Chapter 1

Fracture Mechanisms by Fatigue

1.1. Introduction

In the not so distant past, we often built not from precise calculations, but by intuition. Carpenters did not question the resistance of the wood they used to build their ships. However, there is no doubt that it is necessary to calculate beforehand, if we are to combine safety with economy in our works of art and professional projects. That is not to say that we can be one hundred percent sure about calculations, because they merely turn out to be the product of the transformation of figures that we put in. The figures themselves can also be marred by diverse mistakes, or not correspond with reality. Moreover, if we forget to calculate a particular part of the problem, there is no automatic mechanism which signals this omission. Therefore, we must go by calculations to obtain a satisfactory level of safety, bearing in mind the imprecision of figures, the irregularities of behavior in constructions, and even the defects of theoretical hypotheses.

The main problem, then, is to study how the construction of stability is modified by the random characteristics of the variables that govern it. We will first attempt to point out the importance of proper usage of materials and their propensity to crack. In this chapter, several important points concerning the analysis of the factors of cracking will be presented. Examples are based on a law renowned for being “simple”, but which is representative of crack propagation in zone II (see Figure 1.18). For a greater understanding of behaviors during diverse fracture mechanisms, we will refer to works specialized in continuum mechanics. The reasons governing our choice of welded structures can be explained by their practical importance in metallic works and installations (offshore, building, cars, and other devices assembled by welding). Also, welded structures show a considerable amount
of sensitivity in terms of failure (damage), due to fatigue of the notch stemming from potential penetration lacunae ($L$), located at the root of the weld bead.

Fatigue, like a succession of mechanisms, constitutes a process (distortions, loading), which modifies the properties of a material. This causes cracks that, over time, tend toward the fracturing of a material and/or of the structure. Although the range of stress is smaller than the resistance of the traction, the fact remains that it has a considerable influence on the reliability of the structure. There are stages which occur over time, ranging from activation, slow propagation to final fracture, used to predict the behavior of the structure. These phases are taken into consideration by most of the models concerned with cracking. This is, in fact, the reason we thought this chapter would be useful in a work dedicated to reliability and quality control. In fatigue, damage occurs in zones where the alternating stress is at its most intense: diverse cavities, notches, blowholes once they have been welded, strong heterogeneity of the material, etc.

Moreover, from microscopic examination of fracture, it is clear that typical facies are parallel to the crack propagation, followed by a tear, which is the final fracture. The most significant life expectancy (slow to moderate speed of crack propagation) corresponds to the activation of the crack. Life expectancy is relatively low. The problem lies with activation. In this phase, the material is subjected to damage, which cannot be detected by the naked eye. However, since the structure is not constantly under the microscope, it is beneficial to predict these phenomena using reliability calculations. It is this link between cause and effect which justifies reliability calculations. The stress intensity factor ($s.i.f., \Delta K$) starts at the foot of the weld bead [MAD 71, WAT 73, LAS 92]. The structure remains sensitive in terms of its resistance, presenting a high risk in fatigue.

1.2. Principal physical mechanisms of cracking by fatigue

1.2.1. Fracture mechanics

This section is intended to support the calculations of reliability indexes. Experimentally, it has been demonstrated that the presence of a crack in a part (structure) considerably modifies its resistance [GRO 98, LAS 92]. Additionally, we know that a crack could become unstable during loading. This crack could be propagated in increasing measures, before a brutal fracture occurs. To evaluate the residual strength of a cracked compound, fracture mechanics should be employed. The calculations of cracked solids are based on the crack (sometimes microscopic) in terms of a tributary surface discontinuity with forced decohesion between the neighboring atoms. Among the numerous studies on the topic, the work of Griffith [GRI 21], who pioneered the model on the crack resistance, is the most important. This original paper concerned fragile materials (e.g. glass). Irwin et al., in 1948,
applied Griffith’s work to solid components (structures). The following chart provides a short overview of the work on cracked components.

![Diagram of fatigue mechanisms](image)

**Figure 1.1. Simplified illustration of fragile fracture in mechanics**

Estimates of the longevity of fatigue are based on rigorous calculations, hence the method employed for finite elements (Figure 1.4). Of course, there are other analytical methods for simple cases (boundary integral equations, photoelasticity, or extensometers), as experimental approaches. With the aim of determining the number of $da/dN$ cycles per fracture, many laboratory tests were carried out on smooth test pieces during periodic loading. The literature confirms that during traction, at each one-fourth of a cycle, testing for traction gives a result which correlates to the maximum stress. Wöhler’s curve can then be used to find the link between the alternating stress and the number of cycles per fracture, from which the relation of load ($R$) indicates the quotient between the minimum stress and the maximum stress.

Resistance to fatigue is often modified by a host of factors, such as concentration of stress, temperature, loading, the topography of the surface (rugosity), and random phenomena (wind, ice, waves, etc.). This inevitably leads us to additive considerations of conventional and classic calculations for the resistance of materials. It now becomes even more necessary to consider the statistical aspect of the test results for fatigue. For example:

i) During classic fatigue, dispersion is greater than during low-cycle fatigue.

ii) The minimal endurance limit equates to about half of the endurance limit.

iii) In the low-cycle domain, longevity is defined according to a realistic interval of $[\mu - 3\sigma; \mu + 3\sigma]$ in the reputedly controlled domains (aeronautics or “building”).

At first glance, the problem of the depth of the initial crack ($a_0$) is simple. It becomes complicated when the variable representing the initial crack not only remains random but is also dependent on other parameters of the crack law:
\[
\frac{da}{dN} = C \times (\Delta K)^m
\]  \hspace{1cm} \text{[1.1]}

where:

\( \frac{da}{dN} \) is the geometric ratio of the number of cycles (mm/cycles);

\( C \) and \( m \) are intrinsic parameters of the material (adimensional);

\( \Delta K \) is the s.i.f. (MPa \((m)^{1/2} \approx \) tenacity).

\[
\Delta K = \Delta \sigma \times \sqrt{\pi \cdot a} \times \xi(a)
\]  \hspace{1cm} \text{[1.2]}

where:

\( \Delta \sigma \) is the stress amplitude in the normal direction of the crack (MPa);

\( a \) (or \( a_0/T \)) is the crack size (mm);

\( \xi(a) \) is the indication of corrected geometry (form factor).

The questions we ask, the traditional objectives of fracture mechanics, can be summarized as follows:

– What will be the residual resistance of a cracked component (structure)?

– What critical crack \((a_0/T)\) dimension would be tolerated (see Fig.1.19) by a given loading?

– How long does it take (see Fig.8.42, Chapter 8) for a microscopic crack to reach a critical length?

– Is there an effective experiment (recording gauges) to compare to the reliability model \((a \ priori\ distribution\ law)\)?

– How frequently should a component and/or a structure be inspected?

– Which metrology is implemented for detection (non-destructive control, of course)?

1.2.2. Criteria of fracture (plasticity) in mechanics

It is easy to want to stay in the comfort zone of elastics \((E)\), because the domain is so well-researched. Though, of course, this is not always the case. If a part is plastically deformed or remains unchanged during a given loading, there are always criteria to explain the nature of, what is called, plastic flow. The two most well-known criteria are TRESCA [TRE 81] and von Mises’ (1913) criteria. It is customary
to represent the elastic limit as $R_e$, which appears after the limit of a plastic deformation, in the case of traction according to a single axis ($xx$). The stress is written as $\sigma_{xx}$ and is inferior to $R_e$. Once modified by the safety factor ($s$), the stress becomes $\sigma_{xx} \leq R_e/s$. This is the acceptable limit. Once applied to materials and to structures, this relation will take into account the diverse variations (forms, notches, and fillets) which are essentially the origin of concentrations of stress, hence the relation [1.2].

When the elastic limit is exceeded, plastic deformation occurs. It can therefore be shown by $\sigma_{xx} \geq R_e$. These cases are encountered in the manufacturing process, during folding, stamping, laminating, or forging. The explanation for this resides in plastic deformation, which is shearing because the atoms slide and cause what is known as scission. The latter is maximal if the sliding angle ($\lambda$) is at 45° relative to the traction axis ($xx$).

![Diagram](image)

**Figure 1.2.** Simplified illustration of fracture criteria for mechanical plasticity
For a material that is stressed during loading, which has a known critical value ($K_{IC}$ or $G_{IC}$), in mode I the graph takes into account external factors such as the rate of loading and temperature, to name but two examples. These factors are not tributary to the geometry of the solid component. In reality, for a crack component, the tenacity $K_{IC}$ depends on the degree of biaxiality (see Figure 1.2 (top)), and even on the degree of triaxiality and of the stress of the cfr (crack front). In fact, this depends on the capacity of the solid component to endure plastic deformation in the cfr.

1.2.2.1. Tresca’s elastic limit criteria (1864)

The elastic and plastic domains are separated by a hypersurface with five dimensions (isostatic system). The principal stresses are represented by $[\sigma_1, \sigma_2, \sigma_3]$. The hypersurface takes the form $f(\sigma_1, \sigma_2, \sigma_3) = 0$. Since plastic deformation occurs during shearing, Mohr’s circle would normally be used to explain that for planar stress, the condition of elastic deformation being $R_e \geq |\sigma_1 - \sigma_2|$. In three dimensions, the stresses are shown as follows: $\{R_e \geq |\sigma_1 - \sigma_2|; R_e \geq |\sigma_1 - \sigma_3|; R_e \geq |\sigma_2 - \sigma_3|\}$.

1.2.2.2. von Mises’ elastic limit criteria (1913)

von Mises’ criteria translates the energy of elastic deformation $U = (\sigma \cdot \varepsilon)/2$ into traction–compression. During shearing, $U = (\tau \times \gamma)/2$ is used. Considering we often want to remain in the elastic domain, the energy must not exceed a maximum limit which can be formulated as follows: $R_e \geq \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2}$.

This expresses the ellipse equation, hence the use of Mohr’s circle. This topic could be developed further; however, it does not fall within the scope of this work. Therefore, the manuals concerned with the resistance of materials will be considered. To summarize, the literature proposes the following effective stresses:

- Tresca’s stress: $\sigma_e = \max\{ |\sigma_1 - \sigma_2|; R_e \geq |\sigma_1 - \sigma_3|; R_e \geq |\sigma_2 - \sigma_3| \}$

- von Mises’ stress: $\sigma_e = \frac{1}{2} \sqrt{(|\sigma_1 - \sigma_2|)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$

For planar bidirectional stress:

- Tresca’s stress: $\sigma_e = \sqrt{\sigma_1^2 + 4\tau^2}$

- von Mises’ stress: $\sigma_e = \sqrt{\sigma_1^2 + 3\tau^2}$

- The surface that determines the plastic domain of the elastic: $\sigma_e = R_e$

- Tresca’s stress: $\sigma_e \prec R_e$; von Mises’ stress: $\sigma_e \succ R_e$
Finite element modeling (see Figure 1.3) represents the equivalent field of stress with a color chart. The metallic parts which are subjected to repeated or alternating efforts can break, even if the maximum effort is inferior to the elastic limit. The life span of these parts far exceeds that of the lowest effort (Wöhler or $S–N$ curves).

Fatigue tests are carried out by subjecting a metallic test piece to traction/compression or alternating bending efforts. For most steels there is a critical effort, below which the fracture appears only after a considerable amount of time. This effort is the fatigue limit of steel.

The *rupture* originates from a minuscule crack which progressively expands until a brutal fracture occurs. We calculate the metallic parts subjected to repeated efforts, so that at no point the effort, by a square millimeter, exceeds the fatigue limit. This requires the parts of different sections to be connected to a fillet with a large radius of curvature and the state of the surface to be cared for.

For each cycle of the law of cracking, it is possible to say whether the structure has broken or not. This leads to separating the space into two distinct regions, as is shown in the following figures of models by finite elements (software ANSYS). In conditions of speed, deformation, and temperature, materials show plastic deformation at the tip of the crack, which is sufficiently small to be handled with linear elastic theory.

Paris’ law takes into account the stage of slow crack propagation by fatigue (activation stage of the crack in propagation phase). The crack is likely to propagate in three directions, which are linked to the applied efforts. Three modes of deformation can be distinguished as shown in Figure 1.3. According to the mode of crack propagation, three s.i.f. $K$ can be defined. In the singular zone, the stress field shows a singularity in $(1/\sqrt{r})$ at the tip of the crack.

### 1.3. Modes of fracture

It is generally accepted that the crack propagates due to a combination of stresses, according to the three following modes:

*Mode I* or opening: The normal traction stress is applied to the plane of the crack. In mode $I$, $K_I$ corresponds to the s.i.f. in the mode of opening of the crack edges (this fracture is extremely dangerous).

*Mode II* or straight slip: The shearing stress works in parallel to the plane of the crack and is perpendicular to the front of the crack. In mode $II$, $K_{II}$ corresponds to the s.i.f. in the mode of shearing on the plane of the crack edges.
Mode III or screw slip: The shearing stress works in parallel to the plane of the crack and in parallel to the front of the crack. In mode III, \( K_{III} \) corresponds to the s.i.f. in the mode outside the plane of the crack edges.

