1 What This Book Is About and How to Read It

1.1 “Exercises” vs. “Problems”

This is a book about mathematical problem solving. We make three assumptions about you, our reader:

- You enjoy math.

- You know high-school math pretty well, and have at least begun the study of “higher mathematics” such as calculus and linear algebra.

- You want to become better at solving math problems.

First, what is a problem? We distinguish between problems and exercises. An exercise is a question that you know how to resolve immediately. Whether you get it right or not depends on how expertly you apply specific techniques, but you don’t need to puzzle out what techniques to use. In contrast, a problem demands much thought and resourcefulness before the right approach is found. For example, here is an exercise.

Example 1.1.1 Compute $5436^3$ without a calculator.

You have no doubt about how to proceed—just multiply, carefully. The next question is more subtle.

Example 1.1.2 Write

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}$$

as a fraction in lowest terms.

At first glance, it is another tedious exercise, for you can just carefully add up all 99 terms, and hope that you get the right answer. But a little investigation yields something intriguing. Adding the first few terms and simplifying, we discover that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{3},$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4},$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{4}{5}.$$
which leads to the conjecture that for all positive integers $n$,
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.
\]
So now we are confronted with a problem: is this conjecture true, and if so, how do we prove that it is true? If we are experienced in such matters, this is still a mere exercise, in the technique of mathematical induction (see page 42). But if we are not experienced, it is a problem, not an exercise. To solve it, we need to spend some time, trying out different approaches. The harder the problem is, the more time we need. Often the first approach fails. Sometimes the first dozen approaches fail!

Here is another question, the famous “Census-Taker Problem.” A few people might think of this as an exercise, but for most, it is a problem.

**Example 1.1.3** A census-taker knocks on a door, and asks the woman inside how many children she has and how old they are.

“I have three daughters, their ages are whole numbers, and the product of the ages is 36,” says the mother.

“That’s not enough information,” responds the census-taker.

“I’d tell you the sum of their ages, but you’d still be stumped.”

“I wish you’d tell me something more.”

“Okay, my oldest daughter Annie likes dogs.”

What are the ages of the three daughters?

After the first reading, it seems impossible—there isn’t enough information to determine the ages. That’s why it is a problem, and a fun one, at that. (The answer is at the end of this chapter, on page 11, if you get stumped.)

If the Census-Taker Problem is too easy, try this next one (see page 72 for solution):

**Example 1.1.4** I invite 10 couples to a party at my house. I ask everyone present, including my wife, how many people they shook hands with. It turns out that everyone questioned—I didn’t question myself, of course—shook hands with a different number of people. If we assume that no one shook hands with his or her partner, how many people did my wife shake hands with? (I did not ask myself any questions.)

A good problem is mysterious and interesting. It is mysterious, because at first you don’t know how to solve it. If it is not interesting, you won’t think about it much. If it is interesting, though, you will want to put a lot of time and effort into understanding it.

This book will help you to investigate and solve problems. If you are an inexperienced problem solver, you may often give up quickly. This happens for several reasons.

- You may just not know how to begin.
- You may make some initial progress, but then cannot proceed further.
- You try a few things, nothing works, so you give up.

An experienced problem solver, in contrast, is rarely at a loss for how to begin investigating a problem. He or she\(^1\) confidently tries a number of approaches to get started. This may not solve the problem,

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\(^1\)We will henceforth avoid the awkward “he or she” construction by choosing genders randomly in subsequent chapters.
1.2 The Three Levels of Problem Solving

Some branches of mathematics have very long histories, with many standard symbols and words. Problem solving is not one of them. We use the terms strategy, tactics, and tools to denote three different levels of problem solving. Since these are not standard definitions, it is important that we understand exactly what they mean.

