CHAPTER 5

ANALYSIS OF PERIODIC STRUCTURES

5.1 INTRODUCTION

Periodic structures play an important role in microwaves and optics, e.g. gratings are used to design special filters, phase or delay elements or equalisers. Let us first look on some examples of such periodic structures.

![Fig. 5.1 Bragg grating (COST 240) (Reproduced by permission of Elsevier)](image1)

![Fig. 5.2 Polarisation converter (Reproduced by permission of Elsevier)](image2)

First, a Bragg grating is shown in Fig. 5.1. The next example is a polarisation converter, shown in Fig. 5.2. Here we have additional different sections at the input and output of the structure. The meander line shown in Fig. 5.3a is a typical example of a periodic structure at microwave frequencies. One period of this periodic structure is sketched in Fig. 5.3b.
Different types of fibre Bragg grating are shown in Fig. 5.4. The first two consist of two different sections each and may have symmetric periods. In the third example, however, we have three different sections and unsymmetric periods.

Modelling symmetric periodic structures takes much less numerical effort than analysing unsymmetric structures. Therefore, when examining a periodic structure we should try to consider symmetric periods. E.g. to achieve this, we could take a period defined by the two vertical dashed lines for the Bragg grating shown in Fig. 5.1. Generally, we should take a single period that starts and ends in the centre of a homogeneous section.

Fig. 5.5a shows a period of the structure shown in Fig. 5.4 as an example. The periodic part may of course be more complicated. In Fig. 5.5b we have a Sine grating, which is divided in the presented example into four homogeneous sections. Yet, it still looks the same from the left as from the right. The grating presented in Fig. 5.4c, however, cannot be divided into such symmetric periods and a more complicated algorithm will be required for its analysis.

The above examples all show structures with a one-dimensional periodicity. Two- and three-dimensional periodic structures are a current topic of research. These structures are called photonic crystals (PC) or bandgap structures.

In Fig. 5.6 a structure consisting of dielectric rods in air is sketched as an example of such a PC. PCs prohibit the propagation of electromagnetic waves (in our case in the $x$–$y$-plane) of some frequencies. By introducing defects (e.g. by removing some of the rods), various circuits may be designed. Fig. 5.7 shows such a defect waveguide as an example. One of the problems of interest is the determination of the bandgaps in such structures. We will show how the band diagram can be determined with the MoL.

In the following we will first describe the analysis of general one-dimensional periodic structures. The results are the background for the algorithms that are developed for 2D and 3D structures. After considering
the 1D case we will show how symmetric periodic structures can be analysed very efficiently [1], and give a description of unsymmetric periodic structures. We will then give numerical results for several of the examples above. We will finish by describing in detail the examination of photonic crystal structures and of circuits consisting of PCs.

5.2 PRINCIPLE BEHAVIOUR OF PERIODIC STRUCTURES

Before we describe the analysis procedures for arbitrary periodic structures, let us begin with their general behaviour. For this purpose we shall examine
The electric and magnetic fields of the fundamental TEM mode are constant in each cross-section. The characteristic quantities are:

\[ \beta = nk_o \quad Z_F = \eta_o/n \quad \eta_o = \sqrt{\mu_o/\varepsilon_o} = 120\pi \Omega \]  

where \( \beta \) is the propagation constant, \( k_o \) the free space wave number, \( Z_F \) the characteristic field impedance and \( n \) the effective refractive index of the dielectric. With \( \theta = \beta l \), we describe the relation of the fields between ports \( i \).
**Fig. 5.7** Defect waveguide in a photonic crystal (A. Barcz, S. Helfert and R. Pregla, 'Modeling of 2D photonic crystals by using the Method of Lines', in *ICTON Conf.* © 2002 Institute of Electrical and Electronics Engineers (IEEE))

![Defect waveguide in a photonic crystal](image)

**Fig. 5.8** Periodic parallel-plate structure

and \( i + 1 \) by:

\[
\begin{bmatrix}
    E_i \\
    H_i
\end{bmatrix} = [M] \begin{bmatrix}
    E_{i+1} \\
    H_{i+1}
\end{bmatrix}
\]

\[
[M] = \begin{bmatrix}
    \cos \theta_1 & j Z_{F1} \sin \theta_1 \\
    j Z_{F1}^{-1} \sin \theta_1 & \cos \theta_1
\end{bmatrix} \begin{bmatrix}
    \cos \theta_2 & j Z_{F2} \sin \theta_2 \\
    j Z_{F2}^{-1} \sin \theta_2 & \cos \theta_2
\end{bmatrix}
\]

(5.2)

The two matrices in this product in \([M]\) are nothing other than the well-known transfer matrices of the TEM transmission line sections. With \( \zeta = Z_{F1}/Z_{F2} \), we obtain for this product:

\[
[M] = \begin{bmatrix}
    \cos \theta_1 \cos \theta_2 - \zeta \sin \theta_1 \sin \theta_2 & j Z_{F1} (\sin \theta_1 \cos \theta_2 + \zeta^{-1} \sin \theta_2 \cos \theta_1) \\
    j Z_{F1}^{-1} (\sin \theta_1 \cos \theta_2 + \zeta \sin \theta_2 \cos \theta_1) & \cos \theta_1 \cos \theta_2 - \zeta^{-1} \sin \theta_1 \sin \theta_2
\end{bmatrix}
\]

(5.3)
With the ansatz according to Floquet’s theorem:

\[
\begin{bmatrix}
E_{i+1} \\
H_{i+1}
\end{bmatrix} = \begin{bmatrix}
e^{-j\psi} \\
e^{-j\psi}
\end{bmatrix} \begin{bmatrix}
E_i \\
H_i
\end{bmatrix}
\]

(5.4)

we obtain the following equation for the phase \(\psi\): (The determinant of the matrix in the resulting eigenvalue problem must be zero.)

\[e^{j2\psi} - e^{j\psi} (2 \cos \theta_1 \cos \theta_2 - (\zeta + \zeta^{-1}) \sin \theta_1 \sin \theta_2) + 1 = 0\]

or:

\[\cos \psi = \cos \theta_1 \cos \theta_2 - a \sin \theta_1 \sin \theta_2\]

(5.5)

with:

\[a = \frac{1}{2} (\zeta + \zeta^{-1}) = \frac{1}{2} \left( \frac{Z_1}{Z_2} + \frac{Z_2}{Z_1} \right) = \frac{1}{2} \left( \frac{n_2}{n_1} + \frac{n_1}{n_2} \right) \geq 1\]

We now choose sections of equal electrical length (i.e. \(l_1n_1 = l_2n_2\)), leading to \(\theta_1 = \theta_2 = \theta\). Therefore, we obtain:

\[\cos \psi = \cos^2 \theta - a \sin^2 \theta\]

(5.6)

In Fig. 5.9 the curves of \(\cos \psi\) and \(\psi\) as functions of \(\theta\) are drawn. The periodic structure shows filter behaviour. We have a transmission band (TB) for \(|\cos \psi| \leq 1\) ist. In the stop band (SB) (or band gap) the phase is constant and equal to \(\pi\). To show the change from the homogeneous line to the periodic one, the phase curve \(\psi = 2\theta\) of the homogeneous line is introduced into the diagram, too.