![Modes of deformation of a cracked body](image)

**Figure 1.3. Modes of deformation of a cracked body**

Factor \( K_I \) varies according to the nominal stress \( \sigma_n \) applied to a part which is half the length \( a \) of the crack. In the case of an infinite elastic medium, we use:

\[
K_I = \sigma_N \times \sqrt{\pi a}
\]

[1.3]

For parts with finite dimensions, it has been demonstrated that:

\[
K_I = \sigma_N \times \sqrt{\pi a} \times \xi(a)
\]

[1.4]
where $\xi(a)$ is a corrected coefficient of the geometry that allows $K_I$ to take the following corrected $K_{IC}$ values:

- ordinary steel: $K_{IC} \approx 10$ at 250 MPa m$^{1/2}$
- steel with very high resistance: $K_{IC} \approx 30$ at 100 MPa m$^{1/2}$

Using Irwin’s theory of elasticity, we present, in deformation or in planar stress, displacements $u_i$ and stresses $\sigma_{ij}$, in the singular zone, according to the mode considered.

**Mode I** is a mode of opening the crack, where the displacements are parallel to the direction of propagation. The following equations can be used:

$$
\begin{align*}
&U_1(x,y) = U_1(x,-y) \\
&V_1(x,y) = -V_1(x,-y)
\end{align*}
$$

$$
\begin{align*}
&U_1 = \frac{1}{2}[U(x,y) + U(x,-y)] \\
&V_1 = \frac{1}{2}[V(x,y) - V_1(x,-y)]; \quad W_1 = 0
\end{align*}
$$

**Mode II** is a mode of opening the shearing on the plane, where the displacements of the crack are parallel to the direction of propagation. We use:

$$
\begin{align*}
&U_2(x,y) = -U_2(x,-y) \\
&V_2(x,y) = V_2(x,-y)
\end{align*}
$$

$$
\begin{align*}
&U_2 = \frac{1}{2}[U(x,y) - U(x,-y)] \\
&V_1 = \frac{1}{2}[V(x,y) + V_1(x,-y)]; \quad W_2 = 0
\end{align*}
$$

**Mode III** is a mode of opening the anti-planar (outside-planar) shear, where the displacements of the crack are defined by $U_i, V_i, W_i$, where $i$ is the index indicating the elementary mode of fracture, that is, $i = I, II, or III$, for example:

$$
\begin{align*}
&U_2 = 0; \quad U_2 = 0; \quad W_2 = 0
\end{align*}
$$

Fracture can be mixed. In this case, we proceed to the additivity of the displacements. The combination of modes I and II gives, for example:

$$
\begin{align*}
&U = U_1 + U_2 = U(x,y) \quad \text{and} \quad V = V_1 + V_2 = V(x,y)
\end{align*}
$$
The mathematical equations of displacements $U_i$ and the stresses $\sigma_{ij}$, in Irwin’s sense, are written as follows:

\[
\begin{align*}
U_1 &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \cos\left(\frac{\theta}{2}\right) \left(\kappa - \cos \theta\right) + \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left(\kappa + \cos \theta + 2\right) \\
U_2 &= \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left(\kappa - \cos \theta\right) - \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left(\kappa + \cos \theta - 2\right)
\end{align*}
\]

\[\text{[1.9]}\]

The stress equations, according to Irwin, are written as follows:

\[
\begin{align*}
\sigma_{11} &= \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{30}{2}\right)\right] - \frac{K_{II}}{\sqrt{2\pi r}} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left(2 + \cos \frac{\theta - 30}{2}\right) \\
\sigma_{12} &= \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{30}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{30}{2}\right)\right] \\
\sigma_{22} &= \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{30}{2}\right)\right] + \frac{K_{II}}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{30}{2}\right)
\end{align*}
\]

\[\text{[1.10]}\]

with $\kappa = (3 - \nu)/(3 + \nu)$ in planar deformation and $\kappa = (3 - \nu)/(3 + \nu)$ in planar stress.

The shearing modulus is therefore

$$\mu = E/2(1+\nu).$$

where:

- $r$ and $\theta$ are the radius and the angle, respectively, in polar coordinates;
- $\nu$ and $E$ are Poisson’s coefficient and Young’s modulus, respectively.

It is worth pointing out that in the case of anti-planar loading, the only displacement component remains $U_3$. The respective expressions of displacements and stresses are therefore the following:

\[
\begin{align*}
U_3 &= \frac{2K_{III}}{\mu} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \\
\sigma_{13} &= -\left(\frac{2K_{III}}{\sqrt{2\pi r}}\right) \sin\left(\frac{\theta}{2}\right) \\
\sigma_{23} &= -\left(\frac{2K_{III}}{\sqrt{2\pi r}}\right) \cos\left(\frac{\theta}{2}\right)
\end{align*}
\]

\[\text{[1.11]}\]

The s.i.f. ($K_I$, $K_{II}$, and $K_{III}$) remain independent of $r$ and $\theta$. They are distribution functions of external efforts and crack geometry. Griffith’s theory was the first energetic approach on a cracked body. Moreover, there are other means of characterizing the singularity of the stress field in the neighborhood of the crack.
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front \((n.c.f.)\) or of studying the contour integral, which is Rice’s \([RIC\ 68]\) integral. The aforementioned concepts are only useful for isotropic materials, which have an elastic behavior. Factors \(K_I, K_{II},\) and \(K_{III}\) characterize both the detail of the geometry and of the crack, as well as the nature of the stress. Preventing fracture by fatigue means mastering parameters such as:

- conception of the part (materials and structure);
- some knowledge of loading;
- mastering residual stresses and the elaboration process of a material;
- accurately predicting life expectancy through inspections;
- appropriate dimensioning, periodic checks, and validating hypotheses.

By means of an example, for offshore structures, it is worth emphasizing fatigue inherent to waves, winds, and corrosion.

- For bridges and roads, it is advisable to keep a strict eye on Miner’s damage when dimensioning the roads.
- Ensure that the calculations take into account the stressed materials.
- In aeronautics, cyclic stresses (vibrations and variations of temperature) are at the root of mechanic’s particular interest in fracture.

1.3.1. Directed works

In this section we calculate the principal maximum and minimum stresses, as well as the normal stress and the shearing stress along the plane \((\delta = \text{determined by } e_j)\).

- normal stress on the \(x\)-axis: \(\sigma_x = 100 \times 10^3 \text{ Pa}\)
- normal stress on the \(x\)-axis: \(\sigma_x = 417 \times 10^3 \text{ Pa}\)
- shearing stress on \(xy\): \(\tau_{xy} = -47 \times 10^3 \text{ Pa}\)
- angle of planar stress: \(\theta = 100 \times 10^3 \text{ Pa}\)

1.3.1.1. Conditions and questions

- The normal stress on x- and y-axes are \(\sigma_x\) and \(\sigma_y\).
- The shearing stresses are \(\tau_{xy}\) and \(\tau_{xy}\).
- What are the principal stresses, \(\sigma_{\text{max}}\) and \(\sigma_{\text{min}}\), as well as the shearing stresses on the plane determined by \(AB\)?
Schematization of the problem:

Figure 1.4. Calculation of stresses

Solution:

In accordance with the fundamental principles of mechanics of solid materials, we consider the following.

For planar stress, the normal stress is written as

\[ \sigma_n(\theta) = \frac{\sigma_y + \sigma_x}{2} + \frac{\sigma_y - \sigma_x}{2} \cos(2\theta) + \tau_{xy} \times \sin(2\theta); \text{therefore } \sigma_n(\theta) = 3.171 \times 10^5 \text{ Pa} \]

i) Shearing stress on (xy)

\[ \tau_n(\theta) = \frac{\sigma_y - \sigma_x}{2} \sin(2\theta) - \tau_{xy} \times \cos(2\theta); \text{therefore } \tau_n(\theta) = 1.490 \times 10^5 \text{ Pa} \]

ii) Maximum principal stress

\[ \sigma_{\text{major}} = \frac{\sigma_y + \sigma_x}{2} + \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}; \text{therefore } \sigma_{\text{major}} = 4.242 \times 10^5 \text{ Pa} \]

iii) Maximum principal stress

\[ \sigma_{\text{minor}} = \frac{\sigma_y + \sigma_x}{2} - \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}; \text{therefore } \sigma_{\text{minor}} = 1.098 \times 10^5 \text{ Pa} \]
iv) Maximum and minimum shearing stress

\[
\tau_{\text{max}} = \frac{\sigma_{\text{Major}} - \sigma_{\text{minor}}}{2} = 1.572 \times 10^5 \text{ Pa}; \text{ and } \tau_{\text{min}} = -\tau_{\text{max}} = -1.572 \times 10^5 \text{ Pa}
\]

The relation between the stresses and the tilt angle of the plane is demonstrated in the following graph, with an angle range of 0° to 360°. The range (R) of the angles \( \theta = 0\degree, 1\degree, \ldots, 360\degree \).

![Graph showing principal stresses](image15)

**Figure 1.5. Graph showing principal stresses**

![Graph showing the maximum and minimum shearing stresses](image16)

**Figure 1.6. Graph showing the maximum and minimum shearing stresses**

1.4. Fatigue of metals: analytical expressions used in reliability

What we call fatigue or damage by fatigue is the modification of materials due to the application of effort cycles, which, through repetition, lead to the fracture of component parts. As soon as there is an applied effort over time, what we call
fatigue occurs. Fractures can, therefore, appear for stresses which are often lower than the fracture limit of the material, and even the elasticity limit. Damage is accompanied with no apparent modification of form or the aspect of the part.

The origin of the fracture is due to a progressive crack which stretches until the remaining transversal section can no longer support the applied effort. When we subject test pieces to the cycles of periodic stress, to maximum amplitude (s) of constant frequency, for every fracture, we call $N$ the number of cycles. $(N, \sigma)$ – Stress number of cycles (endurance curve). The three domains that have been previously detailed can be distinguished here.

1.4.1. Wöhler’s law

It is believed by numerous authors that Wöhler’s law (1860) is the oldest [WÖH 1860]. It allows a good representation that is completed by the middle part of the $S–N$ curve. This can be explained by the fact that most curves have a low inflexion point in the neighborhood to which they are rectilinear. The general speed of the $S–N$ diagram is described by the equiprobability curves of fracture, for which this is the expression:

$$\log N = a - (b \times \sigma)$$

[1.12]

where:

- $N$ is the number of cycles;
- $a$ and $b$ are two constants;
- $\sigma$ is the amplitude of the applied stress.

Wöhler’s curves are established with experiments. They depend on numerous factors, such as the given maximum state, the loading, the environment, and the nature of the material. The $S–N$ curve does not, in reality, represent a sharp “bend”, but is a progressive curve connecting to the horizontal branch. Calculating the structures gives each cycle an average stress and an alternating stress, which are often calculated in elasticity. The inherent $S–N$ curves are taken into consideration for each value attributable to the load ratio ($R$) where $\alpha$ and $\beta$ are the characteristics of the material under average stress in a controllable temperature. In the laboratory the expression [1.3], taken from the literature, is:

$$N = \kappa (R) \times (R) \times S_{\alpha}^{-\beta(R)} ; \text{ when } S_{\alpha} > S_{\alpha D}(R)$$

[1.13]
where:

- $S_{ad}$ is the endurance limit;
- $K$ is a load factor;
- $R$ is the load ratio, or $R = \left(\frac{\sigma_{\text{min}}}{\sigma_{\text{max}}}\right)$.

In Chapter 8, which wholly dedicated to the valuation of Monte Carlo’s (MC) method, a case study will be presented. However, it is worth concentrating on Wöhler’s curve ($S$–$N$), because in terms of probability, it is true that the model we are looking to make more reliable is the one that results in Wöhler’s famous curve. The objective of the probabilization of this model is, therefore, to obtain a network of $pS$–$N$ curves, digitally generated by MC simulation. This causes uncertainties, which in turn contribute toward the variability of the curve $S$–$N$. This essentially depends upon:

- the state of the local stress $\sigma$, in MPa;
- the applied loading $S(t)$;
- the intrinsic parameters of the materials;
- the criteria of fatigue;
- rugosity (state of the surface), etc.

In our experimental opinion, metallurgy plays an important role in the variability of the $S$–$N$ curves. It is advisable to bear this in mind in the spreading of uncertainties (see Chapter 3, volume 1).

### 1.4.2. Basquin’s law (1910)

Basquin’s equation represents a hyperbolic form from which a branch asymptotically links to the axis $N$ (loading), as follows:

$$\log N = a - b \times \log(c)$$  \[1.14\]

This type of curve, like Wöhler’s, cannot provide a fatigue limit. With $a$, $b$, and $\delta$ being constant, it can also be written as:

$$c = \left(\frac{a}{N}\right)^{\delta} \quad \text{when} \quad A = e^a \quad \text{and} \quad \delta = \left(\frac{1}{b}\right)$$  \[1.15\]
1.4.3. Stromayer’s law (1914)

Stromayer proposes another relation for greater precision. His law represents the logarithm of the number of cycles in a function of logarithms of applied stress, which however does not have a value to render endurance. It is presented as follows:

\[
\log(N) = a - b \times \log(c - \rho_e)
\]  

[1.16]

where:
- \(\rho_e\) represents the endurance limit;
- \(N\) is the number of cycles;
- \(a\) and \(b\) are constants;
- \(c\) is the amplitude of the applied stress.