A Mountaineering Analogy

You are standing at the base of a mountain, hoping to climb to the summit. Your first strategy may be to take several small trips to various easier peaks nearby, so as to observe the target mountain from different angles. After this, you may consider a somewhat more focused strategy, perhaps to try climbing the mountain via a particular ridge. Now the tactical considerations begin: how to actually achieve the chosen strategy. For example, suppose that strategy suggests climbing the south ridge of the peak, but there are snowfields and rivers in our path. Different tactics are needed to negotiate each of these obstacles. For the snowfield, our tactic may be to travel early in the morning, while the snow is hard. For the river, our tactic may be scouting the banks for the safest crossing. Finally, we move onto the most tightly focused level, that of tools: specific techniques to accomplish specialized tasks. For example, to cross the snowfield, we may set up a particular system of ropes for safety and walk with ice axes. The river crossing may require the party to strip from the waist down and hold hands for balance. These are all tools. They are very specific. You would never summarize, “To climb the mountain we had to take our pants off and hold hands,” because this was a minor—though essential—component of the entire climb. On the other hand, strategic and sometimes tactical ideas are often described in your summary: “We decided to reach the summit via the south ridge and had to cross a difficult snowfield and a dangerous river to get to the ridge.”

As we climb a mountain, we may encounter obstacles. Some of these obstacles are easy to negotiate, for they are mere exercises (of course this depends on the climber’s ability and experience). But one obstacle may present a difficult miniature problem, whose solution clears the way for the entire climb. For example, the path to the summit may be easy walking, except for one 10-foot section of steep ice. Climbers call negotiating the key obstacle the crux move. We shall use this term for mathematical problems as well. A crux move may take place at the strategic, tactical, or tool level; some problems have several crux moves; many have none.

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2In fact, there does not even exist a standard name for the theory of problem solving, although George Pólya and others have tried to popularize the term heuristics (see, for example, [24]).
From Mountaineering to Mathematics

Let’s approach mathematical problems with these mountaineering ideas. When confronted with a problem, you cannot immediately solve it, for otherwise, it is not a problem but a mere exercise. You must begin a process of investigation. This investigation can take many forms. One method, by no means a terrible one, is to just randomly try whatever comes into your head. If you have a fertile imagination, and a good store of methods, and a lot of time to spare, you may eventually solve the problem. However, if you are a beginner, it is best to cultivate a more organized approach. First, think strategically. Don’t try immediately to solve the problem, but instead think about it on a less-focused level. The goal of strategic thinking is to come up with a plan that may only barely have mathematical content, but which leads to an “improved” situation, not unlike the mountaineer’s strategy, “If we get to the south ridge, it looks like we will be able to get to the summit.”

Strategies help us get started, and help us continue. But they are just vague outlines of the actual work that needs to be done. The concrete tasks to accomplish our strategic plans are done at the lower levels of tactic and tool.

Here is an example that shows the three levels in action, from a 1926-Hungarian contest.

Example 1.2.1 Prove that the product of four consecutive natural numbers cannot be the square of an integer.

Solution: Our initial strategy is to familiarize ourselves with the statement of the problem, i.e., to get oriented. We first note that the question asks us to prove something. Problems are usually of two types—those that ask you to prove something and those that ask you to find something. The Census-Taker Problem (Example 1.1.3) is an example of the latter type.

Next, observe that the problem is asking us to prove that something cannot happen. We divide the problem into hypothesis (also called “the given”) and conclusion (whatever the problem is asking you to find or prove). The hypothesis is:

Let \( n \) be a natural number.

The conclusion is:

\[ n(n + 1)(n + 2)(n + 3) \text{ cannot be the square of an integer.} \]

Formulating the hypothesis and conclusion isn’t a triviality, since many problems don’t state them precisely. In this case, we had to introduce some notation. Sometimes our choice of notation can be critical.

Perhaps we should focus on the conclusion: how do you go about showing that something cannot be a square? This strategy, trying to think about what would immediately lead to the conclusion of our problem, is called looking at the penultimate step.\(^3\) Unfortunately, our imagination fails us—we cannot think of any easy criteria for determining when a number cannot be a square. So we try another strategy, one of the best for beginning just about any problem: get your hands dirty. We try plugging in some numbers to experiment with. If we are lucky, we may see a pattern. Let’s try a few different values for \( n \). Here’s a table. We use the abbreviation \( f(n) = n(n + 1)(n + 2)(n + 3) \).

\(^3\)The word “penultimate” means “next to last.”
Notice anything? The problem involves squares, so we are sensitized to look for squares. Just about everyone notices that the first two values of $f(n)$ are one less than a perfect square. A quick check verifies that additionally,

$$f(3) = 19^2 - 1, \quad f(4) = 29^2 - 1, \quad f(5) = 41^2 - 1, \quad f(10) = 131^2 - 1.$$ 

We confidently conjecture that $f(n)$ is one less than a perfect square for every $n$. Proving this conjecture is the penultimate step that we were looking for, because a positive integer that is one less than a perfect square cannot be a perfect square since the sequence $1, 4, 9, 16, \ldots$ of perfect squares contains no consecutive integers (the gaps between successive squares get bigger and bigger). Our new strategy is to prove the conjecture.

