]$</td>
</tr>
<tr>
<td>$1(t)$</td>
<td>$\frac{1}{2} \delta(f) + \frac{1}{j2\pi f}$</td>
</tr>
<tr>
<td>$\text{sgn}(t)$</td>
<td>$\frac{1}{j\pi f}$</td>
</tr>
<tr>
<td>$\text{rect}\left(\frac{t}{T}\right)$</td>
<td>$T \text{sinc}(fT)$</td>
</tr>
<tr>
<td>$\text{sinc}\left(\frac{t}{T}\right)$</td>
<td>$T \text{rect}(fT)$</td>
</tr>
<tr>
<td>$\text{triang}\left(\frac{t}{T}\right)$</td>
<td>$T \text{sinc}^2(fT)$</td>
</tr>
<tr>
<td>$e^{-at} 1(t)$, with $a &gt; 0$</td>
<td>$\frac{1}{a + j2\pi f}$</td>
</tr>
<tr>
<td>$t e^{-at} 1(t)$, with $a &gt; 0$</td>
<td>$\frac{1}{(a + j2\pi f)^2}$</td>
</tr>
<tr>
<td>$e^{-a</td>
<td>t</td>
</tr>
<tr>
<td>$e^{-at^2}$, with $a &gt; 0$</td>
<td>$\sqrt{\frac{\pi}{a}} e^{-\pi^2 f^2/a}$</td>
</tr>
</tbody>
</table>

Example 1.1 C The signal with constant amplitude $A$ and support from $t_1$ to $t_2$

$$x(t) = \begin{cases} A, & t_1 \leq t < t_2 \\ 0, & \text{elsewhere} \end{cases}$$

has the following Ff:

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt = \int_{t_1}^{t_2} A e^{-j2\pi ft} dt = jA \frac{e^{-j2\pi ft_2} - e^{-j2\pi ft_1}}{2\pi f}. $$

The same result can be obtained by writing $x(t)$ as a rectangular pulse, centered at $t_0 = (t_1 + t_2)/2$ and having width $T = t_2 - t_1$, that is

$$x(t) = A \text{rect}\left(\frac{t - t_0}{T}\right)$$
with the rect function introduced in Example 1.1 A. Starting from the \( \text{rect} \leftrightarrow \text{sinc} \) signal/transform pair given in Table 1.3, we write
\[
x_1(t) = \text{rect} \left( \frac{t}{T} \right) \xrightarrow{\mathcal{F}} X_1(f) = T \text{sinc}(Tf).
\]
Then by applying the time shift rule in Table 1.2
\[
x_2(t) = x_1(t - t_0) = \text{rect} \left( \frac{t - t_0}{T} \right) \xrightarrow{\mathcal{F}} X_2(f) = e^{-j2\pi ft_0} X_1(f) = T \text{sinc}(Tf) e^{-j2\pi ft_0}.
\]
Eventually, linearity yields
\[
x(t) = Ax_2(t) = A \text{rect} \left( \frac{t - t_0}{T} \right) \xrightarrow{\mathcal{F}} X(f) = A X_2(f) = AT \text{sinc}(Tf) e^{-j2\pi ft_0}.
\]
The signal and its transform are shown in Figure 1.5.

**Figure 1.5** Graphical representations of the signal and its transform in Example 1.1 C, in the case \( t_2 = 5t_1 \).

### 1.1.2 FREQUENCY DOMAIN REPRESENTATION FOR PERIODIC SIGNALS

For periodic signals, that is signals having the property
\[
x(t) = x(t + T_p)
\]for some period \( T_p \), the frequency domain representation (1.10) fails, since the integral over the whole time axis would not converge, at least in the ordinary sense. Indeed for periodic signals it is common to consider the expansion of \( x(t) \) into its Fourier series of complex exponentials with frequencies that are multiples of \( F_p = 1/T_p \) (called the fundamental
frequency of the signal). Thus we can write

\[ x(t) = \sum_{\ell=-\infty}^{+\infty} X_\ell e^{j2\pi \ell F_p t}, \quad (1.13) \]

where the Fourier coefficients \( \{X_\ell\} \) are determined as

\[ X_\ell = \frac{1}{T_p} \int_{0}^{T_p} x(t) e^{-j2\pi \ell F_p t} \, dt. \quad (1.14) \]

Observe that the Fourier coefficients of a signal have the same physical dimension as the signal.

**Example 1.1 D** A square pulse repetition signal with amplitude \( A \), period \( T_p \) and duty cycle \( d \), with \( 0 < d < 1 \), has the expression

\[ x(t) = \sum_{n=-\infty}^{+\infty} A \text{rect} \left( \frac{t - nT_p}{dT_p} \right). \]

Its Fourier coefficients are calculated as the integral over any period, that is any interval of length \( T_p \),

\[ X_\ell = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi \ell F_p t} \, dt = \frac{1}{T_p} \int_{-dT_p/2}^{dT_p/2} A e^{-j2\pi \ell F_p t} \, dt \]
\[ = j \frac{A}{T_p} \frac{e^{-j\pi \ell F_p dT_p} - e^{j\pi \ell F_p dT_p}}{2\pi \ell F_p} \]
\[ = Ad \text{sinc}(\ell d). \]

For the particular case \( d = 1/2 \) we get

\[ X_\ell = \begin{cases} A, & \ell = 0 \\ (-1)^{(\ell-1)/2} A \ell \pi, & \ell \text{ odd} \\ 0, & \ell \text{ even}. \end{cases} \]

Based on the Fourier series expansion (1.13) and the Fourier transform of the complex exponential in Table 1.3, it is possible to circumvent the convergence problem in (1.10) and define a Fourier transform for periodic signals as a sum of Dirac impulses, regularly spaced by \( F_p \), each weighted by the corresponding Fourier coefficient

\[ \mathcal{X}(f) = \sum_{\ell=-\infty}^{+\infty} X_\ell \delta(f - \ell F_p). \quad (1.15) \]

Conversely, if the Ftf of a signal only contains Dirac impulses at multiples of a frequency \( F_p \) as in (1.15), we can write the signal as a linear combination of complex exponentials as in (1.13). This implies that the signal is periodic with period \( T_p = 1/F_p \).
1.1.3 DISCRETE TIME SIGNALS

Analogously to (1.1) we have the following definition.

**Definition 1.2** A discrete time signal or sequence is a complex function of an integer variable

\[ x : \mathbb{Z} \rightarrow \mathbb{C}. \]  

(1.16)

The integer variable has no physical dimensions attached to it, but it is usually associated with the flowing of time through some constant \( T \) called *quantum*. Thus, we write \( x(nT) \) or equivalently \( x_n \) to represent the value of the quantity \( x \) observed at the instant \( nT \). As in continuous time signals, \( x \) may represent a physical quantity (voltage, current, temperature, pressure), or unlike continuous time signals it may represent just a number. The discrete nature of the time domain may either be inherent to the signal (think for example of a sequence of bits representing a text file), or come from the observation of an analog quantity at regularly spaced instants (see Section 1.3.3 on sampling) in which case \( T \) is called the *sampling period*.

A common graphical representation for a real-valued discrete time signal \( \{x(nT)\} \) is via a sequence of points at coordinates \((nT, x_n)\) stemming out from the horizontal (time) axis, as in Figure 1.6. We stress the fact that \( x \) is not defined at instants that are not multiples of \( T \).

\[ x(nT) \]  

![Figure 1.6 Graphical representation of a real-valued discrete time signal.](image)

For complex-valued signals two-plot representations are used, either real/imaginary part or amplitude/phase plots as in Figure 1.7.

The frequency domain representation of a discrete time signal \( \{x(nT)\} \) is given by its Fourier transform

\[ \mathcal{X} : \mathbb{R} \rightarrow \mathbb{C}, \quad \mathcal{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi fnT}, \]  

(1.17)

which turns out to be a periodic function of frequency with period \( F = 1/T \), as obtained by a linear combination of complex exponentials with periods that are submultiples of \( F \). The inverse transform in this case is given by the formula

\[ x(nT) = T \int_0^F \mathcal{X}(f) e^{j2\pi fnT} \, df. \]  

(1.18)
The following example is the discrete time analog of Example 1.1 C.

**Example 1.1 E**  The signal with constant amplitude $A$ and support from $n_1 T$ to $n_2 T$

$$x(nT) = \begin{cases} A, & n_1 \leq n \leq n_2 \\ 0, & \text{elsewhere} \end{cases}$$

has the Fourier transform

$$\mathcal{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT) e^{-j2\pi fnT} = \sum_{n=n_1}^{n_2} A e^{-j2\pi fnT} = A \sum_{n=n_1}^{n_2} (e^{-j2\pi f T})^n.$$
By the algebraic identity, holding for any $z \in \mathbb{C}$,

$$\sum_{n=n_1}^{n_2} z^n = \frac{z^{n_2+1} - z^{n_1}}{z - 1},$$

we obtain the result

$$X(f) = A \frac{e^{-j2\pi f(n_2+1)T} - e^{-j2\pi f n_1 T}}{e^{-j2\pi f T} - 1} = A e^{-j2\pi n_0 f T} \frac{\sin(\pi N f T)}{\sin(\pi f T)},$$

where $n_0 = (n_1 + n_2)/2$ is the midpoint between $n_1$ and $n_2$, and $N = n_2 - n_1 + 1$ is the number of nonzero samples of the signal. The signal and its transform are shown in Figure 1.8.

**Figure 1.8** Graphical representations of the signal and its Fourier transform in Example 1.1 E, in the case $n_1 = 2, n_2 = 8, n_0 = 5, N = 7$.

**Example 1.1 F** In discrete time signals the equivalent role of the Dirac delta function for continuous time signals is played by the Kronecker delta,

$$\delta_m = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases} \quad (1.19)$$

having as $\text{Ft}f$ the constant value 1.
1.2 ENERGY AND POWER

1.2.1 ENERGY AND ENERGY SPECTRAL DENSITY

The energy of a continuous time signal \( x(t) \) is defined as

\[
E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt,
\]
providing the integral exists and is finite.

From the definition (1.20) we see that the energy of a signal is always real and nonnegative, even for complex signals. Moreover, if \( x(t) \) represents a voltage, \( E_x \) will be measured in \( V^2 \text{s} \).

For a pair of signals \( x, y \) their cross energy is defined as

\[
E_{xy} = \int_{-\infty}^{+\infty} x(t)y^*(t) \, dt,
\]
providing the integral exists and is finite. Unlike the energy of a signal, the cross energy may be negative, or even complex for complex signals. Moreover, since

\[
E_{yx} = \int_{-\infty}^{+\infty} y(t)x^*(t) \, dt = E_{xy}^*,
\]
the cross energy is not commutative in general. However, if both \( x \) and \( y \) are real-valued, then \( E_{xy} = E_{yx} \). Observe that the definition of energy (1.20) can be obtained as a particular case of cross energy by choosing \( y(t) = x(t) \) in the definition (1.21).

In Chapter 4 we will see that the cross energy between two signals satisfies all the axioms of the inner product in a signal space over the complex field \( \mathbb{C} \). In this context, we now state a very useful property: if \( x \) and \( y \) have finite energy, their cross energy is guaranteed to be finite and is bounded by the Schwarz inequality

\[
|E_{xy}| \leq \sqrt{E_x E_y}.
\]

Moreover, two distinct signals whose cross energy is zero are said to be orthogonal.

A fundamental result relating energies and Fourier transforms is given next, whose proof can be found in [1].

**Theorem 1.1 (Parseval’s theorem)** Let \( x, y \) be two continuous time signals with finite energy and \( X, Y \) their Fourier transforms. Then the cross energy of the signals equals the cross energy of their transforms, that is

\[
E_{xy} = \int_{-\infty}^{+\infty} x(t)y^*(t) \, dt = \int_{-\infty}^{+\infty} X(f)Y^*(f) \, df = E_{XY}.
\]

In particular for the energy of a signal we have

\[
E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, df = E_X.
\]
The above results (1.24) and (1.25) allow us to define the energy spectral density of a signal and the cross energy spectral density of two signals as
\[ E_x(f) = |X(f)|^2, \quad E_{xy}(f) = X(f)Y^*(f) \]
respectively, so that the energies are obtained by integrating such densities
\[ E_x = \int_{-\infty}^{+\infty} E_x(f) \, df, \quad E_{xy} = \int_{-\infty}^{+\infty} E_{xy}(f) \, df. \]
(1.27)

**Remark** We observe that the function \( E_x(f) (E_{xy}(f)) \) describes how the signal energy (cross energy) is distributed in the frequency domain. It thus provides more complete information than just the total of energy given by \( E_x (E_{xy}) \).

---

**Example 1.2 A** The energy of a causal decreasing exponential signal
\[ x(t) = A e^{-t/\tau_1} \, 1(t), \quad t \in \mathbb{R} \]
with \( A = 20 \, \text{mV} \) and \( \tau_1 = 100 \, \mu\text{s} \) can be calculated as
\[ E_x = \int_0^\infty A^2 e^{-2t/\tau_1} \, dt = \frac{A^2 \tau_1}{2}. \]
By substituting the parameter values, the above expression yields \( E_x = 2 \cdot 10^{-8} \, \text{V}^2 \cdot \text{s} \). We can also check Parseval’s theorem in this particular case, as
\[ X(f) = \frac{A \tau_1}{1 + j2\pi f \tau_1} \quad \Rightarrow \quad E_x(f) = \frac{A^2 \tau_1^2}{1 + (2\pi f \tau_1)^2} \]
and indeed we get
\[ \int_{-\infty}^{+\infty} \frac{A^2 \tau_1^2}{1 + (2\pi f \tau_1)^2} \, df = A^2 \tau_1^2 \left[ \frac{1}{2\pi \tau_1} \arctan(2\pi f \tau_1) \right]_{-\infty}^{+\infty} = A^2 \tau_1^2 \frac{1}{2\pi \tau_1} \pi = \frac{A^2 \tau_1}{2}. \]
This example is illustrated in Figure 1.9.

---

**Example 1.2 B** Suppose we wish to calculate the energy of the signal
\[ x(t) = A \, \text{sinc}(t/T), \quad t \in \mathbb{R} \]
PRELIMINARIES ON DETERMINISTIC AND RANDOM SIGNALS

Figure 1.9 Illustration of Example 1.2 A: (a), (b) in the time domain; (c), (d) in the frequency domain. Parseval’s theorem assures that the shaded areas on plots (b) and (d) are equal.

with $A = 3 \text{ V}$ and $T = 1 \text{ ms}$. By the change of variable $u = \pi t / T$ we can write

$$E_x = \int_{-\infty}^{+\infty} A^2 \text{sinc}^2(t/T) \, dt = A^2 \frac{T}{\pi} \int_{-\infty}^{+\infty} \frac{\sin^2 u}{u^2} \, du. \quad (1.28)$$

However the integral on the right-hand side is not straightforward to evaluate, since an integral function of $\sin^2 u / u^2$ is not easily expressed in terms of elementary functions.

Observing from Table 1.3 that

$$X(f) = AT \text{rect}(Tf)$$

we get

$$E_x(f) = A^2 T^2 \text{rect}(Tf).$$

Hence, by resorting to Parseval’s theorem, we get

$$E_x = \int_{-\infty}^{+\infty} A^2 T^2 \text{rect}(Tf) \, df = A^2 T^2 \int_{-1/(2T)}^{1/(2T)} dt = A^2 T. \quad (1.29)$$

By substituting the values for $A$ and $T$, the result is $E_x = 9 \cdot 10^{-3} \text{ V}^2 \text{ s}$.

Equating (1.28) and (1.29) we also get the solution to a difficult integral

$$\int_{-\infty}^{+\infty} \frac{\sin^2 u}{u^2} \, du = \pi.$$
1.2.2 INdniANOEUS AND AVERAGE POWER

For a signal $x(t)$, the quantity $|x(t)|^2$ represents the instantaneous power of the signal. The average power is defined as the average over time of the instantaneous power, that is

$$M_x = \lim_{u \to \infty} \frac{1}{2u} \int_{-u}^{u} |x(t)|^2 \, dt$$  \hspace{1cm} (1.30)

as illustrated in Figure 1.10. The reader should not confuse this concept with the notion of power from physics. Here, the power of a signal has the same physical dimension as the square signal (e.g. $V^2$ for a voltage signal), and not W.

![Figure 1.10](image)

Figure 1.10 Illustration of the average power of a signal. The shaded areas above and below the $M_x$ line are equal.

The following proposition gives a very important result.

**Proposition 1.2** The average power of a signal with finite energy is zero.

**Proof** Let $E_x$ be the finite energy of the signal $x$. For any $u$, we have

$$0 \leq \int_{-u}^{u} |x(t)|^2 \, dt \leq E_x$$

and if we divide both sides by $2u$ and take the limit as in (1.30) we get

$$0 \leq M_x \leq \lim_{u \to \infty} \frac{E_x}{2u}.$$

Since $E_x$ is finite and fixed, the limit on the right-hand side is zero, and so is $M_x$.

This proposition shows that, for signals with finite energy it is meaningless to consider the average power, as it is zero. On the other hand, for signals with nonzero average power it is meaningless to consider the energy, as it will turn out to be infinite.
Analogously to (1.30) we can define the \textit{average cross power} for a pair of signals \(x\) and \(y\) as
\[
M_{xy} = \lim_{u \to \infty} \frac{1}{2u} \int_{-u}^{u} x(t)y^*(t) \, dt. \tag{1.31}
\]
The average cross power has the same physical dimension as the product of the two signals.

For \textit{periodic signals} with period \(T_p\), the above definitions simplify and lead to the consideration of the average power within a single period
\[
M_x = \frac{1}{T_p} \int_0^{T_p} |x(t)|^2 \, dt \tag{1.32}
\]
and similarly for the cross power between periodic signals \(x\) and \(y\) with the same period \(T_p\),
\[
M_{xy} = \frac{1}{T_p} \int_0^{T_p} x(t)y^*(t) \, dt. \tag{1.33}
\]

We apply the above result in the following example.

\textbf{Example 1.2 C} We want to find the average power of the sinusoidal signal
\[
x(t) = A \cos(2\pi f_0 t + \varphi_0)
\]
with \(A = 2\, \text{V}, f_0 = 60\, \text{Hz}\) and \(\varphi_0 = \pi/3\).

The signal \(x\) is periodic with \(T_p = 1/f_0\), thus by (1.32) we get
\[
M_x = f_0 \int_0^{T_p} A^2 \cos^2(2\pi f_0 t + \varphi_0) \, dt.
\]
From Table 1.1, by applying the trigonometric formula \(\cos^2 x = (1 + \cos 2x)/2\) we get
\[
M_x = f_0 \int_0^{T_p} \frac{A^2}{2} \, dt + \int_0^{T_p} \frac{A^2}{2} \cos(4\pi f_0 t + \varphi_0) \, dt.
\]
As the second integral vanishes we get
\[
M_x = \frac{A^2}{2}, \tag{1.34}
\]
and substituting the value of \(A\), \(M_x = 2\, \text{V}^2\).

Observe that in the case of a sinusoidal signal the average power has a particularly simple expression that depends only on the amplitude \(A\), and does not depend on either its frequency \(f_0\) or its phase \(\varphi_0\).

The result in the previous example can be extended to an arbitrary sum of sinusoids, where the overall power is the sum of the power of each sinusoid. For example, if
\[
x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \varphi_i)
\]
with distinct frequencies, \( f_i \neq f_j \), for \( i \neq j \), it is

\[
M_x = \sum_{i=1}^{N} \frac{A_i^2}{2}.
\] (1.35)

In general, for a periodic signal we have the following useful theorem.

**Theorem 1.3 (Parseval’s theorem for periodic signals)** Let \( x, y \) be two periodic signals with the same period \( T_p \), and let \( \{X_\ell\}, \{Y_\ell\} \) be the coefficients of their Fourier series. Then the average cross power between \( x \) and \( y \) is

\[
M_{xy} = \sum_{\ell = -\infty}^{+\infty} X_\ell Y_\ell^*.
\] (1.36)

Correspondingly for a periodic signal

\[
M_x = \sum_{\ell = -\infty}^{+\infty} |X_\ell|^2.
\] (1.37)

**Proof** We start from expression (1.33) for average cross power in a period. By replacing \( x(t) \) and \( y(t) \) with their Fourier series we get

\[
M_{xy} = \frac{1}{T_p} \int_{0}^{T_p} \left( \sum_{\ell = -\infty}^{+\infty} X_\ell e^{j2\pi \ell F_p t} \right) \left( \sum_{m = -\infty}^{+\infty} Y_m e^{j2\pi m F_p t} \right)^* dt
\]

\[
= \frac{1}{T_p} \int_{0}^{T_p} \sum_{\ell = -\infty}^{+\infty} \sum_{m = -\infty}^{+\infty} X_\ell Y_m^* e^{j2\pi (\ell-m) F_p t} dt.
\]

Then by exchanging the order of the sums and the integral,

\[
M_{xy} = \frac{1}{T_p} \sum_{\ell = -\infty}^{+\infty} \sum_{m = -\infty}^{+\infty} X_\ell Y_m^* \int_{0}^{T_p} e^{j2\pi (\ell-m) F_p t} dt
\]

and making use of the fact that

\[
\int_{0}^{T_p} e^{j2\pi (\ell-m) F_p t} dt = \begin{cases} T_p, & \ell = m \\ 0, & \ell \neq m \end{cases}
\]

we prove the statement. \( \square \)

**Example 1.2 D** Consider the two complex exponential signals

\[
x(t) = A_1 e^{j(2\pi f_0 + \varphi_1)}, \quad y(t) = A_2 e^{j(2\pi f_0 + \varphi_2)}
\]
with the parameters \(A_1, A_2, f_0, \varphi_1, \varphi_2\) all real-valued. Both signals are periodic with fundamental frequency \(F_p = f_0\) so that their Fourier coefficients are easily seen by inspection to be

\[
\hat{X}_\ell = \begin{cases} 
A_1 e^{j\varphi_1}, & \ell = 1 \\
0, & \ell \neq 1
\end{cases} \quad \text{and} \quad \hat{Y}_\ell = \begin{cases} 
A_2 e^{j\varphi_2}, & \ell = 1 \\
0, & \ell \neq 1.
\end{cases}
\]

Then by applying (1.37) we get

\[
M_x = |\hat{X}_1|^2 = A_1^2, \quad M_y = |\hat{Y}_1|^2 = A_2^2
\]

and by (1.36)

\[
M_{xy} = \hat{X}_1\hat{Y}_1^* = A_1 A_2 e^{j(\varphi_1 - \varphi_2)}.
\]

Observe that if the nonzero Fourier coefficients of \(x\) and \(y\) had different indices, the product \(\hat{X}_\ell \hat{Y}_\ell^*\) would be identically zero for all \(\ell\), so their cross power would vanish.

**Example 1.2 E** Consider the following sum of sinusoids

\[
x(t) = A_1 \cos(2\pi f_1 t + \varphi_1) + A_2 \cos(2\pi f_2 t + \varphi_2) + A_3 \sin(2\pi f_1 t + \varphi_3)
\]

with \(f_2 = 2f_1\). Then \(x(t)\) is a periodic signal with period \(T_p = \text{lcm}(1/f_1, 1/f_2) = 1/f_1\) and hence \(F_p = 1/T_p = f_1\), or equivalently its fundamental frequency is \(F_p = \text{gcd}(f_1, f_2) = f_1\). By using the Euler identity of Table 1.1 we can rewrite \(x(t)\) as

\[
x(t) = \frac{A_1}{2}(e^{j2\pi f_1 t + j\varphi_1} + e^{-j2\pi f_1 t - j\varphi_1}) + \frac{A_2}{2}(e^{j2\pi f_2 t + j\varphi_2} + e^{-j2\pi f_2 t - j\varphi_2})
\]

\[
- j\frac{A_3}{2}(e^{j2\pi f_1 t + j\varphi_3} - e^{-j2\pi f_1 t - j\varphi_3})
\]

\[
= \frac{1}{2}(A_1 e^{j\varphi_1} - jA_3 e^{j\varphi_3}) e^{j2\pi F_p t} + \frac{1}{2}(A_1 e^{-j\varphi_1} + jA_3 e^{-j\varphi_3}) e^{-j2\pi F_p t}
\]

\[
+ \frac{1}{2}A_2 e^{j\varphi_2} e^{j2\pi 2F_p t} + \frac{1}{2}A_2 e^{-j\varphi_2} e^{-j2\pi 2F_p t}.
\]

The only nonzero Fourier coefficients are then

\[
\hat{X}_1 = \frac{1}{2} (A_1 e^{j\varphi_1} - jA_3 e^{j\varphi_3}), \quad \hat{X}_{-1} = \hat{X}_1^*, \quad \hat{X}_2 = \frac{1}{2}A_2 e^{j\varphi_2}, \quad \hat{X}_{-2} = \hat{X}_2^*.
\]

Thus from (1.37) we get

\[
M_x = |\hat{X}_1|^2 = |\hat{X}_{-1}|^2 = |\hat{X}_2|^2 = |\hat{X}_{-2}|^2
\]

\[
= 2 |\hat{X}_1|^2 = 2 |\hat{X}_2|^2
\]

\[
= 2 \cdot \frac{1}{2}[A_1^2 + A_3^2 + 2\Re(jA_1A_3 e^{j(\varphi_1 - \varphi_3)})] + 2 \cdot \frac{1}{2}A_2^2
\]

\[
= \frac{1}{2}[A_1^2 + A_3^2 + A_2^2 - 2A_1A_3 \sin(\varphi_1 - \varphi_3)].
\]

We note that the above result is different from (1.35) since there are two sinusoids with the same frequency.
1.3 SYSTEMS AND TRANSFORMATIONS

The transformation, or system, is the mathematical model of many devices where we can apply a signal at a certain point in the device, named the input, and correspondingly observe a signal at some other point, named the output. The output signal is also called the response of the system to the input signal. An example would be an electrical network where we can apply a time-varying voltage at two terminals (the input) and measure the current flowing through a conductor cross section at some point (the output). In this sense the transformation can be thought of as acting on its input signal to produce the output signal. It is then identified by the class of possible input signals $\mathcal{I}$, the class of possible output signals $\mathcal{O}$, and the map

$$\mathcal{M} : \mathcal{I} \mapsto \mathcal{O}$$

so that if $x \in \mathcal{I}$ is the input signal, the output is

$$y = \mathcal{M}[x].$$

Equation (1.39) which describes the mapping $\mathcal{M}$ is named the input–output relationship of the system.

In our context we will use the terms ‘system’ and ‘transformation’ as synonyms, although in other contexts a distinction is made between the two. Graphically we will represent a system as in Figure 1.11 with a box, with input and output signals indicated on top of an incoming and an outgoing arrow, respectively.

![Figure 1.11 Graphical representation of systems: (a) continuous time; (b) discrete time.](image)

1.3.1 PROPERTIES OF A SYSTEM

According to the map $\mathcal{M}$, a system may have certain properties, such as the following.

**Linearity** A system is said to be linear if it satisfies both the following properties:

(i) *additivity*: for all pairs of input signals $x_1, x_2$

$$\begin{cases}
\mathcal{M}[x_1] = y_1 \\
\mathcal{M}[x_2] = y_2
\end{cases} \quad \text{then} \quad \mathcal{M}[x_1 + x_2] = y_1 + y_2; \quad (1.40)$$

(ii) *homogeneity*: for any input signal $x$ and any complex constant $\alpha$

$$\begin{cases}
\mathcal{M}[x] = y \\
\end{cases} \quad \text{then} \quad \mathcal{M}[\alpha x] = \alpha y. \quad (1.41)$$
In other words we can say that in a linear system superposition and multiplication by a complex scalar of inputs and outputs hold true. An example of a linear system is the Fourier transformation, with $x$ regarded as input, and $\mathcal{X}$ as output.

**Time invariance** A system is said to be *time-invariant* if any time shift in the input signal is reflected to an identical shift in the output. That is, given any input signal $x(t)$ and any shift $t_0 \in \mathbb{R}$, if $y = \mathcal{M}[x]$ is the system response to $x(t)$, then the system response to $x_1(t) = x(t - t_0)$ is $\mathcal{M}[x_1] = y_1$ with $y_1(t) = y(t - t_0)$. In other words a time-invariant system will always respond in the same manner to a given input signal, independently of the instant at which the input signal is applied, that is the system behavior remains identical over time.

**Causality** A system is said to be *causal* if the output value at an arbitrary instant $t_0$ is determined by the input values $\{x(t)\}$ for $t \leq t_0$ only. In other words, in a causal system the present value of the response only depends on past and present values of the input signal and not on future input values.