The field decay or ‘transmission loss’ in the stop band is obtained from:

\[\cos(\pi - j\alpha \Lambda) = -\cosh \alpha \Lambda = \cos^2 \theta - a \sin^2 \theta\]

\[\cosh(\alpha \Lambda) = a \sin^2 \theta - \cos^2 \theta\]

(5.7)

with \(\Lambda = l_1 + l_2\). The centre of the stop band is determined as \(\theta_o = \pi/2\). The related frequency \(f_o\) (\(\theta = k_o nl\)) is given by \(f_o = c/(4nl)\). The maximum of the transmission loss (at the centre) is equal to:

\[\Lambda \alpha_{max} = \ln (a + \sqrt{a^2 - 1}) = \left| \ln \frac{n_1}{n_2} \right| \approx \frac{\Delta n}{n}\]

where \(n = (n_1 + n_2)/2\) and \(|n_1 - n_2| = \Delta n\). The approximate value for the example in Fig. 5.9 is given by \(\Lambda \alpha_{max} = 0.5\), which is in good agreement with the exact value of 0.5108. The band edges of the stop band are determined from the condition \(\cos \psi = -1\). Therefore, we obtain:

\[(a + 1) \sin^2 \theta_c = 2\]

(5.8)
Fig. 5.9 Curves of $\cos \psi$ and $\psi$ as a function of $\theta$

With $\theta = \frac{\pi f}{2 f_o}$ and $\theta_{c1,2} = \frac{\pi f_o \pm \Delta f}{2 f_o}$ we obtain:

$$(a + 1) \cos^2 \left( \frac{\pi \Delta f}{2 f_o} \right) = 2$$

or:

$$\frac{2 \Delta f}{f_o} = \frac{4}{\pi} \arcsin \sqrt{\frac{a - 1}{a + 1}} = \frac{4}{\pi} \arcsin \left| \frac{n_1 - n_2}{n_1 + n_2} \right|$$  (5.9)
For small band stop filters we have \(|n_1 - n_2| \ll n_1 + n_2\). Therefore, the following approximation may be introduced for the relative bandwidth:

\[
\frac{2\Delta f}{f_0} \approx \frac{2}{\pi} \frac{\Delta n}{n}
\]  

(5.10)

The exact value of the relative bandwidth is 0.315 in our example and the approximate one is 0.3183. Thus the approximations are very good even for relatively large values of \(\Delta n/n\).

### 5.3 GENERAL THEORY OF PERIODIC STRUCTURES

#### 5.3.1 Port relations for general two ports

As is well known, we can describe a two port with open-circuit or short-circuit matrix parameters (see Section 2.5 and [3]). Therefore, we can also describe a periodic structure in such a way. Hence, we may write for one period:

\[
\begin{bmatrix}
E_A \\
E_B \\
H_A \\
H_B
\end{bmatrix} =
\begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix}
\begin{bmatrix}
H_A \\
-H_B
\end{bmatrix} =
\begin{bmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{bmatrix}
\begin{bmatrix}
E_A \\
E_B \\
H_A \\
H_B
\end{bmatrix}
\]  

(5.11)

From eq. (5.11) we easily obtain the following transfer matrix relations:

\[
\begin{bmatrix}
E_B \\
H_B
\end{bmatrix} =
\begin{bmatrix}
z_{22}z_{12}^{-1} & z_{21} - z_{22}z_{12}^{-1}z_{11} \\
-z_{12}^{-1} & z_{12}^{-1}z_{11}
\end{bmatrix}
\begin{bmatrix}
E_A \\
H_A
\end{bmatrix}
\]  

(5.12)

\[
\begin{bmatrix}
E_A \\
H_A
\end{bmatrix} =
\begin{bmatrix}
z_{11}z_{21}^{-1} & z_{11}z_{21}^{-1}z_{22} - z_{12} \\
z_{21}^{-1} & z_{21}^{-1}z_{22}
\end{bmatrix}
\begin{bmatrix}
E_B \\
H_B
\end{bmatrix}
\]  

(5.13)

\[
\begin{bmatrix}
E_A \\
H_A
\end{bmatrix} =
\begin{bmatrix}
-y_{21}^{-1}y_{22} & -y_{21}^{-1} \\
y_{12} - y_{11}y_{21}^{-1}y_{22} & -y_{11}y_{21}^{-1}
\end{bmatrix}
\begin{bmatrix}
E_B \\
H_B
\end{bmatrix}
\]  

(5.14)

\[
\begin{bmatrix}
E_B \\
H_B
\end{bmatrix} =
\begin{bmatrix}
-y_{12}^{-1}y_{11} & y_{12}^{-1} \\
y_{22}y_{12}^{-1}y_{11} - y_{21} & -y_{22}y_{12}^{-1}
\end{bmatrix}
\begin{bmatrix}
E_A \\
H_A
\end{bmatrix}
\]  

(5.15)

The vectors \(E_{A,B}\) and \(H_{A,B}\) contain the transversal field components at ports A and B (see e.g. Fig. 5.1). Instead of the fields in the original domain, we can use the ones in the transformed domain.

#### 5.3.2 Floquet modes for symmetric periods

The analysis of periodic structures can be simplified for symmetric (with respect to the periodicity) periods. In this case we may introduce \(z_{22} = z_{11}, z_{12} = z_{21}\) \((y_{22} = y_{11}, y_{12} = y_{21})\). Now, by using the Floquet modal matrices \(S_E\) and \(S_H\), we perform a transformation to Floquet modes according to:

\[
E_{A,B} = S_E \tilde{E}_{A,B} \quad H_{A,B} = S_H \tilde{H}_{A,B}
\]  

(5.16)
5.3.2.1 Floquet modes from transfer matrix equations

By using these Floquet modal matrices $S_E$ and $S_H$ we obtain from eqs. (5.12) and (5.15):

$$
\begin{bmatrix}
\tilde{E}_B \\
\tilde{H}_B
\end{bmatrix}
=
\begin{bmatrix}
S_E^{-1}z_{11}z_{12}^{-1}S_E & -S_E^{-1}(z_{11}z_{12}^{-1}z_{11} - z_{12})S_H \\
-S_H^{-1}z_{12}^{-1}S_E & S_H^{-1}z_{12}^{-1}z_{11}S_H
\end{bmatrix}
\begin{bmatrix}
\tilde{E}_A \\
\tilde{H}_A
\end{bmatrix}
$$

(5.17)

and:

$$
\begin{bmatrix}
\tilde{E}_B \\
\tilde{H}_B
\end{bmatrix}
=
\begin{bmatrix}
-S_E^{-1}y_{12}^{-1}y_{11}S_E & S_E^{-1}y_{12}^{-1}S_H \\
S_H^{-1}(y_{11}y_{12}^{-1}y_{11} - y_{12})S_E & -S_H^{-1}y_{11}y_{12}^{-1}S_H
\end{bmatrix}
\begin{bmatrix}
\tilde{E}_A \\
\tilde{H}_A
\end{bmatrix}
$$

(5.18)

By using Floquet’s theorem, we transform the period structure into an equivalent transmission line. Therefore, eqs. (5.17) and (5.18) should be equivalent to the following transmission line relation for the Floquet modes:

$$
\begin{bmatrix}
\tilde{E}_B \\
\tilde{H}_B
\end{bmatrix}
=
\begin{bmatrix}
\cosh \Gamma_F & -\tilde{Z}_0 \sinh \Gamma_F \\
-\tilde{Y}_0 \sinh \Gamma_F & \cosh \Gamma_F
\end{bmatrix}
\begin{bmatrix}
\tilde{E}_A \\
\tilde{H}_A
\end{bmatrix}
$$

(5.19)

$\Gamma_F$ is the matrix of phase constants and $\tilde{Z}_0$ ($\tilde{Y}_0^{-1} = \tilde{Z}_0^{-1}$) is the matrix of characteristic impedances. To induce this equivalence we have to solve the following eigenvalue problems:

$$
-y_{12}^{-1}y_{11}S_E = z_{11}z_{12}^{-1}S_E = S_E \lambda_E \\
-y_{11}y_{12}^{-1}S_H = z_{12}^{-1}z_{11}S_H = S_H \lambda_H
$$

(5.20)

which result in:

$$
\lambda_E = \lambda_H = \lambda_C = \cosh \Gamma_F
$$

(5.21)