To do so, we need help at the tactical/tool level. We wish to prove that for each $n$, the product $n(n+1)(n+2)(n+3)$ is one less than a perfect square. In other words, $n(n+1)(n+2)(n+3) + 1$ must be a perfect square. How to show that an algebraic expression is always equal to a perfect square? One tactic: factor the expression! We need to manipulate the expression, always keeping in mind our goal of getting a square. So we focus on putting parts together that are almost the same. Notice that the product of $n$ and $n+3$ is “almost” the same as the product of $n+1$ and $n+2$, in that their first two terms are both $n^2 + 3n$. After regrouping, we have

$$[n(n+3)][(n+1)(n+2)] + 1 = (n^2 + 3n)(n^2 + 3n + 2) + 1. \quad (1.2.1)$$

Rather than multiply out the two almost-identical terms, we introduce a little symmetry to bring squares into focus:

$$(n^2 + 3n)(n^2 + 3n + 2) + 1 = (n^2 + 3n + 1 - 1) (n^2 + 3n + 1 + 1) + 1.$$ 

Now we use the “difference of two squares” factorization (a tool!) and we have

$$((n^2 + 3n + 1) - 1) ((n^2 + 3n + 1) + 1) + 1 = (n^2 + 3n + 1)^2 - 1 + 1 = (n^2 + 3n + 1)^2.$$

We have shown that $f(n)$ is one less than a perfect square for all integers $n$, namely

$$f(n) = (n^2 + 3n + 1)^2 - 1,$$

and we are done.

Let us look back and analyze this problem in terms of the three levels. Our first strategy was orientation, reading the problem carefully and classifying it in a preliminary way. Then we decided on a strategy to look at the penultimate step that did not work at first, but the strategy of numerical experimentation led to a conjecture. Successfully proving this involved the tactic of factoring, coupled with a use of symmetry and the tool of recognizing a common factorization.
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The most important level was strategic. Getting to the conjecture was the crux move. At this point the problem metamorphosed into an exercise! For even if you did not have a good tactical grasp, you could have muddled through. One fine method is substitution: Let \( u = n^2 + 3n \) in equation (1.2.1). Then the right-hand side becomes \( u(u + 2) + 1 = u^2 + 2u + 1 = (u + 1)^2 \). Another method is to multiply out (ugh!). We have

\[
(n + 1)(n + 2)(n + 3) + 1 = n^4 + 6n^3 + 11n^2 + 6n + 1.
\]

If this is going to be the square of something, it will be the square of the quadratic polynomial \( n^2 + an + 1 \) or \( n^2 + an - 1 \). Trying the first case, we equate

\[
n^4 + 6n^3 + 11n^2 + 6n + 1 = (n^2 + an + 1)^2 = n^4 + 2an^3 + (a^2 + 2)n^2 + 2an + 1
\]

and we see that \( a = 3 \) works; i.e., \( n(n + 1)(n + 2)(n + 3) + 1 = (n^2 + 3n + 1)^2 \). This was a bit less elegant than the first way we solved the problem, but it is a fine method. Indeed, it teaches us a useful tool: the method of undetermined coefficients.

1.3 A Problem Sampler

The problems in this book are classified into three large families: recreational, contest, and open-ended. Within each family, problems split into two basic kinds: problems “to find” and problems “to prove.” Problems “to find” ask for a specific piece of information, while problems “to prove” require a more general argument. Sometimes the distinction is blurry. For example, Example 1.1.4 above is a problem “to find,” but its solution may involve a very general argument.

What follows is a descriptive sampler of each family.

Recreational Problems

Also known as “brain teasers,” these problems usually involve little formal mathematics, but instead rely on creative use of basic strategic principles. They are excellent to work on, because no special knowledge is needed, and any time spent thinking about a recreational problem will help you later with more mathematically sophisticated problems. The Census-Taker Problem (Example 1.1.3) is a good example of a recreational problem. A gold mine of excellent recreational problems is the work of Martin Gardner, who edited the “Mathematical Games” department for *Scientific American* for many years. Many of his articles have been collected into books. Two of the nicest are perhaps [9] and [8].