**Memorylessness** A system is said to be *memoryless* if the output value at an arbitrary instant $t_0$ is determined by the input value at the same instant $x(t_0)$ only. In other words, in a memoryless system the present value of the response only depends on the present value of the input signal and not on past or future input values.

### 1.3.2 FILTERS

A system that is both linear and time-invariant (LTI) is called a *filter*. Its input–output relationship can always be written as

$$y(t) = \int_{-\infty}^{+\infty} g(t-u)x(u) \, du. \quad (1.42)$$

The operation performed by the integral in (1.42) is named the *convolution* between the signals $x$ and $g$ and is more compactly indicated as

$$y = g * x. \quad (1.43)$$

Hence, the value of $y$ at instant $t$ is also written as

$$y(t) = (g * x)(t). \quad (1.44)$$

The function $g$ in (1.42) uniquely identifies the filter and fully describes it from an input–output point of view: $g$ is called the *impulse response* of the system. The graphical representation of a filter by its input–output map (Figure 1.11) is often replaced by the impulse response (Figure 1.12).

Some important properties of convolution are summarized in Table 1.4. In particular associativity ensures that the cascade of two filters is equivalent to a single filter having as impulse response the convolution of the two impulse responses of the original filters. Also,
the Dirac impulse \( \delta \) is the unit element of convolution. So, a filter with impulse response \( \delta \) represents the identity transformation as the output coincides with the input. Moreover, the application of the impulse \( \delta \) as input to any filter yields a response \( y \) coinciding with \( g \), thus justifying the name ‘impulse response’ for \( g \).

**Table 1.4** Some properties of convolution.

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>associativity</td>
<td>((x \ast g) \ast h = x \ast (g \ast h))</td>
</tr>
<tr>
<td>commutativity</td>
<td>(x \ast g = g \ast x)</td>
</tr>
<tr>
<td>unit element</td>
<td>(x \ast \delta = x, \quad \delta \ast g = g)</td>
</tr>
<tr>
<td>distributivity</td>
<td>((x_1 + x_2) \ast g = (x_1 \ast g) + (x_2 \ast g))</td>
</tr>
<tr>
<td>homogeneity</td>
<td>((\alpha x) \ast g = \alpha(x \ast g), \quad \alpha \in \mathbb{C})</td>
</tr>
<tr>
<td>shift invariance</td>
<td>(y = g \ast x)</td>
</tr>
<tr>
<td></td>
<td>(\left{ \begin{array}{l} y_1 = g \ast x_1 \ y_1(t) = y(t - t_0) \end{array} \right.)</td>
</tr>
</tbody>
</table>

In the frequency domain the input–output relationship (1.42) becomes (see the convolution property in Table 1.2)

\[
\mathcal{Y}(f) = \mathcal{G}(f)\mathcal{X}(f)
\]

so that the Ftf of the output at any frequency \( f \) can be obtained by multiplying the Ftf of the input and of the filter impulse response. The Fourier transform \( \mathcal{G} \) of the impulse response is called the *frequency response* of the filter. The reason for this name is well understood by considering the following examples.

**Example 1.3 A** Consider a filter with frequency response \( \mathcal{G}(f) \) and suppose the input signal is a complex exponential with complex amplitude \( A \in \mathbb{C} \) and frequency \( f_0 \in \mathbb{R} \),

\[
x(t) = A e^{j2\pi f_0 t}.
\]

The input Fourier transform is thus

\[
\mathcal{X}(f) = A\delta(f - f_0)
\]

and by applying (1.45) and the properties of the Dirac impulse we get

\[
\mathcal{Y}(f) = \mathcal{G}(f)A\delta(f - f_0) = A\mathcal{G}(f_0)\delta(f - f_0).
\]
Taking the inverse Fourier transform of $\mathcal{Y}(f)$ yields the output signal

$$y(t) = B e^{j2\pi f_0 t}, \quad \text{with } B = A \mathcal{G}(f_0).$$

Thus, we see that the filter responds to a complex exponential at frequency $f_0$ with another complex exponential at the same frequency and with complex amplitude given by the product of the input amplitude and the value of the frequency response at $f_0$.

Mathematically we can say that a complex exponential is an eigenfunction of the filter with eigenvalue $\mathcal{G}(f_0)$.

A filter can be seen as a device that in the frequency domain analyses each frequency component of the input signal and weights it by the corresponding value of the frequency response $\mathcal{G}(f)$. Hence, by properly designing $\mathcal{G}(f)$ we can amplify or attenuate the different frequency components of the input signal.

A filter whose impulse response $g$ is real-valued will always respond to a real input with a real output. Such a filter is called real, and its frequency response will always exhibit the Hermitian symmetry

$$\mathcal{G}(-f) = \mathcal{G}^*(f). \quad (1.46)$$

Equivalently, we see that its frequency response exhibits even symmetry in its amplitude

$$|\mathcal{G}(-f)| = |\mathcal{G}(f)| \quad (1.47)$$

and odd symmetry in its phase

$$\angle \mathcal{G}(-f) = -\angle \mathcal{G}(f). \quad (1.48)$$

**Example 1.3 B** Suppose the input to a real filter with frequency response $\mathcal{G}(f)$ is a sinusoid with amplitude $A > 0$, frequency $f_0 \geq 0$ and phase $\varphi_0$,

$$x(t) = A \cos(2\pi f_0 t + \varphi_0). \quad (1.49)$$

Its Fft is

$$\mathcal{X}(f) = \frac{A}{2} [e^{j\varphi_0} \delta(f - f_0) + e^{-j\varphi_0} \delta(f + f_0)]$$

and, similarly to what we did in Example 1.3 A, we get

$$\mathcal{Y}(f) = \frac{A}{2} [e^{j\varphi_0} \mathcal{G}(f_0) \delta(f - f_0) + e^{-j\varphi_0} \mathcal{G}(-f_0) \delta(f + f_0)].$$

If we express $\mathcal{G}(f_0)$ in amplitude and phase by writing $\mathcal{G}(f_0) = |\mathcal{G}(f_0)| e^{j\angle \mathcal{G}(f_0)}$ and take into account that, since the filter is real, $\mathcal{G}$ has the Hermitian symmetry, we get

$$\mathcal{G}(-f_0) = \mathcal{G}^*(f_0) = |\mathcal{G}(f_0)| e^{-j\angle \mathcal{G}(f_0)}.$$

Thus we can write

$$\mathcal{Y}(f) = \frac{A |\mathcal{G}(f_0)|}{2} [e^{j[\varphi_0 + \angle \mathcal{G}(f_0)]} \delta(f - f_0) + e^{-j[\varphi_0 + \angle \mathcal{G}(f_0)]} \delta(f + f_0)].$$
and obtain the output signal by inverse Fourier transform as

\[ y(t) = B \cos(2\pi f_0 t + \varphi_1), \quad \text{with } B = A |G(f_0)|, \quad \varphi_1 = \varphi_0 + \angle G(f_0). \]  

(1.50)

A real filter responds to a sinusoid having frequency \( f_0 \) with a sinusoid having the same frequency but with amplitude multiplied by the amplitude of the frequency response at \( f_0 \), and phase increased by the phase of the frequency response at \( f_0 \).

**Energy spectral densities** From the frequency domain relation (1.45) we can relate input and output energy spectral densities (1.26) as follows:

- **cross energy spectral density** between input and output signals

\[
E_{xy}(f) = X(f)Y^*(f) = X(f)X^*(f)G^*(f) = G^*(f)E_x(f), \\
E_{yx}(f) = Y(f)X^*(f) = G(f)X(f)X^*(f) = G(f)E_x(f).
\]  

(1.51)

- **energy spectral density** of the output signal

\[
E_y(f) = |Y(f)|^2 = |G(f)|^2 |X(f)|^2 = E_g(f)E_x(f),
\]  

(1.52)

that is the energy spectral density of the output signal is the product of the energy spectral density of the input signal and the energy spectral density of the filter impulse response.

**Remark** In most applications (1.52) is a very useful tool for computing the energy of a filter output given the input energy spectral density. In fact, given \( E_x \) and \( E_g \), we need to evaluate first \( E_y \) by (1.52) and then determine \( E_y \) by integration of \( E_y \). The alternative would be to evaluate first \( y \) as the convolution \((x \ast g)\) and then determine \( E_y \) by integration of \(|y|^2\). However, it turns out that most of the time the convolution \((x \ast g)\) is very difficult to evaluate, especially for continuous time systems.

---

### 1.3.3 SAMPLING

Sampling is the operation of obtaining a discrete time signal \( \{y_n\} \) from a continuous time signal \( x(t) \) by simply looking up the values of \( x \) at instants that are multiples of a quantum \( T_s \), called the **sampling period**, \n
\[ y_n = x(nT_s), \quad n \in \mathbb{Z}. \]  

(1.53)

It can be viewed as a transformation in which the input signal is continuous time and the output is discrete time, as illustrated in Figure 1.13, and it is the mathematical model of the acquisition or measurement of a quantity at regularly spaced time instants. The inverse of the sampling period, \( F_s = 1/T_s \), is called the **sampling frequency** or **sampling rate** and represents the number of samples per unit time, and is thus measured in Hz.
In the frequency domain the input–output relation is given by the following duality theorem.

**Theorem 1.4** If the discrete time signal \( \{y_n\} \) is obtained by sampling the continuous time signal \( x(t) \), \( t \in \mathbb{R} \), with period \( T_s \), its Fourier transform can be obtained from \( X \) as

\[
\mathcal{Y}(f) = F_s \sum_{\ell = -\infty}^{+\infty} X(f + \ell F_s).
\]  

**Proof** We prove the theorem by taking the inverse Ftf of (1.54). From (1.18), and by exploiting the relation \( F_s = 1/T_s \), we have

\[
y_n = T_s \int_{0}^{F_s} \left[ \sum_{\ell = -\infty}^{+\infty} X(f + \ell F_s) e^{j2\pi f n T_s} \right] df
\]

\[
= \sum_{\ell = -\infty}^{+\infty} \int_{0}^{F_s} X(f + \ell F_s) e^{j2\pi f n T_s} df
\]

\[
= \sum_{\ell = -\infty}^{+\infty} \int_{\ell F_s}^{(\ell + 1) F_s} X(f) e^{j2\pi f n T_s} e^{-j2\pi \ell n} df
\]

\[
= \int_{-\infty}^{+\infty} X(f) e^{j2\pi f n T_s} df = x(n T_s),
\]

which proves the result. \( \square \)

Observe that the sum on the right-hand side of (1.54) is made up of shifted versions, also called *images*, of the input Ftf for all shifts that are a multiple of the sampling rate. The relation between \( x \) and \( y \) is illustrated in Figure 1.14, in both the time and frequency domains.

### 1.3.4 INTERPOLATION

Interpolation is the operation of constructing a continuous time signal \( y(t) \) from a discrete time signal \( \{x_n\} \) with quantum \( T \). If we want the transformation to satisfy both linearity and time invariance, the input–output relationship must be of the type

\[
y(t) = \sum_{n = -\infty}^{+\infty} g(t - n T) x_n.
\]  

(1.55)
The transformation (1.55) is called the **interpolate filter** and the continuous time function \( g \) is its impulse response. In the frequency domain (1.55) corresponds to the following result.

**Theorem 1.5** If \( y(t) \) is obtained from \( \{x_n\} \) through the interpolate filter \( g \), its Fourier transform is

\[
Y(f) = G(f)X(f).
\]  

**Proof** We proceed analogously to the proof of Theorem 1.4 and obtain

\[
Y(f) = \int_{-\infty}^{+\infty} y(t) e^{-j2\pi ft} \, dt \\
= \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} g(t-nT)x_n e^{-j2\pi ft} \, dt \\
= \sum_{n=-\infty}^{+\infty} x_n \int_{-\infty}^{+\infty} g(t-nT) e^{-j2\pi ft} \, dt.
\]

By use of the time shift property of Table 1.2, it is

\[
Y(f) = \sum_{n=-\infty}^{+\infty} x_n G(f) e^{-j2\pi fnT} = G(f)X(f).
\]

The relation (1.56), analogous to (1.45), justifies both the terms interpolate filter for the transformation, and frequency response for \( G(f) \). Observe that in (1.56) \( X(f) \) is a periodic
function with period $1/T$, whereas $G(f)$ and $Y(f)$ are not periodic, in general. Thus, the multiplication by $G(f)$ destroys the periodicity of $X(f)$. Figure 1.15 shows the input and output signals in the time and frequency domains.

![Diagram](image)

**Figure 1.15** The relation between discrete time input and continuous time output in an interpolate filter: (a) in the time domain; (b) in the frequency domain.

In the common sense interpretation of the term ‘interpolation’ one would expect that the values of $y$ at instants $nT$ coincide with $x_n$, and the values in the interval $(nT, nT + T)$ connect two consecutive points $x_n, x_{n+1}$ in some arbitrary fashion. However, this is not guaranteed when using an arbitrary interpolate filter. A necessary and sufficient condition on $g$ for the output to coincide with the input at instants $nT$ for any input signal is given by the following proposition.

**Proposition 1.6** Consider an interpolate filter with input–output relation (1.55). Then, for any input signal $\{x_n\}$,

$$y(nT) = x_n, \quad n \in \mathbb{Z}$$  \hspace{1cm} (1.57)

if and only if $g(t)$ is such that

$$g(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$  \hspace{1cm} (1.58)
Proof: We first prove that condition (1.58) is sufficient. Assume (1.58) holds, then for any \( x \) we have from (1.55) that

\[
y(nT) = \sum_{k=\infty}^{+\infty} g(nT - kT)x_k = g(0)x_n + \sum_{i\neq0} g(iT)x_{n-i} = x_n.
\]

To prove that (1.58) is also necessary assume that (1.57) holds for any input signal \( x \). Then it must hold in particular for the signal

\[
x_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}
\]

and since

\[
y(nT) = \sum_{k=\infty}^{+\infty} g(nT - kT)x_k = g(nT)x_0 = g(nT)
\]

it must be

\[
g(nT) = x_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}
\]

Observe that condition (1.58) does not uniquely determine the function \( g \) since its values at instants that are not multiples of \( T \) are not specified. A few examples of interpolate filters that satisfy (1.58) are shown in Figure 1.16.

1.4 BANDWIDTH

**Definition 1.3** The full band of a continuous time signal \( x \) is the support of its Ftf, i.e. the set of frequencies where its Ftf is nonzero,

\[
\mathcal{B}_x = \{ f \in \mathbb{R} : \mathcal{X}(f) \neq 0 \}.
\]  

(1.59)

For real-valued signals, due to the Hermitian symmetry of the Fourier transform, \( \mathcal{X}(-f) \neq 0 \) if and only if \( \mathcal{X}(f) \neq 0 \) so that \( \overline{\mathcal{B}_x} \) is a symmetric set with respect to the origin. It is therefore common to use the following definition.

**Definition 1.4** The band of a continuous time real-valued signal \( x \) is the subset of nonnegative frequencies where its Ftf is nonzero,

\[
\mathcal{B}_x = \{ f \geq 0 : \mathcal{X}(f) \neq 0 \}.
\]  

(1.60)

Clearly, for real-valued signals we have \( \overline{\mathcal{B}_x} = \mathcal{B}_x \cup (-\mathcal{B}_x) \).

When dealing with discrete time signals, due to the periodicity of their Ftf it is convenient to define the band within one period, as follows.
holding interpolation

\[ g(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases} \]

linear interpolation

\[ g(t) = \text{triang} \left( \frac{t}{T} \right) \]

cubic interpolation

\[ g(t) = \begin{cases} p_1 \left( \frac{|t|}{T} \right), & 0 \leq |t| < T \\ p_2 \left( \frac{|t|}{T} \right), & T \leq |t| < 2T \\ 0, & |t| \geq 2T \end{cases} \]

\[ p_1(x) = \frac{3}{2}x^3 - \frac{5}{2}x^2 + 1 \]

\[ p_2(x) = -\frac{1}{2}x^3 + \frac{5}{2}x^2 - 4x + 2 \]

ideal lowpass interpolation

\[ g(t) = \text{sinc} \left( \frac{t}{T} \right) \]

raised cosine interpolation

\[ g(t) = \text{ircos} \left( \frac{t}{T}, \rho \right) \]

Figure 1.16 Examples of interpolate filters that satisfy condition (1.58).

**Definition 1.5** The full band of a discrete time signal \( x \) with quantum \( T \) is the subset of \([-1/(2T), 1/(2T)]\) where its \( \mathcal{F}f \) is nonzero,

\[ \mathcal{B}_x = \left\{ f \in \left[ -\frac{1}{2T}, \frac{1}{2T} \right] : \mathcal{X}(f) \neq 0 \right\}. \tag{1.61} \]

Again, for discrete time real-valued signals, thanks to the Hermitian symmetry, we define their band as a subset of \([0, 1/(2T)]\).
Once the band $B_x$ of a real-valued signal $x$ has been established, we can define its bandwidth $B_x$ as the measure of $B_x$, i.e.

$$B_x = \int_{B_x} df.$$ 

We notice that for a discrete time signal the bandwidth cannot be greater than half the rate, $B_x \leq 1/(2T)$, whereas for continuous time signals the bandwidth may be arbitrarily large and even infinite.

The concepts of band and bandwidth can also be introduced with regard to systems. In particular the band of a filter is defined as the band of its impulse response $g$ and it is easy to see that it includes the bands of all possible output signals $y$. Therefore the filter bandwidth is also the maximum bandwidth of all filter output signals.

### 1.4.1 Classification of Signals and Systems

Signals and systems are usually grouped into classes according to their band as follows.

**Definition 1.6** A signal $x$ is said to be:

- **band-limited**, if its band $B_x$ is a limited set;
- **baseband**, if $B_x$ is a limited set containing a neighborhood of the origin;
- **passband**, if $B_x$ is a limited set excluding a neighborhood of the origin;
- **narrowband**, if it is passband and its bandwidth is much smaller than its maximum frequency, i.e. $B_x \ll \max\{f; f \in B_x\}$.

Figure 1.17 shows examples of the above definitions.

**Definition 1.7** A filter with impulse response $g$ is said to be:

- **lowpass (LPF)**, if its band $B_g$ is a limited set containing a neighborhood of the origin;
- **highpass (HPF)**, if $B_g$ includes a neighborhood of infinity, excluding a neighborhood of the origin;
- **bandpass (BPF)**, if $B_g$ is a limited set excluding a neighborhood of the origin;
- **narrowband (NBF)**, if it is bandpass and its bandwidth is much smaller than its maximum frequency, i.e. $B_g \ll \max_{f \in B_g} |f|$;
- **notch (NTF)**, if its band is the complement to that of a narrowband filter;
- **allpass**, if its gain $|G(f)|$ is nearly constant over the whole frequency axis.

The above definitions are illustrated in Figure 1.18.
1.4.2 UNCERTAINTY PRINCIPLE

An important result in signal theory is the so-called uncertainty principle for continuous time signals, which states the following.

**Theorem 1.7** A continuous time signal $x$ cannot have both a limited support and a limited band.

The proof of this theorem is far beyond the scope of this book. It relies on the properties of analytical functions. The interested reader can see [1] or books on complex analysis.

The above result states that a continuous time signal can be either time-limited, band-limited, or none of the two, but not both.

1.4.3 PRACTICAL DEFINITIONS OF BAND

Most real life signals (and system impulse responses) have a finite support due to the fact that physical phenomena start and expire within a finite time interval. Strictly speaking, such signals cannot have a limited band, as stated by Theorem 1.7. However, it is quite convenient to assume that their band is ‘practically’ limited in the sense that their Ftf is negligible (i.e. sufficiently small) outside a properly chosen interval $(f_1, f_2)$ that is called the practical band of the signal. The corresponding practical bandwidth of the signal is given by $B = f_2 - f_1$. Such a definition is not unique, depending on what we mean by ‘negligible’. Some common criteria for real-valued signals are given in the following.
Figure 1.18 Examples of frequency domain representation of real filters: (a) lowpass; (b) highpass; (c) bandpass; (d) narrowband; (e) notch; (f) allpass.

**Amplitude** Given a parameter \( \varepsilon \), with \( 0 < \varepsilon < 1 \), we define the band of the passband signal \( x \) as the smallest interval \( (f_1, f_2) \) such that the amplitude of \( X(f) \) outside \( (f_1, f_2) \) does not exceed a fraction \( \varepsilon \) of its maximum, that is

\[
|X(f)| \leq \varepsilon X_{\text{max}}, \quad f \geq 0, f \not\in (f_1, f_2)
\]

where

\[
X_{\text{max}} = \max_f |X(f)|.
\]

Typical values of \( \varepsilon \) in different applications may be \( 1/\sqrt{2} \) (3 dB band), \( 10^{-2} \) (40 dB band), or \( 10^{-3} \) (60 dB band).

**Energy** Given a parameter \( \varepsilon \), with \( 0 < \varepsilon < 1 \), we define the band of the passband signal \( x \) as the smallest interval \( (f_1, f_2) \) such that the signal energy within \( (f_1, f_2) \), obtained by
integrating the energy spectral density, differs by no more than a fraction \( \varepsilon \) from the total signal energy, that is

\[
2 \int_{f_1}^{f_2} E_x(f) \, df \geq (1 - \varepsilon) \int_{-\infty}^{+\infty} E_x(f) \, df = (1 - \varepsilon) E_x. \tag{1.64}
\]

Typical values of \( \varepsilon \) in applications are 10\%, or 1\%.

**First zero** Let \( f_0 \) be the frequency corresponding to the maximum value of \( |\mathcal{X}(f)| \),

\[
f_0 = \operatorname{argmax}_{f \geq 0} |\mathcal{X}(f)|. \tag{1.65}
\]

We define the band of the passband signal \( x \) as the interval \((f_1, f_2)\) containing \( f_0 \) where the transform is nonzero,

\[
f_1 = \max\{0 \leq f < f_0, |\mathcal{X}(f)| = 0\}
\]

\[
f_2 = \min\{f > f_0, |\mathcal{X}(f)| = 0\}. \tag{1.66}
\]

These three criteria can be particularized to baseband signals by considering \( f_1 = 0 \). In fact, they are illustrated in Figure 1.19 for the signal \( x(t) = \text{rect}(t/T) \) that would not be band-limited in a strict sense. Observe that it can be called a baseband signal in the practical sense, as its lower frequency is always \( f_1 = 0 \) and its practical bandwidth is \( B = f_2 \).

The same definitions can clearly be applied to the band of filters, by means of their impulse response. We underline that the value of the practical bandwidth \( B \) will depend on the criterion we choose and the value of the parameter \( \varepsilon \). In general, a smaller \( \varepsilon \) corresponds to a larger bandwidth.

### 1.4.4 HEAVISIDE CONDITIONS

Consider a continuous time or discrete time system with input signal \( x \) and output signal \( y \), which may represent the cascade of many transformations that the input signal undergoes along a communication system. Suppose we want the output \( y \) to resemble the input \( x \) as closely as possible, that is without distortion. In particular, if we require \( y \) to be a possibly attenuated or amplified and/or shifted version of \( x \), i.e.

\[
y(t) = A_0 x(t - t_0) \tag{1.67}
\]

with \( A_0 > 0, t_0 \in \mathbb{R} \), we say that we are seeking the Heaviside conditions for the absence of distortion in the system.

It is evident that if we want (1.67) to hold for *any possible input signal*, the system must be linear and time-invariant, i.e. a filter, and must have the impulse response

\[
h(t) = A_0 \delta(t - t_0) \tag{1.68}
\]

or equivalently, by taking the Ftf of (1.68), with frequency response

\[
\mathcal{H}(f) = A_0 e^{-j2\pi ft_0}. \tag{1.69}
\]
Figure 1.19 Illustration of the practical bandwidth of the signal \( x(t) = \text{rect}(t/T) \) according to:
(a) the amplitude criterion; (b) the energy criterion; (c) the first zero criterion.

Such a filter has

(i) **constant amplitude**

\[
|H(f)| = A_0, \quad f \in \mathbb{R};
\]  

(ii) **linear phase** (better, proportional to the frequency)

\[
\angle H(f) = -2\pi f t_0, \quad f \in \mathbb{R};
\]  

(iii) **constant group delay**, also called **envelope delay**

\[
\tau(f) = -\frac{1}{2\pi} \frac{d}{df} \angle H(f) = t_0, \quad f \in \mathbb{R}.
\]
Conditions (i)–(ii) are known as the Heaviside conditions for the absence of distortion, whereas (iii) is simply a consequence of (ii). These are quite strong requirements to be met by real-life systems, and are too restrictive indeed.

However, for a system with band-limited input signals having a full band \( \mathcal{B} \), we may require absence of distortion only within \( \overline{\mathcal{B}} \). In fact, by writing the requirement (1.67) in the frequency domain, we have

\[
\mathcal{Y}(f) = A_0 e^{-j2\pi ft_0} \mathcal{X}(f), \tag{1.73}
\]

and equating it with the general filter relation \( \mathcal{Y}(f) = \mathcal{H}(f)\mathcal{X}(f) \), we get that the system frequency response \( \mathcal{H}(f) \) must be such that

\[
\mathcal{H}(f)\mathcal{X}(f) = A_0 e^{-j2\pi ft_0} \mathcal{X}(f). \tag{1.74}
\]

We observe that, for values \( f \not\in \overline{\mathcal{B}} \), it is \( \mathcal{X}(f) = 0 \) and (1.74) is an identity, satisfied by any \( \mathcal{H}(f) \). Hence the requirement (1.69) becomes

\[
\mathcal{H}(f) = \begin{cases} 
A_0 e^{-j2\pi ft_0}, & f \in \overline{\mathcal{B}} \\
\text{arbitrary}, & f \not\in \overline{\mathcal{B}}.
\end{cases} \tag{1.75}
\]

Then also the Heaviside conditions (i)–(ii) can be relaxed to hold only for \( f \in \overline{\mathcal{B}} \). Clearly, in this case the impulse response \( h \) does not necessarily have the expression (1.68).

In other words, if we are only interested in transmitting undistorted signals with full band \( \overline{\mathcal{B}} \), it is sufficient that the Heaviside conditions (i)–(ii) are verified within \( \mathcal{B} \); outside \( \mathcal{B} \), the system frequency response may be arbitrary. We show in Figure 1.20 the frequency response of a real bandpass filter that satisfies the Heaviside conditions within the band \( \mathcal{B} = (f_1, f_2) \).

![Figure 1.20](image)

**Figure 1.20** Frequency response of a real filter satisfying the conditions for the absence of signal distortion in the full band \( \overline{\mathcal{B}} = \mathcal{B} \cup (-\mathcal{B}) = (f_1, f_2) \cup (-f_2, -f_1) \). The behavior shown with dotted lines is arbitrary.

**Further insight** Indeed, to satisfy the Heaviside conditions for the absence of distortion for signals that are band-limited within \( \overline{\mathcal{B}} \), it is not necessary that the whole system be a filter. It is sufficient that it acts as a filter on these signals. An example which clarifies this observation can be found in Problem 1.39.
1.4.5 SAMPLING THEOREM

Consider the problem of sampling a continuous time signal, and subsequently reconstruct all its original values through interpolation, as illustrated in Figure 1.21.

![Figure 1.21](image)

Figure 1.21 Sampling of a continuous time signal and subsequent reconstruction of its values through interpolation.

Sufficient conditions for this problem to be effectively solved are given by the following theorem.

**Theorem 1.8 (Shannon–Whittaker sampling theorem)** Consider the system of Figure 1.21 where the input $x(t)$ is a real-valued continuous time signal. If

(i) $x(t)$ is band-limited with $B_x \subset [0, B)$,

(ii) the sampling rate is $F_s \geq 2B$,

(iii) the interpolate filter has frequency response

$$G(f) = \begin{cases} 
T_s, & |f| < B \\
\text{arbitrary}, & B \leq |f| \leq F_s - B \\
0, & |f| > F_s - B,
\end{cases}$$

(1.76)

then the input signal is perfectly reconstructed at the output, that is

$$\tilde{x}(t) = x(t), \quad t \in \mathbb{R}.$$  

(1.77)

**Proof** We proceed to work out the proof in the frequency domain. By combining (1.56) with (1.54) the Ftf of the output signal $\tilde{x}$ is written as

$$\tilde{X}(f) = G(f)X(f) = F_s \sum_{\ell=-\infty}^{+\infty} G(f)X(f - \ell F_s).$$

(1.78)

Now if conditions (i)–(iii) are satisfied, all the terms with $\ell \neq 0$ in the sum vanish, since $G(f)$ and $X(f - \ell F_s)$ have disjoint supports (see the illustration in Figure 1.22) and we are left with only the term for $\ell = 0$, i.e.

$$\tilde{X}(f) = F_s G(f)X(f).$$

(1.79)

By observing that $G$ has constant gain $T_s$ in the band of $x$, we get

$$\tilde{X}(f) = X(f),$$

(1.80)

which corresponds to (1.77) in the frequency domain.
Figure 1.22 Frequency domain illustration of the sampling and perfect reconstruction of the original signal according to the Shannon–Whittaker sampling theorem.

