OVERVIEW OF DISCRETE DYNAMICAL MODELING AND MAPLE™

1.1. INTRODUCTION TO MODELING AND DIFFERENCE EQUATIONS

In this section, we introduce dynamical systems, discuss discrete dynamical systems versus continuous dynamical systems, and informally define a mathematical model.

1.1.1. Model 1: Population Dynamics—A Discrete Dynamical System

Consider the population of a city with a constant growth rate per year. The populations are counted at the end of each year. For simplicity, we assume that there is no immigration to or emigration from the city.

i. Model the population dynamic and predict the long-term behavior of the system.

ii. The population of a city is 100,000 in year 2010. The natural annual growth rate of the population is 1% per year. Predict the city population in 2020. Find the population over the next 30 years and graph it. What is the long-term behavior of the population?
Discussion

i. We will measure the population at discrete time intervals in one year units. Let

\[ p_n = \text{population size at the end of time period (year) } n \]
\[ p_0 = \text{the initial population size} \]
\[ r = \text{the constant growth rate per period (year)} \]

The relationship between the current population, \( p_n \), and the next population, \( p_{n+1} \), is

\[
p_{n+1} = p_n + rp_n
\]
\[
p_{n+1} = (1 + r)p_n
\]

Therefore, the population dynamics can be modeled by equation (1.1).

Equation (1.1) is a difference equation (or recurrence equation). The system (1.1) and the initial value \( p_0 \) represent the population dynamic. Since the population changes over time, this system is a dynamical system. Since this dynamical system changes over discrete time intervals, the system is called a discrete dynamical system. We say that the population dynamics is modeled by the discrete dynamical system (or the difference equation 1.1).

To find \( p_k \), use \( p_0 \) in equation (1.1) to find \( p_1 \), then use \( p_1 \) to find \( p_2 \) and so on until \( p_k \). This process is called iteration of the difference equation (1.1); and the sequence (1.2),

\[ p_0, p_1, p_2, \ldots, p_k \]

for any value of \( k \) (positive integer) is called a solution or numerical solution of the given difference equation (1.1).

From equation (1.1), if the current value of \( p_n \) is known, the next value \( p_{n+1} \) can be calculated. For example, if we have \( p_5 \), we can calculate \( p_6 \). However, if we have \( p_0 \), equation (1.1) does not allow us to calculate, for example, \( p_6 \) in one step. Therefore, we are in need of a closed form to calculate \( p_n \) in one step if we know the values of \( p_0 \) and \( n \). It can be easily proven that

\[ p_n = (1 + r)^n p_0 \]

Equation (1.3) is called the analytical solution of the difference equation (1.1).

Equation (1.3) is an exponential function and will grow or decay exponentially depending on the value of \( r \). If \( r > 0 \), then \( (1 + r) > 1 \), and therefore population size \( p_n \) grows unbounded when \( n \) is very large. If \( r < 0 \), then \( (1 + r) < 1 \), and consequently, the population size approaches zero when \( n \) is very large.

ii. Let us apply the aforementioned model to the given information where \( r = 0.01 \) and \( p_0 = 100,000 \). The city population is modeled by the system

\[ p_{n+1} = 1.01p_n, \quad p_0 = 100,000 \]
To find the population in 2020 (10 years from 2010), we use equation (1.3) with $n = 10$ and $p_0 = 100,000$. We are looking for $p_{10}$. We have

$$p_{10} = (1.01)^{10} \times 100,000 = 110,462.$$  

One way to find the values of $p_1, p_2, \ldots, p_{30}$ is to iterate equation (1.4). Then graph the ordered pairs $(n, p_n)$. To illustrate how the iteration works, let us see how to calculate, for example, $p_3$. Set $n = 0$ in (1.4) to get $p_1$,

$$p_1 = 1.01p_0 = 1.01(100,000) = 101,000.$$  

Then set $n = 1$ in (1.4) to get $p_2$,

$$p_2 = 1.01p_1 = 1.01(101,000) = 102,010.$$  

Finally, setting $n = 2$ in (1.4) gives $p_3$,

$$p_3 = 1.01p_2 = 1.01(102,010) = 103,030.$$  

In particular, to find $P_{10}$, the answer to the original question, we can use the difference equation (1.4) and the initial condition to find the sequence $p_0, p_1, p_2, \ldots, p_{10}$ which is 100,000; 101,000; 102,010; 103,030; 104,060; 105,101; 106,152; 107,213; 108,285; 109,368; 110,462. Thus $p_{10} = 110,462$.

Usually we use Maple to iterate a difference equation. We will introduce Maple in Section 1.3. The graph of $p_n, n = 0, 1, 2, \ldots, 30$ versus $n$ is shown in Figure 1.1.

The analytical solution and numerical solution of (1.4) show that the city population slowly grows unbounded as $n$ becomes very large.

1.1.2. Model 2: Population Dynamics—A Continuous Dynamical System

Consider the following situation. There are some bacteria in a tube with nutritive solution. As time progresses, the bacteria reproduce by splitting and dying. Assuming that there is enough food and space for the bacteria, model the dynamics of the bacteria. Investigate the long-term behavior of the model.

**Discussion**

Let $p(t)$ be the bacteria’s population size (number of bacteria) at time $t$ and $p(0) = p_0$. Assume that the growth rate $r = b - d$, where $b$ is the birth rate and $d$ is death rate. The assumption that there is enough food and space means that there is no restriction on the increasing number of bacteria. Therefore, the rate of change of bacteria’s population size $(dp/dt)$ is proportional to the bacteria’s population $p$. Consequently, the dynamic of the bacteria is modeled by the dynamical systems (1.5) and (1.6)

$$\frac{dp}{dt} = (b - d)p = rp$$  

(1.5)
Equation (1.5) is a first-order ordinary differential equation, and equation (1.6) is called the initial value (condition). From the basics of differential calculus or differential equations, the solution of the systems (1.5) and (1.6) is

\[
 p(t) = e^{rt}p_0 \tag{1.7}
\]

Knowing the values of \( r \) and \( p_0 \), the population \( p(t) \) can be evaluated at any time \( t \). For \( r > 0 \), equation (1.7) implies that the population size \( p(t) \) increases and grows unbounded as \( t \to \infty \), while \( r < 0 \) implies that the population size decreases and approaches zero as \( t \to \infty \).