1.3.1 A monk climbs a mountain. He starts at 8 AM and reaches the summit at noon. He spends the night on the summit. The next morning, he leaves the summit at 8 AM and descends by the same route that he used the day before, reaching the bottom at noon. Prove that there is a time between 8 AM and noon at which the monk was at exactly the same spot on the mountain on both days. (Notice that we do not specify anything about the speed that the monk travels. For example, he could race at 1000 miles per hour for the first few minutes, then sit still for hours, then travel backward, etc. Nor does the monk have to travel at the same speeds going up as going down.)

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These two terms are due to George Pólya [24].
1.3.2 You are in the downstairs lobby of a house. There are three switches, all in the “off” position. Upstairs, there is a room with a lightbulb that is turned off. One and only one of the three switches controls the bulb. You want to discover which switch controls the bulb, but you are only allowed to go upstairs once. How do you do it? (No fancy strings, telescopes, etc. allowed. You cannot see the upstairs room from downstairs. The lightbulb is a standard 100-watt bulb.)

1.3.3 You leave your house, travel one mile due south, then one mile due east, then one mile due north. You are now back at your house! Where do you live? There is more than one solution; find as many as possible.

Contest Problems

These problems are written for formal exams with time limits, often requiring specialized tools and/or ingenuity to solve. Several exams at the high-school and undergraduate levels involve sophisticated and interesting mathematics.

American High School Math Exam (AHSME) Taken by hundreds of thousands of self-selected high-school students each year, this multiple-choice test has questions similar to the hardest and most interesting problems on the SAT.\(^5\)

American Invitational Math Exam (AIME) The top 2000 or so scorers on the AHSME qualify for this three-hour, 15-question test. Both the AHSME and AIME feature problems “to find,” since these tests are graded by machine.

USA Mathematical Olympiad (USAMO) The top 150 AIME participants participate in this elite three-and-a-half-hour, five-question essay exam, featuring mostly challenging problems “to prove.”\(^6\)

American Regions Mathematics League (ARML) Every year, ARML conducts a national contest between regional teams of high-school students. Some of the problems are quite challenging and interesting, roughly comparable to the harder questions on the AHSME and AIME and the easier USAMO problems.

Other national and regional olympiads Many other nations conduct difficult problem solving contests. Eastern Europe in particular has a very rich contest tradition, including very recently China and Vietnam have developed very innovative and challenging examinations.

International Mathematical Olympiad (IMO) The top USAMO scorers are invited to a training program, which then selects the six-member USA team that competes in this international contest. It is a nine-hour, six-question essay exam, spread over two days.\(^7\) The IMO began in 1959, and takes place in a different country each year. At first it was a small event restricted to Iron Curtain countries, but recently the event has become quite inclusive, with 75 nations represented in 1996.

\(^5\)Recently, this exam has been replaced by the AMC-8, AMC-10, and AMC-12 exams, for different targeted grade levels.

\(^6\)There now is an exam for younger students, the USA Junior Mathematical Olympiad (USAJMO).

\(^7\)Starting in 1996, the USAMO adopted a similar format: six questions, taken during two sessions.
Putnam Exam The most important problem solving contest for American undergraduates, a 12-question, six-hour exam taken by several thousand students each December. The median score is often zero.

Problems in magazines A number of mathematical journals have problem departments, in which readers are invited to propose problems and/or mail in solutions. The most interesting solutions are published, along with a list of those who solved the problem. Some of these problems can be extremely difficult, and many remain unsolved for years. Journals with good problem departments, in increasing order of difficulty, are Math Horizons, The College Mathematics Journal, Mathematics Magazine, and The American Mathematical Monthly. All of these are published by the Mathematical Association of America. There is also a journal devoted entirely to interesting problems and problem solving, Crux Mathematicorum, published by the Canadian Mathematical Society.

Contest problems are very challenging. It is a significant accomplishment to solve a single such problem, even with no time limit. The samples below include problems of all difficulty levels.

1.3.4 (AHSME 1996) In the $xy$-plane, what is the length of the shortest path from $(0, 0)$ to $(12, 16)$ that does not go inside the circle $(x - 6)^2 + (y - 8)^2 = 25$?

1.3.5 (AHSME 1996) Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have?