Some comments on the statement and proof of the theorem are appropriate. The interpolate filter gain is set to $T_s$ to obtain equality (1.77); a different gain would yield as $\tilde{x}$ a scaled version of $x$, but in practice the difference is immaterial. Hypothesis (ii) is called the non-aliasing condition and assures that in the summation (1.54) the different shifted versions of $X$ are kept spaced apart and do not overlap with each other. Lastly, condition (iii) on the frequency response allows the interpolate filter to gather the desired unshifted component undistorted while rejecting all other images. Observe that if $F_s > 2B$, condition (iii) allows a certain degree of freedom in selecting the filter specifications. For example the choice that yields the minimum bandwidth filter is given by the ideal filter

$$g(t) = 2BT_s \text{sinc}(2Bt).$$

Specifications (1.81) are also the necessary choice when $F_s = 2B$. In general, other choices for the interpolate filter are possible. When choosing the sampling frequency for a band-limited signal one must consider the trade-off between choosing $F_s$ as small as possible to minimize the number of samples per unit time, and allowing some freedom in the design of the interpolate filter.
Further insight  Indeed, the practical implementation of the interpolate filter requires a transition band between the passband, where \( G(f) \approx T_s \), and the stopband, where \( G(f) \approx 0 \). Moreover, the narrower the transition band, the more challenging the implementation of the filter. Given the constraint (1.76), the transition band must be included within the interval \( (B, F_s - B) \) and its width is bounded by \( F_s - 2B \).

It is understood that if either (i) or (ii) are not satisfied, i.e. the input signal is not even band-limited or the sampling frequency is too small, perfect reconstruction (1.77) is not possible and we must take into account a reconstruction error. For this reason, as illustrated in Figure 1.23, the sampler is preceded by an anti-aliasing filter \( d \) with band \([0, B]\) where \( 2B \leq F_s \). Ideally, the frequency response is

\[
\mathcal{D}(f) = \text{rect} \left( \frac{f}{2B} \right).
\] (1.82)

Figure 1.23  Insertion of an anti-aliasing filter in sampling and interpolation transformations.

Thus, the output \( x \) of the anti-aliasing filter is band-limited in \((-B, B)\) and can be perfectly reconstructed by the interpolate filter to yield \( \tilde{x}(t) = x(t) \). However, with respect to the initial input \( x_1 \) we have introduced an error

\[
e(t) = \tilde{x}(t) - x_1(t) = x(t) - x_1(t)
\] (1.83)

given by the difference between \( x \) and \( x_1 \), and caused by the anti-aliasing filter removing the out-of-band components of \( x_1 \). In the case of an ideal anti-aliasing filter (1.82) and assuming \( \tilde{x} = x \), the energy spectral density of \( e \) is given by

\[
\mathcal{E}_e(f) = |X_i(f) - X(f)|^2
= |1 - \mathcal{D}(f)|^2 |X_i(f)|^2
= \begin{cases} 0, & |f| \leq B \\ \mathcal{E}_{X_i}(f), & |f| > B \end{cases}
\]

and we can calculate the error energy

\[
E_e = \int_{-\infty}^{+\infty} \mathcal{E}_e(f) \, df = 2 \int_{B}^{+\infty} \mathcal{E}_{X_i}(f) \, df.
\] (1.84)
1.4.6 NYQUIST CRITERION

Given the discrete time signal \( \{x_n\} \) with quantum \( T \), in this section we consider the problem of interpolating it and subsequently recover its original discrete time sample values through filtering and sampling, as illustrated in Figure 1.24. It is evident that if the interpolate filter \( g \) satisfies condition (1.58), then \( y(nT) = x_n \) and we only need to sample \( y \) with period \( T \) to get back the original values. In fact \( \tilde{x}_n = z(nT) = y(nT) \). In this case we can remove the filter \( h \).

![Figure 1.24](image)

Interpolation of a discrete time signal and subsequent recovery of its values through filtering and sampling.

In general, we must consider that the cascade of the interpolate filter \( g \) and the filter \( h \) is equivalent to a single interpolate filter with impulse response \( g_1(t) = (g * h)(t) \) and frequency response

\[
G_1(f) = G(f)H(f).
\]  

This can be seen by examining the frequency domain input–output relationship of the cascade, which by (1.45) and (1.56) yields

\[
Z(f) = H(f)Y(f) = H(f)G(f)X(f) = G_1(f)X(f).
\]  

We can therefore state the following theorem which gives a necessary and sufficient condition for the perfect recovery of any discrete time input signal.

**Theorem 1.9 (Nyquist criterion, time domain formulation)** Consider the system in Figure 1.24 with \( g_1(t) = (g * h)(t) \). Perfect reconstruction is assured for all discrete time input signals \( \{x_n\} \) with quantum \( T \) if and only if

\[
g_1(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.
\]  

**Proof** It is sufficient to apply Proposition 1.6 to the equivalent interpolate filter having impulse response \( g_1 \).

In the frequency domain the above theorem can be formulated as follows.

**Theorem 1.10 (Nyquist criterion, frequency domain formulation)** In the system of Figure 1.24, perfect reconstruction is assured for all discrete time input signals \( \{x_n\} \) with quantum \( T \) if and only if

\[
\sum_{\ell=-\infty}^{+\infty} G(f + \ell/T)H(f + \ell/T) = T, \quad f \in \mathbb{R}.
\]  

(1.88)
Proof. We prove that condition (1.88) is equivalent to (1.87). Let \( q_n = g_1(nT) \) as in (1.87). Its Ftf is \( Q(f) = 1 \). Moreover, from (1.54) we have

\[
Q(f) = \frac{1}{T} \sum_{\ell=-\infty}^{+\infty} G_1(f + \ell/T) = \frac{1}{T} \sum_{\ell=-\infty}^{+\infty} G(f + \ell/T) \mathcal{H}(f + \ell/T). 
\]

(1.89)

and the theorem is proved.

Although apparently more cumbersome, the frequency domain formulation (1.88) lends itself to an effective characterization of the filter \( h \). A possible two-step procedure is as follows:

1. Choose the frequency response of the equivalent filter \( G_1(f) \) so that

\[
B_{g_1} \subset B_g
\]

and

\[
\sum_{\ell=-\infty}^{+\infty} G_1(f + \ell/T) = T.
\]

(1.91)

For example it could be \( G_1(f) = T \text{rect}(Tf) \) or \( G_1(f) = T \text{rcos}(Tf, \rho) \).

2. Let (see (1.85))

\[
\mathcal{H}(f) = \begin{cases} 
G_1(f) / G(f), & f \in B_g \\
\text{arbitrary}, & f \not\in B_g.
\end{cases}
\]

(1.92)

Observe that from (1.90), we must have \( B_{g_1} \leq B_g \). Moreover, for \( Q(f) \) in (1.89) to be nonzero over the whole frequency axis, the bandwidth of \( g_1 \) must be at least \( 1/(2T) \), as shown in Figure 1.25. Therefore, a necessary condition for the Nyquist criterion to hold in the system of Figure 1.24 for some choice of \( h \) is that the interpolate filter \( g \) has bandwidth

\[
B_g \geq \frac{1}{2T}.
\]

(1.93)

For the same reasons, also \( h \) must have a bandwidth greater than \( 1/(2T) \). The Nyquist criterion has important applications in the field of digital transmissions, where the input \( \{x_n\} \) represents the transmitted data sequence, the interpolator \( g \) represents the cascade of the transmit filter and channel, whereas \( h \) represents the receive filter.

1.5 REPRESENTATION OF PASSBAND SIGNALS

Real-valued passband signals have an intrinsic redundancy in the frequency domain. As illustrated in Figure 1.26, such signals could as well be equivalently represented by complex-valued counterparts with baseband real and imaginary components. This motivates the present section.
The definition of the analytic signal stems from the observation that the Ftf of a real-valued signal $x$ has Hermitian symmetry. If we define the positive and negative frequency components of its Ftf, respectively, as

$$
\mathcal{X}^+(f) = \begin{cases} 
\mathcal{X}(f), & f > 0 \\
0, & f < 0 
\end{cases} \quad \text{and} \quad \mathcal{X}^-(f) = \begin{cases} 
0, & f > 0 \\
\mathcal{X}(f), & f < 0 
\end{cases}
$$

(1.94)

they are related by

$$
\mathcal{X}^-(f) = [\mathcal{X}^+(f)]^*.
$$

(1.95)

In other words, dropping for example the negative frequency components of a real signal we are still able to reconstruct the entire signal.

Based upon these considerations, we give the following definition.
Definition 1.8  The analytic signal $x^{(a)}$ associated to a real signal $x$ is defined in the frequency domain as

$$\mathcal{X}(f) = 2\mathcal{X}^{(+)}(f) = \begin{cases} 
2\mathcal{X}(f), & f > 0 \\
\mathcal{X}(0), & f = 0 \\
0, & f < 0.
\end{cases} \quad (1.96)$$

That is, it is obtained from $x$ by doubling the amplitude of its positive frequency components, and removing the negative frequency components.

In general, $\mathcal{X}(f)$ does not have the Hermitian symmetry, and hence $x^{(a)}$ is complex-valued. However, the analytic signal $x^{(a)}$ retains all the information of $x$, in the sense that we can always recover the original signal $x$ from $x^{(a)}$ by simply taking its real part, as shown by the following proposition.

Proposition 1.11  Taking the real part of the analytic signal $x^{(a)}$ yields $x$,

$$\mathfrak{R}[x^{(a)}(t)] = x(t). \quad (1.97)$$

Proof  We prove the result in the frequency domain, that is we calculate the FT of $\mathfrak{R}[x^{(a)}]$ and show it turns out to be $\mathcal{X}(f)$. By the real part transform rule of Table 1.2 we get

$$\mathcal{F}[\mathfrak{R}[x^{(a)}]](f) = \frac{1}{2}[\mathcal{X}^{(a)}(f) + \mathcal{X}^{(a)*}(-f)]$$

$$= \begin{cases} 
\frac{1}{2}[2\mathcal{X}(f) + 0], & f > 0 \\
\frac{1}{2}[\mathcal{X}(0) + \mathcal{X}^{*}(0)], & f = 0 \\
\frac{1}{2}[0 + 2\mathcal{X}^{*}(-f)], & f < 0
\end{cases}$$

$$= \begin{cases} 
\mathcal{X}(f), & f > 0 \\
\mathfrak{R} [\mathcal{X}(0)], & f = 0 \\
\mathcal{X}^{*}(-f), & f < 0
\end{cases}$$

$$= \mathcal{X}(f)$$

where in the last step we have used the Hermitian symmetry of $\mathcal{X}(f)$ and the fact that $\mathcal{X}(0)$ is real.  \qed
Further insight  A complete characterization of the analytic signal is given by the following result.

**Proposition 1.12**  For any real-valued signal \( x \), the corresponding analytic signal \( x^{(a)} \) is the only complex-valued signal having the following properties:

(i) its real part yields \( x \), \( \Re[x^{(a)}] = x \), i.e.

\[
x(t) = \frac{x^{(a)} + x^{(a)*}}{2};
\]

(ii) it only has nonnegative frequency components, \( \mathcal{X}^{(a)}(f) = 0, f < 0 \);

(iii) its DC component is real, \( \Im[\mathcal{X}^{(a)}(0)] = 0 \).

**Proof**  Property (i) has been proved in Proposition 1.11 above. It is easy to derive from its definition (1.96) that \( \mathcal{X}^{(a)}(f) \) satisfies (ii), and also (iii), since \( \mathcal{X}(0) = \int_{-\infty}^{+\infty} x(t) \, dt \) is real.

We now show that any signal \( y(t) \) that has the above properties coincides with \( x^{(a)}(t) \). Indeed, let \( d(t) = y(t) - x^{(a)}(t) \). By property (i), it must be that

\[
\Re[d(t)] = \Re[y(t)] - \Re[x^{(a)}(t)] = x(t) - x(t) = 0,
\]

so that \( d(t) \) is purely imaginary and its Fourier transform must have the skew-Hermitian symmetry \( D(f) = -D^*(-f) \) with \( D(0) \) purely imaginary. On the other hand, by property (ii) we have

\[
D(f) = \mathcal{Y}(f) - \mathcal{X}^{(a)}(f) = 0, \quad f < 0
\]

and by the skew-Hermitian symmetry we also have \( D(f) = 0, f > 0 \). Eventually, \( D(0) = \Im[D(0)] \). However by property (iii) \( \Im[D(0)] = \Im[\mathcal{Y}(0)] - \Im[\mathcal{X}^{(a)}(0)] = 0 \) and hence \( D(0) = 0 \). We have thus proved that \( D(f) = 0, f \in \mathbb{R} \), that is \( d(t) \) is the null signal and hence \( y \) coincides with \( x^{(a)} \).

The analytic signal \( x^{(a)} \) can also be obtained from \( x \) through the analytic filter, also called phase splitter, that is a filter with frequency response

\[
\mathcal{H}^{(a)}(f) = 2 \cdot 1(f) = \begin{cases} 2, & f > 0 \\ 1, & f = 0 \\ 0, & f < 0 \end{cases}
\]

and hence impulse response

\[
h^{(a)}(t) = \delta(t) + j \frac{1}{\pi t}.
\]

This relation is illustrated in Figure 1.27.
Example 1.5 A Given the signal \( x(t) = A \cos(2\pi f_0 t + \phi_0) \), with \( f_0 > 0 \), its analytic signal is \( x^{(a)}(t) = A e^{j(2\pi f_0 t + \phi_0)} \). In fact,

\[
\mathcal{X}(f) = \frac{A}{2} e^{j\phi_0} \delta(f - f_0) + \frac{A}{2} e^{-j\phi_0} \delta(f + f_0),
\]

where the first term represents \( \mathcal{X}^{(+)}(f) \), so that the Ftf of the analytic signal results

\[
x^{(a)}(f) = A e^{j\phi_0} \delta(f - f_0),
\]

which corresponds to the signal \( x^{(a)}(t) = A e^{j(2\pi f_0 t + \phi_0)} \).

We observe that

\[
\Re[A e^{j(2\pi f_0 t + \phi_0)}] = x(t).
\]

Moreover, \( A e^{j(2\pi f_0 t + \phi_0)} \) has only positive frequency components. Then, by Proposition 1.12, it is the analytic signal associated with \( x \).

Example 1.5 B Let \( x(t) = a(t) \cos(2\pi f_0 t + \phi_0) \), with \( a(t) \) a real-valued signal that varies slowly with respect to the sinusoid, that is \( a \) is band-limited with bandwidth \( B_a < f_0 \). Then by the modulation rule in Table 1.2 we get

\[
\mathcal{X}(f) = \frac{1}{2} A(f - f_0) e^{j\phi_0} + \frac{1}{2} A(f + f_0) e^{-j\phi_0}.
\]

---

**Figure 1.27** The analytic signal \( x^{(a)} \) obtained via the analytic filter (phase splitter) \( h^{(a)} \): (a) block diagram representation and (b) frequency domain illustration.
We observe that \( x(t) \) is a real-valued passband signal. The first term in the above equation has support with positive frequencies, while the second has support with negative frequencies. Thus the Ftf of the analytic signal is

\[
\mathcal{X}^{(a)}(f) = 2\mathcal{X}^{(+)}(f) = A(f - f_0)e^{j\phi_0} \quad (1.106)
\]

from which we obtain the analytic signal

\[
x^{(a)}(t) = a(t)e^{j(2\pi f_0 t + \phi_0)}. \quad (1.107)
\]

As we did in Example 1.5 A, we observe that \( x^{(a)}(t) \) could be easily obtained by substituting \( \cos(2\pi f_0 t + \phi_0) \) with \( e^{j(2\pi f_0 t + \phi_0)} \) in the expression of \( x(t) \). This is again justified by Proposition 1.12, as we obtain a complex-valued signal whose full band \( B = (f_0 - B_a, f_0 + B_a) \) contains nonnegative frequencies, and whose real part is \( x(t) \).

**Example 1.5 C** We want to find the analytic signal associated with

\[
x(t) = A \text{sinc}(2Bt) \cos(\pi Bt). \quad (1.108)
\]

The Ftf of \( x \) is

\[
\mathcal{X}(f) = \frac{A}{4B} \left[ \text{rect} \left( \frac{f - B/2}{2B} \right) + \text{rect} \left( \frac{f + B/2}{2B} \right) \right] = \begin{cases} 
\frac{A}{2B}, & |f| < \frac{B}{2} \\
\frac{A}{4B}, & \frac{B}{2} < |f| < \frac{3B}{2} \\
0, & |f| > \frac{3B}{2}
\end{cases} \quad (1.109)
\]

and hence we get the Ftf of the analytic signal as

\[
\mathcal{X}^{(a)}(f) = 2\mathcal{X}^{(+)}(f) = \begin{cases} 
\frac{A}{B}, & 0 < f < \frac{B}{2} \\
\frac{A}{2B}, & \frac{B}{2} < f < \frac{3B}{2} \\
0, & f < 0, f > \frac{3B}{2}
\end{cases} \quad (1.110)
\]

Observe that \( \mathcal{X}^{(a)}(f) \) can be written as a sum of two rect pulses,

\[
\mathcal{X}^{(a)}(f) = \frac{A}{2B} \text{rect} \left( \frac{f - \frac{3B}{2}}{B} \right) + \frac{A}{2B} \text{rect} \left( \frac{f - \frac{1B}{2}}{B} \right). \quad (1.111)
\]
from which we derive the analytic signal

\[ x^{(a)}(t) = \frac{3}{4} A \text{sinc}(\frac{3}{2} B t) e^{j\frac{\pi}{2} B t} + \frac{1}{4} A \text{sinc}(\frac{1}{2} B t) e^{j\frac{\pi}{2} B t}. \] (1.112)

We remark that, unlike the two previous examples, the calculation of \( x^{(a)} \) cannot be done by simply substituting the cosine with the complex exponential in the expression of \( x \), since the resulting signal \( \text{sinc}(2 B t) e^{j\pi B t} \) has its full band \( B = (-B, B) + B/2 = (-B/2, 3B/2) \) partly with negative frequencies. As a matter of fact, compared to the case of Example 1.5 B, here \( B_a = B \) is not smaller than \( f_0 = B/2 \).

### 1.5.2 BASEBAND EQUIVALENT

The baseband equivalent \( x^{(bb)}(t) \) of a real-valued passband signal \( x(t) \) is obtained by a frequency shift of its analytic signal.

**Definition 1.9** Given a real-valued (passband) signal \( x(t) \) and a reference frequency \( f_0 \), we call the baseband equivalent (or complex envelope) of \( x \), \( x^{(bb)}(t) \), with respect to the frequency \( f_0 \), the signal having the Fourier transform

\[ \mathcal{X}^{(bb)}(f) = \mathcal{X}^{(a)}(f + f_0) \] (1.113)

where \( \mathcal{X}^{(a)} \) is the Ftf of the analytic signal.

The above definition is illustrated in Figure 1.28.

![Figure 1.28](https://via.placeholder.com/150)

**Figure 1.28** Frequency domain illustration of the baseband equivalent signal derivation.

In the time domain the corresponding relationship is

\[ x^{(bb)}(t) = x^{(a)}(t) e^{-j2\pi f_0 t} \] (1.114)

and is represented by the block diagram in Figure 1.29(a). Using (1.114) in (1.97), we can recover the signal \( x \) from its baseband equivalent as

\[ x(t) = \Re[x^{(bb)}(t) e^{j2\pi f_0 t}], \] (1.115)

as shown by the block diagram in Figure 1.29(b).
Remark  Note that unlike the analytic signal, the definition of the baseband equivalent is somewhat arbitrary, since it requires the choice of a reference frequency $f_0$. In the case of modulated signals (see Chapter 3), the carrier frequency is usually chosen as the reference frequency.

Example 1.5 D  For the signal $x(t) = A \cos(2\pi f_0 t + \varphi_0)$, the baseband equivalent signal with reference frequency $f_0$ is the complex constant $A e^{j \varphi_0}$. In fact, in Example 1.5 A the analytic signal was found

$$x^{(a)}(t) = A e^{j (2\pi f_0 t + \varphi_0)}.$$

Hence by (1.114)

$$x^{(bb)}(t) = e^{-j 2\pi f_0 t} x^{(a)}(t) = A e^{j \varphi_0}.$$

If we choose a different reference frequency $f_1 \neq f_0$ we get a different baseband equivalent signal

$$x^{(bb)}(t) = A e^{j [2\pi(f_0-f_1)t+\varphi_0]}.$$

Example 1.5 E  For the signal in Example 1.5 B,

$$x(t) = a(t) \cos(2\pi f_0 t + \varphi_0), \quad (1.116)$$

with $B_a < f_0$, the baseband equivalent signal with reference frequency $f_0$ is

$$x^{(bb)}(t) = a(t) e^{j \varphi_0}, \quad (1.117)$$

obtained from the analytic signal found in Example 1.5 B, again by means of (1.114). We show this result in the frequency domain, in the case $a(t) = \text{sinc}(2Bt)$ and $\varphi_0 = 0$. 

---
Definition 1.10 Given a real-valued (passband) signal \( x(t) \), and a reference frequency \( f_0 \), we call the baseband components of \( x \) with respect to the frequency \( f_0 \), the signals

\[
\begin{align*}
\text{in-phase component} & \quad x_{1}^{(bb)}(t) = \Re[x^{(bb)}(t)] \\
\text{quadrature component} & \quad x_{Q}^{(bb)}(t) = \Im[x^{(bb)}(t)].
\end{align*}
\]

From (1.115) we can also write

\[
x(t) = x_{1}^{(bb)}(t) \cos(2\pi f_0 t) - x_{Q}^{(bb)}(t) \sin(2\pi f_0 t).
\]

It is interesting to observe the relation between \( x_{1}^{(bb)} \), \( x_{Q}^{(bb)} \) and \( x^{(a)} \) in the frequency domain. From (1.113) it is

\[
\begin{align*}
\chi_{1}^{(bb)}(f) & = \frac{1}{2} [\chi^{(a)}(f + f_0) + \chi^{(a)*}(-f + f_0)] \\
\chi_{Q}^{(bb)}(f) & = \frac{1}{2j} [\chi^{(a)}(f + f_0) - \chi^{(a)*}(-f + f_0)].
\end{align*}
\]

1.5.3 BASEBAND EQUIVALENT OF A TRANSFORMATION

Given a transformation involving passband signals, it is often useful to determine an equivalent transformation between the baseband equivalent of the input and output signals which is called the baseband equivalent transformation of the original transformation. Some transformations that will occur often are given in the following paragraphs (see Figure 1.30).

**Mixer with baseband input** As shown in Figure 1.30(a), if the input \( x \) is a baseband signal, the corresponding output is a passband signal, and the corresponding baseband equivalent transformation is the multiplication by the complex constant \( e^{j\varphi_0} \). In fact, we have

\[
\mathcal{Y}(f) = \frac{1}{2} \chi(f - f_0) e^{j\varphi_0} + \frac{1}{2} \chi(f + f_0) e^{-j\varphi_0}
\]

whose analytic signal is \( \mathcal{Y}^{(a)}(f) = \chi(f - f_0) e^{j\varphi_0} \) provided that \( B_x < f_0 \). Finally, the baseband equivalent is, from (1.113),

\[
\mathcal{Y}^{(bb)}(f) = \chi(f) e^{j\varphi_0}.
\]

**Mixer with bandpass input followed by an LPF** As shown in Figure 1.30(b), if the input \( x \) is a passband signal of bandwidth \( 2B \) and central frequency \( f_0 \), the corresponding output is a signal which contains both a baseband component of bandwidth \( B \) and a passband component around the frequencies \( -2f_0 \) and \( 2f_0 \). After the lowpass signal the signal is baseband with

\[
y(t) = \frac{1}{2} \Re[e^{-j\varphi_1} x^{(bb)}(t)].
\]
Figure 1.30 Passband transformations and their baseband equivalent: (a) mixer with baseband input; (b) mixer with passband input; (c) bandpass filter.

Bandpass filter. Last, the baseband equivalent of a passband filter with impulse response $h$ is a baseband filter with impulse response given by $\frac{1}{2}h^{(bb)}$, as shown in Figure 1.30(c). The proof is easily obtained in the frequency domain. From (1.113) and (1.96), the baseband equivalent of the output signal $y$ is

$$Y^{(bb)}(f) = 2Y^{(+)}(f + f_0).$$

(1.121)

From the input–output relation of a filter, it is

$$Y^{(+)}(f) = X^{(+)}(f)\mathcal{H}^{(+)}(f),$$

(1.122)

which gives

$$2Y^{(+)}(f + f_0) = 2X^{(+)}(f + f_0)\mathcal{H}^{(+)}(f + f_0).$$

(1.123)

Using (1.121) we have

$$Y^{(bb)}(f) = X^{(bb)}(f)\frac{1}{2}\mathcal{H}^{(bb)}(f),$$

(1.124)

which corresponds to the output of a filter with impulse response $\frac{1}{2}h^{(bb)}$. Moreover, the filter in Figure 1.30(c) with impulse response $h^{(bb)}(t)$ has in-phase and quadrature components, $h_I^{(bb)}$ and $h_Q^{(bb)}$, respectively, that are related to $h^{(a)}$ by an expression equivalent to (1.120). Consequently, if $\mathcal{H}^{(a)}$ has Hermitian symmetry around $f_0$, then

$$\mathcal{H}_I^{(bb)}(f) = \mathcal{H}^{(a)}(f + f_0), \quad \mathcal{H}_Q^{(bb)}(f) = 0.$$
In other words \( h^{(bb)} = h^{(bb)}_1 \) is real. In practice this condition is verified by imposing that the filter \( h^{(a)} \) has symmetrical frequency specifications around \( f_0 \).

### 1.5.4 HILBERT TRANSFORM

**Definition 1.11** The [Hilbert transform](https://en.wikipedia.org/wiki/Hilbert_transform) of a real-valued signal \( x(t) \) is the real-valued signal \( x^{(h)}(t) \) obtained as output of the Hilbert filter whose frequency response is given by

\[
\mathcal{H}^{(h)}(f) = -j \text{sgn}(f) = \begin{cases} 
  -j, & f > 0 \\
  0, & f = 0 \\
  j, & f < 0 
\end{cases}
\]  
(1.126)

By writing the expressions of its amplitude and phase

\[
|\mathcal{H}^{(h)}(f)| = \begin{cases} 
  1, & f \neq 0 \\
  0, & f = 0 
\end{cases} \quad \text{and} \quad \angle \mathcal{H}^{(h)}(f) = \begin{cases} 
  -\pi/2, & f > 0 \\
  \pi/2, & f < 0 
\end{cases}
\]  
(1.127)

we see that it is an allpass filter (apart from the DC component that is removed). Also, it introduces a phase shift of \(-\pi/2\) in the spectral components at positive frequency and of \(\pi/2\) in the components at negative frequency. For this reason it is also called a \(\pi/2\) phase shift filter. From Table 1.2 the corresponding impulse response of the Hilbert filter is

\[
h^{(h)}(t) = \frac{1}{\pi t}.
\]  
(1.128)

The frequency and impulse response of the Hilbert filter are shown in Figure 1.31.

It is indeed customary to define mathematically the Hilbert transform through the relation given by the Hilbert filter in the time domain, that is

\[
x^{(h)}(t) = (x * h^{(h)}) (t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} x(u) \frac{t}{t-u} du.
\]  
(1.129)

On the other hand, the frequency domain approach is simpler to analyze.

It is interesting to see the relation between \(x^{(h)}\) and the analytic signal \(x^{(a)}\).