By using the addition theorems for the hyperbolic functions we can also write:

$$
\sinh \Gamma_F = \sqrt{\lambda_C^2 - 1}
$$

The modal matrices $S_E$ and $S_H$ are not independent of each other. Because they are determined from the product of $z_{11}$ and $z_{12}^{-1}$, just the order of these matrices is exchanged (see eq. (5.20)). Therefore, the following relations hold:

$$
S_E = z_{11}S_H \delta_1 \\
S_H = z_{12}^{-1}S_E \delta_2
$$

(5.22)

We introduce the diagonal matrices $\delta_{1,2}$ because the amplitudes of eigenvectors can be chosen arbitrarily. The selection of these values $\delta_{1,2}$ is shown in the next section. For self-consistency, they have to fulfil the condition:

$$
\delta_1 \delta_2 = \lambda_C^{-1}
$$

(5.23)
We will now introduce eq. (5.22) into the off-diagonal submatrices in eq. (5.17). We start with the lower-left submatrix. We obtain:

$$S^{-1}_{H}z_{12}^{-1}S_{E} = S^{-1}_{H}z_{12}^{-1}z_{11}S_{H}\delta_{1} = \lambda_{C}\delta_{1}$$

$$= \delta_{2}^{-1}S^{-1}_{E}z_{12}z_{12}^{-1}S_{E} = \delta_{2}^{-1}$$

(5.24)

We substitute $S_{E}$ with the left eq. (5.22) (first line) or $S_{H}$ (second line) with the right equation (5.22). Since both expressions on the right side must be identical, we have again $\delta_{1}\delta_{2} = \lambda_{C}^{-1}$. We proceed analogously for the right upper matrix in eq. (5.17) and obtain:

$$S^{-1}_{E}(z_{11}z_{12}^{-1}z_{11} - z_{12})S_{H} = S^{-1}_{E}((z_{11}z_{12}^{-1})^2 - I)S_{E}\delta_{2} = (\lambda_{C}^2 - I)\delta_{2}$$

$$= \delta_{1}^{-1}S^{-1}_{H}(z_{12}^{-1}z_{11} - z_{11}^{-1}z_{12})S_{H}$$

$$= \delta_{1}^{-1}(\lambda_{C} - \lambda_{C}^{-1})$$

(5.25)

Since the expressions on the right side (first and third line) must be identical, we obtain again:

$$\delta_{1}\delta_{2} = \lambda_{C}^{-1}$$

Remember, $\lambda_{C}$ and $\delta_{1,2}$ are diagonal matrices, therefore we may exchange the order of the products. $S_{E}$ and $S_{E}$ were replaced through use of eq. (5.22). Furthermore, we take into account:

$$S^{-1}_{H}z_{11}^{-1}z_{12}S_{H} = (S^{-1}_{H}z_{12}^{-1}z_{11}S_{H})^{-1} = \lambda_{C}^{-1}$$

(5.26)

As we know, the off-diagonals of the matrix in eq. (5.17) should be equal to $\bar{Y}_{0} \cdot \sinh \Gamma_{F}$ and $\bar{Z}_{0} \cdot \sinh \Gamma_{F}$, respectively. By using eqs. (5.24) and (5.25), we obtain:

$$\lambda_{C}\delta_{1} = \bar{Y}_{0}\sqrt{\lambda_{C}^2 - I} \quad \text{and} \quad (\lambda_{C}^2 - I)\delta_{2} = \bar{Z}_{0}\sqrt{\lambda_{C}^2 - I}$$

Now, we have various possibilities when choosing the quantities $\delta_{1,2}$ in conjunction with $\bar{Z}_{0}$ and $\bar{Y}_{0}$. Two important ones will be described here.

1. By choosing $\bar{Z}_{0} = \bar{Y}_{0} = I$ we obtain:

$$\delta_{1} = \sqrt{\lambda_{C}^2 - I}/\lambda_{C} \quad \delta_{2} = I/\sqrt{\lambda_{C}^2 - I}$$

(5.27)

2. We can also choose $\delta_{1} = \delta_{2} = \delta$ and obtain:

$$\delta = \lambda_{C}^{-\frac{1}{2}} \quad \bar{Z}_{0} = ((\lambda_{C}^2 - I)/\lambda_{C})^{\frac{1}{2}} = \bar{Y}_{0}^{-1}$$

(5.28)

The problem now is to calculate the eigenvalues $\lambda_{C}$ and the eigenvectors $S_{E,H}$ in an efficient way.
5.3.2.2 Floquet modes from impedance/admittance equations

The last section shows how Floquet modes can be introduced as eigenmodes of periodic structures and how they are related to modes in homogeneous sections. However, for a computational determination of these Floquet modes with eq. (5.20) we must invert the transfer impedance (or admittance) $z_{12}$ (or $y_{12}$). This may cause problems if we have long sections and/or high losses.

We will now show how these problems can be avoided. We will use the open-circuit and short-circuit parameter formulation in eq. (5.11) to determine the Floquet-modes. Then, using the transformation according to eq. (5.16) instead of eq. (5.11), we can write for the equivalent homogeneous lines:

$$
\begin{bmatrix}
\tilde{E}_A \\
\tilde{E}_B
\end{bmatrix} = \tilde{Z}_0 \begin{bmatrix} (\tanh \Gamma_F)^{-1} & (\sinh \Gamma_F)^{-1} \\
(\sinh \Gamma_F)^{-1} & (\tanh \Gamma_F)^{-1}
\end{bmatrix} \begin{bmatrix}
\tilde{H}_A \\
-\tilde{H}_B
\end{bmatrix} \tag{5.29}
$$

$$
\begin{bmatrix}
\tilde{H}_A \\
-\tilde{H}_B
\end{bmatrix} = \tilde{Y}_0 \begin{bmatrix} (\tanh \Gamma_F)^{-1} & -(\sinh \Gamma_F)^{-1} \\
-(\sinh \Gamma_F)^{-1} & (\tanh \Gamma_F)^{-1}
\end{bmatrix} \begin{bmatrix}
\tilde{E}_A \\
\tilde{E}_B
\end{bmatrix} \tag{5.30}
$$

$\Gamma_F$ and $\tilde{Z}_0 = \tilde{Y}_0^{-1}$ are the Floquet modes’ phase and characteristic impedance matrix, respectively. Comparing the submatrices in eqs. (5.29) and (5.30) with the transformed submatrices in eq. (5.11), we see that the following relations hold:

$$
S_{E}^{-1}z_{11}S_{H} = \tilde{Z}_0(\tanh \Gamma_F)^{-1} \quad S_{H}^{-1}y_{11}S_{E} = \tilde{Y}_0(\tanh \Gamma_F)^{-1} \tag{5.31}
$$

From these two equations we can determine $\Gamma_F$ and $\tilde{Z}_0 = \tilde{Y}_0^{-1}$. The multiplication of these two equations results in:

$$
\tanh \Gamma_F = I/\sqrt{S_{E}^{-1}z_{11}y_{11}S_{E}} = I/\sqrt{S_{H}^{-1}y_{11}z_{11}S_{H}} = I/\sqrt{\lambda_{os}} \tag{5.32}
$$

where $\lambda_{os}$ is the eigenvalue matrix of $z_{11}y_{11}$ or $y_{11}z_{11}$. $\lambda_{os}$ is related to $\lambda_C$ by:

$$
\lambda_{os} = \lambda_C^2 / (\lambda_C^2 - I) \quad \lambda_C = \sqrt{\lambda_{os} / (\lambda_{os} - I)} \tag{5.33}
$$

The following relations hold for the new eigenvector matrices $S_{E}$ and $S_{H}$:

$$
S_{E} = z_{11}S_{H}\delta_a \quad S_{H} = y_{11}S_{E}\delta_b \tag{5.34}
$$

As before, we introduce the diagonal matrices $\delta_a, \delta_b$ because the eigenvectors are unique up to multiplication by a non-zero constant. For self-consistency, the condition $\delta_a \delta_b = \delta_b \delta_a = I/\lambda_{os}$ has to be fulfilled. By dividing the left expression in eq. (5.31) by the right one, we obtain:

$$
\tilde{Z}_0 = \sqrt{S_{E}^{-1}z_{11}S_{H}S_{E}^{-1}y_{11}^{-1}S_{H}} \tag{5.35}
$$

$$
= \sqrt{(S_{E}^{-1}z_{11}y_{11}S_{E})\delta_b\delta_a^{-1}(S_{H}^{-1}y_{11}z_{11}S_{H})^{-1}} = \sqrt{\delta_b\delta_a^{-1}}
$$
The selection $\delta_a = \delta_b$ results in $\tilde{Z}_0 = I$.