Since the change in the system (1.5) is continuous, this system is called a continuous dynamical system. Usually continuous dynamical systems are represented by one or more ordinary or partial differential equations. Note that the models discussed in this text are restricted to discrete dynamical systems. In other words, we will discuss in this text only models represented by difference equations.

**FIGURE 1.1.** Graph of a city’s population after \( n \) years, \( p_n \), versus time \( n \) in years. The population is modeled by the difference equation \( p_{n+1} = 1.01p_n \), with the initial population \( p_0 = 100,000 \), and \( n = 0, 1, \ldots, 30 \).
1.1.3. Why Modeling with Difference Equations Is Adopted

There are compelling reasons to restrict models in this text to difference equations, including:

1. Modeling with difference equations is a very powerful tool, and yet a simple one, to model dynamical systems in biology, ecology, the environment, and chemistry.

2. Modeling with difference equations requires knowledge of algebra and does not require knowledge of differential calculus, while modeling with differential equations requires a course of differential equations. This text targets freshman and sophomore life sciences and mathematics majors, many of whom did not have a differential equations course.

1.1.4. What Is a Mathematical Model?

The following is a possible informal definition of a mathematical model:

A mathematical model is a translation of a real-world problem into mathematics notation by forming a mathematics problem corresponding to the real-world problem. Then mathematics tools, ideas, concepts, and techniques are utilized to solve the mathematics problem. The obtained solution is translated back into the real-world problem context.

1.1.5. Basic Terminology of Difference Equations

The order of a difference equation is equal to the difference between the largest and smallest indices in the difference equation. For example, in the difference equations (1.1) and (1.4), the difference between the largest subscript, \( n + 1 \), and the smallest subscript, \( n \), is 1. Therefore, each of equations (1.1) and (1.4) is a first-order difference equation. The difference equation (1.8) is a second-order difference equation.

\[
x_{n+2} = x_{n+1} + x_n
\] (1.8)

Examples of first-order difference equations are:

\[
y_n = 0.8y_{n-1} + 20
\] (1.9)

\[
p_n = 1.02p_{n-1}
\] (1.10)

\[
x_{n+1} = 1.2x_n - 0.15n
\] (1.11)

\[
z_n = n^2z_{n-1} + 2^n
\] (1.12)

\[
B_{n+2} = \frac{4}{n}B_{n+1} - n^{1/2}
\] (1.13)

\[
y_{n+1} = 3y_n^2 + 3n
\] (1.14)

\[
x_n = \sqrt{x_{n-1}} - 23
\] (1.15)

\[
y_{n+1} = y_{n+1}y_n + 6
\] (1.16)
A difference equation is called **linear** if its terms are not raised to a power other than one and if the terms are not multiplied together. Otherwise, the difference equation is called **nonlinear**. For example, the difference equations (1.1), (1.8), (1.9), (1.10), (1.11), (1.12), and (1.13) are linear. The difference equation (1.14) is nonlinear since \( y_n \) is raised to power 2. Equation (1.15) is nonlinear because \( x_{n-1} \) is raised to power \( \frac{1}{2} \). In equation (1.16), two terms, \( y_{n+1} \) and \( y_n \) are multiplied together, therefore (1.16) is nonlinear.

**A first-order linear difference equation** can be represented by the equation

\[
y_{n+1} = a_n y_n + b_n
\]

(1.17)

where \( a_n \) and \( b_n \) are two known sequences. \( a_n \) is a coefficient. Note that each of the sequences \( a_n \) and \( b_n \) can be a constant sequence or can depend on \( n \). When the sequence \( a_n \) is a constant, such as \( a \), the difference equation

\[
y_{n+1} = a y_n + b_n
\]

(1.18)

is called a **first-order linear difference equation with constant coefficients**. When \( b_n = 0 \), the difference equation is called homogeneous and equation (1.17) becomes

\[
y_{n+1} = a_n y_n
\]

(1.19)

Equation (1.19) is called a **first-order linear homogeneous difference equation**. If \( a_n = a \), where \( a \) is a constant, equation (1.19) becomes

\[
y_{n+1} = a y_n
\]

(1.20)

and is called a **first-order linear homogeneous difference equation with constant coefficients**. For example, the difference equations (1.1), (1.4), and (1.10) are first-order linear homogeneous difference equations with constant coefficients, and equations (1.9) and (1.11) are first-order linear difference equations with constant coefficients.

**Exercise 1.1**

1. Suppose that the initial population of a species is 10,000 and the growth rate is 5% per year.
   A. Model this situation by a difference equation.
   B. Find the first five (5) populations.
   C. Find the population size after 20 years.

2. Assume that the population of a country is 100 million and the natural growth rate is 2% per year. Assume that 20,000 immigrants are allowed to immigrate into the country every year.
   A. Model this situation by a difference equation.
   B. Find the population for the next four years.
C. Find the population after 10 years.
D. Can you predict the long-term behavior of the system?

3. Assume that the kidneys remove 25% of an anesthetic substance from the body every hour. A patient is injected with 600 mg of the anesthetic substance before a minor surgery on her tooth.
   A. Model this situation by a difference equation.
   B. Determine the amount of the substance in the patient’s body after the first four hours.
   C. What is the long-term behavior of the system?

4. In 2000, a lake contained 800 lbs. of contamination. In the same year, a new plant started to dump 120 lbs. of the contaminant into the lake every year. Assume that 15% of the contaminant in the lake is naturally removed from the lake every year. Assume that this trend continues.
   A. Model the amount of contaminant in the lake at any year after 2000.
   B. Determine the amount of the contaminant in the lake in years 2001–2006.
   C. Can you determine the long-term behavior of the system?