**Proposition 1.13** The Hilbert transform of a real-valued signal \(x(t)\) is the imaginary part of the analytic signal \(x^{(a)}\) associated to \(x\),

\[
x^{(h)}(t) = \Im x^{(a)}(t), \quad x^{(a)}(t) = x(t) + j x^{(h)}(t).
\]  
(1.130)

**Proof** By taking the Ftf of the complex-valued signal \(x(t) + j x^{(h)}(t)\) we get

\[
\mathcal{X}(f) + j \mathcal{X}^{(h)}(f) = \mathcal{X}(f)[1 + \text{sgn}(f)] = 2 \mathcal{X}(f) 1(f)
\]

which is the Ftf of \(x^{(a)}\). \(\square\)
Example 1.5 F Consider the signal $x(t) = A \cos(2\pi f_0 t + \varphi_0)$, with $f_0 > 0$. In Example 1.5 A we found its analytical signal $x^{(a)}(t) = A e^{j(2\pi f_0 t + \varphi_0)}$. Then, from (1.130) we get its Hilbert transform as

$$x^{(h)}(t) = \Im\{A e^{j(2\pi f_0 t + \varphi_0)}\} = A \sin(2\pi f_0 t + \varphi_0).$$

We see that the Hilbert transform has phase shifted the sinusoid $\cos(2\pi f_0 t + \varphi_0)$ by $\pi/2$.

The same result can be generally applied for a sum of sinusoids. For example, if

$$x(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t + \pi/3) + A_3 \sin(2\pi f_3 t)$$

we can directly derive

$$x^{(h)}(t) = A_1 \cos(2\pi f_1 t - \pi/2) + A_2 \cos(2\pi f_2 t - \pi/6) + A_3 \cos(2\pi f_3 t - \pi/2)$$

Example 1.5 G Consider the signal $x(t) = a(t) \cos(2\pi f_0 t + \varphi_0)$ in Example 1.5 B. Under the hypothesis that $B_a < f_0$, its analytical signal is $x^{(a)}(t) = a(t) e^{j(2\pi f_0 t + \varphi_0)}$. Then, under the same hypothesis we can derive its Hilbert transform from (1.130) as

$$x^{(h)}(t) = a(t) \sin(2\pi f_0 t + \varphi_0).$$

Note that the Hilbert transform simply introduces a phase shift in the sinusoidal term.
Example 1.5 H  For the signal in Example 1.5 C,

\[ x(t) = A \text{sinc}(2Bt) \cos(\pi Bt), \]

we can write its Hilbert transform as

\[ x^{(h)}(t) = \Im \left[ x^{(a)}(t) \right] = \frac{3}{4}A \text{sinc} \left( \frac{3}{2}Bt \right) \sin \left( \pi \frac{3}{2}Bt \right) + \frac{1}{4}A \text{sinc} \left( \frac{1}{2}Bt \right) \sin \left( \pi \frac{1}{2}Bt \right). \]

The Fft of \( x^{(h)} \) is purely imaginary and its plot is given below.

We now state some properties of the Hilbert transform that will be used in Chapter 3.

**Proposition 1.14** Any real-valued signal \( x(t) \) and the corresponding Hilbert transform \( x^{(h)}(t) \) are orthogonal, i.e. their cross energy is zero,

\[ E_{x^{(h)}x} = 0. \]  

(1.131)

Moreover, if the signal \( x \) has finite energy, its energy is equal to the energy of its Hilbert transform

\[ E_x = E_{x^{(h)}}. \]  

(1.132)

**Proof**  The proof of both properties is carried out in the frequency domain. We observe that \( x \) and \( x^{(h)} \) can be seen as input and output signals in the Hilbert filter \( h^{(h)} \). Then, by the energy spectral density relationship (1.51) we can write

\[ E_{x^{(h)}x} = \int_{-\infty}^{+\infty} E_{x^{(h)x}}(f) \, df = \int_{-\infty}^{+\infty} \mathcal{H}^{(h)}(f) E_x(f) \, df = 0, \]

where the last integral vanishes since the frequency response \( \mathcal{H}^{(h)}(f) \) is odd, while \( E_x(f) = |X(f)|^2 \) is even, as \( x \) is real-valued. Also, by (1.52) and using the unit amplitude of \( \mathcal{H}^{(h)} \) we get

\[ E_{x^{(h)}}(f) = |\mathcal{H}^{(h)}(f)|^2 E_x(f) = \begin{cases} E_x(f), & f \neq 0 \\ 0, & f = 0 \end{cases} \]

and if \( E_x(f) \) does not have a Dirac impulse at \( f = 0 \), the integral of \( E_{x^{(h)}} \) over the frequency axis is the same as that of \( E_x \).
In a similar way, for signals with nonzero average power we can state the following proposition.

**Proposition 1.15** Any real-valued signal \( x(t) \) and the corresponding Hilbert transform \( x^{(h)}(t) \) have zero cross power,

\[
M_{x^{(h)}x} = 0.
\]

Moreover, if the signal \( x \) has no DC component, its average power is equal to the average power of its Hilbert transform,

\[
M_x = M_{x^{(h)}}.
\]

### 1.5.5 ENVELOPE, INSTANTANEOUS PHASE AND FREQUENCY

For a sinusoidal waveform

\[
x(t) = A \cos(2\pi f_0 t + \varphi_0),
\]

it is intuitive to identify the amplitude \( A \), the phase at time \( t \), \( \varphi_x(t) = 2\pi f_0 t + \varphi_0 \), and the frequency \( f_0 \).

We now extend the above concepts to a general real-valued signal by means of the notions of *envelope*, *instantaneous phase* and *instantaneous frequency*, that are based upon the analytic signal.

**Definition 1.12** Given a real-valued signal \( x(t) \), with associated analytic signal \( x^{(a)}(t) \), we define:

- **the envelope of** \( x \) as
  \[
  e_x(t) = |x^{(a)}(t)|;
  \]
- **the instantaneous phase of** \( x \) as
  \[
  \varphi_x(t) = \angle x^{(a)}(t);
  \]
- **the instantaneous frequency of** \( x \) as
  \[
  f_x(t) = \frac{1}{2\pi} \frac{d}{dt} \varphi_x(t).
  \]

The above defined signals are illustrated in Figure 1.32.

Hence from (1.97) an alternative expression for \( x(t) \) is given by

\[
x(t) = e_x(t) \cos(\varphi_x(t)).
\]

**Example 1.51** We let the reader verify that in the case of a sinusoidal signal \( x(t) = A \cos(2\pi f_0 t + \varphi_0) \) with \( A > 0 \), the notions of envelope, instantaneous phase and frequency given in (1.136)–(1.138)
Figure 1.32 Illustration of signals associated with a real-valued bandpass signal \( x \): (a) the signal \( x \); (b) the envelope \( e_x \); (c) the instantaneous phase \( \varphi_x \); (d) the instantaneous frequency \( f_x \).

coincide with the customary amplitude, phase and frequency, yielding \( e_x(t) = A, \varphi_x(t) = 2\pi f_0 t + \varphi_0, f_x(t) = f_0 \).

**Example 1.5 J** We examine the signal \( x(t) = a(t) \cos(2\pi f_0 t) \) with \( B_a < f_0 \) introduced in Example 1.5 B. There we calculated its analytic signal, from which we can derive the envelope of \( x \),

\[
e_x(t) = |a(t)e^{j(2\pi f_0 t + \varphi_0)}| = |a(t)|,
\]

and the instantaneous phase of \( x \),

\[
\varphi_x(t) = 2\pi f_0 t + \varphi_0 + \angle a(t),
\]

where we should observe that, since \( a(t) \) is real-valued, \( \angle a(t) \) is a piecewise constant signal that can only take the values 0 or \( \pi \). By neglecting the \( \pi \) jumps in \( \angle a(t) \), we find that the instantaneous frequency of \( x \) is

\[
f_x(t) = f_0.
\]

Starting from the definitions of instantaneous phase and frequency, we also define the phase and frequency deviations of \( x(t) \) with respect to the sinusoid at frequency \( f_0 \).

- The **instantaneous phase deviation** is

\[
\Delta \varphi_x(t) = \varphi_x(t) - 2\pi f_0 t = \angle x^{(bb)}(t).
\]
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- The instantaneous frequency deviation is

\[
\Delta f_x(t) = f_x(t) - f_0 = \frac{1}{2\pi} \frac{d}{dt} \Delta \varphi_x(t).
\]

(1.144)

Observe that the phase and frequency deviation of \(x(t)\) can be calculated from the baseband equivalent signal \(x^{(bb)}\) with reference frequency \(f_0\). Moreover, using (1.114), the envelope of \(x\) can also be calculated from \(x^{(bb)}\) as

\[
e_x(t) = |x^{(bb)}(t)|.
\]

(1.145)

We show in Figure 1.33 the phase and frequency deviation of the signal in Figure 1.32 with respect to the reference frequency \(f_0 = 4.5\) MHz.

![Figure 1.33 Illustration of: (a) the phase deviation \(\Delta \varphi_x\) and (b) the frequency deviation \(\Delta f_x\), for the signal in Figure 1.32(a) with respect to \(f_0 = 4.5\) MHz.](image)

1.6 Random Variables and Vectors

The present section is an abridgment of useful results, by no means complete, and is ancillary to the introduction of random processes (which model probabilistic phenomena) in Section 1.7.

We start with the following definition.

**Definition 1.13** A random variable (rv) \(x\) is a real-valued function defined as

\[
x : \Omega \mapsto \mathbb{R}
\]

(1.146)

where \(\Omega\) is called the sample space in a probability space \([2, 3]\).

We should therefore write \(x(\omega)\) with \(\omega\) an arbitrary point in \(\Omega\), but it is customary to drop the dependence on \(\omega\) from the notation. Each of the possible values of \(x\) corresponding to a given \(\omega\) is called a realization of the variable \(x\).

The fact that \(x\) is defined on a probability space means that there is a probability function \(P[\cdot]\) defined on the subsets of \(\Omega\) so that for any reasonable subset \(A\) of \(\mathbb{R}\) (made as the union,
possibly countable, of intervals and single points) we can determine the probability that the quantity \( x \) takes values in \( A \) as

\[
P[x(\omega) \in A] = P[x^{-1}(A)]
\]

where \( x^{-1}(A) \) denotes the inverse range of \( A \).

A rv is usually classified, according to its possible values, as:

**discrete-valued or discrete** if \( x \) can only take a countable number of values, the set \( A_x \) of possible values then being called the *alphabet* of the rv \( x \);

**continuous-valued or continuous** otherwise.

Examples of discrete rvs are: the number of correctly received bits in the transmission of a digital message over a noisy channel, the intensity level of a pixel in a digital picture, etc. For examples of continuous rvs one can take a noisy voltage at some instant, the pitch frequency of a voiced segment in an audio signal, etc.

### 1.6.1 Statistical Description of Random Variables

A complete statistical description for a *discrete* rv \( x \) is given by its *probability mass distribution* (PMD) which is a real function defined on the alphabet of \( x \),

\[
p_x : A_x \mapsto \mathbb{R},
\]

so that \( p_x(a) \) gives the probability that \( x \) takes exactly the value \( a \),

\[
p_x(a) = P[x = a] = P[x \in \{a\}].
\]

It is evident from (1.148) that \( p_x \) can only take values in the interval \([0, 1]\), since it represents a probability. Then from the knowledge of the PMD we can calculate the probability that \( x \) takes value in any set \( C \subset \mathbb{R} \) as

\[
P[x \in C] = \sum_{a \in A_x \cap C} p_x(a)
\]

thus justifying the expression ‘complete statistical description’. Besides being limited between 0 and 1, \( p_x \) must also satisfy the following *normalization condition*:

\[
\sum_{a \in A_x} p_x(a) = P[x \in A_x] = 1.
\]

A peculiar case is that of a rv taking a single value \( a_0 \) with probability 1: such a variable is called *almost surely (a.s.) constant*. In terms of the above definitions it is

\[
A_x = \{a_0\}, \quad p_x(a_0) = 1.
\]
For continuous rvs a complete description is given by the \emph{probability density function} (PDF), a real function of real variable
\[ p_x : \mathbb{R} \mapsto \mathbb{R}, \]  
defined as
\[ p_x(a) = \frac{d}{da} P[x \leq a]. \]  
Then, the probability of \( x \) taking values in some set \( C \subset \mathbb{R} \) can be calculated as
\[ \mathbb{P}[x \in C] = \int_C p_x(a) \, da. \]  
The peculiarities of this function are less evident than for its discrete counterpart. However, as the probability \( \mathbb{P}[x \leq a] \) can be shown to be a monotonically nondecreasing function of \( a \), we must have
\[ p_x(a) \geq 0, \quad a \in \mathbb{R} \]  
and the normalization condition now reads
\[ \int_{-\infty}^{+\infty} p_x(a) \, da = \mathbb{P}[x \in \mathbb{R}] = 1. \]  

**Further insight** Observe that \( a \) has the same physical dimensions as \( x \), hence from (1.156) we see that for continuous rvs \( p_x \) must have the dimension of \( 1/x \). For example, if \( x \) is a random voltage, \( p_x \) will be measured in \( V^{-1} \).

---

**Example 1.6 A** A rv \( x \) is called \emph{Gaussian} or \emph{normal} if its PDF has the form
\[ p_x(a) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(a-m)^2}{2\sigma^2}} \]  
where \( m \) and \( \sigma \) are real parameters (with \( \sigma > 0 \)). It is said to be \emph{normalized} if \( m = 0 \) and \( \sigma = 1 \). We want to find the probability that \( x \) takes values in the interval \((b, c)\). By (1.154) we get
\[ \mathbb{P}[b < x < c] = \int_b^c \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(a-m)^2}{2\sigma^2}} \, da. \]  
We face the nontrivial problem of evaluating the above integral. By the change of variable \( u = \frac{a - m}{\sigma} \), and defining the \emph{normalized complementary Gaussian distribution} function as the integral of the normalized Gaussian PDF
\[ Q(y) = \int_y^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du, \]  
we can express the above probability as
\[ \mathbb{P}[b < x < c] = \int_{(b-m)/\sigma}^{(c-m)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du = Q\left( \frac{b-m}{\sigma} \right) - Q\left( \frac{c-m}{\sigma} \right), \]
as illustrated in Figure 1.34. The same expression can be used for evaluating the probabilities on unlimited intervals, by using the limit values

$$Q(-\infty) = 1, \quad Q(+\infty) = 0.$$ (1.159)

![Figure 1.34](image)

Figure 1.34  Plots of: (a) the normalized Gaussian PDF; (b) a Gaussian PDF with $m = 2, \sigma^2 = 1/2$.

Gaussian rvs, introduced in the above example, play a very important role in the theory of transmission over noisy channels. To indicate that a rv $x$ is Gaussian with parameters $m$ and $\sigma$ we will write $x \sim \mathcal{N}(m, \sigma^2)$. Values of $Q(y)$ are given in Table 1.A.1 of the Appendix to this chapter.

1.6.2 EXPECTATION AND STATISTICAL POWER

An important statistical parameter for a rv $x$ is its mean (also called mean value or expectation) indicated with $m_x$ or $E[x]$. It is a real value, dimensionally homogeneous with $x$, representing the mass center of $p_x$ and it is calculated as

$$E[x] = \begin{cases} \int_{-\infty}^{+\infty} a \ p_x(a) \ da, & \text{for continuous rvs} \\ \sum_{a \in A_x} a \ p_x(a), & \text{for discrete rvs}. \end{cases}$$ (1.160)

In other words, expectation represents an integration of the variable (in $\omega$ over $\Omega$) weighted by the probability measure. The reader should not be misled by the term ‘expectation’: the mean is not the most likely value of the random variable $x$, it is just a weighted average of its possible values. Indeed, for a discrete $x$, $m_x$ may not even belong to the alphabet $A_x$.

Given an arbitrary real function of real variable $g : \mathbb{R} \mapsto \mathbb{R}$ we can compose it with a rv $x$ obtaining a new rv $y$ given by

$$y(\omega) = g(x(\omega))$$ (1.161)
whose statistical description can be obtained from that of $x$. In particular, for discrete rvs we get
\[ A_y = g(A_x), \quad p_y(b) = \sum_{a \in g^{-1}(b)} p_x(a), \quad b \in A_y \]  
(1.162)
whereas for continuous rvs, by exploiting the identity
\[ P[y \leq b] = P[x \in g^{-1}(C_b)], \quad \text{where } C_b = (-\infty, b] \]  
(1.163)
we get
\[ p_y(b) = \frac{d}{db} \int_{g^{-1}(C_b)} p_x(a) \, da. \]  
(1.164)

However, sometimes we might be interested only in determining the mean of the new r.v. $y = g(x)$, and not in finding its complete statistical description. The following theorem, that we just state, gives a method to calculate the mean of $y$, directly from the statistical description of $x$.

**Theorem 1.16 (Fundamental theorem for expectation)** Given the statistical description of a r.v. $x$, we can calculate the expectation of any other r.v. $y = g(x)$ that is a function of $x$ as
\[ E[y] = E[g(x)] = \begin{cases} 
\int_{-\infty}^{+\infty} \sum_{a \in A_x} g(a) \, p_x(a) \, da, & \text{for continuous rvs} \\
\sum_{a \in A_x} g(a) \, p_x(a), & \text{for discrete rvs}. 
\end{cases} \]  
(1.165)

We illustrate the above theorem with a couple of examples.

---

**Example 1.6 B** Let $x$ be a discrete r.v with alphabet
\[ A_x = \{-2, -1, 0, 1, 2\} \]
and PMD
\[ p_x(-2) = p_x(2) = 1/8, \quad p_x(-1) = p_x(1) = p_x(0) = 1/4 \]
and $y = g(x) = x^2$. Then by applying Theorem 1.16 we can calculate
\[ E[y] = E[x^2] = \sum_{a \in A_x} a^2 p_x(a) = 4 \cdot \frac{1}{8} + 1 \cdot \frac{1}{4} + 0 + 1 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} = \frac{3}{2}. \]

**Example 1.6 C** Let $\varphi$ be a continuous r.v with PDF
\[ p_\varphi(a) = \frac{1}{2\pi} \text{rect} \left( \frac{a}{2\pi} \right) \]
and \( y = g(\varphi) = \cos^2 \varphi \). By Theorem 1.16 we get

\[
E[y] = E[\cos^2 \varphi] = \int_{-\infty}^{+\infty} \cos^2(a) p_x(a) \, da = \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos^2(a) \, da = \frac{1}{2}.
\]

Two important statistical parameters for \( x \) are obtained from Theorem 1.16 by respectively choosing \( g(x) = x^2 \) and \( g(x) = (x - m_x)^2 \), that is:

- **Statistical power of** \( x \)
  \[
  M_x = E[x^2] = \begin{cases} 
  \int_{-\infty}^{+\infty} a^2 p_x(a) \, da, & \text{for continuous rvs} \\
  \sum_{a \in A_x} a^2 \, p_x(a), & \text{for discrete rvs};
  \end{cases} 
  \tag{1.166}
  \]

- **Variance of** \( x \)
  \[
  \sigma^2_x = E[(x - m_x)^2] = \begin{cases} 
  \int_{-\infty}^{+\infty} (a - m_x)^2 p_x(a) \, da, & \text{for continuous rvs} \\
  \sum_{a \in A_x} (a - m_x)^2 \, p_x(a), & \text{for discrete rvs}.
  \end{cases} 
  \tag{1.167}
  \]

The square root of the variance, \( \sigma_x \), is also an important parameter and is called the **standard deviation** of \( x \). It can be seen by exploiting the linearity of the expectation that statistical mean, power and variance of a rv are related by

\[
\sigma^2_x + m_x^2 = M_x. \tag{1.168}
\]

### 1.6.3 Random Vectors

We extend the notion of a random variable and introduce a random vector (rve) \( \mathbf{x} \) as an \( N \)-ple of random variables, i.e. \( \mathbf{x} = [x_1, x_2, \ldots, x_N]^T \) (the superscript \( T \) denotes transposition). The relevant case is when the rvs that make the vector are related to each other, in some sense; therefore, we consider the variables in the vector to be defined on a common probability space. In this sense, the rve is a function

\[
\mathbf{x} : \Omega \mapsto \mathbb{R}^N \tag{1.169}
\]

and we are interested in a joint statistical description. As was done for rvs, we can apply the distinction into discrete and continuous rves, as well as the notion of alphabet, which is now a countable subset of \( \mathbb{R}^N \), namely \( A_{\mathbf{x}} \). Thus we can define the PMD as

\[
p_{\mathbf{x}}(\mathbf{a}) = P[\mathbf{x} = \mathbf{a}] = P[x_1 = a_1, \ldots, x_N = a_N], \tag{1.170}
\]
and the PDF as

\[ p_\mathbf{x}(\mathbf{a}) = \frac{d}{d\mathbf{a}} \mathbb{P}[\mathbf{x} \leq \mathbf{a}] = \frac{\partial^N}{\partial a_1 \cdots \partial a_N} \mathbb{P}[x_1 \leq a_1, \ldots, x_N \leq a_N]. \quad (1.171) \]

For example, the PDF allows us to calculate the probability that \( \mathbf{x} \) takes values in a subset \( C \subseteq \mathbb{R}^N \) in the same manner as described by (1.154) where the integral is now \( N \)-dimensional.

**Example 1.6 D** Given the PDF of the rve \( \mathbf{x} = [x_1, x_2] \)

\[ p_\mathbf{x}(\mathbf{a}) = \beta \gamma e^{-\beta a_1 - \gamma a_2} \]

we calculate the probability \( \mathbb{P}[x_1 > x_2] \). First we rewrite the event as

\[ \{x_1 > x_2\} = \{\mathbf{x} \in C\}, \quad \text{with} \quad C = \{(a_1, a_2) \in \mathbb{R}^2 : a_1 > a_2\}. \]

Then we write the two-dimensional integral of \( p_\mathbf{x} \) over \( C \) as

\[ \mathbb{P}[x_1 > x_2] = \int_C p_\mathbf{x}(\mathbf{a}) d\mathbf{a} = \int_{-\infty}^{+\infty} \int_{-\infty}^{a_1} p_\mathbf{x}(a_1, a_2) da_2 da_1 \]

and we observe that \( p_\mathbf{x}(\mathbf{a}) \) is nonzero only for \( a_1 \geq 0, a_2 \geq 0 \), so we can calculate

\[
\begin{align*}
\mathbb{P}[x_1 > x_2] &= \int_{0}^{+\infty} \int_{0}^{a_1} \beta \gamma e^{-\beta a_1 - \gamma a_2} da_2 da_1 \\
&= \int_{0}^{+\infty} \beta e^{-\beta a_1} \int_{0}^{a_1} \gamma e^{-\gamma a_2} da_2 da_1 \\
&= \int_{0}^{+\infty} \beta e^{-\beta a_1} (1 - e^{-\gamma a_1}) da_1 \\
&= 1 - \frac{\beta}{\beta + \gamma} = \frac{\gamma}{\beta + \gamma}.
\end{align*}
\]

An important special case is that of rves made of independent rvs.

**Definition 1.14** \( N \) rvs \( x_1, \ldots, x_N \) are said to be statistically independent if their joint statistical description can be factored into the product of the \( N \) single rv descriptions. For example using the joint PDF we have

\[ p_\mathbf{x}(\mathbf{a}) = p_{x_1 \cdots x_N}(a_1, \ldots, a_N) = \prod_{i=1}^{N} p_{x_i}(a_i) \quad (1.172) \]

In this case, also the probability that \( \mathbf{x} \in C \), with \( C \) the Cartesian product \( C = C_1 \times \cdots \times C_N \) and each \( C_i \subseteq \mathbb{R} \), can be factored as

\[ \mathbb{P}[\mathbf{x} \in C] = \mathbb{P}[x_1 \in C_1, \ldots, x_N \in C_N] = \prod_{i=1}^{N} \mathbb{P}[x_i \in C_i]. \quad (1.173) \]
An example follows.

**Example 1.6 E** Suppose that two continuous rvs have joint PDF

\[
p_{x_1x_2}(a_1, a_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[-\frac{(a_1 - m_1)^2}{2\sigma_1^2} - \frac{(a_2 - m_2)^2}{2\sigma_2^2}\right].
\]

Since we can factor the PDF as

\[
p_{x_1x_2}(a_1, a_2) = p_{x_1}(a_1)p_{x_2}(a_2), \quad \text{with} \quad p_{x_i}(a_i) = \frac{1}{\sqrt{2\pi\sigma_i}} \exp\left[-\frac{(a_i - m_i)^2}{2\sigma_i^2}\right]
\]

we realize that \( x_1 \) and \( x_2 \) are two statistically independent Gaussian rvs, with mean \( m_1, m_2 \) and variance \( \sigma_1^2, \sigma_2^2 \), respectively. To calculate the probability \( P[x_1 > a, b < x_2 < c] \) we can either integrate their joint PDF over the set

\[
C = \{(a_1, a_2) : a_1 > a, b < a_2 < c\} = (a, +\infty) \times (b, c)
\]

or, simply by making use of independence

\[
P[x_1 > a, b < x_2 < c] = P[x_1 > a]P[b < x_2 < c]
\]

and using the results in Example 1.6 A, provide

\[
P[x_1 > a, b < x_2 < c] = Q\left(\frac{a - m_1}{\sigma_1}\right)\left[Q\left(\frac{b - m_2}{\sigma_2}\right) - Q\left(\frac{c - m_2}{\sigma_2}\right)\right].
\]

When two rvs are not statistically independent we are interested in expressing how the knowledge of the realization of one rv may influence our statistical information on the other rv. This can be formally expressed by using the conditional PDF of a rv \( x \), given that the rv \( y \) is known to take a given value \( b \), defined as

\[
p_{x|y}(a|b) = \frac{p_{xy}(a, b)}{p_y(b)}.
\]

The conditional PDF enjoys all the properties seen in Section 1.6.1 for a PDF, such as the normalization condition

\[
\int_{-\infty}^{+\infty} p_{x|y}(a|b) \, da = P[x \in \mathbb{R}|y = b] = 1,
\]
and allows calculation of probabilities of the rv $x$ conditioned by the value of $y$ as

$$P[x \in C|y = b] = \int_C p_{x|y}(a|b) \, da. \quad (1.176)$$

We can observe that if $x$ and $y$ are independent rvs, then by factoring their joint PDF, the conditional PDF becomes

$$p_{x|y}(a|b) = \frac{p_x(a)p_y(b)}{p_y(b)} = p_x(a)$$

so that knowledge of the value taken by $y$ does not change the probability of the values that $x$ can take. This intuitively justifies the adjective ‘independent’ for the two rvs.

We recall an important result relating unconditioned and conditional probabilities.

**Theorem 1.17 (Total probability theorem)** Given two rvs $x$ and $y$, the PDF of $x$ can be derived by averaging the joint PDF of $x$ and $y$ with respect to the values taken by $y$,

$$p_x(a) = \begin{cases} \int_{-\infty}^{+\infty} p_{xy}(a, b) \, db, & \text{for } y \text{ a continuous rv} \\ \sum_{b \in A_y} p_{xy}(a, b), & \text{for } y \text{ a discrete rv.} \end{cases} \quad (1.177)$$

Equivalently, by (1.174), we can also derive $p_x$ from the conditional PDF $p_{x|y}$ and $p_y$ as

$$p_x(a) = \begin{cases} \int_{-\infty}^{+\infty} p_{x|y}(a|b)p_y(b) \, db, & \text{for } y \text{ a continuous rv} \\ \sum_{b \in A_y} p_{x|y}(a|b)p_y(b), & \text{for } y \text{ a discrete rv.} \end{cases} \quad (1.178)$$

The relationships (1.177) are also known as marginal rules.

The following important result allows the roles between the conditioning and conditioned rvs to be exchanged.

**Proposition 1.18 (Bayes’ rule)** The conditional PDF of $y$ given $x = a$ and the conditional PDF of $x$ given $y = b$ are related by

$$p_{y|x}(b|a) = p_{x|y}(a|b) \frac{p_y(b)}{p_x(a)}. \quad (1.179)$$

**Proof** Multiply both sides of (1.179) by $p_x(a)$ to obtain

$$p_{y|x}(b|a)p_x(a) = p_{x|y}(a|b)p_y(b)$$

where the two sides represent two equivalent expressions for the joint PDF $p_{xy}(a, b)$. □
1.6.4 SECOND ORDER DESCRIPTION OF RANDOM VECTORS, AND GAUSSIAN VECTORS

For a pair of rvs we can consider their second order description that, besides their means \( m_x, m_y \), statistical powers \( M_x, M_y \) and variances \( \sigma_x^2, \sigma_y^2 \), includes also two joint statistical parameters,

- **correlation**
  \[
  r_{xy} = \mathbb{E}[xy] = \begin{cases} 
  \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ab p_{xy}(a, b) \, db \, da, & \text{for continuous rvs} \\
  \sum_{(a,b) \in \mathcal{A}_{xy}} ab p_{xy}(a, b), & \text{for discrete rvs}
  \end{cases} 
  \]  
  (1.180)

  where \( \mathcal{A}_{xy} \) is the alphabet of the rve \([x, y]\);

- **covariance**
  \[
  k_{xy} = \mathbb{E}[(x - m_x)(y - m_y)] = \begin{cases} 
  \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (a - m_x)(b - m_y) p_{xy}(a, b) \, db \, da, & \text{for continuous rvs} \\
  \sum_{(a,b) \in \mathcal{A}_{xy}} (a - m_x)(b - m_y) p_{xy}(a, b), & \text{for discrete rvs.}
  \end{cases} 
  \]  
  (1.181)

From (1.181) it is easy to see that correlation and covariance of two rvs are related to their means as

\[
  r_{xy} = k_{xy} + m_x m_y. \tag{1.182}
\]

**Definition 1.15** If \( r_{xy} = m_x m_y \) or equivalently \( k_{xy} = 0 \), the two rvs are said to be uncorrelated. If \( r_{xy} = 0 \) they are said to be orthogonal. Obviously, the two definitions coincide when at least one of the two means is zero.