The curve that is obtained is identical to Basquin’s curve. It shows, in addition, a horizontal asymptote of ordinates \(\rho_e\), often not confused with the life expectancy axis. Specialists often believe that Stromayer’s law is more realistic than Wöhler or Basquin’s law. Nevertheless, it causes several difficulties of adjustment.

1.4.4. Palmgren’s law

To avoid adjustment problems, Palmgren (1924) introduced the following:

\[
C = \rho_e + \left( \frac{A}{N + B} \right) \text{ when } A = e^a
\]  

[1.17]

where:
- \(\rho_e\) represents the endurance limit;
- \(N\) is the number of cycles;
- \(B\) is the constant parameter;
- \(C\) is the amplitude of applied stress.

For some authors, this law is realistic because it allows us to adjust the quality for a given data. The law can be written in two forms. The first is similar to Palmgren’s law:

\[
\log(N + B) = a - b \times \log(c - \rho_e)
\]  

[1.18]
The classic way of tackling the problem of variable amplitude loading, by Palmgren and Miner, consists of using damage accumulation laws. Palmgren assumes that the total damage of a structure is the sum of damages sustained during each of the loading levels of aptitude.

1.4.5. Corson’s law (1949)

Corson [COR 49] tackled the damage accumulation problem. His relation is based on experiments, summarized in the following equation:

\[ N = \frac{A}{(S - E) \times d^{(S - E)}} \quad [1.19] \]

where:
- \( S \) is the stress representing the endurance limit;
- \( N \) is the number of cycles;
- \( E, A, \) and \( d \) are constants.

By using \( c = \text{Log}(d) \), equation [10.19] becomes:

\[ N = \frac{A \times \exp^{-C(S - E)}}{(S - E)} \quad [1.20] \]

This law, though more recent than Palmgren’s, deals with the problem by linking the number of cycles to stresses. The coefficients it integrates means that the unknown factors are added to the calculations, resulting in imprecisions. It is therefore rarely used.

1.4.6. Bastenaire’s law

In the 1960s, Bastenaire [BAS 60] proposed a formula, which seems the most precise:

\[ N = \left( \frac{A}{C - \rho_e} \right) \times \exp^{-(C - \rho_e)} \quad [1.21] \]
where:
\[ \rho_e \] represents the endurance limit;
\[ N \] is the number of cycles;
\[ A \] and \[ B \] are constant parameters;
\[ C \] is the amplitude of the applied stress.

In terms of a critique, this law does not differ much in its formulation from the other laws that have been described here. Its credit lies in its being a relatively recent law; however, it brings very little new to the table. In fact, it integrates the same parameters (\( C \) and \( \rho_e \)).

### 1.4.7. Weibull’s law

Weibull’s law [WEI55] does not provide any more explanation than Palmgren’s. For reference, it is written as follows:

\[
\log(N + B) = \frac{a - b(S - \rho_e)}{(R - \rho_e)} \tag{1.22}
\]

where:
\[ \rho_e \] represents the endurance limit (constant);
\[ N \] is the number of cycles;
\[ R \] is the resistance to the traction of the material.
\[ a, b, \text{ and } B \] are constant parameters, which take into account the form, the position, and the scale of the probability density functions curve.

Weibull’s law is often used in statistics to represent the life expectancy of structures whose rate of fracture depends on the parameter \( \beta \). It will be studied in greater detail in Chapter 1, statistics and reliability.

### 1.4.8. Henry’s law

This law expresses the damage for a level of stress and is written as follows:

\[
D = \left( \frac{n}{N} \right) \times \frac{1}{1 + \alpha \left( 1 - \frac{n}{N} \right)} \quad \text{and} \quad \alpha = \frac{\rho_e}{\sigma - \rho_e} \tag{1.23}
\]
where:

\( \rho_e \) represents the fatigue limit (endurance);

\( n \) is the constant parameter.

\( \alpha \) takes into account the fact that the speed of damage is faster for greater efforts and due to increasing the number of imposed cycles \( N \) to the level of stress \( (\sigma) \).

### 1.4.9. Corten and Dolen’s law

In this law, damage is seen to result from the germination of pores, which have developed from cracking. It is written as follows:

\[
N_g = \frac{N_1}{\alpha_1 + \alpha_2 \left( \frac{C_2}{C_1} \right) \times d + \alpha_3 \left( \frac{C_3}{C_1} \right) \times d + \ldots + \alpha_n \left( \frac{C_n}{C_1} \right) \times d}
\]

[1.24]

where:

\( N_g \) represents the number of stress cycles leading to fracture, for the chronological increase in stress amplitude;

\( N_1 \) is the number of cycles at the highest level of stress before fracture;

\( \alpha_1, \alpha_2, \) and \( \alpha_n \) are the ratios of the number of cycles applied to the level of stress for the total number of applied cycles;

\( C_1 > C_2 > C_n \) are different levels of alternating stress (applied amplitude);

\( d \) is the reverse of Wöhler’s linear sloping partition.

Analytical expressions from the domain of low-cycle fatigue and other random aspects of the phenomena of fatigue consider that metal can become harder, soften, or remain stable under cyclic stress (plastic domain). From the evolution in mechanics of hysteresis loops, a strain-hardening curve has been determined [LIE 73, LIE 82], which is given by the following expression:

\[
C_a = K' \times \left( \frac{\Delta \epsilon_r \rho}{2} \right) \times n'
\]

[1.25]

where:

\( C_a \) represents the rational alternating stress \( (C_r) \) and \( K' \) is the coefficient;

\( \Delta \epsilon_r \) is the amplitude of rational plastic deformation \( (\epsilon_r) \);

\( (n') \) is the coefficient for cyclic strain-hardening.
1.4.10. Manson–Coffin’s law

Characterized by considerable deformation and a relatively low life expectancy, that is less than 10,000 cycles, Manson–Coffin’s law is used in low-cycle fatigue. The facies of the fracture surfaces are drawn more closely together from those obtained during a static fracture, than from the facies of a fracture surface by fatigue. This means that this law is rarely used. The mechanisms which operate the phenomenon of facies differ to those governing ordinary fatigue. In fact, the amplitude of the deformation is considered, rather than the stress amplitude. Manson–Coffin’s law assumes that the alternating number to fracture \( N_f \) is linked to the amplitude of the plastic deformation \( \Delta \varepsilon_p \)

\[
\left( \Delta \varepsilon_p \right) \times \left( N_f^\alpha \right) = \beta
\]

where:

\( \alpha \) represents the coefficient between 0.5 and 0.7.

\( \beta \) is the constant linked to real deformation while the fracture is in traction, and it can also be expressed as follows:

\[
\beta = 2^{(1-\alpha)} \times \log \left( \frac{S_0}{S} \right)
\]

where \( S \) and \( S_0 \) are the initial damage sections for a traction test.

Manson–Coffin’s law is generally proven when the amplitude of plastic deformation is greater than 1/100. This sensitivity limits its use. The life expectancy expressed in number of cycles leading to fracture is therefore provided by the Manson–Coffin relation in plastic deformation. For total deformation, another relation called Morrow’s relation is employed:

\[
\frac{\Delta \varepsilon_p}{2} = \Delta \varepsilon_p' \times \left( 2N_f \right)^{-c} \text{ in plastic deformation}
\]

Morrow’s relation, for total deformation, is expressed as follows:

\[
\frac{\Delta \varepsilon_t}{2} = \Delta \varepsilon_p' \times \left( 2N_f \right)^{-c} + \left( \frac{C'}{E} \right) \times \left( 2N_f \right)^{b}
\]

where:

\( E \) is Young’s modulus;

\( N_f \) is the number of cycles leading to fracture;
Δε is the amplitude of the total deformation;
ε′ is the coefficient of ductility in fatigue;
C′f is the coefficient of resistance to fatigue.

Morrow’s relations demonstrate that ductile materials which have high coefficients show a good resistance to large deformations. The two essential factors for resistance to fatigue will lead us to introduce the probabilistic concepts for both fatigue and resistance. The main reasons for this being:

– stress fluctuations of a part during service;

– the dispersion of resistance characteristics in fatigue.

The causes of fatigue can originate from external factors and are due to the implementation of the material (thermal treatments, strain-hardening, etc.). The parameters affecting the resistance to fatigue of a material amount to the endurance limit. According to Shigley, the resistance of any part can be expressed as follows:

\[ \rho_e = K_a \times K_b \times K_c \times K_d \times K_e \times K_f \times (\rho_e') \]  

[1.30]

where:

\( \rho_e' \) is the endurance limit on a smooth test piece in rotary bending;

\( K_a \) represents the scale effect;

\( K_b \) represents the surface effect;

\( K_c \) represents the temperature effect;

\( K_d \) represents the reliability effect;

\( K_e \) represents the notch effect;

\( K_f \) represents other effects.

The scale effect \( K_a \) allows the endurance of machine parts and the endurance of test pieces to be compared. The majority of fractures by fatigue \( K_b \) begin at the surface, hence the result machines fracture. According to Shigley, the temperature effect is written as:

\[ K_c = 620/(460 + T), \text{ when } T > 160^\circ Fahrenheir \]  

[1.31]

When the temperature rises, the elasticity limit and the resistance to traction are reduced. The same is true of \( \rho_e \). For example, on one of Wöhler’s classic curves, the endurance limit corresponds to a fracture probability of ½. When the endurance limit corresponds to a survival probability greater than ½, the number of standard deviations
must be subtracted from the endurance limit. Working from the hypothesis that standard deviation on \( \rho_e \approx 80\% \), we consider:

\[
K_d = \{1 - 0.08 \times D\}, \text{ with } D \text{ diameter in mm or inches}\]  \[1.32\]

Factor \( K_d \) is expressed by means of stress concentrations, caused by changes in the section (holes, porosities, heterogeneities, notches, etc.). For welded structures, it is advisable to take the weld bead and the superficial or dense faults situated at the foot of the weld bead (micro-geometry) into consideration. According to the authors I.F.C. Smith and R.A. Smith [SMI 83], the sensitivity of the notch is calculated by the following relation:

\[
q = (K_u - 1)/(K_t + 1)\]  \[1.33\]

To calculate \( K_t \), Peterson proposes the following relation:

\[
K_t = K_i \times \sqrt{1 - \nu \times \left( \frac{K_t - 1}{K_i} \right) + \nu \times \left( \frac{K_t - 1}{K_i} \right)^2}\]  \[1.34\]

Where \( K_i \) is the stress concentration factor.

The mathematical theory of elasticity provides many valuable solutions involving the stress distributions in bodies of simple geometries and loadings. A common use of these solutions is the determination of stress concentration factors (\( K_t \)) resulting from discontinuities or other localized disturbances in the stress field of the solid body.

\( \nu \) is Poisson’s coefficient;

\( K_u \) is the ratio of endurance limits on smooth test pieces and notched test pieces, in which;

\( K_u < K_t \) on account of the possibility of the material adapting;

\( K_u \rightarrow K_t \) for materials with very high elastic limits.

1.5. Reliability models commonly used in fracture mechanics by fatigue

Eyring’s models of stress acceleration are simple. They tend to be used in the domains of applied chemistry and quantum mechanics, because stresses are heavily involved in the process. His model has several characteristics:

– a theoretical basis of chemistry and of quantum mechanics;
– a chemical process (diffusion, corrosion, migrations, etc.) is the cause of degradation which leads to damage due to varying rates of degradation with stress;
Fracture Mechanisms by Fatigue

– a temperature that influences relevant stresses: similar case to Arrhenius’ relation which is an empirical model (activation energy necessary to cross an energy barrier and to start a reaction).

\[
\tau_f = AT^\alpha \exp \left( \frac{\Delta H}{\kappa T} + \left( \frac{B + C}{T} \right) \times S_1 \right)
\]

[1.35]

\( S_1 \) is a function of current–voltage, and the parameters \( \alpha, H, B, \) and \( C \) serve to determine the acceleration between the combination of stresses. As in Arrhenius’ model, \( \kappa \) is Boltzmann’s constant and \( \Delta T \) is Kelvin’s variation of degrees of temperature. Arrhenius’ model predicts failure by increasing (acceleration) the temperature. One of the early models of acceleration predicts the variation of temperature as in the following formula:

\[
\tau_f = \gamma \times \exp \left( \frac{\Delta H}{\kappa T} \right)
\]

[1.36]

With temperature \( T \) in Kelvin (+273.16°C) at the moment, the fracture occurs and \( \kappa \) as Boltzmann’s constant (8.617 × 10⁻⁵ in eV/K). The constant \( \gamma \) is a scale factor which takes into account the calculation of acceleration factors, and \( \Delta H \) designating activation energy, which is the critical parameter in this model. Indeed in this model, Arrhenius’ activation energy, \( \Delta H \), is the factor that must be known to calculate the acceleration of temperature. \( \Delta H \) generally varies between 0.3 and 1.5 and depends upon the fracture mechanism and the materials involved. The acceleration factors between the two temperatures exponentially increase with the rate of increase of \( \Delta H \). The acceleration factor between a higher temperature \( T_2 \) and a lower temperature \( T_1 \) is given by:

\[
f_{acc} = \exp \left( \frac{\Delta H}{\kappa} \left( \frac{1}{T_1} + \frac{1}{T_2} \right) \right)
\]

[1.37]

By substitution the value for Boltzmann’s constant \( \kappa \), the expression of AF is written as a function of \( T \) in °C as follows:

\[
f_{acc} = \exp \left( \Delta H \times 11605 \times \left( \frac{1}{T_1 + 273.16} + \frac{1}{T_2 + 273.16} \right) \right)
\]

[1.38]

Arrhenius’ model is normally used to model the life expectancies of cases when the failing mechanisms depend upon chemical reactions, diffusion processes,
or a migration process (metallurgy). By adding a term to the stress (non-thermal), Eyring’s model becomes:

\[
\tau_f = AT^\alpha \exp \left\{ \frac{\Delta H}{kT} + \left( B + \frac{C}{T} \right) \times S_1 + \left( B + \frac{E}{T} \right) \times S_2 \right\}
\]  

[1.39]

Figure 1.7. Acceleration factor of Eyring’s model in function of \( \Delta H \)

Most of the models used in mechanics do not comprise interaction terms. As well as temperature interactions, Eyring’s model takes the stress into account. In the models without interaction, the acceleration factors for each stress can be calculated. Eyring’s model is often complicated to use in its most general form, and must be simplified for every mechanism of a particular failure. It turns out to be disadvantageous on account of the numerous parameters it includes. Even with just two stresses, there are five parameters to estimate. Each additional stress adds another two unknown parameters. Certain parameters have a secondary effect. For example, when \( \alpha = 0 \), the model works quite well since the temperature term brings us closer to Arrhenius’s model. Moreover, the constants \( C \) and \( E \) are only necessary if there is an interaction effect of temperature, with respect to other stress factors.