5. Identify the following difference equations as linear or nonlinear.
   A. \( x_{n+1} = 3x_n + 2 \)
   B. \( y_n = 2y_{n-1} - 4n \)
   C. \( y_{n+1} = 4\sqrt{y_n} + 5 \)
   D. \( x_{n+1} = x_n x_{n-1} + 6 \)
   E. \( z_{n+2} = 2z_{n+1} - 3 \)
   F. \( x_{n+1} = 3x_n^2 + 4 \)

6. Determine the order of the following difference equations.
   A. \( x_{n+1} = 2x_n + 6n \)
   B. \( x_{n+1} = 4x_n - 2x_{n-1} \)
   C. \( y_{n+2} = 4\sqrt{y_n} + 5 \)
   D. \( z_{n+2} = 2nz_{n+1} - 4z_n \)
   E. \( z_{n+2} = 2z_{n+1} - 3 \)
   F. \( y_n = y_{n-2} + 4 \)

7. Determine whether the following difference equations are homogeneous or nonhomogeneous
   A. \( x_{n+1} = 3x_n + 6n \)
   B. \( y_{n+2} = 4y_{n+1} - 2y_n \)
   C. \( x_{n+1} = 2x_n + n^2 + 5n + 2 \)
   D. \( y_{k+3} = 5y_{k+1} + 6 \)
E. \( z_n = n^2z_{n-1} - nz_{n-2} \)
F. \( z_{n+2} = n^2z_{n+1} + 4z_n + 2n \)

8. Identify the order, linearity, and homogeneity of each of the following difference equations.

A. \( x_{n+3} = 2x_{n+1} + 3x_n - n^2 \)
B. \( y_{n+2} = 4x_{n+1}x_n - 4 \)
C. \( z_{n+1} = 3z_n + 4n \)
D. \( P_{n+2} = 2nP_n \)
E. \( P_{n+1} = P_n + aP_n - bP_n^2 \), where \( a \) and \( b \) are constants
F. \( x_{n+1} = rx_n(1 - x_n) \), where \( r \) is a constant

1.2. THE MODELING PROCESS

It is useful to view mathematical modeling as a process as illustrated in Figure 1.2. The modeling process is represented by a loop, where the starting point is step 1, located in the box in the upper left-hand corner of Figure 1.2.

Step 1: Formulate the Real-World Problem
In this step, get information pertaining to the system under consideration and identify the question(s) to be answered. The question should be neither too general nor too narrow. If the formulated question is too general, it is difficult to manage the problem; and if the question is too narrow, the problem might become trivial. Since the question will be translated into mathematics notation, it should be stated in precise mathematics terms.

FIGURE 1.2. The mathematical modeling process.
As an example, we consider two species in a forest, rabbits and foxes, where foxes eat rabbits and there is enough food for rabbits. We are interested in

1. Determining whether the rabbits and foxes could coexist in this environment.
2. Finding the equilibrium values of the system and determining whether these equilibrium values are stable, unstable, or semi-stable.
3. Modeling the dynamic of the interaction between the predator foxes and the prey rabbits, so we can predict the populations of rabbits and foxes at any year.
4. Investigating the long-term behavior of the two species.

Step 2: Make Assumptions

It is very important to state clearly the assumptions. The construction of a mathematical model greatly depends on the assumptions. It is clear that a change in the assumptions would result in a different model. It is advisable to simplify assumptions, at least at the beginning, in order to make the model manageable. Therefore, some assumptions are made to simplify the model.

We will use the predator–prey example to illustrate step 2. Since the forest is a complex ecosystem, we need to make assumptions to simplify the predator–prey model. We employ the following assumptions:

i. There is enough food for rabbits and the population of rabbits increases by a constant rate. That is, the rabbit’s population increases exponentially.
ii. The population of rabbits decreases as a result of the interactions between rabbits and foxes.
iii. The rabbits are the only source of food for foxes. Therefore, in the absence of rabbits the population of foxes decreases by a constant rate and dies out. That is, the foxes’ population decreases exponentially.
iv. The population of foxes increases as a result of the interactions between rabbits and foxes.
v. The rabbits and foxes live in a closed environment. Which means that there is no interaction between these two species and other species, there is no emigration/immigration from/to the forest, and there is no harvesting or hunting.

Step 3: Formulate the Mathematical Problem

In this step, we enter the mathematics world. In the first part of this step, we need to choose mathematics symbols for the variables and parameters. Recall that variables are quantities that change within the problem, while the parameters are constant within a problem.

For example, in the predator–prey example we might choose the following variables:

\[ R_n = \text{the population of the prey rabbits at the time period } n \]
\[ F_n = \text{the population of predator foxes at time period } n \]
\[ n = \text{the time period in years, } n = 0, 1, 2, \ldots, k \]
For the parameters, we might choose

\[ a = \text{the natural growth rate of rabbits in the absence of foxes and } a > 0. \]
\[ b = \text{the death rate of rabbits as a result of the presence of foxes, and } b > 0. \]
\[ c = \text{the natural decay rate of foxes in the absence of rabbits and } c > 0. \]
\[ d = \text{the growth factor of foxes due to the presence of rabbits and } d > 0. \]

In the second part of step 3 utilize assumptions in step 2 and the variables and parameters defined in the first part of step 3 to formulate the problem in mathematics notation. As a result of the formulation, the problem might be represented by a single algebraic, difference, differential, matrix equation or a system of algebraic, difference, or differential equations. The problem might be represented by an algorithm, and so on.

For example, the predator–prey model may be represented by a system of the following linear difference equations

\[ R_{n+1} = R_n + aR_n - bF_n \]  
\[ F_{n+1} = F_n - cF_n + dR_n \]  

or the following system of two nonlinear difference equations

\[ R_{n+1} = R_n + aR_n - bR_nF_n \]  
\[ F_{n+1} = F_n - cF_n + dR_nF_n \]

It is required to use the selected representation to answer the posed questions on the real-world problem.

Step 4: Solve the Mathematical Problem (Model)
In this step, we use appropriate available mathematical, computational, or graphical tools and techniques to solve the mathematical problem (model). The solution might be an analytical solution (a closed-form mathematical expression), a numerical solution, or a graph. It might also be the implementation of an algorithm, or running/testing a simulation.

For example, there is an analytical solution of the predator–prey model represented by the linear equations (1.21) and (1.22) in the form

\[ X_n = T^n X_0 \]

where \( T = \begin{bmatrix} 1 + a & -b \\ d & 1 - c \end{bmatrix} \), \( X_n = \begin{bmatrix} R_n \\ F_n \end{bmatrix} \), and where \( X_0 = \begin{bmatrix} R_0 \\ F_0 \end{bmatrix} \) is the initial distribution vector.

Matrix algebra allows us to fully investigate the solution. However, the modeler will realize that the linear representations (1.21) and (1.22) of the predator–prey model are unrealistic.
The predator–prey model is a realistic one if it is represented by the nonlinear difference equations (1.23) and (1.24). Therefore, we will focus our attention to answer the questions of the real-world problem using the nonlinear representation. However, for nonlinear equations, there is no analytical solution.