An important property regarding the second order description of statistically independent rvs is the following.

**Proposition 1.19** If two rvs \( x \) and \( y \) are statistically independent they are also uncorrelated.

**Proof** By factoring the joint PDF \( p_{xy}(a, b) = p_x(a)p_y(b) \) we can also factor the two-dimensional integral (1.180) into

\[
x_{xy} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ab p_x(a)p_y(b) \, db \, da = \int_{-\infty}^{+\infty} a p_x(a) \, da \int_{-\infty}^{+\infty} b p_y(b) \, db = m_x m_y. \quad \square
\]

**Further insight** Also related to the second order description of a pair of variables is the concept of equality in the mean square sense.
Definition 1.16 Two rvs \(x, y\) are said to be equal in the mean square sense (we will write \(x \overset{m.s.}{=} y\)) if

\[
E[|x - y|^2] = 0.
\]

(1.183)

The above definition states that \(e = x - y\), representing the difference between \(x\) and \(y\), has null statistical power. This does not mean that \(e\) is identically zero, but rather that it is almost surely null as \(P[e = 0] = 1\). Hence in this case \(x\) and \(y\) take the same value with probability 1.

By extending the above description to rves of arbitrary length \(N\) we can obtain their second order description, made of:

- a mean vector, \(m_x = E[x] = [m_1, \ldots, m_N]^T\);
- a correlation matrix holding power and correlation values

\[
r_x = E[xx^T] = \begin{bmatrix}
m_{x1} & r_{x1x2} & \cdots & r_{x1xN} \\
r_{x2x1} & m_{x2} & \cdots & r_{x2xN} \\
\vdots & \vdots & \ddots & \vdots \\
r_{xNx1} & r_{xNx2} & \cdots & m_{xN}
\end{bmatrix};
\]

- a covariance matrix holding variance and covariance values

\[
k_x = E[(x - m_x)(x - m_x)^T] = \begin{bmatrix}
s_{x1}^2 & k_{x1x2} & \cdots & k_{x1xN} \\
k_{x2x1} & s_{x2}^2 & \cdots & k_{x2xN} \\
\vdots & \vdots & \ddots & \vdots \\
k_{xNx1} & k_{xNx2} & \cdots & s_{xN}^2
\end{bmatrix}.
\]

Although quite informative, the second order description is not a complete statistical description for a vector, which would be given by the PDF. An exceptional case in this respect is given by Gaussian vectors, where the PDF can always be written in terms of the mean vector and covariance matrix as

\[
p_x(a) = \frac{1}{\sqrt{2^N\pi^N} \det k_x} e^{-\frac{1}{2}(a - m_x)^T k_x^{-1}(a - m_x)}. \tag{1.184}
\]

As an example, for \(N = 2\) we get

\[
p_{x1x2}(a_1, a_2) = \frac{1}{2\pi \sqrt{\sigma_1^2\sigma_2^2 - k^2}} \times \exp \left[-\frac{(a_1 - m_1)^2\sigma_2^2 + (a_2 - m_2)^2\sigma_1^2 - 2(a_1 - m_1)(a_2 - m_2)k}{2\sigma_1^2\sigma_2^2 - 2k^2}\right], \tag{1.185}
\]

where for compactness we have written \(m_1 = m_{x1}, m_2 = m_{x2}, \sigma_1^2 = \sigma_{x1}^2, \sigma_2^2 = \sigma_{x2}^2, k = k_{x1x2}^2\).
Beside being an excellent model for noise phenomena, as will be seen in the next chapters, Gaussian vectors are particularly welcome for both their mathematical tractability and the following properties.

**Proposition 1.20** Any linear transformation \( y = Ax + b \) of a Gaussian vector \( x \) is itself a Gaussian vector with mean \( m_y = Am_x + b \) and covariance matrix \( k_y = Ak_xA^T \).

Observe that, in the above proposition, if \( x \) and \( y \) are \( N \)- and \( M \)-dimensional vectors, respectively, then \( A \) is an \( M \times N \) real matrix and \( b \) a column of \( \mathbb{R}^N \).

**Proposition 1.21** If a Gaussian vector has uncorrelated rvs (that is \( k_{x_ix_j} = 0 \), for all \( i \neq j \)) then its rvs are also statistically independent.

**Proof** In the case of a rve with uncorrelated rvs the covariance matrix is diagonal and

\[
\det k_x = \prod_{i=1}^{N} \sigma_i^2, \quad k_x^{-1} = \begin{bmatrix}
1/\sigma_{x_1}^2 & 0 & \cdots & 0 \\
0 & 1/\sigma_{x_2}^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1/\sigma_{x_N}^2
\end{bmatrix}
\]

so that we can write (1.184) as

\[
p_x(a) = \frac{1}{\sqrt{2^N \pi^N \sigma_{x_1}^2 \cdots \sigma_{x_N}^2}} \exp \left[ -\frac{(a_1 - m_{x_1})^2}{2\sigma_{x_1}^2} - \cdots - \frac{(a_N - m_{x_N})^2}{2\sigma_{x_N}^2} \right]
\]

\[
= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_{x_i}} \exp \left[ -\frac{(a_i - m_{x_i})^2}{2\sigma_{x_i}^2} \right]
\]

and thus obtain the factorization (1.172).

The above result represents the converse of Proposition 1.19, and it only holds for Gaussian vectors.

### 1.6.5 COMPLEX-VALUED RANDOM VARIABLES

A complex-valued rv is a function

\[
x : \Omega \mapsto \mathbb{C}.
\]

Actually, it is modeled as a pair of real-valued rvs, \( x_1 = \Re[x] \) and \( x_Q = \Im[x] \), and thus a real-valued two-dimensional rve. The PDF of \( x \) is defined as the joint PDF of \( x_1 \) and \( x_Q \), which for convenience we write as

\[
p_x(a) = p_{x_1,x_Q}(a_1,a_Q), \quad a = a_1 + ja_Q.
\]
Example 1.6 F Suppose we want to find the probability $P[|x| < 1]$ with $x$ a continuous complex-valued rv. Then we must solve the integral

$$P[|x| < 1] = \int_{|a| < 1} p_x(a) \, da.$$  \hspace{1cm} (1.188)

By writing the subset as

$$\{ |a| < 1 \} = \{ a^2 + a_Q^2 < 1 \},$$  \hspace{1cm} (1.189)

we can calculate the probability as the double integral

$$P[|x| < 1] = \int_{-1}^{1} \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} p_{x_1 x_Q}(a_1, a_Q) \, da_1 \, da_Q.$$  \hspace{1cm} (1.190)

Alternatively we can resort to the PDF of $|x|$ 

$$P[|x| < 1] = \int_{0}^{1} p_{|x|}(\rho) \, d\rho.$$  \hspace{1cm} (1.191)

Example 1.6 G To calculate the expectation of a continuous complex-valued rv we have

$$E[x] = \int_C a p_x(a) \, da = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (a_1 + ja_Q)p_{x_1 x_Q}(a_1, a_Q) \, da_1 \, da_Q$$  \hspace{1cm} (1.192)

and by applying the marginal rule (1.177)

$$E[x] = \int_{-\infty}^{+\infty} a_1 p_{x_1}(a_1) \, da_1 + j \int_{-\infty}^{+\infty} a_Q p_{x_Q}(a_Q) \, da_Q = E[x_1] + j E[x_Q].$$  \hspace{1cm} (1.193)

Observe that the result in Example 1.6 G could also be obtained by applying the linearity of expectation

$$m_x = E[x_1 + j x_Q] = m_{x_1} + j m_{x_Q},$$  \hspace{1cm} (1.194)

The second order description of a complex-valued rv includes means, statistical powers, variances, correlation and covariance of the pair $x_1, x_Q$. However, usually we are interested in the statistical power and variance of the complex rv $x$ given by

$$M_x = E[|x|^2], \quad \sigma_x^2 = E[|x - m_x|^2].$$  \hspace{1cm} (1.195)

Observe that both parameters are real and nonnegative. It is easy to derive the relations

$$M_x = E[x_1^2 + x_Q^2] = M_{x_1} + M_{x_Q},$$  \hspace{1cm} (1.196)

$$\sigma_x^2 = E[(x_1 - m_{x_1})^2 + (x_Q - m_{x_Q})^2] = \sigma_{x_1}^2 + \sigma_{x_Q}^2.$$  \hspace{1cm} (1.197)
\[ M_x = \sigma_x^2 + |m_x|^2. \]  

(1.198)

**Example 1.6 H** A complex-valued rv \( x \) is called complex Gaussian if its components \( x_I \) and \( x_Q \) form a Gaussian vector. Hence, its PDF is of the type (1.185). In particular we are interested in complex Gaussian rvs that have uncorrelated components \( (k_{x_1x_Q} = 0) \) with equal variance \( \sigma_{x_I}^2 = \sigma_{x_Q}^2 = \frac{1}{2} \sigma_x^2 \). Hence, the PDF can be written as

\[
p_x(a) = \frac{1}{\sqrt{2\pi \sigma_x^2}} \exp\left(-\frac{(a - m_x)^2}{2\sigma_x^2}\right), \quad a \in \mathbb{C}.
\]

(1.199)

A rv with the above PDF is called circularly symmetric since \( p_x \) exhibits a circular symmetry in the complex plane around the point \( m_x \). We indicate that a rv \( x \) is circularly symmetric complex Gaussian by writing \( x \sim \mathcal{CN}(m_x, \sigma_x^2) \).

In this case, if \( m_x = 0 \), then \( |x| \) is Rayleigh distributed with PDF

\[
p_{|x|}(\rho) = \frac{\rho}{\sigma_x^2} e^{-\rho^2/\sigma_x^2} 1(\rho).
\]

(1.200)

As an example we calculate the probability \( P[|x| > R] \) with \( R > 0 \), for \( x \sim \mathcal{CN}(0, \sigma_x^2) \). We get

\[
P[|x| > R] = \int_{\rho>R} p_{|x|}(\rho) \, d\rho = e^{-R^2/\sigma_x^2}.
\]

(1.201)

Analogously, we can define the second order description of a pair of complex-valued rvs \( x \) and \( y \) by giving their correlation

\[
r_{xy} = E[xy^*] = (r_{x_1y_1} + r_{x_Qy_Q}) + j(r_{x_Qy_1} - r_{x_1y_Q})
\]

(1.202)

and covariance

\[
k_{xy} = E[(x - m_x)(y - m_y)^*] = (k_{x_1y_1} + k_{x_Qy_Q}) + j(k_{x_Qy_1} - k_{x_1y_Q})
\]

which are both complex-valued, in general. Moreover,

\[
r_{xy} = k_{xy} + m_x m_y^*.
\]

(1.203)

In a similar way the description of an \( N \)-dimensional complex-valued rve \( x = [x_1, \ldots, x_N]^T \) can be derived from that of the corresponding \( 2N \)-dimensional real-valued rve

\[
x' = [x_{1,1}, x_{1,Q}, \ldots, x_{N,1}, x_{N,Q}]^T.
\]

### 1.7 RANDOM PROCESSES

Random signals, more properly called random processes (rps), are a key topic in the study of communication systems, where the amount of information is related to the degree of
unpredictability of the transmitted signals. We therefore devote this section to the definition and analysis of random processes. In this context the signals defined according to (1.1) or (1.11) will be called deterministic as the counterpart of random processes.

1.7.1 DEFINITION AND PROPERTIES

Definition 1.17 A continuous time real-valued rp is a function of two variables

\[ x : \mathbb{R} \times \Omega \mapsto \mathbb{R}, \tag{1.204} \]

where \( \Omega \) is the sample space of some probability space.

It should thus be denoted as \( x(t, \omega), t \in \mathbb{R}, \omega \in \Omega \) but it is customary to drop the dependence on \( \omega \) in the notation and simply write \( x(t), t \in \mathbb{R} \). Analogously we introduce discrete time random processes.

Definition 1.18 A discrete time real-valued rp is a function of two variables

\[ x : \mathbb{Z} \times \Omega \mapsto \mathbb{R}. \tag{1.205} \]

Again, we point out that it is customary to omit the \( \omega \) dependence and write \( x(nT) \) or \( x_n, n \in \mathbb{Z} \).

A rp can be interpreted as a collection of deterministic signals, each one identified by a different point \( \omega \) in the sample space \( \Omega \), as illustrated in Figure 1.35. Each of the deterministic signals composing \( x \) is called a realization of \( x \). From another point of view a rp can be interpreted as a collection of rvs, each one identified by a different instant \( t \in \mathbb{R} \), or index \( n \in \mathbb{Z} \).

![Figure 1.35](image)

Three realizations of a continuous time real-valued random signal.

The first-order statistical description of a rp is given by the first order PDF

\[ p_x(a; t) = p_{x(t)}(a), \quad t \in \mathbb{R}, \quad a \in \mathbb{R}. \]
**Definition 1.19** A continuous time complex-valued rp is a function

\[ x : \mathbb{R} \times \Omega \mapsto \mathbb{C}. \]  

(1.206)

As we did for a complex rv, we can view it as a pair of real-valued rps, its real and imaginary parts,

\[ x_I(t) = \Re x(t), \quad x_Q(t) = \Im x(t), \quad x(t) = x_I(t) + jx_Q(t), \]

and its first order statistical description is given by the function

\[ p_x(a; t) = p_{x_I}(t)(a_I, a_Q), \quad t \in \mathbb{R}, \quad a \in \mathbb{C}, \quad a_I, a_Q \in \mathbb{R} \]  

(1.207)

i.e. by the joint description of \( x_I(t) \) and \( x_Q(t) \). From now on, for the sake of generality, all the rps we deal with are assumed to be complex-valued unless otherwise stated.

**Definition 1.20** The mean of a rp \( x \) is a deterministic signal, that at each time instant \( t \) takes the value of the statistical mean of the rv \( x(t) \),

\[ m_x : \mathbb{R} \mapsto \mathbb{C}, \quad m_x(t) = \mathbb{E}[x(t)]. \]  

(1.208)

Analogously, we have the following definitions.

**Definition 1.21** The statistical power of a rp \( x \) is a deterministic signal, that at each time instant \( t \) takes the value of the statistical power of the rv \( x(t) \),

\[ M_x : \mathbb{R} \mapsto \mathbb{R}, \quad M_x(t) = \mathbb{E}[[x(t)]^2], \]  

(1.209)

and it turns out that \( M_x(t) \) is a real-valued nonnegative signal.

It is also interesting to consider some joint statistical descriptions of pairs of rvs taken from the same rp at different instants. Thus we make the following definitions.

**Definition 1.22** The autocorrelation function of a rp \( x \) is a function of two real variables, the reference time \( t \) and the delay time (lag) \( \tau \). It takes the value of the correlation between two rvs in the process, one taken at the instant \( t \), and the other at the instant \( t - \tau \):

\[ r_x(t, \tau) = \mathbb{E}[x(t)x^*(t - \tau)]. \]  

(1.210)

It is easy to see that the autocorrelation function, evaluated at \( \tau = 0 \), yields the statistical power as

\[ r_x(t, 0) = \mathbb{E}[|x(t)|^2] = \mathbb{E}[|x(t)|^2] = M_x(t). \]  

(1.211)

An important class is that of Gaussian rps.

**Definition 1.23** A real-valued rp is Gaussian if all the rves that can be built by taking an arbitrary number of variables in the rp are Gaussian vectors in the sense of (1.184).

In particular all the rvs of a Gaussian process are Gaussian, \( x(t) \sim \mathcal{N}(m_x(t), \sigma_x^2(t)) \), so that, for example, we can calculate, starting from the mean \( m_x(t) \) and power \( M_x(t) \), the probability

\[ P[b < x(t) < c] = Q \left( \frac{b - m_x(t)}{\sigma_x(t)} \right) - Q \left( \frac{c - m_x(t)}{\sigma_x(t)} \right), \]  

(1.212)

where \( \sigma_x(t) = \sqrt{M_x(t) - m_x^2(t)} \) and \( Q \) is defined in (1.157).
**Definition 1.24** A complex-valued rp $x(t)$ is said to be circularly symmetric Gaussian if $x(t)$ is a circularly symmetric Gaussian rv for any $t$, $x(t) \sim \mathcal{CN}(m_x(t), \sigma^2_x(t))$.

### 1.7.2 STATIONARY AND ERGODIC RANDOM PROCESSES

**Definition 1.25** A rp $x(t)$ is said to be stationary with respect to a statistical description (e.g. mean, autocorrelation or PDF) if such a description is invariant to any time shift of the signal, that is all rps $x_1(t) = x(t - t_0)$ obtained by an arbitrary time shift $t_0$ have the same description.

In particular, it can be easily seen that for a rp to be stationary in its mean, $m_x(t)$ is a constant since we find

$$m_x(t) = m_{x_1}(t) = E[x_1(t)] = E[x(t - t_0)] = m_x(t - t_0), \quad t, t_0 \in \mathbb{R}. \quad (1.213)$$

Analogously, for a rp stationary in statistical power or variance, these quantities are constant, so that we can drop the time dependence in the notation and write simply $m_x, M_x$ and $\sigma^2_x$ for the mean, power and variance, respectively.

In a similar manner, for a rp stationary in its autocorrelation, $r_x(t, \tau)$ is only a function of the lag $\tau$ and is independent of the reference time instant $t$, so that we can simply write $r_x(\tau)$.

**Definition 1.26** A rp that is stationary in both its mean and autocorrelation (and consequently in its power and variance also) is called wide sense stationary (WSS).

Hence, for a WSS rp $x$,

$$E[x(t)] = m_x, \quad E[x(t)x^*(t - \tau)] = r_x(\tau). \quad (1.214)$$

**Remark** It should be clear that stationarity does not imply that the signal realizations are constant over time, only that the signal statistical description does not depend on time. A constant rp is of course also stationary, but not the converse. A rough illustration of the realizations of a stationary process (in mean) is given in Figure 1.36.

A generalization of the stationarity property is *cyclostationarity*, defined as follows.

**Definition 1.27** A rp $x(t)$ is said to be cyclostationary with respect to a statistical description if such a description is invariant to a time shift of the signal by some fixed quantity $T_c$, which is called the cyclostationarity period of $x$.

In other words the definition states that the rp $x_1(t) = x(t - T_c)$ has the same description as $x(t)$. By mimicking (1.213) we can see that for a rp that is cyclostationary in its mean (or similarly in its power or variance) with period $T_c$, the mean must satisfy

$$m_x(t) = m_{x_1}(t) = E[x_1(t)] = E[x(t - T_c)] = m_x(t - T_c), \quad t \in \mathbb{R}, \quad (1.215)$$
Figure 1.36 Realizations of: (a) a stationary process; (b) a nonstationary process.

and thus it is a periodic signal with period $T_c$. Similarly for a rp that is cyclostationary in its correlation, the function $r_x(t, \tau)$ is periodic in $t$ with period $T_c$. Again, cyclostationarity of a rp does not imply that its realizations are periodic of $T_c$, only that its statistical description is periodic. A rp with periodic realizations is of course also cyclostationary, but not the converse. Figure 1.37 gives an intuitive illustration of this property.

Example 1.7 A Let $a(t)$ be a WSS rp, and $c(t)$ be a deterministic periodic signal with period $T_0$. Then their product $x(t) = a(t)c(t), t \in \mathbb{R}$, is a rp with mean

$$m_x(t) = E[a(t)c(t)] = c(t) E[a(t)] = m_a c(t)$$

and statistical power

$$M_x(t) = E[|a(t)c(t)|^2] = |c(t)|^2 E[|a(t)|^2] = M_a |c(t)|^2.$$ 

Since $c(t)$ and $|c(t)|^2$ are periodic with period $T_0$, so are both $m_x(t)$ and $M_x(t)$. As regards the autocorrelation

$$r_x(t, \tau) = E[a(t)c(t)a^*(t-\tau)c^*(t-\tau)] = c(t) c^*(t-\tau) E[a(t)a^*(t-\tau)] = \rho_a(\tau) c(t) c^*(t-\tau),$$

it is periodic in $t$ with period $T_0$. Thus $x(t)$ is cyclostationary in mean, power and autocorrelation with period $T_0$. 

---
Another property that is strongly linked to stationarity is \textit{ergodicity}.

\textbf{Definition 1.28} A stationary rp (in mean) is said to be ergodic in its mean if the time average converges to its statistical mean, in some sense to be specified:

$$\frac{1}{2u} \int_{-u}^{u} x(\omega,t) \, dt \xrightarrow{u \to \infty} m_{x}, \quad \omega \in \Omega. \quad (1.216)$$

Observe that the quantity on the left-hand side of (1.216), call it $\bar{x}_{u}$, is itself a rv, its value depending on the particular realization of $x(t)$, that is on the sample point $\omega \in \Omega$. On the contrary, thanks to the stationarity of $x(t)$, $m_{x}$ is a constant value. As defined in (1.216), ergodicity states that in some sense, as $u$ grows, $\bar{x}_{u}$ will converge to $m_{x}$.

Similarly we can state the property of ergodicity in correlation or statistical power for rps that are stationary in these descriptions.

We observe that ergodicity of a rp is a very welcome property since it relates a \textit{statistical description} of the rp to a measure on a \textit{single realization}. All the rps we will deal with in this book can be assumed ergodic unless otherwise stated. That is why we use the same notation for the ‘average power’ of a deterministic signal (see (1.30)) and the ‘statistical power’ of a rp. Indeed ergodicity is the property that justifies the procedure of gathering values from a realization as time elapses, to empirically infer some statistical properties of the associated rp.
1.7.3 SECOND ORDER DESCRIPTION OF A WSS PROCESS

We recall that for a WSS random process the autocorrelation function $r_x(\tau)$ is only a function of the lag $\tau$. Thanks to this property we can make the following definition.

**Definition 1.29** The power spectral density (PSD) of a WSS rp is the Fourier transform of its autocorrelation function:

\[
\mathcal{P}_x(f) = \begin{cases} 
\int_{-\infty}^{+\infty} r_x(\tau) e^{-j2\pi f \tau} \, d\tau, & \text{for continuous time signals} \\
T \sum_{k=-\infty}^{+\infty} r_x(kT) e^{-j2\pi f kT}, & \text{for discrete time signals.} 
\end{cases}
\]  

(1.217)

Observe that for discrete time rps, the Fourier transform defined in (1.17) is multiplied by the quantum $T$.

The PSD enjoys the following properties:

1. It is real-valued, $\mathcal{P}_x(f) \in \mathbb{R}$, even for complex-valued rps.
2. It is nonnegative, $\mathcal{P}_x(f) \geq 0$.
3. Its integral yields the rp statistical power

\[
\int_{-\infty}^{+\infty} \mathcal{P}_x(f) \, df = M_x, \quad \text{for continuous time rps}
\]

\[
\int_{0}^{1/T} \mathcal{P}_x(f) \, df = M_x, \quad \text{for discrete time rps with quantum } T.
\]  

(1.218)

4. For real rps it is even, $\mathcal{P}_x(f) = \mathcal{P}_x(-f)$.

The first and fourth properties are straightforward consequences of the symmetries in the correlation function and properties of the Fourier transform. The second property is not easy to prove and we provide some justifications for it later on (see (1.265)). The third property comes from relating $r_x(0)$ with $\mathcal{P}_x(f)$ and recalling (1.211). This property intuitively justifies the name ‘statistical power spectral density’. A deeper motivation for this name will be seen in Section 1.8.1.

Mean and correlation (or PSD) give the second order description of a WSS rp. It is of course an incomplete statistical description; nevertheless it is very useful. For Gaussian rps, for example, it allows one to derive any statistical description, as illustrated in the following example.

**Example 1.7 B** Let $x(t)$ be a continuous time real-valued stationary Gaussian rp with mean $m_x = 0$ and power spectral density $\mathcal{P}_x(f) = P_0 e^{-T_0 |f|}$, with $P_0 = 10^{-4} \text{ V}^2 / \text{Hz}$ and $T_0 = 100 \mu\text{s}$. We want to find the probabilities $P[x(t) > A]$ and $P[x(t) > x(t + T_0) + A]$ with $A = 1 \text{ V}$. From the PSD and (1.218) we calculate the statistical power

\[
M_x = 2P_0/T_0, 
\]  

(1.219)
which coincides with the variance since \( x(t) \) has zero mean. Therefore, in terms of the complementary Gaussian distribution function (1.157) we have

\[
P[x(t) > A] = Q\left( \frac{A - m_x}{\sigma_x} \right) = Q\left( A \sqrt{\frac{T_0}{2P_0}} \right) = Q(1/\sqrt{\sigma}) \approx 0.24. \tag{1.220}
\]

To calculate the second probability we consider the \( z(t) = x(t) - x(t + T_0) \). Being a linear combination of jointly Gaussian variables, \( z(t) \) is itself a Gaussian rp with mean

\[
E[x(t)] - E[x(t + T_0)] = m_x - m_x = 0,
\]

i.e. \( m_z = 0 \), and variance

\[
E[x^2(t)] + E[x^2(t + T_0)] - 2E[x(t)x(t + T_0)] = 2[M_x - r_x(-T_0)], \tag{1.222}
\]

i.e. \( \sigma_z^2 = M_z = 2[M_x - r_x(-T_0)] \). By taking the inverse Fourier transform of \( P_x(f) \) we obtain the autocorrelation function

\[
r_x(\tau) = \frac{2P_0 T_0}{T_0^2 + (2\pi \tau)^2} \Rightarrow r_x(-T_0) = \frac{2P_0}{T_0(1 + 4\pi^2)}. \tag{1.223}
\]

Hence we get

\[
\sigma_z^2 = \frac{2P_0}{T_0} \left( 1 - \frac{2}{1 + 4\pi^2} \right). \tag{1.224}
\]

and the probability we are looking for is

\[
P[x(t) > x(t + T_0) + A] = P[z(t) > A] = Q\left( \frac{A - m_z}{\sigma_z} \right) = Q\left( A \sqrt{\frac{T_0}{2P_0} \frac{4\pi^2 + 1}{4\pi^2 - 1}} \right),
\]

giving

\[
P[x(t) > x(t + T_0) + A] = \sqrt{\frac{1}{2} \frac{4\pi^2 + 1}{4\pi^2 - 1}} \approx 0.234.
\]

**Example 1.7 C** Consider a WSS discrete time rp \( x(nT) \), whose rvs are statistically independent and have the same statistical distribution. Such variables are called independent and identically distributed (iid), and the PSD of \( x \) takes a particular form. Indeed, as distinct rvs are uncorrelated, the autocorrelation function is

\[
r_x(mT) = \begin{cases} E[|x(nT)|^2] = M_x, & m = 0 \\ E[x(nT)]E[x^*(nT - mT)] = |m_x|^2, & m \neq 0, \end{cases} \tag{1.225}
\]

which can be written in a more compact form as

\[
r_x(mT) = |m_x|^2 + \sigma_x^2 \delta_m. \tag{1.226}
\]

By taking its Fourier transform times \( T \) we obtain

\[
P_x(f) = T \sigma_x^2 + |m_x|^2 \sum_{\ell=-\infty}^{+\infty} \delta(f - \ell/T), \tag{1.227}
\]
which is composed of a constant part $\sigma^2 T$ and a periodic sequence of Dirac impulses (also called spectral lines) spaced by $1/T$. Correlation and PSD for such a discrete time rp with iid variables are both illustrated in Figure 1.38.

![Graphical representation of (a) the autocorrelation function and (b) the PSD for a discrete time rp $x(nT)$ with iid rvs.](image)

Spectral lines in the PSD  In many applications it is important to detect the presence of sinusoidal components in a rp. With this intent we give the following theorem.