1.5.1. Coffin–Manson’s model for the analysis of crack propagation

Coffin–Manson’s model is suitable for the evaluation of crack propagation in fatigue of materials. A model of this type, known as Coffin–Manson’s model, has been successfully used for crack propagation in metals assembled by welding. The model takes the following form:

\[
N_f = \frac{1}{f_{acc}} \times \Delta T^{-\beta} \times G(\tau_{\text{max}})
\]  

[1.40]
where:

- $N_f$ is the number of cycles to fracture;
- $f$ is the cycle frequency;
- $\Delta T$ is the temperature during a cycle;
- $G(\tau_{\text{max}})$ is a factor valued at the maximum temperature reached each cycle.

The typical values for the exponent of cycle frequency ($\alpha$) and the exponent of temperature range ($\beta$) vary between $-1/3$ and 2.

$\Delta H$ is the term of activation energy when $G(\tau_{\text{max}})$ is about 1.25.

Arrhenius’s model defines damage as a result of acceleration due to an increase in temperature. The early model of acceleration is more successful in predicting the time of fracture (time-to-fail) which varies according to temperature.

**EXAMPLE.**— $G(\tau_{\text{max}}) = 1.25$; $f = 50$ Hz, $\Delta T = 20$ to $50$ °C, $\alpha = -1/3$ and $\beta = 2$, $f_{\text{acc}} = 1$ (acceleration factor), the curve of the number of cycles in function of temperature is found and traced with a unit acceleration factor (=1).

<table>
<thead>
<tr>
<th>$\Delta T$</th>
<th>$N_f(\Delta T)$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0.012</td>
</tr>
<tr>
<td>1</td>
<td>0.011</td>
</tr>
<tr>
<td>2</td>
<td>0.01</td>
</tr>
<tr>
<td>3</td>
<td>9.962×10^{-3}</td>
</tr>
<tr>
<td>4</td>
<td>9.515×10^{-3}</td>
</tr>
<tr>
<td>5</td>
<td>9.096×10^{-3}</td>
</tr>
<tr>
<td>6</td>
<td>8.705×10^{-3}</td>
</tr>
<tr>
<td>7</td>
<td>...</td>
</tr>
</tbody>
</table>

**Figure 1.8.** Number of cycles to fracture in function of the variation of temperature and the acceleration factor in Arrhenius’ model

### 1.5.2. Neuber’s relation (1958)

The classic methods of analysis are based either on Neuber’s [NEU 58] coefficient, or on the local amplitude of deformation. The relations [1.41], [1.42],
and [1.3] are applied for sharp notches, where \( r \) is the radius of the notches and \( \rho' \) is a characteristic of the material.

\[
K_u = 1 + K_i \times \frac{1}{\sqrt{1 + \frac{\rho'}{r}}} \quad [1.41]
\]

The number of cycles at initiation \( N_i \) is expressed by the following relation:

\[
\log(K_u) = 1.2969 + 0.1602 \times \log(N_i) \quad [1.42]
\]

where \( K_u \) is Neuber’s coefficient as a result of relation [1.41].

Neuber found that \( \rho' = 0.48 \) mm as agreed in bending tests. That the relation agrees with its values is also confirmed by Markovin and Moore [SMI 82, SMI 85] on fatigue tests for steel: SAE 1035, SAE 1010 and SAE 1038 (Canada and USA).

\[
K_u = 1 + \frac{K_t - 1}{\left(1 + \frac{\pi}{\pi - \omega} \times \frac{\rho'}{r}\right)} \quad [1.43]
\]

where \( K_u \) is the Neuber’s coefficient which takes into account the opening angle of the notch \( (\omega) \). The formulas [1.40] and [1.43] therefore allow \( K_y \) to be calculated by the notch effect which is partially due to the state of the combined stress, existing at the bottom of the notch. By combining von Mises’ criteria and Neuber’s theory in relation to the distribution of notch stress, we obtain a theoretical coefficient of notch effect \( K_u \). \( K_t \) can therefore be deduced from Heydoo’s relation, which is written as follows:

\[
\left(\frac{K_t}{K_u} - 1\right) = \left(\frac{M}{\sqrt{n \times r}}\right) \quad [1.44]
\]

where

- \( n \) is the function coefficient of the notch type;
- \( M \) is the constant of the material.

To find \( M \) and \( n \), special manuals on the relevant topic will be used. Finally, Neuber and Heydoo explain the relation between \( K_t \) and \( K_u \) by a single parameter \( (r) \)
for a given material and a given notch type. The notch effect is given by the following relation:

\[ K_e = \frac{1}{K_u} \]  

[1.45]

It essentially depends upon:
- cavitation, and/or aging;
- corrosion and contact corrosion;
- residual stresses;
- radiations and other aggressions.

Neuber also presented a method founded on the s.i.f. (= \( \Delta K \)) for a sharp notch (\( \rho < 2/10 \) mm) on a soft steel plate, more than 20 mm in length and 5 mm in thickness. The number of cycles \( N_i \) necessary for initiation of a crack measuring 1/10 mm is as follows:

\[ N_i = \left(2.90 \times 10^8\right) / \Delta K^4 ; \Delta K \text{ in MPa} \sqrt{m} \]  

[1.46]

In general, caution must be paid to the formulations linking \( N_i \) and \( \Delta K \) (intrinsic parameters of material \( C \) and \( m \)). It would seem that the initiation of a fatigue crack would make considerably more parameters intervene than those shown here and in technical literature. In our opinion, intensively using the electronic microscope should allow us to be more specific about these parameters and their actions on the safety of structures. However, mathematical simulations, as sophisticated as they may be, do not replace material observation of resistance phenomena in fracture mechanics by fatigue.

Coffin–Manson’s model is not suitable for the use of Eyring’s models for crack propagation by fatigue. The explanation of this model is as follows.

Coffin–Manson’s model is one which takes into account crack propagation in stage 1 of Ritchie’s representation (see Figure 1.18). It therefore takes into account the phenomena of deformation of materials by fatigue where Irving’s theory (see [1.47]) is not applicable. Therefore, Coffin–Manson’s model is extremely useful to study the stresses of cycles and the frequency of use with variable temperatures. This is a crucial point, since other models sometimes do not integrate the temperature in a significant way, which is the most important parameter in welded structures. The expression is written as follows:

\[ N_r = A \times f^{-\alpha} \times \Delta T^{-\beta} \times G(T_{\text{max}}) \]  

[1.47]
where:

- $N_r$ is the number of cycles to fracture;
- $f$ is the cycle frequency;
- $\Delta T$ is the variation of the temperature during the cycle;
- $G(T_{\text{max}})$ is one of Arrhenius’ terms, valued at the maximum $T \, ^\circ C$ reached during each cycle;

The typical values for each cycle frequency ($\alpha$) and the temperature range ($\beta$) are $\alpha = -1/3$ and $\beta = 2$, respectively.

By reducing the cycle frequency, the number of cycles to fracture also reduces. Literature states that activation energy $\Delta H$ represented by $G(T_{\text{max}})$ is around 1¼.

1.5.3. Arrhenius’ model

Arrhenius’ model is an important model which is used particularly in its capacity to predict the acceleration of failures (damage) due to an increase in temperature. This early model of acceleration predicts the time to fracture (time-to-fail) as a function of the temperature. Arrhenius’ empirical equation is written as:

$$T_r = \alpha \times \text{Exp} \left( \frac{\Delta H}{K \times T} \right)$$

[1.48]

where:

- $T$ is the temperature measured in Kelvin (+273.16 °C) at the point of fracture.
- $\alpha$ is the constant which represents the scale factor, initiated with acceleration factors $\Delta H$ expressing the activation energy (the latter is a critical parameter of Arrhenius’ model).

$\Delta H$ incorporates the range of values (0.3 or 0.4 reaching 1.5 or more). It depends upon the damage mechanism and on the materials, and the acceleration factors between two temperatures exponentially increase as $\Delta H$ increases.).

$K$ is the Stephan–Boltzmann’s constant (= 8.617 $\times$ 10$^{-5}$ in ev/K).

The acceleration factor ($F_a$), for $T_2$ (high temperature) and $T_1$ (low temperature) is expressed as:

$$F_a = \text{Exp} \left( \frac{\Delta H}{K} \times \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \right)$$

[1.49]
Activation energy $\Delta H$ is the only factor needed to calculate the acceleration of temperature. Using value $K$ (in Kelvin) in degree Celsius, we obtain:

$$F_a = \exp \left( \Delta H \times 11605 \times \left( \frac{1}{T_1 + 273.16} - \frac{1}{T_2 + 273.16} \right) \right)$$

[1.50]

As has been previously stated, $\Delta H$ is unknown. Arrhenius’ model has been successfully used to study the damage mechanisms which depend on corrosion (offshore structures assembled by welding and/or bolts), diffusion processes, or migration processes, to name but a few examples. This model is also widely used in the domain of electronic equipment. This work focuses principally on continuum mechanics.

**Digital application:** Let us calculate the acceleration factor $F_a$ for $T_1 = 20 \, ^\circ C$ and $T_2 = 100 \, ^\circ C$ if $\Delta H$ varies between $[\frac{1}{2}$ and 1] with an increment of 0.05. Therefore, $F_a$.

<table>
<thead>
<tr>
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<th>$F_a(\Delta H)$</th>
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</tr>
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<td>$3.17 \times 10^3$</td>
</tr>
<tr>
<td>10</td>
<td>$4.85 \times 10^3$</td>
</tr>
</tbody>
</table>

with $\Delta H := 0.5, 0.55, 1$

**Figure 1.9.** Evolution of the acceleration factor as a function of activation energy $\Delta H$

1.5.4. **Miner’s law (1954)**

With respect to the physical description of damage, the reliability approach means the following problems can be resolved: oversizing, resistance to different stresses, appropriate geometry, and metrology of parts or of metallic assembly [LIN 65]. Statistical analysis takes into account the reliability treatments of dispersion causes, which imply that life expectancy on Wöhler’s curve cannot be
represented by a point, but by the distribution of $N_{\text{cycles}}$. $N_p$ and $N_c$ indicate that the stressed part by $\sigma_p$ has a probability $p$ to break (see Figure 1.7) after $N_p$-cycles. The notion of damage assumes two distinct aspects, one aspect being physical and the other descriptive. It corresponds with variations of the physical properties of materials subjected to different stresses. It also corresponds to a quantitative description of the endurance of materials subjected to different stresses. Miner [MIN 54] proposed a simple damage law. The hypothesis is as follows.

$$D = \left( k \sum_{i=1}^{n_i} \frac{n_i}{N_i} \right) = 1 \quad (D \text{ for damage}) \quad [1.51]$$

If the fraction ($n_i/N_i$) of life expectancy is carried out at a certain level of stress $C_i$, the remaining endurance at another level $C_z$ will be ($n_z/N_z = 1 - Z$). Miner’s law is not very precise, but it is very simple. It takes into account what is called understressing and overstressing.

For example, Wöhler’s law cannot be applied to the study on reservoirs under pressure. We must, therefore, study the phenomenon of cracking causing leakages (in the case of spacecrafts). The main criticism of Miner’s law is that it ignores the order in which the levels of loading occur. Indeed, experiments demonstrate that the cracking rate depends not only on the amplitude of loading at the moment it is considered, but also on the amplitudes during the precedent cycles. The phenomenon of anterior memory must therefore be considered.