It can be easily shown that this system has two equilibrium values, say $R_e$ and $F_e$, where

$$R_e = \frac{c}{d}, \quad F_e = \frac{a}{b}$$

Using Maple it can be concluded that these equilibrium values are unstable. With Maple we create two types of graphs, so called time-series graphs ($R_n$ vs. $n$) and phase-plane graphs ($F_n$ vs. $R_n$ or $R_n$ vs. $F_n$), for different values of the parameters $a$, $b$, $c$, and $d$ and the initial values $R_0$ and $F_0$. Answers to the posed questions and the system’s long-term behavior can be extracted from these graphs.

Step 5: Interpret the Solution
At this stage, the answers to the mathematical problem need to be interpreted in terms of the context of the real-world problem. The modeler must check the answers to ensure that the model answered the original real-world problem within the assumptions made in step 2 and the initial conditions.

In our predator–prey example, the following might be interpretations to solutions obtained in step 4:

- The rabbits and foxes may coexist together without extinction of one of the species. At certain values of the rabbits and foxes, there is no change in the rabbit and fox populations. This interpretation is a conclusion of the existence of equilibrium values of rabbits and foxes.
- The equilibrium values of the rabbits and foxes are sensitive to small change. In other words, a small change to the equilibrium values makes the rabbits and foxes populations diverge from the equilibrium values. This interpretation is a conclusion from the instability of the equilibrium values established in step 4.

Step 6: Verify the Model
At this stage, it is necessary to verify the validity of the model. A common method to do that is to compare the model’s predicted results with known real-world data or with data obtained from an experiment designed to test the model. Be sure that all the necessary variables were utilized and all the assumptions were incorporated.

If the outcome of the verification is unsatisfactory, the modeler needs to refine the model, which is improving it. To refine the model, you need to reexamine the modeling process starting with step 1. Be sure that you did not omit necessary assumptions and variables. Check that the mathematical problem accurately represents the real-world problem. Check for the correctness of solving the mathematical problem.
If the verification of the model in step 6 is satisfactory, the modeler writes a report on the model, if he/she is required to submit one. The report should follow the requested format.

**Exercise 1.2**

In Exercises 1–8, real-world situations are briefly stated without specifics. For each situation, do the following:

A. Formulate the real-world problem.
B. Make assumptions. If you make assumptions to simplify the problem, at least at the beginning, state them.
C. Choose mathematical symbols for the variables and parameters.
D. Formulate the mathematical problem. Note that you are not asked to solve the mathematical problem.

1. The dynamics of the population of the United States.
2. The dynamics of the population of a single species.
3. The dynamics of the population of a single species, such as deer, with hunting.
4. The dynamics of the interaction of two species competing for the same food, such as foxes and wolves who compete for rabbits.
5. The dynamics of two interacting species, predator and prey, such as falcons and rats.
6. Obtaining the Maximum Sustainable Yield of a natural renewable resource such as a colony of whales.
7. Administering a drug, such as antibiotic, for a patient.
8. The dynamics of the spread of a contagious disease, such as flu, among the students of a college.

**1.3. GETTING STARTED WITH MAPLE**

In this section, we introduce the basics to start Maple. This introduction is by no means a reference or manual for Maple. It is simply a quick start with Maple to be able to write simple commands, iterate difference equations, and produce graphs.

**1.3.1. Start Maple**

To start Maple, double click on the Maple icon on the computer. You should see a large window headed by **Untitled (1)** - [Server 1] - Maple 17. The large window is called the *worksheet*, where the commands are entered and evaluated.

The Maple prompt is >
In the worksheet, the prompt > opens the command line. At the prompt, type the instructions. Each command line is followed by a semicolon (or a colon), and then the ‘Enter’ key is pressed. After pressing the ‘Enter’ key, Maple executes the line typed.

1.3.2. Conducting Computations

The basic ways to do computations are

- Enter (type) the commands on the worksheet.
- Use a user-defined program, where the program code is written on a text editor and saved in a file.

**Save Your Work**

To save your work, click on **File**, then click on **Save As...** and save the file as you do for regular files. The file name will appear at the top of the worksheet and will replace the previous file name Untitled (1)*. You are advised to regularly save your work by clicking on **File** then **Save**.

**Get Help**

To get help, you may use one of the following:

- At the prompt, type > ?topic and Enter. For example, to get help on plot, type

  ```maple
  > ?plot
  ```

- Click on **Help** on the toolbar and select from the submenu.

**Math and Text Modes**

Note that you can enter a text by clicking on **Text** on the toolbar. In the text mode, no prompt (>)) is displayed. To switch from the text mode to the mathematics (command) mode, click on **Math** on same toolbar.

1.3.3. Quitting Maple

To quit Maple, click on **File**, then click on **Exit**. You are advised to save your work before you exit Maple.

1.3.4. Simple Arithmetic and Definition of Variables

The basic arithmetic operators are

```
+ addition
- Subtraction
* multiplication
```
Examples
The following are commands and the corresponding output to illustrate the basics of Maple syntax and tools. You are advised to start a worksheet, enter the commands, and think about the output before you press the ‘Enter’ key. Try to understand why you got an output different from your expectation, or why you got an ‘Error’ message.

\[
\begin{align*}
> 2 + 7 - 4; & \quad 5 \\
> 3 * 6/9; & \quad 2 \\
> 3 * 7/9; & \quad \frac{7}{3} \\
> \text{evalf}(%); & \quad 2.333333333 \\
> 3.0 * 7/9; & \quad 2.333333333 \\
> 11/3; 26/6, 11.0/3; \text{evalf}(11/3); & \quad \frac{11}{3}, 26/6, 11.0/3; \text{evalf}(11/3); \\
& \quad 3.666666667 \\
> \sqrt{25}; & \quad 5 \\
> \sqrt{26}, \sqrt{26.0}; & \quad \sqrt{26}, 5.099019514 \\
> 2*\pi, \text{evalf}(2*\pi); & \quad 2\pi, 6.283185308 \\
> x := 3, y := 4; & \quad x := 3, y := 4 \\
> 2*x + 3*y; & \quad 18 \\
> 2*x + 3*y/5, 2.0*x + 3*y/5; & \quad 42/5, 8.400000000 \\
> \sqrt{x^2 + y^2}, (x^2 + y^2)^{1/2}, (x^2 + y^2)^{0.5}; & \quad 5, \sqrt{25}, 5.000000000 \\
> y^{(-1)}, 3*y^{(-1.0)}; & \quad \end{align*}
\]
Note that the assignment sign “:=” assigns the value of the expression on the right-hand side to the variable on the left-hand side of the assignment sign. To reassign a variable, say \( y \), enter the command: \( y := 'y' \);

\[
> y := x^2;
\]
\( y := 16 \)