1.3.6 (AHSME 1994) When $n$ standard six-sided dice are rolled, the probability of obtaining a sum of 1994 is greater than zero and is the same as the probability of obtaining a sum of $S$. What is the smallest possible value of $S$?

1.3.7 (AIME 1994) Find the positive integer $n$ for which

$$[\log_2 1] + [\log_2 2] + [\log_2 3] + \cdots + [\log_2 n] = 1994,$$

where $[x]$ denotes the greatest integer less than or equal to $x$. (For example, $[\pi] = 3$.)

1.3.8 (AIME 1994) For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define $\Delta A$ to be the sequence $(a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots)$ whose $n$th term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_9 = 0$. Find $a_1$.

1.3.9 (USAMO 1989) The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of six games with 12 distinct players.

1.3.10 (USAMO 1995) A calculator is broken so that the only keys that still work are the sin, cos, tan, sin$^{-1}$, cos$^{-1}$, and tan$^{-1}$ buttons. The display initially shows 0. Given any positive rational number $q$, show that pressing some finite sequence of buttons will yield $q$. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

1.3.11 (IMO 1976) Determine, with proof, the largest number that is the product of positive integers whose sum is 1976.

1.3.12 (Russia 1996) A palindrome is a number or word that is the same when read forward and backward, for example, “176671” and “civic.” Can the number obtained by writing the numbers from 1 to $n$ in order (for some $n > 1$) be a palindrome?

1.3.13 (Putnam 1994) Let $(a_n)$ be a sequence of positive reals such that, for all $n$, $a_n \leq a_{2n} + a_{2n+1}$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

1.3.14 (Putnam 1994) Find the positive value of $m$ such that the area in the first quadrant enclosed by the ellipse $x^2/9 + y^2 = 1$, the $x$-axis, and the line $y = 2x/3$ is equal to the area in the first quadrant enclosed by the ellipse $x^2/9 + y^2 = 1$, the $y$-axis, and the line $y = mx$.

1.3.15 (Putnam 1990) Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?
Open-Ended Problems

These are mathematical questions that are sometimes vaguely worded, and possibly have no actual solution (unlike the two types of problems described above). Open-ended problems can be very exciting to work on, because you don’t know what the outcome will be. A good open-ended problem is like a hike (or expedition!) in an uncharted region. Often partial solutions are all that you can get. (Of course, partial solutions are always OK, even if you know that the problem you are working on is a formal contest problem that has a complete solution.)

1.3.16 Here are the first few rows of Pascal’s Triangle.

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1,
\end{array}
\]

where the elements of each row are the sums of pairs of adjacent elements of the prior row. For example, \(10 = 4 + 6\). The next row in the triangle will be

\[1, 6, 15, 20, 15, 6, 1.\]

There are many interesting patterns in Pascal’s Triangle. Discover as many patterns and relationships as you can, and prove as much as possible. In particular, can you somehow extract the Fibonacci numbers (see next problem) from Pascal’s Triangle (or vice versa)? Another question: is there a pattern or rule for the parity (evenness or oddness) of the elements of Pascal’s Triangle?

1.3.17 The Fibonacci numbers \(f_n\) are defined by \(f_0 = 0\), \(f_1 = 1\) and \(f_n = f_{n-1} + f_{n-2}\) for \(n > 1\). For example, \(f_2 = 1\), \(f_3 = 2\), \(f_4 = 3\), \(f_5 = 5\), \(f_6 = 8\), \(f_7 = 13\), \(f_8 = 21\). Play around with this sequence; try to discover as many patterns as you can, and try to prove your conjectures as best as you can. In particular, look at this amazing fact: for \(n \geq 0\),

\[
f_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).
\]

1.3.18 An “ell” is an L-shaped tile made from three \(1 \times 1\) squares, as shown below.

For what positive integers \(a, b\) is it possible to completely tile an \(a \times b\) rectangle using only ells? (“Tiling” means that we cover the rectangle exactly with ells, with no overlaps.) For example, it is clear that you can tile a \(2 \times 3\) rectangle with ells, but (draw a picture) you cannot tile a \(3 \times 3\) with ells. After you understand rectangles, generalize in two directions: tiling ells in more elaborate shapes, tiling shapes with things other than ells.