Because $z_{11}$ and $y_{11}$ are the open-circuit input impedance and short-circuit input admittance, respectively, they can be computed in a numerically stable way. $\Gamma_F$ and $S_{E,H}$ are also obtained numerically stable. We could determine $z_{11}$ and $y_{11}$ by performing an impedance and an admittance transformation along the whole period (i.e. two transformations). To reduce the numerical effort, we will show in the next sections how $\Gamma_F$ and $S_{E,H}$ can be computed from half of the periods, which roughly cuts the numerical effort in half.

5.3.2.3 Symmetric and anti-symmetric excitation
As mentioned in the last section, we can use half of the periods to determine the Floquet modes, in the case of symmetric periods. We obtain from eq. (5.11) for the even and odd excitation case:

1. Even case (magnetic wall in symmetry plane M): $E_A = E_B, H_A = -H_B$

$$E_A = (z_{11} + z_{12})H_A \rightarrow Z_{\text{even}} = z_{11} + z_{12} = z_{11h} \quad (5.36)$$
$$H_A = (y_{11} + y_{12})E_A \rightarrow Y_{\text{even}} = y_{11} + y_{12} = y_{11h}^{-1} \quad (5.37)$$

2. Odd case (electric wall in symmetry plane M): $E_A = -E_B, H_A = H_B$

$$E_A = (z_{11} - z_{12})H_A \rightarrow Z_{\text{odd}} = z_{11} - z_{12} = y_{11h}^{-1} \quad (5.38)$$
$$H_A = (y_{11} - y_{12})E_A \rightarrow Y_{\text{odd}} = y_{11} - y_{12} = y_{11h} \quad (5.39)$$

The subscript $h$ symbolises the half of the period. The matrices $z_{11}$ ($y_{11}$) and $z_{12}$ ($y_{12}$) can now be obtained from $z_{11h}$ and $y_{11h}$:

$$z_{11} = \frac{1}{2}(z_{11h} + y_{11h}^{-1}) \quad z_{12} = \frac{1}{2}(z_{11h} - y_{11h}^{-1}) \quad (5.40)$$
$$y_{11} = \frac{1}{2}(z_{11h}^{-1} + y_{11h}) \quad y_{12} = \frac{1}{2}(z_{11h}^{-1} - y_{11h}) \quad (5.41)$$

5.3.2.4 Determination of Floquet modes from half of the periods
The required quantities for Floquet modes can be calculated very efficiently by using the open- and short-circuit matrix parameters of half of the periods. Using the relations in eqs. (5.40) and (5.41) we obtain e.g.:

$$z_{11}z_{12}^{-1} = (z_{11h} + y_{11h}^{-1})(z_{11h} - y_{11h}^{-1})^{-1}$$
$$= (z_{11h}y_{11h} + I)(z_{11h}y_{11h} - I)^{-1} \quad (5.42)$$
$$-y_{12}^{-1}y_{11} = (y_{11h} - z_{11h}^{-1})^{-1}(y_{11h} + z_{11h}^{-1})$$
$$= (z_{11h}y_{11h} - I)^{-1}(z_{11h}y_{11h} + I) \quad (5.43)$$
$$z_{12}z_{11} = (z_{11h} - y_{11h}^{-1})^{-1}(z_{11h} + y_{11h})$$
$$= (y_{11h}z_{11h} - I)^{-1}(y_{11h}z_{11h} + I) \quad (5.44)$$
$$-y_{11}y_{12}^{-1} = (y_{11h} + z_{11h}^{-1})(y_{11h} - z_{11h}^{-1})^{-1}$$
$$= (y_{11h}z_{11h} + I)(y_{11h}z_{11h} - I)^{-1} \quad (5.45)$$
\( I \) is an adequate unit matrix. We obtain the eigenvalues/eigenvectors by introducing eqs. (5.42) and (5.44) into eq. (5.20):

\[
S_E^{-1} z_{11} z_{12}^{-1} S_E = \lambda_C = S_E^{-1} (z_{11h}y_{11h} + I) S_E S_E^{-1} (z_{11h}y_{11h} - I)^{-1} S_E
\]

or:

\[
S_H^{-1} z_{12}^{-1} z_{11} S_H = \lambda_C = S_H^{-1} (y_{11h} z_{11h} - I)^{-1} S_H S_H^{-1} (y_{11h} z_{11h} + I) S_H
\]

where we defined:

\[
\lambda_C = (\lambda_h + I)(\lambda_h - I)^{-1} = (\lambda_h - I)^{-1}(\lambda_h + I)
\]

This is a fundamental equation for determining the Floquet modes with symmetric periods. As with eq. (5.32) and in contrast to eq. (5.20), we determine the Floquet modes from the input impedance and admittance without the need for an inversion. Therefore, these expressions are numerically stable.

Equivalent equations can be obtained by using eqs. (5.43) and (5.45). The parameters \( \delta_{1,2} \) in eq. (5.27) (with \( \tilde{Z}_0 = \tilde{Y}_0 = I \)) are now given by:

\[
\delta_1 = 2\sqrt{\lambda_h}/(\lambda_h + I) \quad \delta_2 = \frac{1}{2}(\lambda_h - I)/\sqrt{\lambda_h}
\]

Introducing these values into eq. (5.22) by using the relations (5.40) and (5.49), we obtain the following relations for the eigenvector matrices \( S_E \) and \( S_H \):

\[
S_E = z_{11h} S_H / \sqrt{\lambda_h} \quad S_H = y_{11h} S_E / \sqrt{\lambda_h}
\]

These formulas are analogous to those in eq. (5.34), with \( \delta_a = \delta_b = I/\sqrt{\lambda_{av}}. \) Due to the relation \( \cosh x = (1 + \tanh^2 \frac{x}{2})/(1 - \tanh^2 \frac{x}{2}) \), we have the following relation between \( \lambda_h \) and \( \Gamma_F \):

\[
\tanh \left( \frac{1}{2} \Gamma_F \right) = \lambda_h^{-\frac{1}{2}} = I/\sqrt{S_E^{-1} z_{11h} y_{11h} S_E} = I/\sqrt{S_H^{-1} y_{11h} z_{11h} S_H}
\]

This expression is similar to that given in eq. (5.32) for \( \tanh \Gamma_F \). We immediately obtain the result in eq. (5.32) from that in eq. (5.52) if we examine the half of two concatenated periods. The result can also be obtained from \( \tanh \Gamma_F = \sinh \Gamma_F / \cosh \Gamma_F \) by introducing the impedance/admittance matrices in eq. (5.11).