Note that Maple displays the output if the command is ended with “;”. If a command ends with “:”, the command is executed, but the output is suppressed (i.e., the output is not displayed).

\[
> \text{factor}(a^2 - b^2); \text{factor}(a^2 - 2a - 15);
\]
\[
(a - b)(a + b)
\]
\[
(a + 3)(a - 5)
\]

\[
> \text{factor}(a^2 - 2a + 3);
\]
\( a^2 - 2a + 3 \)

\[
> \text{expand}( (x-2)*(x+6) ); \text{expand}( (x+2)^3 );
\]
\[
x^2 - 4x - 12
\]
\[
x^3 + 6x^2 + 12x + 8
\]

\[
> \text{sum} (k, k = 1..5); \text{sum} (k^2, k = 1..5); \text{sum} ( (k/3)^2, k = 1..5 );
\]
\( 15, 55, \frac{55}{9} \)

\[
> \text{sum}(1/k, k = 1..5);
\]
\( \frac{137}{6} \)

\[
> \text{sum}(k, k = 1..n);
\]
\( \frac{1}{2}(n+1)^2 - \frac{1}{2}n - \frac{1}{2} \)

### 1.3.5. Comments in Maple
Comments can be inserted into a worksheet and in a program code. Maple does not evaluate everything written in a command line after the pound sign (#) to the end of the line. Therefore, insert your comments and program documentations after #.
Example

> # The following is the iteration of the difference equation
> # $P(n+1) = 1.1 P(n)$ with $P(0) = 100$
> P := Array(0..10);  # Population array to hold the
# populations
> P[0] := 100;  # Initial population $P_0$

1.3.6. Solving Equations

To solve the quadratic equation $x^2 - 2x - 15 = 0$:

> restart;
> solve(x^2 - 2*x - 15);

5, -3

To solve the quadratic equation $ax^2 + bx + c = 0$

> solve(a*x^2 + b*x + c, x);

\[
\frac{-b + \sqrt{-4ac + b^2}}{2a}, \quad \frac{-b - \sqrt{-4ac + b^2}}{2a}
\]

> solve(x^2 + x - 3/4);

\[
\frac{1}{2} - \frac{3}{2}, \quad \frac{1}{2}
\]

To solve the system of equations $y = x + 1$ and $y = -2x + 7$ in $x$ and $y$:

> solve( {y=x+1, y = 2*x+7}, {x, y} );

\{x = -6, y = -5\}

To solve the system of equations $y = 2x + 1$ and $y = 2x + 5$ in $x$ and $y$:

> solve( {y = 2*x+3, y = 2*x + 5}, {x, y} );

Maple did not return an output, which means that the system has no solution.

To solve the equations $y = x^2$ and $y = 2x + 8$ in $x$ and $y$:

> solve( {y = x^2, y = 2*x + 8}, {x, y} );

\{x = -2, y = 4\}, \{x = 4, y = 16\}

1.3.7. Complex Numbers

The solution of $x^2 + 1 = 0$ is $x = \pm \sqrt{-1}$. In mathematics, we use $i = \sqrt{-1}$, where $i^2 = -1$. Since there is no real value for $i = \sqrt{-1}$, this number is complex (not real).
In general, a complex number \( z \) can be written in the form \( z = a + bi \), where \( a \) and \( b \) are real numbers. For complex arithmetic, review Section 3.5.

Maple uses \( I \) for \( i \) (note it is a capital I and not i).

\[
\begin{align*}
> \sqrt{-1}; & \quad I \\
> I^2; & \quad -1 \\
> \text{solve}(x^2 + 1 = 0); & \quad I, -I \\
> \text{solve}(x^2 - 2x + 5); & \quad 1 + 2i, 1 - 2i \\
> z1 := 2 + 3i; z2 := -5 + 2i; & \quad z1 := 2 + 3i \\
& \quad z2 := -5 + 2i \\
> \text{Re}(z1), \text{Im}(z1); & \quad 2, 3 \\
> \text{Re}(z2), \text{Im}(z2); & \quad -5, 2 \\
> z3 := z1 + z2; z4 := z1*z2; z5 := z1/z2; & \quad z3 := -3 + 5i \\
& \quad z4 := -16 - 11i \\
& \quad z5 := -\frac{4}{29} - \frac{19}{29}i
\end{align*}
\]

1.3.8. Functions and Expressions in Maple

Informally, a function is a relationship between one variable, called the dependent variable, and one or more variables, called independent variables.

The following are functions:

\[
\begin{align*}
f(x) &= x^2 + 1 \\
g(x) &= 2x^3 - \sqrt{x}, \\
v(r, h) &= \pi r^2 h
\end{align*}
\]

Note that \( v \) is the volume of a right circular cylinder, where \( r \) is the radius of the base and \( h \) is the height of the cylinder. \( v \) is a function in two variables \( r \) and \( h \).

The function \( f \) can be defined by the command:

\[
> f := x -> x^2 + 1;
\]

\[
f := x \mapsto x^2 + 1
\]
To evaluate the function $f$ at a specific value of $x$, say $x = 5$, that is, $f(5)$, enter the command:

```maple
d > f(5);  
26
```

```maple
d > f(-3), f(1/2), f(1.5), f(0);  
10, 5/4, 3.25, 1
```

```maple
d > f(a); f(a+h); f(alpha);  
\( a^2 + 1 \)
\( (a + h)^2 + 1 \)
\( \alpha^2 + 1 \)
```

```maple
d > f(sin(x)), f(Pi), evalf(f(Pi));  
\( \sin(x)^2 + 1 \)
\( \pi^2 + 1 \)
\( 10.86960440 \)
```

Note that if several commands are entered on one line and separated by “;”, the output of each command is displayed on a separate line. If several commands are entered on one line and separated by “,”, the outputs of all commands are displayed on one line.