**Theorem 1.22 (Spectral lines in the PSD)** The PSD of a WSS rp $x$, $P_x$, can be uniquely decomposed into a component $P_x^{(c)}$ with no impulses and a discrete component consisting of impulses (spectral lines) $P_x^{(d)}$, so that

$$P_x(f) = P_x^{(c)}(f) + P_x^{(d)}(f), \quad (1.228)$$

where $P_x^{(c)}$ is an ordinary (piecewise continuous) function and

$$P_x^{(d)}(f) = \sum_{i \in I} M_i \delta(f - f_i), \quad (1.229)$$

where $I$ identifies a discrete set of frequencies $\{f_i\}, i \in I$.

The inverse Fourier transform of (1.228) yields the relation

$$r_x(\tau) = r_x^{(c)}(\tau) + r_x^{(d)}(\tau), \quad (1.230)$$

with

$$r_x^{(d)}(\tau) = \sum_{i \in I} M_i e^{j2\pi f_i \tau}. \quad (1.231)$$

The most interesting consideration is that the following rp decomposition corresponds to the decomposition (1.228) of the PSD:

$$x(t) = x^{(c)}(t) + x^{(d)}(t), \quad (1.232)$$

where $x^{(c)}$ and $x^{(d)}$ are orthogonal rps having PSD functions

$$P_{x^{(c)}}(f) = P_x^{(c)}(f) \quad \text{and} \quad P_{x^{(d)}}(f) = P_x^{(d)}(f). \quad (1.233)$$
Moreover, $x^{(d)}$ is given by
\[ x^{(d)}(t) = \sum_{i \in I} x_i e^{j2\pi f_i t}, \] (1.234)
where $\{x_i\}$ are orthogonal rvs with statistical powers $\{M_i\}$.

**Definition 1.30** A WSS rp $x$ is said to be asymptotically uncorrelated if the following two properties hold:

1. $\lim_{\tau \to \infty} r_x(\tau) = |m_x|^2$,
2. $k_x(\tau) = r_x(\tau) - |m_x|^2$ is absolutely integrable.

The property (1) denotes that $x(t)$ and $x(t - \tau)$ become uncorrelated for $\tau \to \infty$.

For such processes, one has
\[ r^{(c)}_x(\tau) = k_x(\tau) \quad \text{and} \quad r^{(d)}_x(\tau) = |m_x|^2. \] (1.236)

**Remark** From (1.236), one has $P^{(d)}_x(f) = |m_x|^2 \delta(f)$, and an asymptotically uncorrelated process exhibits at most one spectral line at the origin. In other words, an asymptotically uncorrelated process can have a spectral line only if $m_x \neq 0$.

By making use of the PSD, we extend the notion of band to rps as follows.

**Definition 1.31** The full band of a WSS rp $x$ is the support of its PSD,
\[ \mathcal{B}_x = \{ f \in \mathbb{R} : P_x(f) > 0 \}. \] (1.237)

For real-valued rps the autocorrelation function $r_x$ is itself real-valued, and its PSD has even symmetry. It is therefore customary, as we did in Section 1.4, to define its band as follows.

**Definition 1.32** The band of a real-valued WSS rp $x$ is the subset of nonnegative frequencies where its PSD is nonzero,
\[ \mathcal{B}_x = \{ f \geq 0 : P_x(f) > 0 \}. \] (1.238)

An important class of rps has constant PSD.

**Definition 1.33** A WSS rp $x$, whose power spectral density is constant over the whole frequency axis, $P_x(f) = P_0$, is called white noise, or simply white.

Observe that in order to have constant PSD, a white rp must have an impulsive autocorrelation,
\[ r_x(\tau) = P_0 \delta(\tau), \quad \text{for continuous time rps} \]
\[ r_x(mT) = P_0/T \delta_m, \quad \text{for discrete time rps}, \]
and in any case \( r_x \) must vanish outside the origin, that is any two distinct rvs belonging to the same white rp are orthogonal (see Definition 1.15). From (1.236), if \( x \) is asymptotically uncorrelated then a white process also has zero mean and all its rvs are uncorrelated. In particular, a zero-mean white Gaussian rp is called white Gaussian noise (WGN) and is made of independent rvs, whereas any zero-mean rp with iid rvs (not necessarily Gaussian) is white.

A peculiarity of continuous time white rps is that their statistical power turns out to be infinite, since
\[
\mathbb{M}_x = \int_{-\infty}^{\infty} \mathcal{P}_0 \, df = \infty,
\]
and one may wonder whether it makes sense to consider rps with infinite power in applications. However, we will see in the following chapter that continuous time white rps are a very convenient model for many random phenomena. On the contrary, this problem is not present for discrete time rps, where
\[
\mathbb{M}_x = \int_{0}^{1/T} \mathcal{P}_0 \, df = \mathcal{P}_0/T
\]
is finite.

### 1.7.4 Joint Second Order Description of Two Random Processes

For a pair of rps \( x(t) \) and \( y(t) \), we consider their cross correlation function
\[
\rho_{xy}(t, \tau) = \mathbb{E}[x(t)y^*(t - \tau)]. \tag{1.239}
\]
By extension of Definition 1.15 two rps are said to be orthogonal if \( \rho_{xy}(t, \tau) = 0 \), for all \( t, \tau \), and uncorrelated if \( \rho_{xy}(t, \tau) = \mathbb{m}_x \mathbb{m}_y^* \), for all \( t, \tau \).

**Definition 1.34** Two WSS rps \( x \) and \( y \) are said to be jointly WSS if their cross correlation function \( \rho_{xy}(t, \tau) \) does not depend on \( t \).

In the case of jointly stationary rps we can thus drop the dependence on \( t \) and simply write \( \rho_{xy}(\tau) \). Correspondingly, the Fourier transform of the cross correlation, \( \mathcal{P}_{xy}(f) \), is called the cross power spectral density of the two rps. Unlike the PSD of a single rp, the cross PSD of two rps may also be negative or complex-valued, similarly to the cross energy spectral density of two deterministic signals.

**Further insight** The PSD \( \mathcal{P}_x(f) \) and the cross PSD \( \mathcal{P}_{xy}(f) \) satisfy the following properties that we give without proof:
\[
\mathcal{P}_{xy}(f) = \mathcal{P}_{yx}^*(f), \tag{1.240}
\]
\[
\mathcal{P}_y(f) = \mathcal{P}_x(-f), \tag{1.241}
\]
\[
0 \leq |\mathcal{P}_{xy}(f)|^2 \leq \mathcal{P}_x(f)\mathcal{P}_y(f). \tag{1.242}
\]
From (1.242) we deduce that if two rps \( x \) and \( y \) have PSDs with disjoint bands, then they are orthogonal.

A case that we frequently encounter in practice is that of a zero-mean WSS (complex-valued) rp \( x \) that is orthogonal with its complex conjugate \( x^* \), that is
\[
\rho_{xx^*}(t, \tau) = \mathbb{E}[x(t)x(t - \tau)] = 0, \quad \text{for all } t, \tau. \tag{1.243}
\]
Proposition 1.23  Let \( x(t) = x_1(t) + j x_Q(t) \) be a zero-mean WSS rp, orthogonal to its complex conjugate \( x^* \). Then the auto- and cross correlations of its components \( x_1, x_Q \) are

\[
\begin{align*}
\rho_{x_1}(\tau) &= \rho_{x_Q}(\tau) = \frac{1}{2} \Re \{ \rho_x(\tau) \}, \\
\rho_{x_Qx_1}(\tau) &= -\rho_{x_1x_Q}(\tau) = \frac{1}{2} \Im \{ \rho_x(\tau) \}.
\end{align*}
\]  

(1.244)

(1.245)

In particular their cross power is zero,

\[
\rho_{x_1x_Q}(0) = 0.
\]  

(1.246)

Proof  To prove (1.244) and (1.245), we simply write

\[
\begin{align*}
x_1(t) &= \frac{x(t) + x^*(t)}{2}, \\
x_Q(t) &= \frac{x(t) - x^*(t)}{2j}
\end{align*}
\]

and apply the linearity of expectation and the hypothesis \( \rho_{xx^*}(t, \tau) = 0 = \rho_{x^*x}(t, \tau) \). Finally, from (1.245) it follows that

\[
\rho_{x_Qx_1}(0) = \frac{1}{2} \Im \{ \rho_x(0) \} = \frac{1}{2} \Im \{ M_x \} = 0.
\]

\( \square \)

Proposition 1.24  A zero-mean WSS complex-valued Gaussian rp \( x \) that is orthogonal to \( x^* \) is also circularly symmetric, with \( x(t) \sim \mathcal{CN}(0, \sigma_x^2) \).

Proof  Consider the two rvs \( x_1(t) \) and \( x_Q(t) \). They both have zero mean and from (1.246) their covariance is null, while from (1.244)

\[
\begin{align*}
M_{x_1} &= \rho_{x_1}(0) = \frac{1}{2} \Re \{ \rho_x(0) \} = \frac{1}{2} \Re \{ M_x \} = \frac{1}{2} M_x, \\
M_{x_Q} &= \rho_{x_Q}(0) = \frac{1}{2} \Re \{ \rho_x(0) \} = \frac{1}{2} \Re \{ M_x \} = \frac{1}{2} M_x.
\end{align*}
\]  

(1.247)

(1.248)

Hence, \( x_1(t) \) and \( x_Q(t) \) are Gaussian (with zero mean), uncorrelated, and hence statistically independent, rvs. Moreover, they have the same variance, so that \( x(t) \) is a circularly symmetric rv for all \( t \).

\( \square \)

1.7.5  SECOND ORDER DESCRIPTION OF A CYCLOSTATIONARY PROCESS

For cyclostationary rps, whose statistical description is periodic in \( t \), with period \( T_c \), we consider average values over a period for mean, statistical power, autocorrelation and PSD:

\[
\begin{align*}
\langle m_x \rangle &= \frac{1}{T_c} \int_0^{T_c} m_x(t) \, dt, \\
\langle M_x \rangle &= \frac{1}{T_c} \int_0^{T_c} M_x(t) \, dt, \\
\rho_x(\tau) &= \frac{1}{T_c} \int_0^{T_c} \rho_x(t, \tau) \, dt, \\
P_x(f) &= \frac{1}{T_c} \int_0^{T_c} P_x(t, f) \, dt,
\end{align*}
\]  

(1.249)

(1.250)

where \( P_x(t, f) \) is the Ftft of \( \rho_x(t, \tau) \) with respect to \( \tau \). It is seen that the Ftft of the average autocorrelation yields the average power spectral density, \( P_x(f) \), whose integral yields the average statistical power \( M_x \).
Example 1.7 D  Let \(a(t)\) be a continuous time WSS process and let \(x(t) = a(t) \cos(2\pi f_0 t + \varphi_0)\). By the results in Example 1.7 A, with \(c(t) = \cos(2\pi f_0 t + \varphi_0)\), we know that \(x(t)\) is cyclostationary with period \(T_c = 1/f_0\). It has time-varying mean
\[
m_x(t) = m_a \cos(2\pi f_0 t + \varphi_0)
\]
and autocorrelation
\[
r_x(t, \tau) = r_a(\tau) \cos(2\pi f_0 t + \varphi_0) \cos(2\pi f_0 (t - \tau) + \varphi_0).
\]
Then, by taking the average of the mean we get
\[
m_x = \frac{1}{T_c} \int_0^{T_c} m_x(t) \, dt = \frac{m_a}{T_c} \int_0^{T_c} \cos(2\pi f_0 t + \varphi_0) \, dt = 0,
\]
since the integral of the cosine function over a period is zero. As regards the autocorrelation, by means of the trigonometric formula (see Table 1.1)
\[
\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)],
\]
we can calculate
\[
r_x(\tau) = \frac{1}{T_0} \int_0^{T_0} r_x(t, \tau) \, dt
\]
\[
= r_a(\tau) \frac{1}{2T_0} \int_0^{T_0} \cos(2\pi f_0 (2t - \tau) + 2\varphi_0) + \cos(2\pi f_0 \tau) \, dt
\]
\[
= \frac{1}{2} r_a(\tau) \cos(2\pi f_0 \tau),
\]
where we have made use of the fact that \(\cos(4\pi f_0 t - 2\pi f_0 \tau - 2\varphi_0)\) is periodic in \(t\) with period \(T_c/2\), and its integral vanishes. By taking the Fft of the last result we get the average PSD
\[
\mathcal{P}_x(f) = \frac{1}{4} [\mathcal{P}_a(f - f_0) + \mathcal{P}_a(f + f_0)].
\]
(1.251)
The average power can be calculated either as \(M_x = r_x(0)\) or by integrating \(\mathcal{P}_x(f)\): in any case we get
\[
M_x = \frac{1}{2} M_a.
\]
(1.252)
This result extends (1.34) and is illustrated in Figure 1.39.

For a cyclostationary rp the definition of band is still given by (1.237), with reference to its average PSD. An apparently surprising result related to the band of a cyclostationary rp is the following, which we give without proof.

Proposition 1.25 If a cyclostationary rp with period \(T_c\) is band-limited with bandwidth \(B < 1/(2T_c)\), then it is also stationary.
**1.8 SYSTEMS WITH RANDOM INPUTS AND OUTPUTS**

Consider a system with map $\mathcal{M}$ and input the rp $x(t)$. Pick any $\omega_0$ in the sample space $\Omega$ on which $x(t, \omega)$ is defined, let the realization $x(t, \omega_0)$ be the input to $\mathcal{M}$, and call the corresponding observed output $y(t, \omega_0)$. By repeating this procedure for all $\omega \in \Omega$ we obtain a collection of deterministic outputs that are the realizations of the output random process $y(t, \omega)$.

The rp $y(t)$ is then defined on the same sample space as $x(t)$, since each realization of the input corresponds to a realization of the output. In this context we face the problem of deriving the statistical description of the output $y$ from that of the input $x$. We will shortly see that this problem has an easy solution in the case of linear systems. A noteworthy result, dealing with Gaussian rps, is the following, which somewhat resembles Proposition 1.20

**Proposition 1.26** The output of a linear system with input a Gaussian rp is itself a Gaussian rp.

**1.8.1 FILTERING OF A WSS RANDOM PROCESS**

In the case of a filter we have two important results.

**Theorem 1.27** In a filter with impulse response $g(t)$ and input a WSS rp $x(t)$, the output rp $y(t)$ is itself WSS and jointly stationary with the input, having:

- **mean**
  \[ m_y = m_x \int_{-\infty}^{+\infty} g(t) \, dt, \]

- **covariance function**
  \[ R_y(t, t') = \int_{-\infty}^{+\infty} R_x(u) g(t-u) g(t'+u) \, du, \]
cross correlation
\[ r_{yx}(\tau) = (g * r_x)(\tau), \quad (1.254) \]

autocorrelation
\[ r_y(\tau) = (g^* * r_{yx})(\tau) = [(g^* * g) * r_x](\tau), \quad (1.255) \]
where \( g^*_-(t) = g^*(-t) \).

**Proof** We first prove (1.253). From the input–output relationship of the filter we have
\[
E[y(t)] = E \left[ \int_{-\infty}^{+\infty} x(t-u)g(u) \, du \right].
\]
Now the key step is to exchange the order of integration in \( t \) and expectation (which is a linear operator, see Section 1.6.2) to obtain
\[
E[y(t)] = \int_{-\infty}^{+\infty} E[x(t-u)g(u)] \, du
\]
and observe that the signal \( g(t) \) is not random, so we can factor it out of the expectation
\[
E[y(t)] = \int_{-\infty}^{+\infty} E[x(t-u)]g(u) \, du.
\]
We now observe that \( E[x(t-u)] = m_x \) by the hypothesis on stationarity of the input \( r_p \) and obtain the result
\[
E[y(t)] = \int_{-\infty}^{+\infty} m_x g(u) \, du,
\]
which is independent of \( t \). To prove (1.254) we proceed analogously by writing
\[
E[y(t)x^*(t-\tau)] = E \left[ \int_{-\infty}^{+\infty} x(t-u)g(u) \, du \, x^*(t-\tau) \right]
\]
\[
= E \left[ \int_{-\infty}^{+\infty} x(t-u)g(u)x^*(t-\tau) \, du \right]
\]
\[
= \int_{-\infty}^{+\infty} E[x(t-u)g(u)x^*(t-\tau)] \, du
\]
\[
= \int_{-\infty}^{+\infty} g(u) E[x(t-u)x^*(t-\tau)] \, du
\]
\[
= \int_{-\infty}^{+\infty} g(u) r_x(\tau-u) \, du.
\]
The proof of (1.255) is similar, and we omit it here. □

In applications, it is important to consider the frequency domain equivalents of the above results.
Proposition 1.28  Given a filter with frequency response \( G(f) \), in terms of the input statistical description, the output second order statistical description is given by:

\[
\begin{align*}
\text{mean} & \quad m_y = m_x G(0), \\
\text{cross PSDs} & \quad \mathcal{P}_{yx}(f) = G(f) \mathcal{P}_x(f), \quad \mathcal{P}_{xy}(f) = G^*(f) \mathcal{P}_x(f), \\
\text{output PSD} & \quad \mathcal{P}_y(f) = G^*(f) \mathcal{P}_{yx}(f) = |G(f)|^2 \mathcal{P}_x(f).
\end{align*}
\]

Proof  Equation (1.256) is just a rewriting of (1.253), with \( G(0) = \int_{-\infty}^{+\infty} g(t) \, dt \). To obtain (1.257) and (1.258) we just take Fourier transforms of both sides in equations (1.254) and (1.255), respectively. The second of (1.257) follows from (1.240).

The result given by (1.258) is by far the most relevant: it states that the power spectral density of the output \( \mathcal{P}_y(t) \) is the product between the power spectral density of the input and the energy spectral density \( \mathcal{E}_y(f) = |G(f)|^2 \) of the filter impulse response.

Remark  Equations (1.257) and (1.258), relating the PSD of WSS \( \mathcal{P} \)s in a filter, are similar to (1.51) and (1.52), relating the energy spectral density of deterministic signals.

Remark  Observe that to evaluate the statistical power at the output of a cascade of filters, we can either calculate the autocorrelation functions or the PSD of the processes along the cascade. Most commonly, the frequency domain approach with calculation of the PSD is much easier, since it requires multiplications (see (1.258)) instead of convolution integrals (see (1.255)).

Further insight  An application of the above result (1.258) is given by the following theorem.

Theorem 1.29 (Irrelevance principle)  If a WSS \( \mathcal{P} \) is the input to a filter whose frequency response has unit gain over \( \overline{B}_x \)

\[
G(f) = \begin{cases} 
1, & f \in \overline{B}_x \\
\text{arbitrary}, & f \notin \overline{B}_x,
\end{cases}
\]

then the output \( \mathcal{P} \) coincides with the input in the mean square sense, that is

\[
E[|y(t) - x(t)|^2] = 0,
\]

and by extending Definition 1.16 we can write \( y(t) \overset{m.s.}{=} x(t) \). A filter satisfying (1.259) is thus called irrelevant for the \( \mathcal{P} \).
Proof Let $e(t) = y(t) - x(t)$. Since for any realization we can write $E(f) = Y(f) - X(f) = [G(f) - 1]X(f)$, we can think of $e(t)$ as output of a filter with input $x(t)$ and frequency response

$$G_e(f) = G(f) - 1 = \begin{cases} 0, & f \in \overline{B}_x \\ \text{unspecified}, & f \not\in \overline{B}_x. \end{cases}$$

Therefore $e$ is stationary and its power spectral density is

$$P_e(f) = |G_e(f)|^2 P_x(f) = 0,$$

vanishing at all frequencies, since $G_e(f)$ is zero on the support of $P_x$. Its power

$$M_e = E[|y(t) - x(t)|^2] = \int_{-\infty}^{+\infty} P_e(f) \, df$$

is thus zero. \hfill \Box

In the above theorem, $e(t)$ represents the difference (or the error signal) between $y(t)$ and $x(t)$. The theorem does not state that all the realizations of $e(t)$ are identically zero so that all the realizations of $x$ and $y$ coincide. It rather states that, at any instant $t$, $e(t)$ is an almost surely zero rv, so that the rvs $x(t)$ and $y(t)$ coincide with probability 1.

It is also noteworthy to consider the result of applying (1.258) when $G$ is an ideal filter with full band $\overline{B}$,

$$G(f) = \begin{cases} 1, & f \in \overline{B} \\ 0, & f \not\in \overline{B} \end{cases}$$

with an arbitrary input $rp \ x(t)$. Then each realization in $y(t)$ will contain only those frequency components of the corresponding realization of $x(t)$ which fall in $\overline{B}$. From (1.258) we also obtain the PSD of $y(t)$ as

$$P_y(f) = \begin{cases} P_x(f), & f \in \overline{B} \\ 0, & f \not\in \overline{B} \end{cases}$$

and its power as

$$M_y = \int_{-\infty}^{+\infty} P_y(f) \, df = \int_{\overline{B}} P_x(f) \, df,$$

so that (1.264) represents the statistical power associated with the components of $x$ whose frequency lies in the filter full band $\overline{B}$. In particular when $\overline{B} = [f_0 - \Delta F/2, f_0 + \Delta F/2]$ is a sufficiently narrow interval around its central frequency $f_0$, so that $P_x(f) \simeq P_x(f_0)$, for all $f \in \overline{B}$, equation (1.264) becomes

$$M_y = \Delta F \ P_x(f_0),$$

so that $P_x(f_0)$ represents the statistical power per unit frequency of the components of the $rp \ x$ at the frequency $f_0$. This justifies furthermore the name ‘power spectral density’ for $P_x$. Moreover, since the statistical power is a real nonnegative quantity, so must be the PSD $P_x(f_0)$ at any frequency $f_0$, as stated by property 2 on page 75.
1.8.2 FILTERING OF A CYCLOSTATIONARY RANDOM PROCESS

On the other hand, it is also reasonable to ask what happens when the filter input is a cyclostationary rp. In this case we have the following result.

**Theorem 1.30** In a filter with impulse response \( g(t) \) having as input a cyclostationary rp \( x(t) \) with period \( T_c \), the output rp \( y(t) \) is cyclostationary (also jointly with the input) with the same period, having:

average mean

\[
\mathbb{m}_y = \mathbb{m}_x \int_{-\infty}^{+\infty} g(t) \, dt, \tag{1.266}
\]

average cross correlation

\[
\mathbb{r}_{yx}(\tau) = (g \ast \mathbb{r}_x)(\tau), \tag{1.267}
\]

average autocorrelation

\[
\mathbb{r}_y(\tau) = (g^* \ast \mathbb{r}_{yx})(\tau) = [(g^* \ast g) \ast \mathbb{r}_x]\, (\tau). \tag{1.268}
\]

**Proof** We only give proof of (1.266). The other relationships can be derived similarly (see also Theorem 1.27).

The time-varying mean of \( y(t) \) can be calculated as

\[
\mathbb{m}_y(t) = \mathbb{E}[y(t)] = \int_{-\infty}^{+\infty} g(u) \mathbb{E}[x(t-u)] \, du = (g \ast \mathbb{m}_x)(t),
\]

and it is therefore periodic with period \( T_c \). Its average within a period is

\[
\mathbb{m}_y = \frac{1}{T_c} \int_0^{T_c} \mathbb{m}_y(t) \, dt = \int_{-\infty}^{+\infty} g(u) \frac{1}{T_c} \int_0^{T_c} \mathbb{E}[x(t-u)] \, dt \, du = \int_{-\infty}^{+\infty} g(u) \mathbb{m}_x \, du,
\]

where we have used the fact that the average of \( \mathbb{m}_x(t) \) over any period is \( \mathbb{m}_x \).

Again by writing these results in the frequency domain we obtain:

average mean

\[
\mathbb{m}_y = \mathbb{m}_x \mathcal{G}(0), \tag{1.269}
\]

average cross PSD

\[
\mathcal{P}_{yx}(f) = \mathcal{G}(f)\mathcal{P}_x(f), \quad \mathcal{P}_{xy}(f) = \mathcal{G}^*(f)\mathcal{P}_x(f), \tag{1.270}
\]

average output PSD

\[
\mathcal{P}_y(f) = \mathcal{G}^*(f)\mathcal{P}_{yx}(f) = |\mathcal{G}(f)|^2 \mathcal{P}_x(f). \tag{1.271}
\]

These relationships coincide with those for a stationary input given by (1.256)–(1.258).
1.8.3 REPRESENTATION OF PASSBAND WSS RANDOM PROCESSES

In Section 1.5 we showed that the signals associated with a real-valued passband signal $x(t)$ (analytic signal, envelope, instantaneous phase and frequency, baseband equivalent and Hilbert transform) can be obtained from suitable transformations having $x$ as input. It is then appropriate to study the second order description of the same transformations when a stationary real-valued passband rp $x(t)$ is applied at their input.

The following results are immediate applications of the results in Proposition 1.28 to the analytic filter $h(a)$ and the Hilbert filter $h(h)$.

**Proposition 1.31** The analytic signal associated to a WSS real-valued rp $x(t)$ is a WSS complex-valued rp $x^{(a)}(t)$ with

$$m_{x^{(a)}} = m_x, \quad P_{x^{(a)}}(f) = \begin{cases} 4P_x(f), & f > 0 \\ P_x(0), & f = 0 \\ 0, & f < 0. \end{cases} \quad (1.272)$$

**Proposition 1.32** The Hilbert transform of a WSS real-valued rp $x(t)$ is a WSS real-valued rp $x^{(h)}(t)$ with

$$m_{x^{(h)}} = 0, \quad P_{x^{(h)}}(f) = \begin{cases} P_x(f), & f \neq 0 \\ 0, & f = 0. \end{cases} \quad (1.273)$$

Concerning the rps envelope $e_x(t)$ and instantaneous phase $\varphi_x(t)$, since they are obtained from $x^{(a)}(t)$ through nonlinear time-invariant memoryless transformations, in general it is not easy to derive their second order description from that of $x^{(a)}$, apart from the fact that

$$M_{e_x} = E[e_{x}^2(t)] = E[|x^{(a)}(t)|^2] = M_{x^{(a)}}. \quad (1.274)$$

Moreover, by observing that the instantaneous frequency $f_x(t)$ can be obtained from $\varphi_x(t)$ through the filter with frequency response $H(f) = jf$, assuming $\varphi_x$ is WSS, also $f_x$ is WSS with PSD that is related to the PSD of $\varphi_x$ by

$$P_{f_x}(f) = |H(f)|^2 P_{\varphi_x}(f) = f^2 P_{\varphi_x}(f). \quad (1.275)$$

Another result about the Hilbert transform of a WSS rp is the following.

**Proposition 1.33** A WSS real-valued rp $x$ and its Hilbert transform $x^{(h)}$ have zero cross power

$$r_{x^{(h)} x}(0) = 0. \quad (1.276)$$

Moreover, if the PSD of $x$ does not have a spectral line at $f = 0$ the statistical powers of $x$ and $x^{(h)}$ coincide,

$$M_x = M_{x^{(h)}}. \quad (1.277)$$

**Proof** The proof is analogous to the proof of Proposition 1.14, with PSDs replacing energy spectral densities.
We now introduce two results that relate a real-valued rp with its complex-valued analytic signal and baseband equivalent.

**Theorem 1.34** The baseband equivalent of a zero-mean WSS real-valued rp \( x(t) \) is a WSS complex-valued rp \( x^{(bb)}(t) \) with PSD

\[
P_{x^{(bb)}}(f) = P_{x^{(a)}}(f + f_0).
\]  

(1.278)

**Proof** First of all *assume that* \( x \) *is stationary* in correlation. Then so is \( x^{(a)} \), by Proposition 1.31. By writing the correlation for \( x^{(bb)} \) we get

\[
r_{x^{(bb)}}(t, \tau) = E\left[ x^{(bb)}(t) x^{(bb)*}(t - \tau) \right]
\]

\[
= E\left[ x^{(a)}(t) e^{j2\pi f_0 t} x^{(a)*}(t - \tau) e^{-j2\pi f_0 (t - \tau)} \right]
\]

\[
= E\left[ x^{(a)}(t) x^{(a)*}(t - \tau) e^{j2\pi f_0 \tau} \right]
\]

\[
= r_{x^{(a)}}(\tau) e^{j2\pi f_0 \tau}.
\]

Then the Ftf of the above result yields (1.278).