In the uniaxial case, the load ratio $R$, at the bottom of the notch, tends toward $-1$ for an average of $N = 1000$ cycles. In the case of deformation imposed on smooth
test pieces under a lower number of cycles \((N)\), there is a relaxation of the average stress \(\sigma_{av}\) and \(R\) therefore tends toward \((-1)\). This is called structural damage in low-cycle fatigue. It is in conventional fatigue that the imposed tests are most suitable for the stability of \(R\) (not of the variation observed in tests). Miner provided a conventional form of damage created by \(N_{cycles}\), as represented in [1.51]. This can be written in detail as follows:

\[
D = \sum_{i=1}^{k} n_i \left( \frac{n_i}{\kappa(R)} \times S_{\alpha}^{\beta(R)} \right) = ||
\]

[1.52]

The effective stress for cycle \((N)\) of fatigue is taken into consideration by the Smith–Watson–Topper (SWT) criteria. The SWT criteria take into account three important factors: alternating deformation \((\varepsilon_{alternating}, \text{MPa})\), maximum stress \((\sigma_{max}, \text{MPa})\), and the elasticity modulus \((E, \text{MPa})\):

\[
\sigma_{effective} = \sqrt{\varepsilon_{alternating} \times \sigma_{max} \times E} \quad \text{in MPa}
\]

[1.53]

Without considering the fracture ratio \((R)\), the number of cycles will be shown as follows:

\[
N = \kappa \left( \frac{1}{1-R} \right)^{-\alpha} \times S_{\alpha} \quad ; \quad \text{when } S_{\alpha} = S_{aD} \times 2 \left( \frac{1}{1-R} \right) ; \quad S_{aD} \neq 0
\]

[1.54]

1.6. Main common laws retained by fracture mechanics

Fracture mechanics helps us to quantitatively study the stages of slow propagation and brutal crack propagation. It helps us to calculate the characteristic parameters of local distribution of cracks and the deformations in the neighborhood of a crack, and also the critical lengths \((a_c)\) of cracks, leading to a fracture for a given load.

For very acute cracks, G.R. Irwin [IRW 64] links the cracking rate (deformation) to the s.i.f. \(K\) as defined in Figure 1.18 representing Ritchie [RIT 79]. If \(K\) reaches \(K_{cr}\), the crack brutally propagates. By knowing \(K_{cr}\), we can find the critical length of the initial crack which leads to brutal fracture. Taking into consideration these effects, Henry established a formula which expresses the damage for level C of the previous relation. Of course, there are several formulas that take various levels of damage into account, Monson, Corten, and Dolen’s theories are a few examples. Although less precise, Miner and Henry’s formulas are often used because of their
simplicity. They allow us to predict the residual life expectancy of a part after a program of varied loads, and sometimes provide a sufficient approximation.

Welding techniques have not only developed, but improved over the years. They have certain advantages; however, they can also be the cause of specific problems. These problems can be very serious. The factors influencing the failures of welded assemblies are as follows:

– factors associated with the execution of welding;
– factors associated with metallurgical alteration due to welding;
– factors associated with the dimensioning of weld beads (geometry);

These three main factors are of a metallurgical, mechanical, and technological order respectively.

Faults can appear according to the way in which welding is carried out. These faults affect the mechanical behavior of the joint. The International Institute of Welding (IIW) has proposed a classification of these faults.

The alterations due to the thermal cycle of the welding operation cause local modifications of the mechanical properties that, in turn, constitute the causes of failure. The main factors are quenching, aging, overheating, and the softening of the solidification structure.

With respect to mechanical factors, the errors are of a conceptual order and not concerned with the execution. The severity of these factors is linked to the stress concentration they create. Methods of finite element analysis have stressed the importance of the shape of the weld bead.

In the domain of fracture by fatigue, good results have been obtained with the help of cracking laws based on fracture mechanics and in particular on Paris’ law. Other laws are also commonly used, such as Fost and Digdale’s law and MacEvily’s law. There are other laws which are used in a different ways; however, this chapter will limit itself to Paris’ law. Experience shows that the life expectancy of a welded structure strongly depends upon the size of the initial crack ($a_0$). For instance, according to Lawrence’s calculations [LAW 73], the life expectancy quadruples when $a_0$ changes from 5 to 0.5 mm. When the weld shows no sign of a penetration fault, $a_0$ must be estimated, that is the half-small axis of the half-ellipse acting as a model, which is generally less than 0.5 mm. The fatigue of metals is tackled in four different ways:

– the reading of Wöhler’s curves;
– Coffin’s methods;
– the study of cyclic strain-hardening;
– measuring crack speed \((da/dN)\).

The first method is the most used because it determines the endurance limit, which if maintained, low, protects from brutal fractures. The determination of Wöhler’s curve allows us to make qualitative judgments with respect to the choice of materials for solid parts. Modeling the cracking phenomenon by fatigue integrates the following main notations:

– \(a\), the crack length;
– \(N\), the number of applied cycles;
– \(N_f\), the number of cycles to fracture;
– \(\sigma\), the nominal stress;
– \(K\), the s.i.f.;
– \(C\) and \(m\), the intrinsic parameters of the material.

From the crack length measurements on test pieces subjected to fatigue, different formulas have been proposed to calculate the propagation rate. The following are the most frequently used expressions:

### 1.6.1. Fost and Dugdale’s law

Fost and Dugdale showed in 1960 that the diameter of the plastic zone \((a)\) is proportional to one (diameter) obtained from ductile fracture in plasticity by integration of:

\[
\frac{da}{dN} = \{\alpha \times A \times \sigma^3\} \quad \text{[1.55]}
\]

where:

\(\alpha\) is the multiplier coefficient which depends on the state of the stress;

\(A\) is the constant, which depends on the material and on the stress;

\(\sigma\) is the amplitude of the stress.

To be precise, the plastic zone depends on the strain-hardening coefficient and on the mode of stress. For plastic deformation, there is an increase, in the peripheral zone, in the density of dislocations and of dislocation residues. So, when the crack opens during each cycle, there is a final rearrangement of the dislocations to accommodate the slip at the bottom of the crack. When the crack is closed, the rearrangement becomes definitive. We will therefore consider McEvily’s law.
1.6.2. McEvily’s law (1979)

This law is a function of the s.i.f. and the stress, which is written as follows:

\[ \frac{da}{dN} = f \left( K_t, \sigma \right) \quad [1.56] \]

where \( K_t \) is the stress concentration coefficient and \( \sigma \) is the nominal stress.

Both Fost–Dugdale and McEvily’s laws take inter-inclusionary fractures into consideration, which occur according to a process of low-cycle fatigue. They are associated with Manson–Coffin’s damage law at the bottom of the crack, which is presented as follows:

\[ \left( 4 \times \Delta N \right) \times \left( \frac{\varepsilon_p}{\varepsilon_f} \right)^{\frac{1}{C}} = 1 \quad [1.57] \]

where:

- \( C \) is the exponent from Manson–Coffin’s law;
- \( \varepsilon_p \) is deformation at plastic limit;
- \( \varepsilon_p \) is deformation at fracture;
- \( \Delta N \) is the range of the number of cycles.

For the study of cracking, we consider the detailed expression of McEvily’s law:

\[ \frac{da}{dN} = \left( \frac{C}{E} \right) \times \left( \Delta K^2 - \Delta K_s^2 \right) \times \left( 1 + \frac{\Delta K}{K_c - \frac{\Delta K}{1 - R}} \right) \quad [1.58] \]

In 1979, McEvily proposed a relation applicable to alloys with low resilience, which is written as follows:

\[ \frac{da}{dN} = \left( \frac{C'}{E} \right) \times \left( \Delta K - \Delta K_s \right)^2 \times \left( 1 + \frac{\Delta K}{K_c - \frac{\Delta K}{1 - R}} \right) \quad [1.59] \]
where:

- $C$ and $C'$ are the intrinsic factors of the material;
- $R$ is the load ratio (or the s.i.f.);
- $K_C$ is the critical s.i.f. which corresponds to brutal fracture;
- $\Delta K_0$ is the s.i.f. threshold (see Figure 1.18, Ritchie later);
- $\Delta K$ is the variation of s.i.f.

### 1.6.3. Paris’s law

The study of the behavior of structures, in linear elasticity, often uses this law (1960). Fracture calculations gives the correlation between the intrinsic factors of the material ($C$ and $m$), because tenacity is dependent upon it, through the means of the s.i.f. The crack propagation law gives the following equation:

$$
\frac{da}{dN} = C \times (\Delta K)^m \quad \text{when} \quad (\Delta K) > 0
$$

where $C$ and $m$ are intrinsic parameters of the material and $\frac{da}{dN}$ expresses the crack propagation ratio. Linearizing expression [1.59] allows:

$$
\log \left( \frac{da}{dN} \right) = \log(C) + m \times \log(\Delta K)
$$

Vicker’s tests have shown that the zone affected by heat is localized at the foot of the weld bead for the four methods of welding\(^1\). The expression of the s.i.f. (or tenacity) is written as follows:

$$
\Delta K = \Delta \sigma \times \xi(a) \times \sqrt{\pi \times a}
$$

The tenacity represents the resistance to deformations and to fracture. It is characterized by the elasticity limit $R_e$, resistance to fracture $R_m$, and the hardness HB, HRC, or HV for resistance to deformations. Work on welded cross-joints [GRO 94, GRO 95, LAS 92] for four different methods of welding has shown that fracture occurs at the foot of the weld bead. By replacing $\Delta K$ (ISO 12737: 1996) with its expression in [1.62], the following is obtained:

---

1 SAW, submerged-cored arc welding; FCAW, flux cored arc welding; SMAW 57 and SMAW 75, submerged metal arc welding.
\[
\frac{da}{dN} = g(a/T)^m C \times \Delta \sigma^m \left(\sqrt{\pi a}\right)^m ; \quad g(a/T) = \xi(a) ; \quad \int_{N_i}^{N_f} \frac{dN}{N} = \int_{N_i}^{N_f} \frac{da}{C(\Delta K)^m} \\
\]

when \(\Delta K = \Delta \sigma \times \xi(a) \times \sqrt{a} \rightarrow dN = \frac{1}{\left(C \pi^{m/2}\right) \times \Delta \sigma^m} \times \int_{a_i}^{a_f} \frac{da}{a^{m/2} \times \left(\xi(a)\right)^m} \quad [1.63]

\[
dN = \frac{1}{C(\xi(a))^m \pi^{m/2} \Delta \sigma^m} \int_{a_i}^{a_f} a^{-m/2} da = \frac{1}{C(\xi(a))^m \pi^{m/2} \Delta \sigma^m} \frac{1 - a_i^{-m/2} - a_f^{-m/2}}{m/2 - 1}
\]

where:

- \(N\) is the number of cycles (by load);
- \(a\) is the length of the crack in mm (or in \(\mu m\));
- \(T\) is the thickness of the stressed metal plate in mm.

The average of the intrinsic parameters of the material \((C, m)\) for the four methods of welding is:

\[
C_{\text{average}} = \frac{6,069 \times 10^{-8}}{24,64 \times m} \text{ MPa} \times \sqrt{m} \quad \text{when } R^2 = 0,963 \quad [1.64]
\]

The following photo shows resilience test piece of a welded cross-joint [LAS 92].

Figure 1.11. Resilience test piece of a welded cross-joint
In other instances, we can only go by previous experience or the correlations between the characteristics of materials and fracture behavior [GRO 94] of components (valuation from ISO 2553).

![Figure 1.12. Average relation of the intrinsic parameters of the material](image)

The procedure for measuring the dimensions of the weld bead (figure 1.11) is based on a procedure used in dentistry. Figure 1.12 illustrates the results from the four methods of welding. To measure the geometry of the weld bead, it was rolled in a dental paste. The imprint left by the bead was then measured. Thereby, the evolution of the geometry \( g(a/T) \) used to calculate fracture parameters was deduced [GUR 78].

![Figure 1.13. Evolution of the correction factor of geometry by welding methods](image)
1.6.3.1. Stress concentration factors

The elements of stress concentration play a predominant role in RDM on machine elements, materials, and structures. Solids under stress load will present stress fields of low- to medium-levels with high gradient zones. At this level, shearing stresses reach dangerous levels. This is called the seat of crack by plasticizing. It leads to fracture by fatigue. Three cases illustrating a stepped shaft are shown in the following:

\[ A = \left( \sqrt{\frac{x}{1 - \delta}} + 1 \right) - 1 \text{ and } B = \sqrt{x} \text{ when } x = \left( \frac{h}{r} \right) \text{ and } \delta = \left( \frac{d}{D} \right) \]

\[ K_{\text{Traction}} = \left( \frac{1}{\sqrt{0.88A}} + \frac{1}{0.843B} \right)^2 + 1; \quad \sigma_{\text{nominal}} = \frac{4F}{\pi d^2} \]

\[ K_{\text{Flexion}} = \left( \frac{1}{\sqrt{0.541A}} + \frac{1}{0.843B} \right)^2 + 1; \quad \sigma_{\text{nominal}} = \frac{32M}{\pi d^3} \]

\[ K_{\text{Torsion}} = \left( \frac{1}{\sqrt{0.263A}} + \frac{1}{0.843B} \right)^2 + 1; \quad \sigma_{\text{nominal}} = \frac{16F}{\pi d^3} \]

The respective relations to calculate stress concentration factors, for one with semi-circular bottom groove, are shown in Figure 1.15.
Figure 1.15. Stress concentration factors for three figures: shaft with semi-circular bottom groove

Observation: It is worth pointing out that for the remaining cases, such as plates with symmetrical steps, reference should be made to manuals concerned with the resistance of materials for more detail.