The function $g$ is defined by

```maple
d > g := x -> 2*x^3 - sqrt(x);  
g := x \rightarrow 2x^3 - \sqrt{x}
```

The function $v$ is defined by

```maple
d > v := (r, h) -> Pi*r^2*h;  
v := (r, h) \rightarrow \pi r^2 h
```

```maple
d > v(2, 10);  
40\pi
```

```maple
d > evalf(%);  
125.6637062
```

Note that % refers to the last result, %% refers to the second previous result, and %%% refers to the third previous result, and so on.

An expression can be defined as a string of constants, variables, mathematical functions, and mathematical operations (such as $+$, $-$, $\ast$, $/$, $\wedge$, $\sin$, $\cos$, $\ln$, ...).

The following are examples of expressions:

\[-x^2 + 4\]
\[4x^3 - 2x + \cos x\]
\[5/4\]

We will use the first expression to illustrate how to define an expression in Maple and how to evaluate it at a specific value of the variable. Let us write the expression $-x^2 + 4$
in the form $y = -x^2 + 4$. It will be defined by a simple assignment. To evaluate the expression at a specific value, we use the Maple's function `subs`:

```
> y := -x^2 +4;
y := -x^2 +4
> subs(x=3, y);
-5
> subs(x=0, y), subs(x=2, y), subs(x=3/2, y), subs(x=1.25, y);
4, 0, 7/4, 2.4375
> subs(x=2*Pi, y), subs(x=cos(beta), y), subs(x = a+b, y);
-4pi^2 + 4, -cos(beta)^2 + 4, -(a + b)^2 + 4
```

Similarly, the other two expressions may be defined by

```
> y2 := 4*x^3 – 2*x + cos(x);
y2 := 4x^3 – 2x + cos(x)
> subs(x=Pi/3, y2);
4/27 - 2/3 + cos(1/3)
> evalf(%);
2.999127368
> y3 := 5/4;
y3 := 5/4
> subs(x=2, y3), subs(x = Pi, y3);
5/4, 5/4
```

1.3.9. Lists and Sets

```
> restart:
> with(LinearAlgebra):
Lists and Sets
> L := [10, 4, 6, 2];
L := [10, 4, 6, 2]
> L[1], L[2], L[4];
10, 4, 2
> L[];
10, 4, 6, 2
> S := {8, 5, 1, 4, 4, 5, 3};
S := {1, 3, 4, 5, 8}
> S[];
1, 3, 4, 5, 8
> S[1], S[2], S[5];
1, 3, 8
```
Note that \( L := [x_1, x_2, \ldots, x_k] \) defines a list of the expressions \( x_1, x_2, \ldots, x_k \). The elements of a list are ordered and to access the \( i \)th element in \( L \) enter \( L[i] \). To get all the elements of the list \( L \) enter \( L[] \).

Note that \( S := \{x_1, x_2, \ldots, x_k\} \) defines a set of the expressions \( x_1, x_2, \ldots, x_k \). The elements of a set are not ordered, and there are no repetitions of elements in a set. To get all the elements of the set \( S \), enter \( S[] \).

\begin{verbatim}
> L1 := [ seq (k^2, k = 1..5 ) ];
L1 := [1, 4, 9, 16, 25]
> S1 := { seq (k^2, k = 1..5 ) };
S1 := \{1, 4, 9, 16, 25\}
> L2 := [ seq (k^2, k = -3..3 ) ];
L2 := [9, 4, 1, 0, 1, 4, 9]
> L2[1], L2[2], L2[4], L2[6], L2[7];
9, 4, 0, 4, 9
> S2 := { seq (k^2, k = -3..3 ) };
S2 := \{0, 1, 4, 9\}
> S2[1], S2[2], S2[3], S2[4];
0, 1, 4, 9
\end{verbatim}

Do you see the difference between \( L_2 \) and \( S_2 \)?

1.3.10. For Loops

Consider the following commands to find sum of squares of integers 1, 2, \ldots, 5.
In other words, we want to find \( 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \).

\begin{verbatim}
> restart:
> Total := 0:
> for k from 1 to 5 do
>     Total := Total + k^2
> end;
Total := 1
Total := 5
Total := 14
Total := 30
Total := 55
> k;
6
\end{verbatim}

We started by having a variable, \( \text{Total} \), that will hold the sum of the squares of the required integers. We must start with \( \text{Total} = 0 \). We used a “for” loop to make the iteration 5 times. Note that \( k \) starts with 1 and increased every time by 1.
In this for loop:

Number of iterations = 5.
The values of $k$ are 1, 2, 3, 4, 5.
The value of $k$ on exit is $k = 6$.

Consider the following commands:

```maple
> k := 'k';
> Total := 0;
> for k from 2 by 5 to 20 do
    Total := Total + k
end;
Total := 2
Total := 9
Total := 21
Total := 38
> k;
22
```

In this for loop:

Number of iterations = 4
The values of $k$ are 2, 7, 12, 17
The value of $k$ on exit is $k = 22$.

Consider the following “for loop”:

```maple
> k := 'k';
> Total := 0;
> for k from 5 to 1 do
    Total := Total + k
end;
> k;
5
```

There is no output from this for loop. Do you know why?

```maple
> k;
```

In this for loop:

Number of iterations = 0
The value of $k$ on exit is $k = 5$. 

```maple
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```
1.3.11. Arrays

Consider the following commands:

> restart:
> with(LinearAlgebra):
> A := Array(1..5);

\[
A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

> Dimension(A);

5

Note that A is an array of 0s and the length of A is 5.

> for k from 1 to 5 do Shift Enter
A[k] := k^2 Shift Enter
end do; Enter

\[
\begin{align*}
A[1] &= 1 \\
\end{align*}
\]

> A[1], A[3], A[5];

1, 9, 25

> B := Array(0..5):
> Dimension(B);

6

Note that B is an array of 0s of length 6. The first slot of B is B[0] and B[1] is the second slot.

> B[0], B[1], B[5];

0, 0, 0

> for k from 0 to 5 do Shift Enter
B[k] := 10*k Shift Enter
end do;

\[
\begin{align*}
B_0 &= 0 \\
B_1 &= 10 \\
B_2 &= 20 \\
B_3 &= 30 \\
B_4 &= 40 \\
B_5 &= 50 \\
\end{align*}
\]

> B[0], B[3], B[5];

0, 30, 50

> C := Array(2..5):
Note that the command Dimension works for arrays but not for lists or sets.

### 1.3.12. Graphing Functions and Expressions in Maple

To graph a defined function, expression, or a set of ordered pairs, use Maple’s functions `plot` or `pointplot`. You may restart the worksheet and load the plots package.

#### Graphing a Function

As an example, graph the function \( f(x) = x^2 + 1 \).