By introducing the result of eq. (5.48) into eq. (5.28) (with \( \delta_1 = \delta_2 \)), we obtain for the wave impedance:

\[
\tilde{Z}_0 = \sqrt{(\lambda_C^2 - I)/\lambda_C} = 2\sqrt{\lambda_h/\lambda_h^2 - I}
\]
5.3.3 Concatenation of $N$ symmetric periods

In this subsection we would like to give the relevant formulas for $N$ concatenated periodic sections (see Fig. 5.1). Each periodic section was described like a homogeneous waveguide. Therefore, if we concatenate $N$ periods we need only replace $\Gamma_F$ with $N\Gamma_F$. Eq. (5.19) now reads:

$$\begin{bmatrix} \tilde{E}_D \\ \tilde{H}_D \end{bmatrix} = \begin{bmatrix} \cosh(N\Gamma_F) & -\tilde{Z}_0 \sinh(N\Gamma_F) \\ -\tilde{Y}_0 \sinh(N\Gamma_F) & \cosh(N\Gamma_F) \end{bmatrix} \begin{bmatrix} \tilde{E}_C \\ \tilde{H}_C \end{bmatrix}$$

(5.53)

Ports C and D are the input and output ports of the whole structure. Furthermore, we can again write the field relations between the generalised two ports C and D with open-circuit impedance or short-circuit admittance parameter ($z$ and $y$)-matrices:

$$\begin{bmatrix} \tilde{E}_C \\ \tilde{E}_D \end{bmatrix} = \begin{bmatrix} \tilde{Z}_0 & 0 \\ 0 & \tilde{Y}_0 \end{bmatrix} \begin{bmatrix} \tilde{H}_C \\ \tilde{H}_D \end{bmatrix}$$

(5.54)

The parameters are defined as:

$$\tilde{z}_1 = \frac{\tilde{Z}_0}{\tanh(N\tilde{\Gamma}_F)} \quad \tilde{z}_2 = \frac{\tilde{Z}_0}{\sinh(N\tilde{\Gamma}_F)}$$

$$\tilde{y}_1 = \frac{\tilde{Y}_0}{\tanh(N\tilde{\Gamma}_F)} \quad \tilde{y}_2 = -\frac{\tilde{Y}_0}{\sinh(N\tilde{\Gamma}_F)}$$

(5.55)

By using $\tilde{E}_{C,D} = \tilde{Z}_{C,D} \tilde{H}_{C,D}$ and $\tilde{H}_{C,D} = \tilde{Y}_{C,D} \tilde{E}_{C,D}$, the impedance/admittance transformation for the whole periodic structure is performed by:

$$\tilde{Z}_C = \tilde{z}_1 - \tilde{z}_2(\tilde{z}_1 + \tilde{Z}_D)^{-1}\tilde{z}_2 \quad \tilde{Y}_C = \tilde{y}_1 - \tilde{y}_2(\tilde{y}_1 + \tilde{Y}_D)^{-1}\tilde{y}_2$$

(5.56)

The relations between the impedances/admittances with tilde ($\sim$) (Floquet impedances) and bar ($\bar{}$) (mode impedances) are given by:

$$\tilde{Z}_D = S^{-1}_E T_E \bar{Z}_{Di} T_H^{-1} S_H = \bar{Y}_D^{-1} \quad \bar{Z}_{Ci} = T_E^{-1} S_E \tilde{Z}_C S_H^{-1} T_H \equiv \bar{Y}_{Ci}^{-1}$$

(5.57)

$\bar{Z}_{Di}$ and $\bar{Z}_{Ci}$ are the values at the inner side of ports D and C in the transformed domain. This expression is very similar to that for transforming impedances/admittances at the interface between two homogeneous sections (eqs. (2.152) and (2.153)). Since the connecting waveguides at the outer side (o) of ports C and D can be completely different from the sections on the inner side (i), a further transformation may be required. This can be done by the known transformation between two different waveguides (see Section 2.5). We have e.g. at port D:

$$\bar{Z}_{Di} = T_{EI}^{-1} J_{i}^{ct} (J_o^{c} T_{Ho} \bar{Y}_{Do} T_{Eo}^{-1} J_{o}^{ct})^{-1} J_{i}^{c} T_{Hi} = \bar{Y}_{Di}^{-1}$$

(5.58)
5.3.4 Floquet modes for unsymmetric periods

Floquet modes can also be calculated for unsymmetric circuits consisting of e.g. concatenated various sections (e.g. see Fig. 5.10). In the general case we must calculate the Floquet modes from the combination of the E and H fields. The procedure is different from that for symmetric periods. We define the supervectors:

\[ \hat{F}_{A,B} = [E_{A,B}^t, H_{A,B}^t]^t \]  \hspace{1cm} (5.59)

![Fig. 5.10 Bragg-grating with unsymmetrical periods](image)

These supervectors are now transformed into Floquet mode supervectors \( \hat{\tilde{F}}_{A,B} \) according to:

\[ \hat{\tilde{F}}_{A,B} = S_{EH} \hat{F}_{A,B} \quad \hat{\tilde{F}}_{A,B} = [\hat{F}_{A,B,f}^t, \hat{F}_{A,B,b}^t]^t \quad \hat{S}_{EH} = \begin{bmatrix} S_{Ef} & S_{Ef} \\ S_{Hb} & S_{Hb} \end{bmatrix} \] \hspace{1cm} (5.60)

The subscripts ‘f’ and ‘b’ indicate ‘forward’ and ‘backward’, respectively.

Let us now introduce Floquet’s theorem into eq. (5.13) according to:

\[ \hat{\tilde{F}}_A = e^{\hat{\Gamma}_F} \hat{\tilde{F}}_B \] \hspace{1cm} (5.61)

The diagonal matrix \( e^{\hat{\Gamma}_F} \) is obtained from the eigenvalue problem:

\[ \hat{S}_{EH}^{-1} \begin{bmatrix} z_{11}z_{21}^{-1} & z_{11}z_{22}^{-1}z_{22} - z_{12} \\ z_{21} & z_{21}z_{22}^{-1} \end{bmatrix} \hat{S}_{EH} = e^{\hat{\Gamma}_F} \] \hspace{1cm} (5.62)

Instead of eq. (5.13) we may use eq. (5.12). The examination of the transfer matrix in eq. (5.62) shows that its determinant can be computed as \( \det(\text{transfer matrix}) = \det(z_{12}^{-1}) \det(z_{21}) = 1 \). To obtain this relation, we used the Schur complement. The matrices \( z_{12} \) and \( z_{21} \) have identical eigenvalues in the isotropic case. The value of 1 was found numerically. This means that the product of all eigenvalues \( \exp(\Gamma_F) \) is also equal to 1 and the sum of \( (\Gamma_F)_i \) results in 0. Now we order \( \hat{\Gamma}_F \) according to \( \hat{\Gamma}_F = \text{diag}(\Gamma_{F1}, -\Gamma_{F2}) \), where the
real parts of $\Gamma_{F1}$ and $\Gamma_{F2}$ are positive. Usually, we introduce ABCs at the side walls, which results in complex values for $\Gamma_{F1,2}$. Let us now assume $N$ periods between the input and the output ports C and D of the finite periodic device (e.g. see Fig. 5.1). The relation between the fields at these ports in Floquet mode domain can be written as:

\[
\begin{bmatrix}
\tilde{F}_{fC} \\
\tilde{F}_{bC}
\end{bmatrix} = 
\begin{bmatrix}
e^{N\Gamma_{F1}} \\
e^{-N\Gamma_{F2}}
\end{bmatrix}
\begin{bmatrix}
\tilde{F}_{fD} \\
\tilde{F}_{bD}
\end{bmatrix}
\] (5.63)

The relation between the fields at port D is given by $\mathbf{E}_D = \mathbf{Z}_D \mathbf{H}_D$, where $\mathbf{Z}_D = \mathbf{T}_E \mathbf{Z}_D \mathbf{T}_H^{-1}$. $\mathbf{Z}_D$ is the input impedance in the transformed domain of the connecting waveguide at the output. $\mathbf{T}_E$ and $\mathbf{T}_H$ are the modal matrices of this waveguide. So we have:

\[
\begin{bmatrix}
\tilde{F}_{fD} \\
\tilde{F}_{bD}
\end{bmatrix} = \mathbf{S}^{-1}_{EH} \begin{bmatrix}
\mathbf{Z}_D \mathbf{H}_D \\
\mathbf{H}_D
\end{bmatrix} = \begin{bmatrix}
\mathbf{S}_{Ef}^i & \mathbf{S}_{Eb}^i \\
\mathbf{S}_{Hf}^i & \mathbf{S}_{Hb}^i
\end{bmatrix} \begin{bmatrix}
\mathbf{Z}_D \mathbf{H}_D \\
\mathbf{H}_D
\end{bmatrix}
\] (5.64)

where the superscript $(i)$ indicates the submatrices of the inverted matrix $\mathbf{S}^{-1}_{EH}$. It should however be clarified that the determination of these submatrices requires the inversion of the whole matrix $\mathbf{S}^{-1}_{EH}$, i.e. we cannot simply invert all the blocks of this matrix. Therefore, $\mathbf{S}_{Ef}$ is not the inverse of $\mathbf{S}_{Ef}$!