In TFT the criteria are as follows:

\[
A = \left( \sqrt{\frac{x,y}{1-\delta}} + 1 \right) - 1 \quad \text{and} \quad B = \sqrt{x} \quad \text{when} \quad x = \left( \frac{h}{r} \right) \quad \text{and} \quad \delta = \left( \frac{d}{D} \right)
\]

\[
K_{\text{traction}} = \frac{1}{\sqrt{\left( \frac{1}{1.197A} \right)^2 + \left( \frac{1}{1.871B} \right)^2}} + 1; \quad \sigma_{\text{nominal}} = \frac{4F}{\pi d^2}
\]

\[
K_{\text{Flexion}} = \frac{1}{\sqrt{\left( \frac{1}{0.715A} \right)^2 + \left( \frac{1}{2B} \right)^2}} + 1; \quad \sigma_{\text{nominal}} = \frac{32M}{\pi d^3}
\]

\[
K_{\text{Torsion}} = \frac{1}{\sqrt{\left( \frac{1}{0.365A} \right)^2 + \left( \frac{1}{B} \right)^2}} + 1; \quad \sigma_{\text{nominal}} = \frac{16F}{\pi d^3}
\]

[1.66]

1.6.4. G.R. Sih’s law

Paris’ law is useful only for simple mode crack propagation. Only one load parameter is involved (stress amplitude) and is only useful in mode I; simple propagation. G.R. Sih [SIH 79] proposes replacing the s.i.f. ($\Delta K$) amplitude of Paris’
law with the amplitude of minimum deformation energy density $\Delta S_{\text{min}}$. Sih’s law is based on the concept of deformation energy density.

$$\frac{da}{dN} = C \times \left( \Delta S_{\text{min}}^m \right)$$ [1.67]

where:

- $C$ and $m$ are the intrinsic factors of the material;
- $\Delta S_{\text{min}}$ is the minimum energy density;
- $\frac{da}{dN}$ is the crack propagation rate.

An approach to fatigue by linear fracture mechanics will now be illustrated. It also involves a host of mathematical expressions. Yamada and Albrecht’s [YAM 77] model will be used to calculate the variation of the s.i.f. and Gurney’s [CUR 78] model for the geometric correction $g(a/T)$.

1.7. Stress intensity factors in fracture mechanics

Dimensioning structures which are stressed in different ways leads to an even greater control of fatigue behavior in welded joints. To calculate the crack propagation, numerous works have recognized the link between the logarithm of the variation of the s.i.f. and the logarithm of propagation rate. Certainly, this is relative to stage II of Ritchie’s [RIT 79] chart (see Figure 1.18), for which Paris proposed a law where the s.i.f. is expressed by relation [1.62], where $\Delta \sigma$ is the stress amplitude and $\xi(a)$ is the geometry of the crack.

1.7.1. Maddox’s model

This model was originally intended for welded cross-joints without penetration fault. Maddox [MAD 75] assumes that the surface fault is a semi-elliptic of half-axes $a$ and $c$. Therefore, he suggests calculating the s.i.f. by the following relation:

$$K = \sigma \times \sqrt{\pi} \times a \times \left( \frac{M_s \times M_t \times M_k}{\Phi_0} \right)$$ [1.68]

where:

- $\sigma$ is the nominal stress (in MPa);
- $a$ is the crack length (in mm or inch);
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$M_s$ is the correction of the free surface (crack surface);

$M_t$ is the correction of finite thickness;

$M_k$ is the correction due to the stress concentration introduced by the geometry of the weld bead;

$\Phi_0$ is the form factor = 1 for $(a/c) = 0$ crack length.

For welded cross-joints, Maddox proposes a connection angle $\theta = 0$ and for $\Phi_0 = 1$:

$$K = \sigma \sqrt{\pi a} \left\{ 1.122 - 0.23 \left( \frac{a}{T} \right) + 10.55 \left( \frac{a}{T} \right)^2 - 21.7 \left( \frac{a}{T} \right)^3 + 33.19 \left( \frac{a}{T} \right)^4 \right\}$$  \[1.69\]

where $T$ is the thickness of the welded sheet metal (in mm or inches). This expression has been applied several times. As early as 1965, B. Gross and J. E. Srawley [GRO65] proposed a model which draws conclusions about the s.i.f.

1.7.2. Gross and Srawley's model

The proposed equation [1.70] is empirical and shows a model for a ratio of $(a/W)$ ranging from 0.1 to 0.35.

$$Y^2 = \frac{K^2 \times B^2 \times W^2}{M^2} = 139 \left( \frac{a}{W} \right) - 221 \left( \frac{a}{W} \right)^2 + 783 \left( \frac{a}{W} \right)^3$$ \[1.70\]

where:

- $Y^2$ is a dimensional quantity which is exclusively the function of $a/W$;
- $M^2$ is the square of the bending moment ($M = P \cdot l/2$);
- $K$ is the s.i.f. in mode I crack opening;
- $B$ and $W$ are the width and the thickness of the stressed test piece by $M$, respectively.

1.7.3. Lawrence’s model

For butt joints, Lawrence [LAW 73] proposed a model from linear elastic mechanics. It comprises three distinct steps:

Step 1: Finite elements are used to determine, the stress field existing in a joint, along the crack. For triangular meshing, in planar deformation, we deduce the stress ratio $(\sigma/S)$ along the crack.
Step 2: The curves are adjusted ($\sigma/S$) by the following polynomial function:

$$\frac{\sigma}{S} = b_0 + b_1 \left(\frac{a}{t}\right) + b_2 \left(\frac{a}{t}\right)^2 + b_3 \left(\frac{a}{t}\right)^3 + b_4 \left(\frac{a}{t}\right)^4$$ \[1.71\]

where:

$t$ is the thickness of the sheet metal;

$b_i$ are constants dependent on the geometry of the weld bead;

$a$ is the crack length;

$\sigma/S$ is the stress ratio.

Step 3: Consists of calculating the s.i.f. by the following relation:

$$K = \sqrt{\pi a} \times \left[1.1\sigma - \int_0^t f\left(\frac{a}{t}\right)\frac{d\sigma}{dt} dt\right]$$ \[1.72\]

The function of a semi-finite crack subjected to a non-uniform load $\sigma(a/t)$ is written as:

$$f\left(\frac{a}{t}\right) = 0.8 \left(\frac{a}{t}\right) + 0.04 \left(\frac{a}{t}\right)^2 + 3.62 \times 10^{-6} \times \text{Exp}^{11.18\left(\frac{a}{t}\right)}$$ \[1.73\]

The advantage of this model is that it replaces calculation using finite elements with calculation by the polynomial function which is dependent on the geometry of the weld bead. Lawrence’s method, for martensitic steel, is used to study faults that are smaller than 0.025 mm. However, his method is hardly convincing with respect to estimating life expectancy. For this, it is advisable to refer to T.R. Gurney’s [GUR 78] approach.

1.7.4. Martin and Bousseau’s model

Martin and Bousseau’s [MAR 76] method is linked to other methods in that it employs finite elements. It suggests determining the stress variation along the crack plane and, for a given type of joint, deducing the s.i.f. ($K_t$) at the foot of the weld bead, beside the crack activation. This method is expressed as follows:

$$\frac{d a}{d N} = C \times \left(1.1 \times \sigma_N \times \sqrt{\pi a} \times K_t\right)^m$$ \[1.74\]
where:

\( C, m \) are intrinsic parameters of the material;

\( da/dN \) is the crack rate (life expectancy);

\( a \) is the crack length;

\( \sigma_N \) is the nominal stress;

\( K_r \) is the nominal stress.

Employing this function leads to a markedly lower number of propagation cycles than those obtained by Lawrence’s model.

1.7.5. Gurney’s model

Gurney’s model, in relation to the figure, proposes the following:

\[
\Delta K = \Delta \sigma \times \sqrt{\pi a} \sec \left( \frac{\pi a}{2b} \right) \tag{1.75}
\]

For Figure 1.13 in zone I of Figure 1.18 we have:

\[
\Delta K = \frac{\sqrt{\Delta P}}{BW} \left\{ 29.6 \sqrt{\frac{a}{W}} - 185.53 \sqrt{\frac{a}{W}} + 655.75 \sqrt{\frac{a}{W}} - 10177 \sqrt{\frac{a}{W}} + 638.99 \sqrt{\frac{a}{W}} \right\} \tag{1.76}
\]

where:

\( \Delta K \) is the variation of the s.i.f.;

\( \Delta \sigma \) is the variation of the nominal stress;

\( a/W \) is the relative crack length \((2B = W)\);

\( B \) is the thickness of the sheet metal.

1.7.6. Engesvik’s model

Knut Engesvik’s [ENG 82] model is based on Yamada and Albrecht’s approach, from which the following can be written:

\[
\Delta K = \Delta \sigma \times \sqrt{\pi a} \times f(a) \quad \text{when} \quad f(a) = f_s \times f_T \times f_E \times f_G \tag{1.77}
\]
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where:

$f_s$ is the correction factor of the free surface;

$F_T$ is the correction factor of the finite thickness;

$F_E$ is the correction factor of the shape at fracture;

$F_G$ is the correction factor of geometry;

$\Delta \sigma$ is the variation of nominal stress.

$f_{(a)}$ coincides with $g(a/T)$. From Yamada and Albrecht’s model we use:

$$f_G = \frac{2}{1.12 \times \pi \times \Delta a} \times \frac{\sigma(x)}{\sqrt{a^2 - x^2}} \times g\left(\frac{x}{a}\right) dx$$  \hspace{1cm} [1.78]

Using Lawrence’s model as a base, we use the correction of geometry expression as follows:

$$g\left(\frac{x}{a}\right) = 0.8 \left(\frac{x}{a}\right) + 0.04 \left(\frac{x}{a}\right)^2 + 3.62 \times 10^{-6} \times \exp\left(11.18 \left(\frac{x}{a}\right)\right)$$  \hspace{1cm} [1.79]

1.7.7. Yamada and Albrecht’s model

This famous model [YAM 78] allows us to calculate $\Delta K$ in the case of a semi-elliptical fault of half-axis $(a)$, given as follows:

$$\Delta K = \frac{1.12}{E_K} \times f_G \times \Delta \sigma \times \sqrt{\pi a} \times \sqrt{\frac{W}{\pi a} \tan\left(\frac{\pi a}{W}\right)}$$  \hspace{1cm} [1.80]

where:

$\Delta \sigma$ is the variation of nominal stress;

$E_K$ is the elasticity modulus;

$W = 2B$ is the relative crack length $W=2B = a/T$ (See Figure 1.13);

$a$ is the crack size;

$f_G$ is the correction factor of the stress concentration effect (s.i.f.) due to the geometry of the weld bead.
Finally, to take into account the influence of short cracks subjected to a field of plastic deformation, El Haddad et al.’s [HAD 78, 80] model suggests using an elastoplastic solution. The relation is given:

$$\Delta K = E \times \Delta \varepsilon \times \sqrt{\pi (l + l_0)}$$  \[1.81\]

where:

- $\Delta \varepsilon$ is a local deformation in plastic regime that is obtained by using the finite element method;
- $E$ is the elasticity modulus;
- $l = l$ is the crack length;
- $l_0 = l_0$ is the constant of the material.

Neuber’s [NEU 58] rule means that the solution can be approached using the following relation:

$$K_{equivalent} \times \Delta \sigma = \sqrt{\Delta \sigma \times \Delta \varepsilon \times E}$$  \[1.82\]

where:

- $\Delta \varepsilon$ and $\Delta \sigma$ are the deformation and the local stress at the tip of the crack, respectively.
- $K_{equivalent}$ is an equivalent stress concentration coefficient, which is written as follows:

$$K_{equivalent} = \frac{M_k \times M_S \times M_t}{\Phi_0}$$  \[1.83\]

where $M_k$, $M_S$, and $M_t$ are correction factors explained in Maddox’s model (a) (see also relation [1.66]).

1.7.8. Tomkins and Scott’s model

Tomkins et al.’s [TOM 78] model is applied to analyze surface cracking around the intersection of a knot, using the following integral:

$$K = 2 \sqrt{\frac{C}{\pi}} \times \int_0^C \frac{\sigma}{\sqrt{C^2 - x^2}} \, dx = Y_\sigma \times Y_S \times \sigma_R \times \sqrt{\pi a}$$  \[1.84\]
where:

- $\sigma$ is the stress which considers the evolution of the initial s.i.f.
- $Y_\sigma$ is a factor which considers the evolution of the stress field;
- $Y_S$ is a factor which considers geometric effects;
- $\sigma_R$ is the radial stress at the hot spot;
- $x$ and $C$ are the large and small axes of an ellipse (geometry), respectively;
- $a$ is the crack length.

### 1.7.9. Harrison’s model

Harrison’s [HAR 78] model is applied in the case of cracking due to welding faults (lack of penetration). It is written as follows:

$$\Delta K = \Delta \sigma \times \sqrt{\pi a} \times \sqrt{\frac{W}{\pi a}} \times \tan \left( \frac{\pi a}{W} \right)$$  [1.85]

For the same application, Lawrence and Munse present the following relation for butt joints:

$$\Delta K = \Delta \sigma \times \sqrt{\pi a} \sqrt{\cos \left( \frac{\pi a}{W} \right)}$$  [1.86]

where:

- $\Delta \sigma$ is the variation of nominal stress;
- $\Delta K$ is the variation of the s.i.f.
- $W$ is the thickness of the sheet metal;
- $a$ is the crack length.