```maple
> restart:
> with(plots):
> f := x -> x^2 + 1;
> plot(f);
```

The graph is shown in Figure 1.3. Note that the default range for \( x \) is from \(-10\) to 10. The required range of \( x \) can be added to plot command as in the following example, where the graph is shown in Figure 1.4:

```maple
> plot(f, -2..3);
```

#### Graphing an Expression

To graph an expression \( y = -x^2 + 4 \), enter the commands:

```maple
> y := -x^2 + 4;
> plot(y);
> plot(y, x=-2..3);
```

The graphs of the last two commands are shown in Figures 1.5 and 1.6, respectively.
FIGURE 1.3. Graph from Maple command `plot(f)`, where the function $f(x) = x^2 + 1$ is defined by $f := x \mapsto x^2 + 1$.

FIGURE 1.4. Graph from Maple command `plot(f, -2..3)`, where the function $f(x) = x^2 + 1$ is defined by $f := x \mapsto x^2 + 1$. 
FIGURE 1.5. Graph from Maple command `plot(y)`, where the expression $y = -x^2 + 4$ is defined by $y := -x^2 + 4$.

FIGURE 1.6. Graph from Maple command `plot(y, x = -2..3)`, where the expression $y = -x^2 + 4$ is defined by $y := -x^2 + 4$. 
Plotting More Than One Curve on the Same Axes

Consider the expressions:

\[ y = x^2 + 1 \]
\[ y = -x^2 + 4 \]

To have the two graphs on the same axes where the range for \( x \) is from \(-3\) to \(3\), enter the following command. The graph is shown in Figure 1.7.

\[ \text{plot( [x^2 + 1, -x^2 + 4], x = -3..3 );} \]

Another way to obtain the same graph is define the two expressions in functional notation. Here are the commands that produce the same graph in Figure 1.7:

\[ f := x \rightarrow x^2 + 1, \quad g := x \rightarrow -x^2 + 4; \]
\[ \text{plot( [f, g], -3..3 );} \]

FIGURE 1.7. The graphs of the expressions \( y = x^2 + 1 \) and \( y = -x^2 + 4 \) on the same axes where the range of \( x \) is from \(-3\) to \(3\). The Maple command is \text{plot( [x^2 + 1, -x^2 + 4], x = -3..3 );}.}
1.3.13. Graphing Arrays, Lists, and Sets

According to a report published by the Center for Disease Control and Prevention (CDC), the number of diagnoses of HIV infection in the United States and dependent areas are given in the following table:

<table>
<thead>
<tr>
<th>Year</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of HIV</td>
<td>49,226</td>
<td>45,470</td>
<td>43,051</td>
<td>42,181</td>
</tr>
</tbody>
</table>

We can graph the number of diagnoses of HIV versus the year. Letting $X$ be a list (array) of years, and $Y$ be a list of number of diagnoses of HIV, and use the plot command.

```maple
> Y := [49226, 45470, 43051, 42181]:
> plot(X, Y, style = point);
```

The point graph is shown in Figure 1.8a. Note that the same graph may be obtained by the command

```maple
> pointplot(X, Y);
```

![Graph showing the number of diagnoses of HIV versus the year](image)

**FIGURE 1.8.** The (a) point and (b) line graphs of the number of diagnoses of HIV in the United States and dependent areas versus the year.
The line graph, shown in Figure 1.8b, may be obtained by the command

```maple
> plot(X, Y);
```

A set of ordered pairs can be graphed as in the following example:

```maple
> S := { [-3, 25], [-1, 18], [0, -5], [1, 4], [4, 14] }:
```

The following command produces the graph of the points (ordered pairs), which is shown in Figure 1.9a:

```maple
> plot(S, style = point);
```
FIGURE 1.9. The (a) point and (b) line graphs from the Maple commands:
Note the command `pointplot(S)` is the same as the command `plot(S, style = point)`.

The following command produces a line graph of the points (ordered pairs), which is shown in Figure 1.9b:

```maple
> plot(S);
```

FIGURE 1.9. (Continued)
You may obtain a graph that combines the given two graphs (Figure 1.9a and b) on one graph, Figure 1.9c, by entering the following commands:

```maple
> P1 := pointplot(S):
> P2 := plot(S):
> display(P1, P2);
```

1.3.14. Some Plot Options

Here are some of the options that can be included in Maple’s plot commands. A list of available options is found by entering the command

```maple
> ?plot
```

**color**

color = c, where c is a color such as black, red, blue, yellow, … for all color options enter

```maple
> ?plot, color
```

We assume that the functions f and g as well as the set of ordered pairs, PopPoints, are already defined.

**Example**

```maple
> plot(f, color = red);
> plot([f, g], color = [black, blue]);
```

**style**

style = s, where s may be a point, line, patch, or patchnogrid. The default is line.

The option style = points makes a graph of discrete disconnected points.

**Example**

```maple
> plot(PopPoints, style = point);
```

**labels**

labels = [xlabel, ylabel], where xlabel and ylabel must be strings.

**Example**