Introducing this relation into eq. (5.63) results in:

\[
\begin{bmatrix}
\tilde{F}_{fC} \\
\tilde{F}_{bC}
\end{bmatrix} = 
\begin{bmatrix}
e^{N\Gamma_{F1}} \\
e^{-N\Gamma_{F2}}
\end{bmatrix}
\begin{bmatrix}
\mathbf{S}_{Ef}^i & \mathbf{S}_{Eb}^i \\
\mathbf{S}_{Hf}^i & \mathbf{S}_{Hb}^i
\end{bmatrix} \begin{bmatrix}
\mathbf{S}_{Ef}^i & \mathbf{S}_{Eb}^i \\
\mathbf{S}_{Hf}^i & \mathbf{S}_{Hb}^i
\end{bmatrix} \begin{bmatrix}
\mathbf{Z}_D \mathbf{H}_D \\
\mathbf{H}_D
\end{bmatrix}
\] (5.65)

From the last equation we obtain:

\[
\tilde{F}_{bC} = \tilde{R}_C \tilde{F}_{fC} \quad \tilde{R}_C = e^{-N\Gamma_{F2}}(\mathbf{S}_{Hf}^i \mathbf{Z}_D + \mathbf{S}_{Hb}^i)(\mathbf{S}_{Ef}^i \mathbf{Z}_D + \mathbf{S}_{Eb}^i)^{-1} e^{-N\Gamma_{F1}}
\] (5.66)

These expressions are similar to the formulas with reflection coefficients given in [4]. At port C we obtain the real fields with:

\[
\begin{bmatrix}
\mathbf{E}_C \\
\mathbf{H}_C
\end{bmatrix} = \begin{bmatrix}
\mathbf{S}_{Ef} & \mathbf{S}_{Eb} \\
\mathbf{S}_{Hf} & \mathbf{S}_{Hb}
\end{bmatrix} \begin{bmatrix}
\tilde{F}_{fC} \\
\tilde{F}_{fC}
\end{bmatrix}
\] (5.67)

The load impedance of the input waveguide $\mathbf{Z}_C$ at port C is now given by:

\[
\mathbf{E}_C = \mathbf{Z}_C \mathbf{H}_C \quad \mathbf{Z}_C = (\mathbf{S}_{Ef} + \mathbf{S}_{Eb} \tilde{R}_C)(\mathbf{S}_{Hf} + \mathbf{S}_{Hb} \tilde{R}_C)^{-1}
\] (5.68)

By using the modal matrices of the input waveguide (they may be different from those of the output waveguide), we can also calculate the impedance matrix $\mathbf{Z}_C$ in the transformed domain (mode domain).

The transfer matrix for ports C and D can easily be determined as:

\[
\begin{bmatrix}
\mathbf{E}_C \\
\mathbf{H}_C
\end{bmatrix} = \begin{bmatrix}
\mathbf{S}_{Ef} & \mathbf{S}_{Eb} \\
\mathbf{S}_{Hf} & \mathbf{S}_{Hb}
\end{bmatrix} \begin{bmatrix}
e^{N\Gamma_{F1}} \\
e^{-N\Gamma_{F2}}
\end{bmatrix} \begin{bmatrix}
\mathbf{S}_{Ef}^i & \mathbf{S}_{Eb}^i \\
\mathbf{S}_{Hf}^i & \mathbf{S}_{Hb}^i
\end{bmatrix} \begin{bmatrix}
\mathbf{E}_C \\
\mathbf{H}_C
\end{bmatrix}
\] (5.69)
or:

\[
\begin{bmatrix}
E_C \\
H_C
\end{bmatrix} = \begin{bmatrix}
V_E & Z_f \\
Y_f & V_H
\end{bmatrix} \begin{bmatrix}
E_D \\
H_D
\end{bmatrix}
\] (5.70)

with:

\[
V_E = S_{Ef}e^{NT_f_1}S_{Ef} + S_{Eb}e^{-NT_f_2}S_{Ef}
\]
\[
V_H = S_{Hf}e^{NT_f_1}S_{Ef} + S_{Hb}e^{-NT_f_2}S_{Ef}
\]
\[
Z_f = S_{Ef}e^{NT_f_1}S_{Ef} + S_{Eb}e^{-NT_f_2}S_{Ef}
\]
\[
Y_f = S_{Hf}e^{NT_f_1}S_{Ef} + S_{Hb}e^{-NT_f_2}S_{Ef}
\] (5.71)

By rearranging the expressions in eq. (5.71), we can determine the general \(z\)-matrix and \(y\)-matrix parameters for ports \(C\) and \(D\) (see eq. (5.11)):

\[
\begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix} = \begin{bmatrix}
V_E Y_f^{-1} & V_E Y_f^{-1} V_H - Z_f \\
Y_f^{-1} & Y_f^{-1} V_H
\end{bmatrix}
\begin{bmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{bmatrix} = \begin{bmatrix}
V_H Z_f^{-1} & Y_f - V_H Z_f^{-1} V_E \\
-Z_f^{-1} & Z_f^{-1} V_E
\end{bmatrix}
\] (5.72)

Special care is required when the exponential terms are numerically determined in the products.

### 5.3.5 Some further general relations in periodic structures

We will now provide some further relations in periodic structures that may be of interest and helpful.

#### 5.3.5.1 Characteristic impedances

We first define wave impedances at the ports. At the output port (B) we choose \(Z_{02}\) as load impedance, i.e. \(E_B = Z_{02}H_B\). The input impedance (at port A) is \(Z_{01}\), with \(E_A = Z_{01}H_B\). The impedances are transformed according to:

\[
Z_{01} = z_{11} - z_{12}(z_{22} + Z_{02})^{-1}z_{21} \quad Z_{02} = -z_{22} + z_{21}(z_{11} - Z_{01})^{-1}z_{12}
\] (5.73)

Analogously, we use port A as output. As load impedance we choose \(Z_{01}\) (\(E_A = -Z_{01}H_A\)). We should now obtain at port B the input impedance \(Z_{02}\), with \(E_B = -Z_{02}H_B\). Here, the impedance transformation formulas are:

\[
Z_{01} = -z_{11} + z_{12}(z_{22} - Z_{02})^{-1}z_{21} \quad Z_{02} = z_{22} - z_{21}(z_{11} + Z_{01})^{-1}z_{12}
\] (5.74)

After some lengthy but not difficult computations we can determine \(Z_{01}\) and \(Z_{02}\):

\[
Z_{01} = \sqrt{y_{11}^{-1}z_{11}} \quad Z_{02} = \sqrt{y_{22}^{-1}z_{22}}
\] (5.75)

In deriving eq. (5.75), we took into account that the open-circuit parameter (\(z\)) matrix is the inverse of the short-circuit parameter (\(y\)) matrix (see eq. (5.11)).
5.3.5.2 Difference equations