There are many models related to calculating s.i.f. according to the design, materials, the environment, and geometry. Some models have been shown here which are necessary for reliability calculations; but reference can also be made to literature on the topic for a deeper understanding of the subject.

### 1.8. Intrinsic parameters of the material (C and m)

There are a number of “relations between C and m” used to calculate the fatigue of welded structures (Paris’ law –See Figure 1.12). For example, Japanese law WES
Tanaka [TAN 81] and Kanazawa et al. [KAN 79] describe this standards using concepts from fracture mechanics of welded assemblies, where $R_m = 100$ Mpa. The norm proposes an average value equal to 2 for $m$. $C$ is expressed by the linear regression of $da/dN$ (mm/cycles) and $\Delta K$ by kP/mm$^{3/2}$. Their relation is given as follows:

$$C = \frac{5.53 \times 10^{-5}}{59.2^m} \text{ (JWES)}$$

There is also a Nordic regulation Dn$V$ for the construction of offshore steel platforms. In mode II (see Figure 1.18) for stable crack propagation in fatigue, the increase in $m$ and the simultaneous decrease in $C$ is often observed, when the elasticity limit increases. Therefore, several authors have proposed relations in $m$ and Log($C$). In our works [GRO 92, GRO 94], based on a large experimental project lead by Professor Tom Lassen [LAS 82, 92], we proposed four types of relations between $C$ and $m$, stemming from mathematical regression. The results accompanied with their correlation coefficients are as follows:

$$\begin{cases}
C_{SAW} = \frac{8.855 \times 10^{-8}}{29.39^m} & R^2 = 0.982; \quad C_{FCAW} = \frac{2.14 \times 10^{-8}}{18.13^m} & R^2 = 0.997 \\
C_{SMAW57} = \frac{4.092 \times 10^{-8}}{20.08^m} & R^2 = 0.975; \quad C_{SMAW76} = \frac{2.263 \times 10^{-8}}{16.23^m} & R^2 = 0.97 
\end{cases}$$

The average relation of the four methods of welding is written as follows:

$$C = 6.069 \times 10^{-5}/24.64^m; \quad R^2 = 0.963 \quad \text{units MPa; } \sqrt{m}$$

We used a simple regression on the semi-Log scale for the experimental values represented by Paris’ relation.

**COMMENTS.**— In stress concentration, zones efforts can cause cracks. Nevertheless, in this domain, a fault (lacuna) or a crack, sometimes extremely small, can be the cause of a fracture. On reflection, it seems that normal calculations from continuum mechanics cannot predict fractures since they do not assume the presence of a fault. It is therefore necessary, before adopting a reliability approach, to accurately calculate the:

- distinct s.i.f. ($\Delta K$) for each method of welding used;
- initial crack lengths ($a_0$);
1.9. Fracture mechanics elements used in reliability

This summary explains the behavior of existing cracks within the material, by estimating their evolution of the limiting conditions and the qualities required by the base material and the filler material. A fault is perceived at a given moment and we observe its slow propagation under a load effect, until the size of the fault becomes critical. Defined by fracture mechanics, the characteristics of stress distribution and deformations in the neighborhood of the crack front (n.c.f.) allow us to quantify the phenomenon of cracking. Fatigue itself is characterized by a large range of stress variation, which is inferior to resistance to traction of the stressed material. The main steps of fatigue are crack activation until final fracture ($a_0$ to $a_f$).

To predict the behavior of fatigue we use, in addition to the number of cycles, the amplitude of the stress, loading or imposed deformation $\Delta \sigma$ in MPa, the surface state (its finish), as well as the medium where the material (or the structure) evolves. The behavior of structures is influenced by different parameters, including geometric characteristics of the structure and weld beads, loading, and the limiting conditions.

In the case of a crack with sharp initiated notches, Irwin [IRW 58, 64] decided that Neuber’s classic calculations were outdated. He therefore proposed a calculus that allows us to obtain the stress state at the crack at a point n.c.f. (crack front).

$$\sigma_{xy} = K \times \left( \frac{f_{xy}}{\sqrt{2\pi \cdot r}} \right)$$  \[1.90\]

where:

$f_{xy} = g(a/T)$ is a function of correction of geometry;

$\sigma_y$ is the applied nominal stress;

$r$ and $\theta$ are the radius and the angle in polar coordinates, respectively.

The s.i.f. ($K$) defines the stress state of the component, by taking into account the global and local geometry of the crack. When $K$ reaches a value of $K_{cr}$, there is a brutal fracture of the part, as illustrated in Figure 1.14. Usually, the notion of brutal fracture is associated with the presence of an initial fault existing in the weld or in the structure. It is therefore worth reiterating the existence of a potential crack propagation from an initial fault ($a_0$).
Fracture can occur during a static load or after the propagation of a fault, under different stresses and until it reaches a critical dimension $a_{cr}$. Since Griffith [GRI 20] proposed his fracture theory, Rice [RIC 68] carried out more polished analyses than Griffith’s, though his fundamental principles remained the same. In our case of singular structures, for example, we have applied Paris’ crack law. Theories which are applicable to numerous materials and which authorize the use of critical fracture stress expressions can be expressed in the following relation:

$$\sigma = \sqrt{(E \cdot G_c)/\pi a}$$  \[1.91\]

where $G_c$ is the rate of releasing energy so that the crack propagates in a unitary length. To characterize the $s.i.f. K$, we consider the following relation:

$$K = \sigma \times \sqrt{\pi a} \times g(a/T)$$  \[1.92\]

This equation is applied when the test piece is thin with very large lateral dimensions, in other words, when the crack length $(2a)$ is very small in relation to its dimensions. As simple as it may be, the load reached in Paris’ law allows us to have a significant influence on the crack length. In 1870, A. Wöhler [WÖH 70] focused on the study of premature fractures of rail car axles. In light of his conclusions, the behavior of materials subjected to fatigue can be determined. To do so, laboratory test pieces which are subjected to simple efforts of rotary flexion are used. There are different amplitude levels of stresses $\sigma_a$. Then, the number of cycles required to fracture the test piece is measured.

Bearing in mind the statistical character of fracture theory in linear elasticity, to study the reaction of the material subjected to fatigue the test should be repeated several times with different stress amplitudes. On a semi-log scale, a curve $\sigma_a = f(N_{cycles})$ is then plotted as shown in Figure 1.17.
When a structure is subjected to a load of varying intensity, it is subjected to a phenomenon of fatigue which leads to its fracture, even though the charge remains constantly below its static resistance. This phenomenon is linked to crack propagation from a fault within the part itself, that is, a set of conditions favorable to a local decohesion of the material, because of a high stress concentration. Generally three steps in the life of a structure can be distinguished: crack initiation, slow crack propagation by fatigue, and finally fracture, as demonstrated in Figure 1.18 [RIT 79].

The three stages previously schematized are defined as follows:

Zone I: low-cycle fatigue zone under strong stress amplitudes where life expectancy is short ($N_{\text{cycles}} < 10^4$ cycles). Before fracture there is a plastic deformation.

Zone II: fatigue zone or limited endurance zone where fracture occurs after a number of cycles, which increases when the stress decreases ($10^4 < N_{\text{cycles}} < 10^6$ cycles).

Zone III: unlimited endurance zone, also known as safety zone, under weak stress amplitudes. Fracture does not occur, even after a high number of cycles ($N_{\text{cycles}} < 10^7$).

Fatigue cracks originate from the surface of parts, where there are numerous faults. Moreover, the surface is subjected to aggression of the environment. Dislocations are also as mobile at the surface as they are at the core. The number of cycles necessary for fatigue crack initiation essentially depends on the sharpness of...
the fault where it originates. In the case of welded cross-structures, it is often said that the crack originates at the foot of the weld bead.

![Figure 1.18. Illustration of the propagation rate da/dN in function of the s.i.f. (Ritchie)](image)

1.10. Crack rate (life expectancy) and s.i.f. ($K_\sigma$)

Materials contain inclusions, fabrication faults, heterogeneity, etc. Components show signs of section changes or surface states, which are more or less perfect. Insofar as these conditions favor the apparition of stress concentrations and consequently fatigue cracks under cyclic-load, not only the possibility of crack activation but also their propagation must be borne in mind. This explains why engineers spend a long time calculating when they need to design structures with cyclic loads. The structures must not only anticipate the possibility of cracks forming, but also evaluate their propagation rates, to be sure that these cracks will not reach a critical length and result in brutal fracture.

In the 1960s, P. Paris and F. Erdogan [PAR 63] proposed a relation for the evolution of a fatigue crack, through the measure of propagation, represented by $da/dN$ and the s.i.f. (parameter that characterizes the fault) which is determined
analytically or by finite elements (see Figure 1.1) When a crack is formed, its length increases with increase in the number of load cycles.

Knowing that $\Delta K = (K_{\text{max}} - K_{\text{min}})$ in a simple case when the stress amplitude is constant, the growth of the crack length ($a$) drives a rise in the value of the s.i.f. Indeed if the opposite varies, so does the s.i.f. In fact, when we represent the variation of crack rate ($da/dN$) as a function of s.i.f. ($\Delta K$) in a logarithmic scale, the curve (see Figure 1.19) represents a linear part in accordance with Paris’ equation, used for its simplicity and success in corroborating many experiment results. The test conditions, for metallic materials tested in an ambient atmosphere, are such that the value of the exponent ($m$) of Paris’ law is generally between 2 and 4. We will attempt to demonstrate this in our case study later in this chapter. Many works have already proposed a number of convincing relations between $C$ and $m$, the results published by [KAN 71, ENG 82, GRO 98].

![Experimental curve: crack depth as a function of the number of applied cycles.](image)

$C$ has often been linked to the characteristics of the material (elasticity limit, resistance to traction, strain-hardening coefficient, lengthening to fracture). These correlations are not satisfactory. In 1971, for $\Delta K$ in $N. \text{mm}^3/2$ and $a$ in mm, Kitagawa and Missumi [KIT] proposed the following:

$$C = \frac{10^{-4}}{2 \times (55^m)}$$  \[1.93\]
This relation is useful for a large number of steels, in particular for metals with central cubic structures. For austenitic steels and non-ferrous metals, however, the relation is less useful. In many cases, the data obtained from experiments are too dispersed to be able to determine the exact values of $C$ and $m$, which are included among the influencing factors for the behavior of solid structures in fatigue. Several typical values are presented in the following:

<table>
<thead>
<tr>
<th>Grade of the material</th>
<th>$M$</th>
<th>$C$ (SI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Martensitic steels with $500 \text{MPa} &lt; S_y &lt; 2000 \text{Mpa}$</td>
<td>$1.35 \times 10^{-10}$</td>
<td>2.25</td>
</tr>
<tr>
<td>Steels with a ferritic-pearlitic structure</td>
<td>$6.90 \times 10^{-12}$</td>
<td>3.00</td>
</tr>
<tr>
<td>Austenitic steels (stainless)</td>
<td>$5.60 \times 10^{-12}$</td>
<td>3.25</td>
</tr>
</tbody>
</table>

Table 1.1. Ideal values of the intrinsic parameters of the materials (without corrosion)
Many factors influence the behavior of solids in fatigue, such as:

– the intrinsic parameters of the material, $C$ and $m$;
– the variation of the $s.i.f.$ ($\Delta K$) and the stress $\sigma$;
– the function of the correction of geometry $g(a/T)$.

In a bid to correct the imperfections involved in calculating $s.i.f.$, factor $g(a/T)$ has often been introduced, which in turn is dependent on several factors [GUR 78, GRO 98]. It is also important to remember that every modification of the microstructure of the base material or of the weld leads to variations in the principal mechanical properties, such as the elasticity limit, the resistance to traction, ductility, and tenacity.

1.10.1. Simplified version of Taylor’s law for machining

Taylor has established a relation which gives the life expectancy of a tool in function of the machining parameters. For example, the intrinsic parameters of the material appear in a simplified version of Taylor’s law for machining.

$$\Delta T = C_v \times \left( V_{cutting} \right)^n$$

[1.94]

where:

$\Delta T$ is the life expectancy of a cutting tool in min;
$C_v$ is a Taylor constant, intrinsic value of the material;
$V_c$ is the cutting rate (m/min or in ft/min);
$n$ is a (negative) value, depending on the type of operation.

Example of application: A carbide tool GC 4015 for which the manufacturer recommends $V_c = 315$ m/min and $f = 0.4$ mm/min for a life expectancy of 15 min. The machining type, carbide turning of a steel gives $n = -8$.