```maple
> plot(PopPoints, labels = ["Time n in years", "Population P(n)"]) ;
```
symbol

symbol = s, where s is one of the following: asterisk, box, cross, circle, diamond, point, solidcircle, solidbox, soliddiamond, solidsphere, or sphere. The default symbol is diamond.

Example

> plot (PopPoints, style = point, symbol = circle, color = red);
> plot (f, style = point, symbol = asterisk, color = blue);

view

view = [xmin..xmax, ymin..ymax], where the range for the x-axis is from xmin to xmax and the range for the y-axis is from ymin to ymax. The default range of x is from −10 to 10.

Example

> plot( x^2 + 10, view = [-5..8, 0..100] );
> plot([-sin(x), 4*cos(x)], view = [-4*Pi..6*Pi, -5..6] );

1.3.15. Iteration

Let us consider the following situation. The population of a species increases every year by 10%. Letting \( P_n \) be the population at the end of \( n \) years and \( P_0 = 100 \) be the initial population, this situation is modeled by the difference equation (or recurrence relation)

\[
P_{k+1} = 1.1P_k, \quad P_0 = 100, \quad k = 0, 1, \ldots, n
\]  

(1.25)

To obtain the sequence \( P_1, P_2, P_3, \ldots, P_6 \), we use \( P_0 \) to get \( P_1 \), use \( P_1 \) to get \( P_2 \), …, use \( P_5 \) to get \( P_6 \). To obtain the first 6 values of \( P_k \) after the initial value \( P_0 \), we use the difference equation (recurrence relation) (1.25) recursively. We have

For \( k = 0 \), \( P_1 = 1.1P_0 = 1.1(100) = 110 \)
For \( k = 1 \), \( P_2 = 1.1P_1 = 1.1(110) = 121 \)
For \( k = 2 \), \( P_3 = 1.1P_2 = 1.1(121) = 133.10 \)
For \( k = 3 \), \( P_4 = 1.1P_3 = 1.1(133.10) = 146.41 \)
For \( k = 4 \), \( P_5 = 1.1P_4 = 1.1(146.41) = 161.051 \)
For \( k = 5 \), \( P_6 = 1.1P_5 = 1.1(161.051) = 177.1561 \)

This process is called iteration and can be obtained by a “for” loop in Maple.

One way to write code to iterate the difference equation (1.25) is to create an array that contains the values of the initial value and the calculated values. Let us call the
array P. It is a good practice to create the array with zeroes, that is, an array with the required length and store zeroes in it.

Here is a possible code (list of commands) for this task:

> restart:
> with(LinearAlgebra):
> with(plots):
> n := 6:
> P := Array(0..n): # Define P as an array of zeroes with length 7
> P[0] := 100: # Store the initial value in the first slot
> for k from 1 to 6 do P[k] := 1.1 * P[k-1] 
end do;
P

1 = 110.0
2 = 121.00
3 = 133.100
4 := 146.4100
5 := 161.05100
6 := 177.156100

If we need to form the ordered pairs (k, P_k), k=0, 1, ..., n and graph them, that is graphing P_k versus k, we can enter the following commands:

> PopPoints := { seq([k, P[k]], k =0..n ) };

PopPoints := {
[0, 100], [1, 110.0], [2, 121.00],
[3, 133.100], [4, 146.4100],
[5, 161.05100], [6, 177.156100]
}

> pointplot(PopPoints);

The graph is shown in Figure 1.10.

1.3.16. Programs

In Maple, we can write programs/procedures. Let us use the given example to illustrate the structure of a procedure and the rules for utilizing it. In the example, the population of a species is modeled by the difference equation

\[ P_{k+1} = 1.1P_k, \quad P_0 = 100, \quad k = 0, 1, ..., n \]

We write a program (procedure) called Population that accepts, as input, the initial population P_0 and the time n. The output of the program/procedure Population is an array of length (n + 1) that holds the populations P_0, P_1, P_2, ..., P_n.

Although a procedure can be written and invoked in the worksheet, it is much easier and more efficient to write the procedure on a text editor and
save it. Then the procedure/program is called and invoked. Here are the suggested steps:

Step1: Write the procedure with documentation using a text editor such as Notepad. Here is the suggested code of the procedure,

```maple
# Program to calculate the population P(n) of a species
# represented by the linear first-order difference equation:
# P(k+1) = 1.1P(k), k = 0, 1, 2, ..., n
# Input:
# Initial population P0
# Time n in years
# Output:
# Array P of (n + 1) slots that contain P0, P1, P2, ..., Pn
Population := proc (P0, n)
    local k, P; # local variables
    P := Array(0..n); # Declaration of P
    P[0] := P0; # Storing P0 in the first slot
    for k from 1 to n do
        P[k] := 1.1*P[k-1] # "for" loop
    end do;
    return P; # The output
end proc;
```

FIGURE 1.10. Point graph of a species population, $P_n$, versus the time $n$ in years, where $P_n = 1.1P_{n-1}$, $P_0 = 100$, and $n = 0, 1, ..., 6$. 

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Save the program on a file. (We created a folder called MyMaplePrograms. In this folder, we created folders for book sections, for example, the folder for this section is Ch1S3. This program is saved as: C:/MyMaplePrograms/Ch1S3/P1_Population.txt.)

Step 2: It is recommended to have a new worksheet or enter the restart command on the current worksheet. Read the file that contains the program. If there are no errors, you can invoke the program; otherwise, you need to make the necessary corrections on the text editor and resave the file and then read it again at the worksheet. Here are the commands:

```maple
restart:
with(LinearAlgebra):
with(plots):
read "C:/MyMaplePrograms/Ch1S3/P1_Population.txt":
P0 := 100:
n := 6:
Y := Population(P0, n):
```

The output of the program Population is assigned to Y. Let us say that we want to form the set of ordered pairs \([k, Y[k]]\), \(k = 0, 1, \ldots, n\), so we can graph these ordered pairs.

```maple
PopPoints := { seq( [k, Y[k]], k = 0..n ) }:
plot(PopPoints, style = point);
```

The graph is shown in Figure 1.10. Now assume that we want have graph of the same problem with initial population, \(P_0 = 10,000\) and \(n = 20\) years, we just need to edit and enter the commands for \(P_0\) and \(n\); update the values of \(Y\), PopPoints; and re-execute the plot command. The graph is shown in Figure 1.11.

If you want to write the program Population in the worksheet, you can do the following:

```maple
restart:
with(LinearAlgebra):
with(plots):
Population := proc (P0, n)
    local k, P;
    P := Array(0..n);
P[0] := P0;
    for k from 1 to n do
        P[k] := 1.1*P[k-1]
    end do;
    return P;
end proc;
```

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Now assume that we want to call this program for \( P_0 = 1000 \) and \( n = 20 \) years, and graph \( P_n \) versus \( n \), we can enter the following code:

\[
\begin{align*}
> & P0 := 1000: \\
> & n := 20: \\
> & Y := \text{Population}(P0, n): \\
> & \text{PopPoints} := \{ \text{seq} \left( \left[ k, Y[k] \right], k = 0..n \right) \}: \\
> & \text{pointplot}(\text{PopPoints}, \text{symbol} = \text{solidcircle}, \\
& \quad \text{labels} = \left[ \text{"Time n in years"}, \text{"Population P(n)"} \right], \text{view} = [0..20, \ 0..8000] ) ; \\
\end{align*}
\]

The graph is shown in Figure 1.11, where the axes are labeled, the view (the range of the axes) is determined, and the symbol is selected.