In principle we could use eq. (5.62) to determine the Floquet modes in unsymmetric periodic structures. However, to construct this transfer matrix we would need to multiply with exponentially increasing terms (the solution of backward propagating modes). Therefore, as before, numerical problems can occur, particularly for long sections and high losses. So we present a method to determine the Floquet modes in a much more stable way here. For this purpose, we start with the equations for two periods of an unsymmetric periodic structure (see Fig. 5.11). From eq. (5.11) we obtain the following general difference equations:

\[ z_{12} H_{n+1} + z_{21} H_{n-1} = (z_{11} + z_{22}) H_n \]  
\[ -y_{12} E_{n+1} - y_{21} E_{n-1} = (y_{11} + y_{22}) E_n \]

\[ z_{12} H_{n+1} + z_{21} H_{n-1} = (z_{11} + z_{22}) H_n \]  
\[ -y_{12} E_{n+1} - y_{21} E_{n-1} = (y_{11} + y_{22}) E_n \]

\[ H_n + H_{n-1} = 2z_{12}^{-1} z_{11} H_n \]  
\[ E_n + E_{n-1} = 2y_{12}^{-1} y_{11} E_n \]

Fig. 5.11 Two periods of an unsymmetrical periodic structure

For symmetrical periods (with \( z_{11} = z_{22} \) and \( z_{12} = z_{21} \)), eq. (5.77) simplifies to:

\[ H_{n+1} + H_{n-1} = 2z_{12}^{-1} z_{11} H_n \]  
\[ E_{n+1} + E_{n-1} = 2y_{12}^{-1} y_{11} E_n \]

\[ H_n = S_H \tilde{H}_n \]  
\[ \tilde{H}_n = e^{-r_F} \tilde{H}_{n-1} \]

we obtain again from eq. (5.78):

\[ \cosh r_F = S_H^{-1} z_{12}^{-1} z_{11} S_H \]

We can determine an analogous equation for the electric field.
For the unsymmetric case, we rewrite eq. (5.77) for the two different ports $n$ and $n + 1$:

$$z_{21}H_{n-1} + z_{12}H_{n+1} - (z_{11} + z_{22})H_n = 0$$
$$z_{21}H_n + z_{12}H_{n+2} - (z_{11} + z_{22})H_{n+1} = 0 \quad (5.81)$$

Now we introduce the wave ansatz according to:

$$H_{n+1} = s_F H_n \quad \text{with} \quad s_F = e^{-r_F} \quad (5.82)$$

and obtain:

$$z_{21}H_n - s_F(z_{11} + z_{22})H_n + s_F z_{12}H_{n+1} = 0$$
$$z_{21}H_n - (z_{11} + z_{22})H_{n+1} + s_F z_{12}H_{n+1} = 0 \quad (5.83)$$

or:

$$\begin{bmatrix} z_{21} & 0 \\ z_{21} & -(z_{11} + z_{22}) \end{bmatrix} - s_F \begin{bmatrix} z_{11} + z_{22} & -z_{12} \\ 0 & -z_{12} \end{bmatrix} \begin{bmatrix} H_n \\ H_{n+1} \end{bmatrix} = 0 \quad (5.84)$$

Instead of using both expressions from eq. (5.81), we can use eq. (5.82). The equivalent problem reads in this case:

$$\begin{bmatrix} 0 & I \\ z_{21} & -(z_{11} + z_{22}) \end{bmatrix} - s_F \begin{bmatrix} I & 0 \\ 0 & -z_{12} \end{bmatrix} \begin{bmatrix} H_n \\ H_{n+1} \end{bmatrix} = 0 \quad (5.85)$$

This is a general eigenvalue problem and is analogous to that described by Helfert in [6]. After a suitable ordering of the eigenvalues we may write:

$$s_F = \text{diag}(s^f_F, s^b_F) = \text{diag}(\exp(-r^f_F), \exp(r^b_F)) \quad (5.86)$$

With:

$$\begin{bmatrix} H_n \\ H_{n+1} \end{bmatrix} = \tilde{S}_H \begin{bmatrix} \tilde{F}^f_F \\ \tilde{F}^b_F \end{bmatrix} \quad (5.87)$$

we may write for the corresponding eigenvector matrix $\tilde{S}_H$:

$$\tilde{S}_H = \begin{bmatrix} s^f_F & s^b_F \\ s^f_H s^f_F & s^b_H s^b_F \end{bmatrix} \quad (5.88)$$

Numerically, it was found that the matrices $\Gamma^f_F$ and $\Gamma^b_F$ are equal to each other with isotropic material. The matrix $\tilde{S}_E$ can be determined from eq. (5.11). Note: a difference equation can also be developed from the transfer matrix expressions given in eqs. (5.12)–(5.15).
Fig. 5.12 Complex phase of one period of the Bragg grating in Fig. 5.1 as a function of the frequency. The frequency is plotted on the ordinate.

Fig. 5.13 Damping of one period of the Bragg grating in Fig. 5.1 with rectangular and sinusoidal shape as a function of the frequency.

5.4 NUMERICAL RESULTS FOR PERIODIC STRUCTURES IN ONE DIRECTION

The complex phase of one period of the Bragg grating (Fig. 5.1) as a function of the frequency is shown in Fig. 5.12. Note: here, the frequency is plotted on the ordinate.

Fig. 5.13 shows the damping of the phase for different shapes.

The basic equations for the analysis of 2D structures are given in Section 5.3.1. Results for the reflectivity ($|S_{11}|^2$) of the fundamental mode of the Bragg grating in Fig. 5.1 are shown in Fig. 5.14 for different numbers...
of periods $N$. It is an interesting phenomenon that the curves are not symmetrical. The results are identical to those published in [7] and [8] – analysed with different algorithms. The curves are calculated with absorbing boundary conditions. Electric and magnetic walls can also be used. In the latter cases, the curves are superimposed by very small oscillation, as can be seen for the case $N = 2000$ m (metallic walls). For a planar Bragg grating, the CPU times were measured in [7]. It was found that the CPU time for analysing a grating with 1000 periods could be reduced from 2.5 h to 2 min if the Floquet approach was used instead of the general impedance transformation. For fibre gratings with constant material parameters in the azimuthal direction (like the ones for which we will show numerical results in this subsection), only a one-dimensional discretisation of the cross-section is required, because the $\varphi$-dependence of the fields can be given analytically. Therefore, the numerical effort is similar to the planar case and a comparable time saving is expected. However, since the analysis without ‘Floquet’ requires a lot of CPU time, this comparison has not been made for fibre structures. The algorithm is a combination of the numerically stable impedance transformation and Floquet’s theorem. Therefore, the numerical effort does not depend on the number of periods and the numerical problems can be minimised. One more advantage is that in this method (as usual in the MoL) the mode coupling due to inhomogenities is automatically taken into account.

![Graph](Image)

**Fig. 5.14** Reflectivity of the Bragg grating in Fig. 5.1 with different numbers of periods as a function of the wavelength (Reproduced by permission of Elsevier)

Results for the reflectivity ($|S_{11}|^2$) of the fibre Bragg grating in Fig. 5.4 are shown in Fig. 5.15a for two different numbers of periods $M$. The numerical examinations were done for a device with constant cladding index that consisted of two different homogeneous regions. Curves for the reflectivity and
transmission as a function of the wavelength and for 2000 and 5000 periods were calculated. The refractive index in the core of the first homogeneous region was \( n_{co1} = 1.45 \). For the difference between core and cladding index we chose \( \Delta = 0.005 \). It is an interesting phenomenon that the curves are in principle not symmetric. This is because the radiation of optical waves in the grating is larger for smaller wavelengths. Similar results are presented in [9]. In Fig. 5.15b numerical results are shown for a fibre Bragg grating with unsymmetric periods. The curve in the middle is identical to that shown in Fig. 5.15a. For the other two curves, section A was split into two subsections A and C each of half the length the former section. Then the core index of section C was chosen as \( n_C = 0.9995n_A \) for the left curve and \( n_C = 1.0005n_A \) for the right one. The dashed line is for anisotropic material \( (n_{cz} = 1.5) \). At the interfaces between the different homogeneous sections, guided modes as well as radiation modes are excited. These radiation modes were taken into account with the introduction of absorbing boundary conditions into the finite difference scheme (see Chapter 2). Anisotropic material was introduced in Figs. 5.16 and 5.17. In this case, the component of the tensor in the direction of propagation is different to the transverse components. The isotropic case was already examined in [2]. The curve with the anisotropic materials is shifted slightly to the right. The gratings have \( N = 5000 \) and \( N = 8000 \) periods. The size of each period \( \Lambda = d_1 + d_2 = 0.528 \) \( \mu \)m and the relative difference between core refractive indices in the two sections \( \Delta n_{co} = (n_{co2} - n_{co1})/n_{co1} \) has been chosen to be 0.001. As mentioned above, the method permits us to analyse a more general periodic fibre structure with varying core and cladding index. In our examinations we kept the core radius constant in both sections. An improvement of the reflectivity can be achieved when the condition:

\[
n_1^e d_1 = n_2^e d_2 = \pi/2
\]

is fulfilled for each interface between the layers in periodic structures. Here \( n_{1,2}^e \) are effective indices in two different sections of the grating. The analysis of apodised and chirped fibre gratings, as well as of the non-linear properties of gratings, can easily be performed.