By use of (1.272) we also have

\[
P_{x^{(bb)}}(f) = \begin{cases} 
4P_x(f + f_0), & f > -f_0 \\
0, & f < -f_0 \\
P_x(0), & f = -f_0 
\end{cases}
\]  

(1.279)

Furthermore we can identify the inverse relation: from the statistical description of \( x^{(a)} \) and \( x^{(bb)} \) determine that of \( x \). From (1.272) we obtain

\[
P_x(f) = \frac{1}{4} \left[ P_{x^{(a)}}(f) + P_{x^{(a)}}(-f) \right],
\]  

(1.280)

while (1.278) yields

\[
P_x(f) = \frac{1}{4} \left[ P_{x^{(bb)}}(f - f_0) + P_{x^{(bb)}}(-f - f_0) \right].
\]  

(1.281)

**Proposition 1.35** Let \( x \) be a zero-mean WSS real-valued rp. Then its analytic signal \( x^{(a)} \) is orthogonal to its conjugate \( x^{(a)*} \)

\[
r_{x^{(a)}x^{(a)*}}(\tau) = 0.
\]  

(1.282)

Moreover, also \( x^{(bb)} \) is orthogonal to its conjugate \( x^{(bb)*} \),

\[
r_{x^{(bb)}x^{(bb)*}}(\tau) = 0.
\]  

(1.283)
Proof We first prove (1.282). Observe from (1.130) that

\[ x(t) = x(t) + jx(t). \]

So, with respect to cross correlation we can write

\[
\begin{align*}
r_{x(a)x(a)\ast}(\tau) &= E[(x(t) + jx(h)(t)) (x(t - \tau) + jx(h)(t - \tau))] \\
&= r_x(\tau) - r_{x(h),x}(\tau) + jr_{x(h),x}(\tau),
\end{align*}
\]

which in the frequency domain gives the cross PSD

\[
\begin{align*}
P_{x(a)x(a)\ast}(f) &= [P_x(f) - P_{x(h)}(f)] + j[P_{x(h),x}(f) + P_{x(h),x}(f)],
\end{align*}
\]

where \( P_{x(h)} \) is given by (1.273), and by recalling (1.126) we also have

\[
\begin{align*}
P_{x(h),x}(f) &= -j \text{sgn}(f) P_x(f), \\
P_{x(h),x}(f) &= j \text{sgn}(f) P_x(f).
\end{align*}
\]

So, by substitution we obtain

\[
P_{x(a)x(a)\ast}(f) = P_x(f) - P_{x(h)}(f) = \begin{cases} P_x(0), & f = 0 \\ 0, & f \neq 0. \end{cases}
\]

Since \( x \) has zero mean, then from (1.236) \( P_x \) does not have spectral lines at \( f = 0 \) and the inverse Ftf of \( P_{x(a)x(a)\ast} \) yields (1.282). Using (1.114), (1.283) simply follows from (1.282).

Further insight As we have seen, a WSS real-valued rp \( x(t) \) assures that the analytic signal \( x(a)(t) \) is WSS and is orthogonal to its conjugate. The inverse relation also holds.

Theorem 1.36 Let \( x \) be a zero-mean real-valued rp. Then \( x \) is WSS if and only if its analytic signal \( x(a) \) is WSS and orthogonal to \( x(a)\ast \) or equivalently if its baseband equivalent \( x(bb) \) is WSS and orthogonal to its conjugate \( x(bb)\ast \).

Proof The direct relation was proved in Theorem 1.34 and Proposition 1.35. The inverse relation is based on deriving the correlation of

\[
x(t) = \frac{x(a)(t) + x(a)\ast(t)}{2},
\]

in terms of \( r_{x(a)}, r_{x(a)}\ast \) and \( r_{x(a)x(a)\ast} \). Based on the above assumptions it turns out that

\[
r_x(t, \tau) = \frac{1}{4} r_x(a)(t, \tau) + \frac{1}{4} r_x(a)\ast(t, \tau) = \frac{1}{4} r_x(a)(\tau) + \frac{1}{4} r_x(a)(-\tau)
\]

and hence \( x \) is WSS. An equivalent result is obtained for \( x(bb) \) by use of (1.114).

An interesting result of Theorem 1.36 is that if the cross correlation between \( x(bb) \) and \( x(bb)\ast \) is not zero, \( r_{x(bb)x(bb)\ast}(\tau) \neq 0 \), then \( x(t) \) turns out to be cyclostationary of period \( 1/(2f_0) \), even if \( x(bb) \) is WSS. However, if we consider the average PSD of \( x \), relation (1.280) still holds true.
Furthermore, let $x^{\text{bb}}_1$ and $x^{\text{bb}}_Q$ be the in-phase and quadrature components, respectively, of $x^{\text{bb}}$. From (1.244) and (1.245) it follows that

$$r_{x^{\text{bb}}_1}(\tau) = r_{x^{\text{bb}}_Q}(\tau) = \frac{1}{2} \Re\{r_{x^{\text{bb}}}(\tau)\},$$

and also that

$$M_{x^{\text{bb}}_1} = M_{x^{\text{bb}}_Q} = \frac{1}{2} M_{x^{\text{bb}}},$$

$$r_{x^{\text{bb}}_1 x^{\text{bb}}_Q}(0) = 0.$$

(1.285)

In the frequency domain (1.285) becomes

$$P_{x^{\text{bb}}_1}(f) = P_{x^{\text{bb}}_Q}(f) = \frac{1}{4} \left[ P_{x^{\text{bb}}}(f) + P_{x^{\text{bb}}}(f) \right],$$

$$P_{x^{\text{bb}}_1 x^{\text{bb}}_Q}(f) = -P_{x^{\text{bb}}_Q x^{\text{bb}}_1}(f) = \frac{1}{4} j \left[ P_{x^{\text{bb}}}(f) - P_{x^{\text{bb}}}(f) \right].$$

(1.287)

We now apply the above results to a couple of important examples.

**Example 1.8 A  Linear modulation.** Consider the real-valued passband signal

$$x(t) = \Re\{a(t) e^{j2\pi f_0 t}\},$$

where $a(t)$ is a zero-mean WSS complex-valued rp, band-limited with full band $(-B_a, B_a)$ and $B_a < f_0$. We want to determine whether $x$ is WSS.

We make use of Theorem 1.36. By a derivation similar to that in Example 1.5 E we find that $a(t)$ is the baseband equivalent of $x(t)$. Since $a(t)$ has zero mean and is WSS, hence $x$ will be WSS if and only if $r_{aa^*}(t, \tau) = E[a(t)a(t-\tau)] = 0$ for all $t$ and $\tau$.

In the case that $a(t)$ is real-valued (e.g. in DSB modulation, as will be seen in Chapter 3) we have $r_{aa^*}(t, \tau) = r_a(\tau)$ and $r_{aa^*}(t, 0) = M_x$. Hence the above orthogonality condition is not satisfied. Hence, $x(t)$ will be cyclostationary (as seen in Example 1.7 D) but not WSS. On the other hand, for an SSB modulation it is seen that indeed $a$ is orthogonal to $a^*$ and $x$ turns out to be WSS.

**Example 1.8 B  Baseband equivalent of passband noise.** Assume that $x$ is a real-valued WSS rp with zero mean and passband PSD given by

$$P_x(f) = \left\{ \begin{array}{cl} N_0/2, & f_1 < |f| < f_2 \\ 0, & \text{elsewhere.} \end{array} \right.$$ 

Then its baseband equivalent $x^{\text{bb}}$, with respect to a given reference frequency $f_0$, is a zero-mean WSS rp with PSD (see (1.279))

$$P_{x^{\text{bb}}}(f) = \left\{ \begin{array}{cl} 2N_0, & f_1 - f_0 < f < f_2 - f_0 \\ 0, & \text{elsewhere.} \end{array} \right.$$
and statistical power $M_{x^{(bb)}} = 2N_0(f_2 - f_1)$. Moreover (1.283) holds true. Furthermore, if $x$ is Gaussian, then $x^{(bb)}$ is complex Gaussian since it is obtained from $x$ through linear transformations. Hence, by Proposition 1.24, we get that $x^{(bb)}(t)$ is a circularly symmetric Gaussian rps, and by (1.286)

$$M_{x^{(bb)}}(bb) = \frac{1}{2} M_{x^{(bb)}} = N_0(f_2 - f_1).$$

If we choose the reference frequency as the midpoint $f_0 = (f_1 + f_2)/2$, $P_{x^{(bb)}}$ turns out to have Hermitian symmetry and hence $r_{x^{(bb)}}$ is real-valued. Then, by (1.285), the rps $x_1^{(bb)}$ and $x_Q^{(bb)}$ are orthogonal to each other (i.e. $r_{x_1^{(bb)}(bb)}(\tau) = 0$ for all $\tau$), and from (1.287) their PSD is given by

$$P_{x_1^{(bb)}}(f) = P_{x_Q^{(bb)}}(f) = \frac{1}{2} P_{x^{(bb)}}(f) = N_0 \text{ rect} \bigg( \frac{f}{f_2 - f_1} \bigg).$$

### 1.8.4 Sampling and Interpolation of Stationary Random Processes

After examining the effect of filters on stationary rps, we now move to the other two linear transformations that were introduced in Section 1.3, sampling and interpolation. Firstly, we consider a continuous time stationary rp $x(t)$, with mean $m_x$ and autocorrelation $r_x(\tau)$, sampled with period $T_s$, yielding a discrete time signal $\{y_n\}$. Then, by the identity $y_n = y(nT_s) = x(nT_s)$ we get straightforwardly the following result.

**Proposition 1.37** The discrete time rp $\{y_n\}$ obtained by sampling a WSS continuous time signal $x(t)$ is itself WSS. Its (constant) mean is the same as that of the input,

$$m_y = m_x,$$

and its autocorrelation function is obtained by sampling the autocorrelation function of $x$,

$$r_y(mT_s) = r_x(mT_s).$$

**Proof** By exploiting the input–output relationship for sampling (1.53) and the stationarity of the input we can write

$$E[y_n] = E[x(nT_s)] = m_x,$$

and

$$E[y_n y_n^*_{n-m}] = E[x(nT_s)x^*(nT_s - mT_s)] = r_x(mT_s).$$

By taking Fourier transforms of both sides of (1.289), times $T_s$, we get the following result.
Proposition 1.38 The (periodic) PSD of the discrete time output $r p y$ can be obtained from the PSD of the continuous time input $r p x$ by

$$\mathcal{P}_y(f) = \sum_{\ell=-\infty}^{+\infty} \mathcal{P}_x(f + \ell F_s).$$  \hspace{1cm} (1.292)

For an interpolator, the framework is complicated by the fact that even with a stationary discrete time input $r p$, the continuous time output $r p$ is cyclostationary in general.

Theorem 1.39 The continuous time $r p y(t)$ obtained by interpolating a discrete time WSS $r p \{x_n\}$ with quantum $T$ through a filter $g$ (see (1.55)) is cyclostationary with period $T$, and it has average mean

$$m_y = \frac{m_x}{T} \int_{-\infty}^{+\infty} g(t) \, dt,$$  \hspace{1cm} (1.293)

and average autocorrelation

$$r_y(\tau) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} (g^* * g)(\tau - mT) r_x(mT).$$  \hspace{1cm} (1.294)

Proof We only show the proof of (1.293). The proof of (1.294) is similar. From the input–output relation (1.55), the time-varying mean of $y(t)$ can be calculated as

$$m_y(t) = E[y(t)]$$

$$= E \left[ \sum_{n=-\infty}^{+\infty} g(t - nT) x_n \right]$$

$$= \sum_{n=-\infty}^{+\infty} g(t - nT) E[x_n]$$

$$= m_x \sum_{n=-\infty}^{+\infty} g(t - nT),$$

and the series in the last step yields a function that is periodic with period $T$. Averaging the above result within a period gives

$$m_y = \frac{1}{T} \int_0^T m_y(t) \, dt$$

$$= \frac{1}{T} m_x \sum_{n=-\infty}^{+\infty} \int_0^T g(t - nT) \, dt$$

$$= \frac{1}{T} m_x \sum_{n=-\infty}^{+\infty} \int_{nT}^{(n+1)T} g(u) \, du$$

$$= \frac{1}{T} m_x \int_{-\infty}^{+\infty} g(u) \, du,$$

where in the third step we have used the change of variable $u = t - nT$ in the integral. \hfill \Box
The corresponding frequency domain result reads as follows.

**Proposition 1.40** The cyclostationary output of an interpolate filter with frequency response $G(f)$ driven by a WSS input has average mean

$$m_y = m_x \frac{1}{T} G(0)$$

and average power spectral density

$$P_y(f) = \frac{1}{T} |G(f)|^2 P_x(f) = \frac{E_g(f)}{T^2} P_x(f).$$

**Proof** The derivation of (1.295) from (1.293) is straightforward. To derive (1.296) we observe that (1.294) is the time domain input–output relation of an interpolate filter having $x$ as input, $y$ as output and $(1/T) \ast g^*$ as impulse response. The corresponding frequency domain relationship is obtained by Theorem 1.5,

$$F[y] = \frac{1}{T} F[g \ast g^*] F[x].$$

Then, since $P_y$ is the Fourier transform of $y$, whereas $P_x$ is the Fourier transform of $x$ multiplied by $T$, we get

$$P_y(f) = \frac{1}{T} |G(f)|^2 \frac{1}{T} P_x(f),$$

and hence the result.

Observe that equations (1.293)–(1.296) are analogous to the corresponding relations (1.256)–(1.258) for ordinary filters, although in the interpolator case we deal with a cyclostationary output and therefore average its statistics within a period. Interesting applications of the above results will be seen in the context of both digital pulse amplitude modulation in Chapter 4 and band-limited interpolate filters in Chapter 6.

A peculiar case is when the interploate filter $g$ is lowpass with bandwidth $B_g \leq 1/(2T)$, driven by a WSS input. Then the average output PSD will turn out to be baseband with bandwidth $B_y \leq 1/(2T)$, and by Proposition 1.25 $y(t)$ will be stationary rather than cyclostationary. In particular, for an ideal lowpass interpolate filter

$$G(f) = T \text{rect}(Tf) = \begin{cases} T, & |f| \leq 1/(2T) \\ 0, & |f| > 1/(2T) \end{cases}$$

the output is WSS with statistical power

$$M_y = \int_{-\infty}^{+\infty} P_y(f) \, df = \int_{-\infty}^{+\infty} \frac{|G(f)|^2}{T^2} P_x(f) \, df = \int_{-1/(2T)}^{1/(2T)} P_x(f) \, df = M_x$$

coinciding with that of the input.
Appendix: The complementary normalized Gaussian distribution function

We give possible solutions to the problem of evaluating the complementary Gaussian distribution function $Q(y)$, which was defined in Example 1.6 A.

In Table 1.A.1 we give values of $Q(y)$ with four significant digits, for many values of $y$ that lie between 0 and 30.

For other values of $Q(y)$ one can resort to good approximations as follows:

- For negative values of $y$, use the complementary symmetry illustrated in Figure 1.A.1(a), $Q(y) = 1 - Q(-y)$.

- If $y$ is small, use linear interpolation, that is, pick the values $y_1, y_2$ that are the closest to $y$ among those given in Table 1.A.1, so that $y_1 < y < y_2$, and obtain the approximation

$$Q(y) \simeq \frac{y-y_1}{y_2-y_1} Q(y_2) + \frac{y_2-y}{y_2-y_1} Q(y_1). \quad (1.A.1)$$

Since $Q(y)$ is a convex function, the linear interpolation (1.A.1) is always an upper bound. Moreover, let $y_i$ be the closest value to $y$ between $y_1$ and $y_2$, then a lower bound is given by

$$Q(y) > Q(y_i) - |y - y_i| \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2}. \quad (1.A.2)$$

By letting $\Delta y = y_2 - y_1$ and $\Delta Q = Q(y_1) - Q(y_2)$, the absolute error in the approximation is thus bounded by

$$|y - y_i| \left( \frac{\Delta Q}{\Delta y} \right) \left( \frac{\Delta y}{\sqrt{2\pi}} \right) e^{-\frac{1}{2}y_i^2}. \quad (1.A.3)$$

For example, to evaluate $Q(3.77)$ we find $y_1 = 3.7$ and $y_2 = 3.8$, and the approximation (1.A.1) yields $Q(3.77) \simeq 0.7 Q(3.8) + 0.3 Q(3.7) \simeq 8.298 \cdot 10^{-5}$. The error is less than $1.88 \cdot 10^{-6}$ (about 2% relative error) so that the exact value lies between $8.11 \cdot 10^{-5}$ and $8.3 \cdot 10^{-5}$.

- If $y$ is large, use the approximation

$$Q(y) \simeq \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}y^2}. \quad (1.A.4)$$

The relative accuracy of the approximation is assured by the fact that $Q(y)$ obeys the bounds

$$\frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \left( 1 - \frac{1}{y^2} \right) < Q(y) < \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, \quad y > 0, \quad (1.A.5)$$

so that the approximation (1.A.4) is always an upper bound and the relative error is less than $1/y^2$. 
Table 1.A.1 Table of values for the complementary normalized Gaussian distribution function $Q(y)$.

<table>
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<th>$y$</th>
<th>$Q(y)$</th>
<th>$y$</th>
<th>$Q(y)$</th>
<th>$y$</th>
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<td>$2.275 \cdot 10^{-2}$</td>
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<td>4.2</td>
<td>$1.335 \cdot 10^{-5}$</td>
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<td>4.3</td>
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<td>$6.117 \cdot 10^{-39}$</td>
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<td>4.4</td>
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<td>4.6</td>
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<td>$9.964 \cdot 10^{-8}$</td>
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<td>$9.866 \cdot 10^{-10}$</td>
<td>30</td>
<td>$4.907 \cdot 10^{-198}$</td>
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</table>
Figure 1.A.1  Plots of the complementary normalized Gaussian distribution function $Q(y)$: (a) on a linear scale; (b) on a logarithmic scale.
For example, to evaluate $Q(\sqrt{200})$ we calculate $e^{-100}/\sqrt{400\pi} \approx 1.05 \cdot 10^{-45}$, with a relative error less than 0.5%. The absolute error is smaller than $5 \cdot 10^{-48}$, that is the exact value is between $1.045 \cdot 10^{-45}$ and $1.055 \cdot 10^{-45}$.

In Figure 1.A.1 we give two plots of the function $Q(y)$ on a linear and a logarithmic scale respectively.

REFERENCES AND FURTHER READING


PROBLEMS

*Time and frequency domain representation*

1.1 Consider the signal $x(t), t \in \mathbb{R}$, and its Fourier transform $\mathcal{X}(f)$. Find the Fourier transform of $y(t) = x(-2t + t_0)$ in terms of $\mathcal{X}(f)$.

1.2 Consider the signal $x(t), t \in \mathbb{R}$, and its Fourier transform $\mathcal{X}(f)$. Given $\mathcal{Y}(f) = -2\mathcal{X}^*(-f + f_0)$, find the relation between $y(t)$ and $x(t)$.

1.3 Find the Fourier transform of $x(t) = 3\cos(2\pi f_0 t) + \cos(2\pi f_1 t)$, with $f_0 = 0.2$ Hz and $f_1 = 100$ Hz. Compute $z(t) = (x * \text{sinc})(t)$.

1.4 Show that for a real-valued *nonnegative* signal $x(t)$, its Fourier transform $\mathcal{X}(f)$ satisfies the following inequalities:

(a) $\mathcal{X}(0) \geq 0$,

(b) $|\mathcal{X}(f)| \leq \mathcal{X}(0)$, for all $f \in \mathbb{R}$.

1.5 Find the Ftf of the following signals and sketch a plot of both the signal and the transform for each pair:

(a) $x(t) = \begin{cases} A_1, & |t| \leq t_1 \\ A_2, & t_1 < |t| \leq t_2 \\ 0, & |t| > t_2 \end{cases}$
(b) \[ y(t) = Ae^{-\pi(t^2+3)}, \]

(c) \[ z(t) = Ae^{-\pi(t+3)^2}. \]

(d) Find the signal whose Fourier transform is
\[ X(f) = e^{2Tf \text{rect}(Tf/5)}. \]

1.6 Give a reason why each of the following signal/Fourier transform pairs is undoubtedly wrong, without performing the calculation:

(a) \[ x(t) = \log \frac{t^2}{t^2 + T^2} \cos \frac{2\pi t}{T} \quad \xrightarrow{\mathcal{F}} \quad X(f) = \frac{2fT^2}{1 + f^4T^4}, \]

(b) \[ x(t) = \frac{t^2T}{t^4 + T^4} \quad \xrightarrow{\mathcal{F}} \quad X(f) = e^{-f^4T^4}, \]

(c) \[ x(t) = 2j \sin 2\pi \left( \frac{t}{T} + \frac{t^3}{T^3} \right) \quad \xrightarrow{\mathcal{F}} \quad X(f) = T \frac{1 + j}{\log(1 + f^2T^2)}. \]

1.7 Show that the inverse Fourier transform of
\[ X(f) = A \text{triang}(f/F_1) \text{rect}(f/F_2), \quad F_2 \leq 2F_1 \]
is the signal
\[ x(t) = \frac{AF_2}{2F_1} \left[ (2F_1 - F_2) \text{sinc}(F_2t) + \frac{F_2}{2} \text{sinc}^2 \left( \frac{F_2t}{2} \right) \right]. \]

1.8 Solve the following problems about the frequency domain representation of periodic signals.

(a) Find the Fourier coefficients of the following signals with period \( T_p \):
\[ x(t) = A \sin^2 \left( \frac{2\pi t}{T_p} \right), \]
\[ y(t) = \begin{cases} A, & kT_p < t < kT_p + T_p/2 \\ 0, & t = kT_p, t = kT_p + T_p/2 \\ -A, & kT_p + T_p/2 < t < kT_p + T_p \end{cases}, \quad k \in \mathbb{Z}, \]
\[ z(t) = 2 \sin(2\pi t/T_p) \cos(2\pi t/T_p) + 1. \]

Then give a graphical representation of their coefficient amplitude.

(b) Why are the following Fourier coefficients of the periodic signal \( x(t) = e^{i \sin 2\pi t/T_p} \) undoubtedly wrong?
\[ X_\ell = \begin{cases} j, & \ell = 0 \\ 1/\ell^2 + j/\ell, & \ell \neq 0. \end{cases} \]
1.9 Solve the following problems about the frequency domain representation of discrete time signals.
(a) Find the Fourier transforms of the following signals

\[ x(nT) = \begin{cases} A e^{-2n}, & -3 \leq n \leq 6 \\ 0, & \text{elsewhere} \end{cases}, \]

\[ y(nT) = 1/2^{|n|}, \quad n \in \mathbb{Z}. \]

(b) What discrete time signal has the Fourier transform

\[ \mathcal{X}(f) = 2A \left( \frac{f}{F} - \left\lfloor \frac{f}{F} \right\rfloor \right), \]

where \( \left\lfloor a \right\rfloor_Q \) indicates the closest integer to \( a \) (e.g. \( \left\lfloor 3.4 \right\rfloor_Q = 3, \left\lfloor 3.7 \right\rfloor_Q = 4 \)).

(c) Why is the following signal/transform pair incorrect?

\[ x(nT) = \frac{1}{n^2 + 1} \quad \frac{\rightarrow}{f} \quad \mathcal{X}(f) = T \arctan(fT). \]

Energy and power

1.10 Plot the signal

\[ x(t) = A e^{-t/T} 1(t + T) + A e^{-t/T} 1(t - T), \quad A = 1 \text{ V}, \ T = 1 \text{ ms} \]
and find its energy \( E_x \) and energy spectral density \( \mathcal{E}_x \).

1.11 Given the continuous time signals

\[ x(t) = A \text{sinc}^2(Ft)e^{j2\pi Ft}, \quad y(t) = A \text{sinc}(2Ft), \]
let \( z = x \ast y \).

(a) Find the expression of \( Z(f) \).

(b) Find the energy of \( z \).

(c) Find the cross energies \( E_{xy} \) and \( E_{yz} \).

1.12 Given the signal

\[ x(t) = (j2\pi t/T - 1) \text{rect}(t/T), \quad T = 2 \mu s. \]

(a) Find the energy of \( x \) and the cross energy between \( x \) and \( x^* \).

(b) Give the expressions of the energy spectral density \( \mathcal{E}_x \) and the cross energy spectral density \( \mathcal{E}_{x^*} \).

1.13 In Section 1.2 we have seen that a continuous time signal cannot have both finite energy and nonzero average power. Thus it must either have infinite energy as in Example 1.2 C, or zero average power as in Example 1.2 A, or both. Provide an example of a continuous time signal that has both infinite energy and zero average power.
1.14 (a) Find the average power of the following signals:

\[ x(t) = \cos \frac{2\pi t}{T_p} + |\cos \frac{2\pi t}{T_p}|, \quad y(t) = \sum_{\ell=0}^{+\infty} \frac{1}{(3+j)^\ell} e^{j2\pi F \ell t}. \]

(b) Find the average power of the signal with Fourier transform

\[ X(f) = f^2 \text{rect} \left( \frac{f}{2B} \right). \]

**Systems and transformations**

1.15 The filter \( h_1 \) relating the input voltage signal \( v_i(t) \) to the output voltage \( v_o(t) \) is shown in the figure, where \( R = 100 \Omega \) and \( C = 100 \text{nF} \).

\[ \begin{array}{c}
\text{R} \\
v_i(t) \searrow \swarrow C \\
v_o(t)
\end{array} \]

Determine:

(a) the frequency response of the filter \( H_1(f) \);

(b) the output signal \( v_o(t) \) when the input signal is \( v_i(t) = V_0 \), with \( V_0 = 2 \text{ V} \);

(c) the output signal \( v_o(t) \) when the input signal is

\[ v_i(t) = 10 \cos(2\pi 10000t + \pi/8) + 5 \sin(2\pi 5000t + \pi/5). \]

1.16 The filter \( H(f) \) is shown in the figure. The input signal is the current \( i_i(t) \) and the output signal is the current \( i_o(t) \), with \( R = 50 \Omega \) and \( L = 100 \mu\text{H} \).

\[ \begin{array}{c}
i_i(t) \searrow \swarrow R \\
i_o(t) \searrow \swarrow L
\end{array} \]

Determine:

(a) the frequency response of the filter, \( H(f) \);

(b) the output signal \( i_o(t) \) when the input signal is

\[ i_i(t) = 20 \sin(2\pi 1000t + \pi/8) + 13 \sin(2\pi 2000t + \pi/4). \]
1.17 Find the power of the signal \( y(t) \) obtained by filtering \( s(t) = 3 \cos(2\pi f_0 t) + 5 \cos(2\pi f_1 t) \) through the filter with frequency response \( \mathcal{H}(f) \) represented in the figure. Assume \( f_0 = 6 \cdot 10^4 \text{Hz}, \) \( f_1 = 10^6 \text{Hz}, \) \( C = 6, \) \( B = 10^5 \text{Hz}. \)

1.18 Below are given the input-output relations of four continuous time transformations with \( x \) the input signal and \( y \) the output signal. For any transformation decide whether or not it is causal, memoryless, time invariant, or linear.

(a) \( y(t) = t^2 x(t) \)
(b) \( y(t) = x(t^3) \)
(c) \( y(t) = \log|x(t)| \)
(d) \( y(t) = Ax(t - 3T) + B \sin 2\pi t/T. \)

1.19 Prove the following statements, where \( x \) and \( y \) are continuous time signals and \( z = x * y \) is their convolution:

(a) if \( x \) is periodic with period \( T_p, \) then so is \( z; \)
(b) if \( x \) has bounded amplitude (that is there exists an \( A \) such that \( |x(t)| \leq A, t \in \mathbb{R} \)) and \( \int_{-\infty}^{+\infty} |y(t)| \, dt \) is finite, then \( z \) has bounded amplitude;
(c) it is not true that if both \( x \) and \( y \) have bounded amplitudes, so does \( z. \) Give a counterexample.

1.20 Given the signals

\[ x(t) = \frac{A}{t^2 + T_1^2}, \quad y(t) = B \text{sinc} \left( \frac{t}{T_1} \right), \]

find their convolution \( z = x * y \) and their cross energy \( E_{xy}. \)

*Hint: Calculate the Ftfs \( X(f) \) and \( Y(f) \) and operate in the frequency domain.*

1.21 The two-port network (filter) shown below is made of an impedance \( Z(f) \) and an admittance \( Y(f). \)

If we take the voltage \( v_i(t) \) as the input signal and the voltage \( v_o(t) \) as the output signal, the frequency response of the filter is

\[ G(f) = \frac{1}{1 + Y(f)Z(f)}. \]
(a) Find the filter impulse response when \( Z(f) \) is a resistance \( R = 100 \, \Omega \) and a capacitance \( C = 1 \, \text{nF} \) in series, that is \( Z(f) = R + 1/(j2\pi f C) \), while the admittance is a capacitance \( C = 1 \, \text{nF} \), that is \( Y(f) = j2\pi f C \).