1.3.19 Imagine a long \(1 \times L\) rectangle, where \(L\) is an integer. Clearly, one can pack this rectangle with \(L\) circles of diameter 1, and no more. (By “pack” we mean that touching is OK, but overlapping is not.) On the other hand, it is not immediately obvious that \(2L\) circles is the maximum number possible for packing a \(2 \times L\) rectangle. Investigate this, and generalize to \(m \times L\) rectangles.

1.4 How to Read This Book

This book is not meant to be read from start to finish, but rather to be perused in a “non-linear” way. The book is designed to help you study two subjects: problem solving methodology and specific
mathematical ideas. You will gradually learn more math and also become more adept at “problem-solvingology,” and progress in one area will stimulate success in the other.

The book is divided into two parts, with a “bridge” chapter in the middle. Chapters 1–3 give an overview of strategies and tactics. Each strategy or tactic is discussed in a section that starts out with simple examples but ends with sophisticated problems. At some point, you may find that the text gets harder to understand, because it requires more mathematical experience. You should read the beginning of each section carefully, but then start skimming (or skipping) as it gets harder. You can (and should) reread later.

Chapters 5–9 are devoted to mathematical ideas at the tactical or tool level, organized by mathematical subject and developed specifically from the problem solver’s point of view. Depending on your interests and background, you will read all or just some of these chapters.

Chapter 4 is a bridge between general problem solving and specific mathematical topics. It looks in detail at three important “crossover” tactics that connect different branches of mathematics. Some of the material in this chapter is pretty advanced, but we place it early in the book to give the reader a quick route to sophisticated ideas that can be applied very broadly.

As you increase your mathematical knowledge (from Chapters 5 to 9), you may want to return to the earlier chapters to reread sections that you may have skimmed earlier. Conversely, as you increase your problem solving skills from the early chapters, you may reread (or read for the first time) some of the later chapters. Ideally, you will read every page of this book at least twice, and read, if not solve, every single problem in it.

Throughout the book, new terms and specific strategy, tactic, and tool names are in italics. Also, when an important point is made, it is indented and printed in italics, like this.

That means, “pay attention!” To signify the successful completion of a solution, we use the “Halmos” symbol, a filled-in square. We used a Halmos at the end of Example 1.2.1 on page 5, and this line ends with one.

Please read with pencil and paper by your side and/or write in the margins! Mathematics is meant to be studied actively. Also—this requires great restraint—try to solve each example as you read it, before reading the solution in the text. At the very least, take a few moments to ponder the problem. Don’t be tempted into immediately looking at the solution. The more actively you approach the material in this book is, the faster you will master it. And you’ll have more fun.

Of course, some of the problems presented are harder than other. Toward the end of each section (or subsection), we may discuss a “classic” problem, one that is usually too hard for the beginning reader to solve alone in a reasonable amount of time. These classics are included for several reasons: they illustrate important ideas; they are part of what we consider the essential “repertoire” for every young mathematician; and, most important, they are beautiful works of art, to be pondered and savored. This book is called The Art and Craft of Problem Solving, and while we devote many more pages to the craft aspect of problem solving, we don’t want you to forget that problem solving, at its best, is a passionate, aesthetic endeavor. If you will indulge us in another analogy, pretend that you are learning jazz piano improvisation. It’s vital that you practice scales and work on your own improvisations, but you also need the instruction and inspiration that comes from listening to some great recordings.

8Named after Paul Halmos, a mathematician and writer who popularized its use.
Solution to the Census-Taker Problem

The product of the ages is 36, so there are only a few possible triples of ages. Here is a table of all the possibilities, with the sums of the ages below each triple.

\[
\begin{array}{cccccccc}
(1,1,36) & (1,2,18) & (1,3,12) & (1,4,9) & (1,6,6) & (2,2,9) & (2,3,6) & (3,3,4) \\
38 & 21 & 16 & 14 & 13 & 13 & 11 & 10 \\
\end{array}
\]

Aha! Now we see what is going on. The mother’s second statement (“I’d tell you the sum of their ages, but you’d still be stumped”) gives us valuable information. It tells us that the ages are either (1, 6, 6) or (2, 2, 9), for in all other cases, knowledge of the sum would tell us unambiguously what the ages are! The final clue now makes sense; it tells us that there is an oldest daughter, eliminating the triple (1, 6, 6). The daughters are thus 2, 2, and 9 years old.