If we plot, on bi-log paper, the life expectancy of a tool as a function of the cutting speed, we obtain the slope of the straight line $n$. For instance, calculating the life expectancy of a tool under the following cutting conditions, $n$ and $C$ are the intrinsic parameters of the material, e.g. coefficients provided by the manufacturer of cutting tools.
1.11. Elements of stress ($S$) and resistance theory ($R$)

In the following section, we are going to present another method known as stress/resistance ($S/R$). This will be accompanied by our own research results [GRO 94, GRO 95]. When fissures appear at the foot of the weld bead of a welded cross-assembly [GRO 98] and are random, they can be described by a probability distribution of initial cracks (variable or non-variable in function of the longevity of the assembly) that the parameters of Paris’s law are themselves random, owing to the variability of the base material (as well as the filler metal for welding), of the dimensional variability, of that of treatments, etc. These can be described by a distribution of the probabilities of the intrinsic parameters of the material ($m$, $C$). The reliability of the assembly can be calculated in function of the life expectancy.

1.11.1. Case study, part 2 – suspension bridge (Cirta)

NOTE.– This case study is undertaken with a mechanical (static) approach. It will be referred to later for reliability calculations (Cornell and de Hosofe–Lind’s reliability index).
PROBLEM.– We are looking for the “correct” environmental factors of the cable to support the tension and the deflection, supporting a uniform load. Secondary effects (wind) will be disregarded. The length of cable $L$ and the deflection ($f$) must support a uniform load $Q$. Let us find, then, the expression and the value of this deflection, environmental factors $H$, and tension $T$.

![Diagram of a suspension bridge](Cirta Bridge)

**Figure 1.21. Example of calculations for a suspension bridge (Cirta Bridge)**

- Uniform load: $Q = 0.01$ (lbf/ft).
- Stretch of the length is the total length of the cable $L = 100$ ft.
- $f$: deflection $f = 2.738$ in. (mm).
- $l$ is the length in mm (or inch).
- $\tau$ is the tension in N (or in lbf).

**SOLUTION.–** The following equations come from the literature (RDM) dedicated to calculating cable design (in the case of suspension bridges with a parabolic shape) in static. The total length of the suspension cable is calculated by:

$$L = \ell \times \left[ 1 + \frac{8}{3} \left( \frac{f}{\ell} \right)^2 - \frac{32}{5} \left( \frac{f}{\ell} \right)^4 + \frac{256}{7} \left( \frac{f}{\ell} \right)^6 \right] \text{ (in ft or m)}$$

[1.95]

The deflection ($f$) of the suspension cable is calculated by:

$$f = \frac{1}{8} \times \left( \frac{Q \times \ell^2}{\tau} \right) \text{ (in ft or m)}$$

[1.96]
The cable tension is expressed as follows:

\[ T = \frac{1}{2} Q \times \ell \times \sqrt{1 + \frac{\ell^2}{16 f^2}} \quad (\text{in } N \text{ or } lb) \quad [1.97] \]

In the case of \( l = 90\% \times L \) (design of the horizontal range). Length \( l \) is written as follows:

\[
L = \ell \times \left[ 1 + \frac{8}{3} \left( \frac{f}{\ell} \right)^2 - \frac{32}{5} \left( \frac{f}{\ell} \right)^4 + \frac{256}{7} \left( \frac{f}{\ell} \right)^6 \right] \rightarrow \ell = 119.854 \text{ ft}
\]

From [1.96], the tension of the environment (\( \tau \)) is deduced as follows:

\[
\tau = \frac{1}{8} \left( \frac{Q \times \ell^2}{f} \right) = 14.028 \text{ lbf} \quad [1.98]
\]

The cable tension (\( T \)) is therefore calculated by [1.97]:

\[
T = \frac{1}{2} \ell \times Q \times \sqrt{1 + \frac{\ell^2}{16 f^2}} = 14.079 \text{ [lbf]}
\]

The range of (\( x \)) values is calculated as follows. The shape of \( y(x) \) parabola is given by the following equation:

\[
y(x) = \frac{f \times x^2}{\left( \frac{\ell}{2} \right)^2} \quad \text{when} \quad x = -\frac{\ell}{2}, \ldots, -\left( \frac{\ell}{2} - \frac{1}{10} \times \frac{\ell}{2} \right), \ldots, \left( \frac{\ell}{2} \right) \quad [1.99]
\]

1.11.2. Case study: failure surface of geotechnical materials

PROBLEM.— This case study establishes the factor of safety (\( FS \)) so that the fracture of the knot (\( A \)) of the slope circle can be managed or even avoided. The classic method, called the layer method, is used to this end (see Figure 1.23):
Figure 1.22. Suspension cable with a parabolic shape for a bridge

Figure 1.23. Safety factor for a failure surface in geotechniques
Initial data: What is the factor of safety (FS) to counter the fracture of the tilt slope, shown in Figure 1.23? The tilt angle is $\beta = 45^\circ$ and the tilted height is $H = 18$ m. The circle has a radius of $R = 25$ m and the center has the coordinates $A (X_c, Y_c)$. The ground has a density of $\rho = 17.56 \times 10^3$ N/m$^3$. The ground grip represented by the cohesion stress is $Cohesion = 18 \times 10^3$ Pa = N/m$^2$ and a friction angle of $\phi = 18^\circ$.

- Density of the ground: $\rho = 17.56 \times 10^3$ N/m$^3$.
- Cohesion: Cohesion = $18 \times 10^3$ Pa (N/m$^2$).
- Friction angle: $\phi = 18^\circ$.
- Tilt angle of the slope: $\beta = 45^\circ$.
- Height of the slope: $H = 18$ m.
- Radius of the circle: $R = 25$ m.
- Coordinates of the center: $A [X_c = 7.56$ m and $Y_c = 25$ m].
- Number of slices: $n = 117$.

Statements

- To express with a matrix the expression of the slope.
- To find, with linear interpolation, the relation which renders $y_{\text{slope}}(x)$ and $y_{\text{fracture}}(x)$ in function of $x$.
- To find the maximum distance $X_{\text{Max}}(x, x_c, y_c, R)$ which is expressed in function of the circle equation.
- To plot the fracture circle, the slope, and the central point of the circle.
- To find the factor of safety $FS$ for the 77 layers, that is $n = 1$ to 77.
- To find the weighted slice $W_i$, the (lowest) angle $\alpha_i$ of the tilted slice, and the friction length by slice $\Delta L_i$.
- For layers 1 to 77, to explain with a graph the behavior of the structure, by emphasizing the $FS$.
- According to results from calculations, at what number layer can a singularity, that is a change of sign from (+) to (−), be observed?
Solutions

We will use the 2D function to develop an image of the failure surface and the slope. The $3 \times 2$ matrix is used to define the coordinates of key points along the slope. The defined points form the knot at the highest point on the slope ($A$).

$$\text{slope}(H, \beta) = \begin{bmatrix}
0 \times H & 0 \times H \\
\frac{H}{\tan(\beta)} & H \\
\frac{3 \times H}{2 \times \tan(\beta)} & H
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
18 & 18 \\
27 & 18
\end{bmatrix} \text{ m} \quad [1.100]$$

The function of linear interpolation, which renders the expression of the slope, is given as follows:

$$y_{\text{slope}}(x) = \text{interp}\left(slope^{(0)}, slope^{(1)}, x\right) \quad [1.101]$$

We use a circle equation to express the function of fracture:

$$y_{\text{fracture}}(x) = y_c - \sqrt{R^2 - (x_c - x)^2} \quad [1.102]$$

We must define a series of $x$ coordinates to plot the graph (see Figure 1.23). The maximum distance, $X_{\text{max}}$, required to plot the graph is the point where the fracture circle crosses the highest point of the slope, which is determined trigonometrically as follows:

$$x_{\text{max}}(H, R, x_c, y_c) = x_c + \sqrt{R^2 - (x_c - H)^2} = 31.56 \text{ m} \quad [1.103]$$

Once the variables of the first series are known, the $x$ values can be calculated by iteration. The range is known by the hypothesis (from 1 to 77) or:

Range of variables of: $i = [1, 2, \ldots, n]$

$$\text{Distance}(x) = \frac{x_i}{m} = x_i = \left(\frac{i}{n}\right) \times x_{\text{max}} \quad [1.104]$$
Digital calculations

\[ y_{\text{slope}}(x_i) = m \]

\[ y_{\text{Fracture}}(x_i) = m \]

\[ x_i = m \]

<table>
<thead>
<tr>
<th>( y_{\text{c}} )</th>
<th>( x_{\text{c}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.27</td>
<td>0.27</td>
</tr>
<tr>
<td>0.539</td>
<td>0.539</td>
</tr>
<tr>
<td>0.809</td>
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<tr>
<td>1.888</td>
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</tr>
<tr>
<td>2.158</td>
<td>2.158</td>
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<tr>
<td>2.428</td>
<td>2.428</td>
</tr>
<tr>
<td>2.697</td>
<td>2.697</td>
</tr>
<tr>
<td>2.967</td>
<td>2.967</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

\[ \frac{y_{\text{c}}}{m} = 25 \]

\[ \frac{x_{\text{c}}}{m} = 7.56 \]

Center of the circle

Slope

Fracture circle

(Knot) Slope of the fracture circle

\[ \frac{x_{\text{c}}}{m}, \frac{x_i}{m}, \frac{x_i}{m} \]

Figure 1.24. Graph showing the slope and the fracture circle (and its precise center)
Calculating the factor of safety $FS$ by the classic method (slices) leads us to using the following equation. The derivative of the equation complies with geotechnical references.

$$FS = \sum_{i=1}^{n} \left( \text{Cohesion} \times \Delta L_i + \Delta \omega_i \times \cos(\alpha_i) \times \tan(\phi) \right)$$

$$\sum_{i=1}^{n} \left( \omega_i \times \sin(\alpha_i) \right)$$

where the weighted slice $\omega_i$ is given by the following expression:

$$\omega_i = \frac{x_{\text{max}}}{n} \times \left[ y_{\text{slope}}(x_i) - y_{\text{fracture}}(x_i) \right] \times \rho$$

The (lowest) angle of the tilted slice is expressed with:

$$\alpha_i = \arctan \left( \frac{x_i - x_c}{y_c - y_{\text{fracture}}(x_i)} \right)$$

The length of friction by slice is written as follows:

$$\Delta L_i = \left( \frac{x_{\text{max}}}{n} \right) \cos(\alpha_i)$$

**Digital calculations**

Digital calculation of the factor of safety: from [1.105], we consider the following equation:

$$FS = \sum_{i=1}^{n} \text{Cohesion} \times \Delta L_i + \Delta \omega_i \times \cos(\alpha_i) \times \tan(\phi) \bigg/ \sum_{i=1}^{n} \omega_i \times \sin(\alpha_i) = 1.085$$

The calculation takes into account changes in safety, such as change in the number of layers (slices) used for the calculation which is changed. This can easily be visualized by redefining the $FS$ in terms of $n$ and plotting (determinant) $FS(n)$ for a series of values from 1 to 77, and results are as follows:
Figure 1.25. Factor of safety in function of number of slices
Behavior can be explained by plotting the numerator and the denominator on a graph:

\[
\text{Numerator}(n) = \sum_{i=1}^{n} \left( \text{cohesion} \times \Delta L_i \right) \times \alpha_i \times \cos(\alpha_i) \times \tan(\phi)
\]

and denominator \( (n) = \sum_{i=1}^{n} \alpha_i \times \sin(\alpha_i) \)

\[ [1.109] \]

At \( n = 40 \), there is a singularity of the \( FS(n) \) owing to the denominator which passes below zero.

After calculations, the following result is obtained:

\[
\begin{array}{c|c|c}
\text{Num}(n) & \text{Den}(n) & \frac{\text{Num}(n)}{10^9} \\
\hline
-6.104 \times 10^6 & 9.615 \times 10^{-3} & 0.011 \\
-9.59 \times 10^6 & -5.412 \times 10^{-4} & 0.017 \\
-1.048 \times 10^7 & -0.014 & 0.019 \\
-8.807 \times 10^6 & -0.03 & 0.016 \\
-4.586 \times 10^6 & -0.048 & \ldots \\
2.156 \times 10^6 & 9.615 \times 10^{-3} & \ldots \\
1.139 \times 10^7 & -5.412 \times 10^{-4} & \ldots \\
2.311 \times 10^7 & -0.014 & \ldots \\
3.727 \times 10^7 & -0.03 & \ldots \\
5.386 \times 10^7 & -0.048 & \ldots \\
7.285 \times 10^7 & \ldots & \ldots \\
9.423 \times 10^7 & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
\end{array}
\]

\( \frac{\text{Den}(n)}{10^5} \) = \( \uparrow \)

**Figure 1.26. Expressions of the denominator and the numerator in function of \( (n) \)**

**COMMENTS.**— Grapho-analytic proof: at the 40th layer, that is \( n = 40 \), there is a singularity = \( (-0.012) \).

\[
\left\{ \frac{\text{Num}(n)}{10^9} = 1.569 \frac{\text{Kg}^2}{s^{-4}} \; \text{and} \; \frac{\text{Den}(n)}{10^5} = -0.012 \frac{\text{Kg}}{s^{-2}} \right\}
\]
1.12. Conclusion

It should be borne in mind that this chapter has been purposefully developed here, even though the main focus of this work is reliability and quality control (volume 3). The reasons are as follows. First the case studies that follow are based on models of fracture mechanics, among others. Second, a high number of mechanical models exist, which are applied differently to materials and to structures. We therefore thought it useful to add this additional chapter, without which the readers would have had to do their own supplementary research. The present chapter has also included practical examples of reliability and uncertainties.

1.13. Bibliography