(b) With \( Z(f) \) as above and \( Y(f) \) a resistance \( R = 100 \, \Omega \) and a capacitance \( C = 1 \, \text{nF} \) in parallel, that is \( Y(f) = 1/R + j2\pi f C \), find the filter output to the input

\[
v_i(t) = V_0 (1 + \cos 2\pi f_0 t + \sin 2\pi f_1 t)
\]

with \( 2\pi f_0 = 10 \, \text{MHz} \) and \( 2\pi f_1 = 1 \, \text{GHz} \).

1.22 Consider the cascade of systems

\[
\begin{array}{ccc}
x(t) & \rightarrow & g_1 \\
& \downarrow & \\
x_1(t) & \rightarrow & g_2 \\
& \downarrow & \text{sampler} \\
x_2(t) & \rightarrow & y_n
\end{array}
\]

where the continuous time filters have frequency responses

\[
G_1(f) = \text{rect}(2T_s f), \quad G_2(f) = \cos(2\pi f T_s).
\]

(a) Plot the Fourier transform of the output signal \( \{y_n\} \) when the input signal is a Dirac impulse, \( x(t) = \delta(t) \).

(b) Write the expression of the output signal \( \{y_n\} \) corresponding to the input signal

\[
x(t) = \cos(2\pi f_0 t) + \cos(4\pi f_0 t), \quad \text{with} \quad f_0 = \frac{1}{6T_s}.
\]

1.23 Consider an interpolate filter with quantum \( T \) and impulse response

\[
g(t) = \cos(2\pi f_0 t) \, \text{rect}(t/\tau).
\]

(a) Find the values of \( \tau \) so that \( g \) yields correct interpolation according to (1.57) for any value of \( f_0 \).

(b) Find the values of \( f_0 \) so that \( g \) yields correct interpolation for \( \tau = 3T \).

1.24 Find for what values of the parameters \( A \) and \( \tau \) the interpolate filter with impulse response

\[
g(t) = (1 - A) \, \text{sinc}(t/T) + A \, \text{sinc}^2(t/\tau), \quad t \in \mathbb{R}
\]

yields correct interpolation according to (1.58) for any discrete time input signal with quantum \( T \). For which of the values found above does the filter have minimum bandwidth?

**Bandwidth**

1.25 The real-valued signal \( x(t), t \in \mathbb{R} \), has bandwidth \( B \). Determine the bandwidth of the following signals:

(a) \( y_1(t) = x(t) \cos(2\pi f_0 t) \),

(b) \( y_2(t) = (x(t) + A) \cos(2\pi f_0 t) \),

(c) \( y_3(t) = [x(t) \cos(2\pi f_1 t)] * [\text{sinc}(t/T) e^{j2\pi t/T} \cos(2\pi f_1 t)] \).

Consider \( B = 1/T = 100 \, \text{Hz} \), \( f_0 = 20 \, \text{Hz} \), and \( f_1 = 30 \, \text{Hz} \).
1.26 Consider the real filter with frequency response (for positive frequencies) \[ H^+(f) = \begin{cases} G_0, & 0 \leq f \leq f_1 \\ G_0/2, & f_1 < f \leq f_2 \\ 0, & f > f_2 \end{cases} \]

with \( G_0 > 0 \). Determine if and under what conditions the filter satisfies the Heaviside condition for the input signal \( x(t) = A \cos(2\pi f_3 t) + B \cos(4\pi f_4 t) \).

1.27 Consider the system with input \( v_i(t) \) and output \( v_o(t) \) implemented as in the figure, with \( L = 0.75 \text{ mH} \) and \( R = 100 \Omega \). Let \( h(t) \) be the impulse response relating \( v_o(t) \) to \( v_i(t) \).

(a) Write and plot the expressions of the amplitude and phase of \( H(f) \).
(b) Discuss if the input signal \( v_i(t) = 6 \sin(2\pi f_0 t + \pi/3) \) with \( f_0 = 40 \text{ kHz} \), is distorted (according to the Heaviside conditions) by \( h(t) \). Provide the expression of \( v_o(t) \) in terms of \( v_i(t) \).
(c) Determine whether the signal \( v_i(t) = 6 \sin(2\pi f_0 t + \pi/3) + 42 \cos(2\pi f_1 t + \pi/4) \) with \( f_0 = 40 \text{ kHz} \), \( f_1 = 50 \text{ kHz} \), is distorted (according to the Heaviside conditions) by \( h(t) \).
(d) Determine the bandwidth of the input signal such that the filter \( h(t) \) satisfies the Heaviside conditions.

1.28 Find the full band and the energy of the signal \( x(t) = \sin^2(t/T) e^{j2\pi t/T} \).

1.29 Find the band and bandwidth of the following signals:
(a) \( x(t) = A \sin(t) \ e^{j\pi F_1 t/2} \)
(b) \( y(t) = A \sin^k(t/T_1) \)
(c) \( s(nT) = A e^{-|n|} \)
(d) \( z(t) = A \sin^3(2\pi t/T_p) \).

1.30 Find the band and bandwidth of the following signals:
(a) \( x(t) = \frac{e^{-j2\pi t/T}}{1 - j2\pi t/T} \)
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(b) \[ y(t) = (x * x^*)(t) \]

(c) \[ z(t) = \begin{cases} 0, & |t| \geq 1 \\ t^2 \log \frac{1}{|t|}, & |t| < 1 \end{cases} \]

1.31 Consider the continuous time real-valued signal \( x(t) = e^{(t_0 - t)/T} 1(t - t_0), t \in \mathbb{R} \).

(a) Find the expression of its practical bandwidth \( B \) based on the amplitude criterion. Calculate the value of \( B \) with \( \varepsilon = -40 \text{ dB}, t_0 = 30 \text{ ms} \) and \( T = 1 \text{ ms} \).

(b) Find the expression of its practical bandwidth \( B \) based on the energy criterion. Calculate the value of \( B \) with \( \varepsilon = 1\% \), \( t_0 = 30 \text{ ms} \) and \( T = 1 \text{ ms} \). Discuss the dependence of the results on the signal parameters \( t_0 \) and \( T \).

1.32 Consider the continuous time real signal \( x(t) = \frac{A}{(T^2 + t^2)}, t \in \mathbb{R} \).

(a) Find the expression of its practical bandwidth \( B \) based on the amplitude criterion. Calculate the value of \( B \) with \( \varepsilon = -20 \text{ dB}, A = 1 \text{ V}, T = 1 \text{ ms} \).

(b) Find the expression of its practical bandwidth \( B \) based on the energy criterion. Calculate the value of \( B \) with \( \varepsilon = 1\% \), \( A = 1 \text{ V}, T = 1 \text{ ms} \).

(c) Repeat 1.32(a) for the signal \( y(t) = x(t) \cos 2\pi f_0 t \), with \( f_0 T > 1 \), and calculate \( B \) for the same parameters as 1.32(a) and \( f_0 = 10 \text{ kHz} \). Discuss the dependence of the results on the signal parameters \( T \) and \( f_0 \).

1.33 A filter has impulse response \[ h(t) = F_2 \text{sinc} [F_1(t - t_1)] \text{sinc} [F_2(t - t_2)], t \in \mathbb{R}. \]

(a) For \( F_2 = 2F_1 = 10 \text{ kHz} \) and \( t_2 = t_1 = 2 \text{ ms} \), find in what band the filter satisfies the Heaviside conditions, and give the values of amplitude and envelope delay.

(b) Prove that for \( 0 < F_1 < F_2 \) and any \( t_2, t_1 \), the filter satisfies the Heaviside conditions for baseband signals with bandwidth \( B \leq (F_2 - F_1)/2 \), and give the values of amplitude and envelope delay.

1.34 A continuous time LTI system \( G \) has the following input–output relation \[ y(t) = 2x(t) + \tau \dot{x}(t), \quad \tau = 1 \mu s, \]

where \[ \dot{x}(t) = \frac{d}{dt} x(t). \]

Find the impulse response \( h \) of a causal filter such that the cascade of the two systems meets the Heaviside conditions with amplitude \( A = 1/2 \) and delay \( t_0 = 2\tau \).

1.35 Design a filter having impulse response \( h \) with the following specifications

(i) It should meet the Heaviside conditions in the full band \( \overline{B} = (-B_1, B_1) \), with \( B_1 = 1 \text{ MHz} \).

(ii) Its bandwidth should be as small as possible.
(iii) The energy of the noncausal part of $h$ should not be more than $\varepsilon = 1/100$ of its total, i.e.

$$\int_{-\infty}^{0} |h(t)|^2 \, dt \leq \varepsilon \int_{-\infty}^{+\infty} |h(t)|^2 \, dt.$$ 

Provide impulse and frequency responses.

Observe that (iii) means that the causal part $h_c(t) = h(t)1(t)$ yields a very good (and physically feasible) approximation of the desired $h$, since the energy of the difference is $E_{h-h_c} \leq \varepsilon E_h$. What is the delay, in order to obtain such an accurate approximation?

1.36 Consider the continuous time signal

$$x(t) = \text{sinc} \left( \frac{t - \tau}{2\tau} \right) + \text{sinc} \left( \frac{t + \tau}{2\tau} \right), \quad \text{with } \tau = 10 \, \text{ms}.$$ 

(a) Find a sampler/interpolator scheme as in Figure 1.21, that allows perfect reconstruction of $x$ with minimum sampling frequency. Give the sampling frequency and the interpolate filter impulse response.

(b) Find the energy of the reconstruction error in a scheme as in Figure 1.23 with $D(f) = G(f) = \text{rect}(f/F_s)$ and $F_s = 1/(3\tau)$.

1.37 The signal $x_i(t) = \text{sinc}^2(F_1 t)$ is fed into an anti-aliasing filter $D(f) = \text{rect}(f/F_s)$, then sampled with rate $F_s$ and reconstructed with an ideal interpolator $G(f) = \text{rect}(f/F_s)$, as in Figure 1.23. Find the minimum sampling frequency $F_s$ which yields a reconstruction error with energy $E_e \leq E_{x_i}/100$.

1.38 Given the signals

$$x(t) = \frac{A}{1 + (2\pi B t)^2}, \quad y(t) = \text{sinc}^2(2B t),$$ 

let $z = x \ast y$.

(a) Find the minimum sampling frequency $F_s$ that allows perfect reconstruction of $z$ through the scheme of Figure 1.21.

(b) Determine the minimum sampling frequency if the interpolate filter has frequency response

$$G(f) = \begin{cases} T_s, & |f| \leq F_s/4 \\ T_s(3/2 - 2|f|/T_s), & F_s/4 < |f| < 3F_s/4 \\ 0, & |f| \geq 3F_s/4. \end{cases}$$

1.39 Consider the transformation of Figure 1.23 where $T_s = 0.1 \, \text{ms}$ and both the filter and the interpolate filter have frequency response

$$\mathcal{H}(f) = \mathcal{G}(f) = \frac{1}{2} \, \text{rcos}(f/F_1, \rho), \quad \text{with } \rho = \frac{1}{4}, \quad F_1 = 9 \, \text{kHz}.$$ 

(a) Prove that the overall transformation satisfies the Heaviside conditions $\tilde{x}(t) = A x_i(t - t_0)$ for all band-limited signals $x_i$ with bandwidth $B_{x_i} \leq (1 - \rho)F_1/2 = 3375 \, \text{Hz}$ and give the appropriate values for $A$ and $t_0$. 
(b) Prove that the overall transformation is not in general equivalent to a filter for signals with bandwidth greater than $F_s - B_g = 4375$ Hz.

(c) Suppose the filter $h$ and the sampler are assigned as above. The task is to design the interpolate filter so that the Heaviside conditions are met for all signals $x_i(t)$ with bandwidth up to $F_1/2$. Determine the interpolate filter frequency response with both the minimum bandwidth and the maximum bandwidth.

1.40 An interpolate filter $g$ has frequency response $G(f) = e^{\pi(Tf)^2} \text{rect}(Tf)$. Find the frequency and impulse response of a filter $h$ that allows perfect recovery of the discrete time input samples $\{x_n\}$ in the scheme of Figure 1.24.

**Representation of passband signals**

1.41 Evaluate the Hilbert transform of the signals:
- (a) $x(t) = \sin(2\pi f_0 t + \pi/4)$;
- (b) $x(t) = \sin(2\pi B t + \pi/8) \cos(2\pi f_0 t)$ with $f_0 > B$;
- (c) $x(t) = \text{sinc}(2Bt)$.

1.42 Consider the signal

$$s(t) = 4 \cos(2\pi f_0 t + 0.2 \cos 2\pi f_0 t).$$

(a) What is the instantaneous phase deviation of $s(t)$?

(b) What is the average power of $s(t)$?

1.43 Consider the signal

$$s(t) = 4 \cos(2\pi f_0 t + 0.2 \cos 2\pi f_1 t + 0.3 \cos 2\pi f_2 t + 0.3 \cos 2\pi f_3 t),$$

with $f_1 = 400$ Hz, $f_2 = 600$ Hz, $f_3 = 800$ Hz.

(a) What is the instantaneous phase deviation of $s(t)$?

(b) What is the average power of the instantaneous phase deviation of $s(t)$?

(c) What is the envelope of $s(t)$, if $f_0 = 100$ kHz?

1.44 Find the instantaneous frequency deviation of the real-valued signal

$$x(t) = A \cos [2\pi (f_0 + \lambda(t)) t]$$

with reference frequency $f_0$. Assume that the signal $y(t) = A e^{j2\pi t \lambda(t)}$ has full band limited within the interval $(-f_0, f_0)$.

1.45 Evaluate the baseband equivalent of the signals:
- (a) $x(t) = \sin(2\pi f_0 t + \pi/4) + 2 \cos(2\pi f_0 t + \pi/8)$;
- (b) $x(t) = \text{sinc}^2(Bt) \cos(2\pi f_0 t + \pi/8)$ with $f_0 > B$;
- (c) $x(t) = \text{rect}(Bt) \sin(2\pi f_0 t + \pi/4)$ with $f_0 \gg B$.

1.46 (a) Find the baseband equivalent of the signal

$$x(t) = \frac{A \cos(2\pi f_0 t)}{1 + (2\pi B t)^2}.$$
(b) As the signal
\[ a(t) = \frac{A}{1 + (2\pi Bt)^2} \]
is practically band-limited (for example assuming a bandwidth based on the energy criterion), we could say that for \( f_0 > B \), \( a(t) \) is an approximate baseband equivalent of \( x(t) \). Evaluate the energy of the error \( e(t) \) we would make by taking \( a(t) \) instead of \( x^{(bb)}(t) \), that is
\[ e(t) = a(t) - x^{(bb)}(t), \]
and express the ratio \( E_e/E_a \) versus the ratio \( f_0/B \).

1.47 Given the signal
\[ x(t) = \Re \left[ \frac{e^{j2\pi f_0 t}}{1 - j2\pi t/T_1} \right], \]
plot its Fourier transform and find the expression of the following signals in the time domain:
(a) the Hilbert transform of \( x \);
(b) the envelope and instantaneous frequency of \( x \);
(c) the baseband equivalent of \( x \) (specify the reference frequency).

**Random variables and vectors**

1.48 Let \( y = x + w \) with \( w \) a Gaussian rv with zero mean and variance \( \sigma_w^2 \). Determine the probability \( P[y > 0 \mid x = -1] \).

1.49 Let \( x_1 \) and \( x_2 \) be two independent Gaussian rvs with zero mean and variance \( \sigma_1^2 \) and \( \sigma_2^2 \), respectively. Determine the probability that the random vector \( [x_1, x_2] \) is within the upper right sub-plane having lower left corner \((1,1)\). Compute the probability that \([x_1, x_2] \) is outside of the above sub-plane.

1.50 (a) Determine the parameters \( K, M \in \mathbb{R} \), such that each of the following functions is a possible PMD of a discrete rv with the given alphabet:
\[ p_1(a) = K^{a+M}, \quad a \in \{0, 1, 2, \ldots\} \]
\[ p_2(a) = K^a + M, \quad a \in \{0, 1, 2, \ldots\} \]
\[ p_3(a) = K^a + M, \quad a \in \{0, 1, 2\} . \]

(b) Determine the parameters \( K, M \in \mathbb{R} \), such that each of the following functions is a possible PDF of a continuous rv:
\[ p_1(a) = \frac{K}{(a + M)^2}1(a) \]
\[ p_2(a) = K \text{rect}(a) + M\delta(a) \]
\[ p_3(a) = e^{Ka^2} + M. \]

1.51 Consider a Gaussian rv \( x \sim \mathcal{N}(1, 1/2) \), and the function
\[ g(a) = \frac{d}{da} \ln[p_x(a)], \]
where $p_x$ is the PDF of $x$.

(a) Find $E[g(x)]$.

(b) Show that the same result holds for any continuous rv with PDF $p_x(a)$ that is continuous, differentiable and strictly positive.

1.52 Let $x$ be a Poisson rv, with alphabet $\mathcal{A}_x = \{0, 1, 2, \ldots \}$ and PMD $p_x(a) = e^{-\Lambda^a}/(a!)$.

(a) Find the value of $\Lambda$ for which $P[x = 2]$ is maximized.

(b) Show that the probability $P[x > 2]$ as a function of $\Lambda$ is continuous and monotonically increasing.

(c) Calculate $E[e^x]$.

1.53 Let $x$ be a Gaussian rv $x \sim N(0, \sigma^2)$.

(a) Find the PDF of $y = x^2 + 1$.

(b) Calculate $E[x^2(x + 1 + \sin x)]$.

1.54 In GSM cellular phone networks, the cells covering wide open environments are approximately circular with a maximum radius of 35 km. Assume that the propagation speed of the waves in air is $c = 3 \cdot 10^8$ m/s. Consider a terminal randomly placed within such a cell, so that with $x$ and $y$ the spatial coordinates of the terminal with respect to the base station, the PDF of the rve $\mathbf{x} = [x, y]$ is constant within a circle centered at the origin and having a radius of 35 km, and zero outside.

(a) Write the expression of $p_{\mathbf{x}}(a_1, a_2)$.

(b) Find the PDF of the distance $r$ between terminal and base station.

(c) With $\tau$ the propagation delay over the distance $r$, find $m_\tau$, $\sigma_\tau^2$ and $P[\tau > m_\tau]$.

1.55 The discrete rve $\mathbf{x} = [x_1, x_2]$ has alphabet

$\mathcal{A}_x = \{(a_1, a_2) \in \mathbb{N} \times \mathbb{N} : 0 \leq a_1 \leq a_2\}$

with $\mathbb{N} = \{0, 1, 2, \ldots \}$ and PMD

$p_x(a_1, a_2) = K(1/2)^{a_1+2a_2}, \quad (a_1, a_2) \in \mathcal{A}_x$.

Find the conditional PMD $p_{x_1|a_1}(a_2|a_1)$.

1.56 The Gaussian rve $\mathbf{x} = [x_1, x_2]$ has mean vector and autocorrelation matrix

$\mathbf{m}_x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{r}_x = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}$.

(a) State whether $x_1$ and $x_2$ are statistically independent.

(b) Find the probability that $\mathbf{x}$ takes values in the set $C$ shown below.
1.57 Let \( \mathbf{x} \) be a Gaussian rve with mean vector and covariance matrix
\[
\mathbf{m}_x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{k}_x = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix},
\]
and let \( z = x_1 + x_2 \). Are \( x_1 \) and \( x_2 \) statistically independent? Write the expression of \( p_z(a) \).

1.58 Consider the rve \( \mathbf{x} = [x_1, x_2] \) having PDF
\[
p_\mathbf{x}(a_1, a_2) = \begin{cases} 
    e^{\lambda(a_1+a_2)+c}, & |a_1| + |a_2| \leq 1 \\
    0, & |a_1| + |a_2| > 1. 
\end{cases}
\]
(a) State whether \( x_1, x_2 \) are statistically independent and give the expression of the conditional PDF \( p_{x_1|x_2}(a_1|a_2) \).
(b) Find the value of \( c \) for \( \lambda = 1 \).
(c) Does there exist a linear invertible transformation \( \mathbf{y} = \mathbf{A}\mathbf{x} \) such that \( \mathbf{y} \) is a Gaussian rve?

**Random processes**

1.59 Consider the rp \( x(t) = A e^{-t/T_1} \mathbf{1}(t) \), where \( A \) is a rv, uniformly distributed between 0 and \( A_{\text{max}} = 1 \text{ V} \), that is with
\[
p_A(a) = \begin{cases} 
    1/A_{\text{max}}, & 0 < a < A_{\text{max}} \\
    0, & \text{elsewhere} 
\end{cases}
\]
while \( T_1 = 1 \text{ s} \). Plot a few realizations of \( x \).
(a) Find its statistical power \( \mathbb{M}_x(t) \) and the probability \( \mathbb{P}[x(2T_1) > 0.1 \text{ V}] \).
(b) Write the expression of \( p_x(a; t) \) and give its graphical representation.

1.60 Consider the rp
\[
x(t) = A + B \cos 2\pi f_0 t, \quad t \in \mathbb{R}
\]
where the rvs \( A \) and \( B \) are iid uniformly distributed in \((0, 1)\). Find its cyclostationarity period and give its average second order description.

1.61 For what values of the parameters \( A, B \in \mathbb{R} \) can the functions below be
(a) PSDs of stationary rps with finite power
\[
\mathcal{P}_1(f) = A \text{rect}(T f) + B \text{triang}(T f/2)
\]
\[
\mathcal{P}_2(f) = e^{A|f|} - e^{B|f|},
\]
(b) autocorrelations of stationary rps
\[
x_3(mT) = \begin{cases} 
    A, & m = 0 \\
    B, & m = \pm 1 \\
    0, & |m| \geq 2 
\end{cases}
\]
\[
x_4(\tau) = A \text{sinc}(\tau/T - B).
\]
*Hint*: Check whether their Fourier transform can be PSDs.
1.62 Consider the binary random process \( x_n, n \in \mathbb{Z} \), with iid random variables and probability mass distribution \( p_x(1) = p_x(-1) = \frac{1}{2} \).

(a) Calculate the mean, autocorrelation, PSD and the first order PMD of the process \( y_n \),
\[
y_n = \frac{x_n + x_{n-1}}{2},
\]
and determine whether it has iid random variables itself.

(b) Determine whether the signal
\[
z_n = x_n x_{n-1}
\]
has iid random variables.

1.63 Consider the continuous time rp
\[
x(t) = a \cos(2\pi \lambda t) + b \sin(2\pi \lambda t)
\]
with \( a, b, \lambda \) iid rvs, all with uniform distribution in the interval \([-1, 1]\). Determine whether \( x \) is WSS, then find its statistical power and PSD.

1.64 Consider a continuous time WSS Gaussian rp \( x(t) \) with mean \( m_x = 1 \) and autocorrelation \( r_x(\tau) = 1 + \text{triang}(\tau/T) \).

(a) Write the expression of its PSD.

(b) Are the rvs \( x(0) \) and \( x(T) \) statistically independent?

(c) Find the probability \( P[x(0) + x(T) < 1] \).

1.65 Consider the rp \( x \), obtained by multiplying a WSS rp \( a(t) \) by a periodic signal \( c(t) \) of period \( T_c \) as in Example 1.7 A. Show that the average PSD of \( x \) can be obtained from that of \( a \) as
\[
\mathcal{P}_x(f) = \sum_{\ell=-\infty}^{+\infty} |C_\ell|^2 \mathcal{P}_a(f - \ell F_c), \quad F_c = 1/T_c,
\]
where \( \{C_\ell\} \) are the Fourier coefficients of \( c \), and its average statistical power is
\[
M_x = M_a M_c.
\]
Give a graphical representation for \( \mathcal{P}_x(f) \) and the value of \( M_x \) in the case
\[
\mathcal{P}_a(f) = \delta(f) + \frac{1}{F_1} \text{triang}(f/F_1)
\]
and \( c(t) = \cos^3(2\pi t/T_c) \), with \( F_1 < F_c \).

1.66 The continuous time Gaussian process \( x \) has zero mean and autocorrelation
\[
r_x(t, \tau) = e^{-\rho_0 |\tau|} \cos(2\pi f_0 t).
\]

(a) Find \( P[x(T_1) > 1] \) and \( P[x(T_1) > x(2T_1)] \), with \( T_1 = 1/(8f_0) \).

(b) Write the expression of the average PSD of \( x \).
1.67 Consider the system with input $v_i(t)$ and output $v_o(t)$ implemented as in the figure, with $L = 0.75 \text{ mH}$ and $R = 100 \Omega$. The signal $v_i(t)$ has uniform amplitude in the interval $[-5, 5]$ V and PSD as in the figure.

Determine the value of $K$ and the PSD of $v_o(t)$.

1.68 Find the PSD of the output $y(t)$ of a filter $h(t)$, given the autocorrelation or PSD of the input $x(t)$, with

(a) $H(f) = \text{rect}(f/2B)$, $r_x(\tau) = \frac{N_0}{2} \delta(\tau)$, where $N_0$ and $B$ are positive constants;

(b) $h(t) = A \exp(-\alpha t) 1(t)$, $P_x(f) = \frac{B}{1 + (2\pi \beta f)^2}$, where $A$, $\alpha$ and $B$ are positive constants.

1.69 Consider the Gaussian random process $s(t)$ having zero mean and PSD

$$P_s(f) = \text{triang}(fT),$$

where $T = 2$. Evaluate the probability $P[s(t) > 1]$.

1.70 Let $s(t)$ be a Gaussian random process with zero mean and PSD

$$P_s(f) = \frac{10^{-2}}{2B} \text{rect}\left(\frac{f}{2B}\right), \quad B = 10^9 \text{ Hz}.$$ 

The process is filtered by a filter with frequency response

$$G_{Ch}(f) = G_0 \text{ rect}\left(\frac{f}{2B_{Ch}}\right),$$

with $G_0 = 0.3$, $B_{Ch} = 10^9$ Hz. Compute the probability that the amplitude of the signal at the filter output, $y(t)$, lies in the range $[-0.6, 0.6]$.

1.71 The continuous time random process $x$, WSS with mean $m_x = 2$ V and PSD

$$P_x(f) = \frac{P_0}{B} \text{triang}(f/B) + P_0 \delta(f - B) + m_x^2 \delta(f), \quad P_0 = 8 \text{ V}^2,$$
is input to a filter with frequency response \( G(f) = G_0 \text{rect}(f/B) \).

(a) Find the mean and statistical power of the filter output \( y \) for \( G_0 = -3 \text{ dB} \).

(b) For \( G_0 = 1 \), define the rv \( z \) as \( z(t) = x(t) - y(t) \). Find mean, PSD and statistical power of \( z \). Is \( z \) a real-valued rv?

1.72 The random voltage signal \( x \) is WSS, with mean \( m_x = 1 \text{ V} \) and autocorrelation \( r_x(\tau) = m_x^2 (1 + e^{-|\tau|/T_1}) \) where \( T_1 = 1 \mu s \). Find mean, PSD and statistical power of the output \( y \), when \( x \) is the input to a filter with impulse response

\[
g(t) = \frac{1}{T_1} e^{-t/T_1} 1(t).
\]

1.73 Let \( \{x_n\} \) be a discrete time white process at the input of an interpolate filter with impulse response \( g \). Show that the continuous time \( y \) at the output has time-varying statistical power

\[
M_y(t) = M_x \sum_{n=-\infty}^{+\infty} |g(t - nT)|^2
\]

and average statistical power

\[
M_y = M_x \frac{1}{T} E_g.
\]

1.74 The continuous time \( y \) is defined as

\[
y(t) = x_n, \quad nT \leq t < nT + T,
\]

where \( \{x_n\} \) is a digital rv with iid variables and alphabet \( A_x = \{0, 1, 2\} \) with the three values equally likely.

Show that \( y \) can be thought of as being obtained from \( x \) through an interpolate filter and find its average PSD and statistical power.

1.75 In the sampling and interpolation scheme of Figure 1.23 the input \( x_i \) is WSS with PSD \( P_{x_i}(f) = P_0 \text{triang}(f/F_s) \) while the anti-aliasing filter and the interpolate filter have frequency responses \( D(f) = \text{rect}(f/F_d) \) and \( G(f) = T_s \text{rect}(f/F_g) \), respectively.

(a) Find for what values of \( F_x, F_d, F_g \) the input \( x_i \) is perfectly reconstructed in the mean square sense.

(b) Find the statistical power of the error in the case \( F_x = F_d = F_g = F_s \).

(c) Find for what values of \( F_x, F_d, F_g \) the output signal \( \hat{x} \) is WSS and write its PSD for \( F_x = F_s, F_d = 2F_s, F_g = F_s/2 \).