As a further example of a cylindrical periodic structure, the magnetron resonator [10] in Fig. 5.18a is shown. This resonator is designed to operate in the fundamental \( \pi/2 \) mode around 38 GHz, with \( N = 16 \) slots of depth 1.385 mm, an anode radius of 2.25 mm and a cathode radius of 1.3 mm. Fig. 5.18b shows the resonant frequencies of the TE\(l_10\) and TE\(l_20\) modes \( (l = 0, 1, 2, \ldots, N/2) \) with axial open-boundary conditions. The results obtained by [10] and the MoL are in very good agreement.

A typical periodic microwave structure is the microstrip meander line. A meander line on RT/duroid microwave laminate with 23 periods (see Fig. 5.3a) has been fabricated, measured and compared with the calculated results. To match the input and output impedances of the meander line to the 50 Ω connecting line of width \( w_1 \), and to minimise insertion loss, the first and last
Fig. 5.15 Reflectivity and transmittivity of the fibre Bragg gratings in Fig. 5.4 (a) with symmetric periods and with two different numbers of periods (b) with unsymmetric periods as a function of the wavelength (R. Pregla, ‘Analysis of gratings with symmetrical and unsymmetrical periods’, in ICTON Conf. © 2004 Institute of Electrical and Electronics Engineers (IEEE))
strips (or the outer half periods) of the meander lines were used as impedance transformers. Therefore, their width \( w_2 \) and the distance \( d_1 \) to the next strips were changed (see Fig. 5.19).

Fig. 5.20a shows the measured values of the scattering parameters in dB (with Hewlett Packard 8720D network analyser) against the numerical results. As we can see, the measured and the calculated results are in very good agreement. Because a lossless structure was assumed in the analysis, the transmission coefficient \( |S_{21}| \) can be computed from the relation \( |S_{11}|^2 + |S_{21}|^2 = 1 \). However, to check the accuracy of the algorithm, we also calculated \( S_{21} \) by transforming the fields from the input to the output. The difference between the theoretical (see above formula) and numerical results is approximately \( 10^{-14} \), which shows the high accuracy of the proposed approach. The measured and the calculated phase of the whole periodic structure are also in very good agreement (Fig. 5.20b).
5.5 ANALYSIS OF PHOTONIC CRYSTALS

5.5.1 Determination of band diagrams

Waveguide devices can be made of photonic crystals for frequency bands in which the dispersion diagram shows a stop band. For the simple example in Fig. 5.8 we obtain a stop band between the values $\theta_{c1}$ and $\theta_{c2}$ or between the
Fig. 5.19 Input part of the analysed meander line with dimensions: \( w_1 = 0.6 \) mm, \( w_2 = 0.4 \) mm, \( w = 0.4 \) mm, \( d_1 = 0.6 \) mm, \( d = 0.4 \) mm, \( l = 6.4 \) mm. Substrate: \( \varepsilon_r = 10.2 \), thickness: \( h = 0.635 \) mm (L. Greda and R. Pregla, ‘MoL – Analysis of Periodic Structures’, in IEEE MTT-S Int. Symp. Dig. © 2003 Institute of Electrical and Electronics Engineers (IEEE))

frequencies \( f_{c1} = f_0 - \Delta f \) and \( f_{c2} = f_0 + \Delta f \). In the diagram shown in Fig. 5.9 we showed only one solution of eq. (5.6). However, due to the symmetry of the trigonometric functions, we obtain the same value \( \psi_1 \) not only for \( \theta_1 \) but for \( \pi - \theta_1 \). Therefore, the curve in Fig. 5.9 has been extended symmetrically to \( \theta = \pi/2 \). (Note that we show \( \theta \) as a function of \( \psi \) in band diagrams.) The band diagram for the simple one-dimensional periodic structure in Fig. 5.8 is given in Fig. 5.21. With \( \Theta = k_o\Lambda/2 \), we may write \( \Theta/\pi = \Lambda/\lambda \).

In photonic crystals we have a periodicity in various directions (e.g. the \( x \)- and \( y \)-direction in Fig. 5.6). Waves can propagate along these main directions, but also under arbitrary angles.

To identify the band gaps we must in principle examine all these possible angles. However, due to the periodicity and the symmetry, it is sufficient to examine only the so-called irreducible Brillouin zone or, more accurately, the edges of this irreducible Brillouin zone (for more details see e.g. [11] or the first chapters of a book about solid state physics like [12]). As an example, Fig. 5.22 shows the irreducible Brillouin zone of the square lattice presented in Fig. 5.6.

Let us examine the direction of propagation on the edge of the triangle. Between points \( \Gamma \) and \( X \) we have \( k_{x_2} = 0 \). This describes a wave propagating in \( x_1 \)-direction with changing propagation constant \( k_{x_1} \). In the \( X \) point we have \( k_{x_2}a = \pi \). On the line between \( X \)- and \( M \)-direction we have propagation in \( x_2 \)-direction with a varying wave number \( k_{x_2} \). For \( k_{x_1} \) we always have \( \pi/a \).

At point \( M \) we have \( k_{x_2} = k_{x_1} = \pi/a \). Finally, for the connection between \( \Gamma \) and \( M \) we have \( k_{x_1} = k_{x_2} \). Therefore, this describes a propagation in diagonal direction. As mentioned before, at the \( M \) point we have \( k_{x_1} = k_{x_2} = \pi/a \).

To compute the band diagrams in \( \Gamma-X \)-, resp. \( X-M \)-direction we use elementary cells of size \( a \times a \). Two examples are shown on the left side of
Fig. 5.20 Meander line in Fig. 5.19 with 23 periods: (a) absolute values of the scattering parameters (b) total phase in degree (L. Greda and R. Pregla, ‘MoL – Analysis of Periodic Structures’, in IEEE MTT-S Int. Symp. Dig. © 2003 Institute of Electrical and Electronics Engineers (IEEE))

Fig. 5.23. For the Γ–X part of the diagram we discretise in vertical direction lines (lines in $x_1$-direction). On the top and bottom of this cell, periodic boundary conditions (PBC) are introduced into the difference operators (see Section A.2.9). Since the phase factor $\beta_x$ (which is equal to $k_{x_2}$) is equal to zero, fields on the top and bottom are identical. We vary the frequency and determine the Floquet modes (as described in Section 5.3.2.4) and their phase
factor $\Gamma_F$. This is different to other methods for computing the band diagrams, where the frequency is determined as a function of both propagation constants $k_{x_1}$ and $k_{x_2}$.

When determining the X–M part of the diagram we proceed in a similar way. We discretise in the horizontal direction and use periodic boundary conditions on the left and on the right. We have to introduce the condition $k_{x_1}a = \pi$. Therefore, the fields on the left and right are the negative of each other. The further steps are identical to those for the $\Gamma$–X-direction. Due to the symmetry of the elementary cell, these two calculations only differ in the choice of the periodic boundaries.

To compute the bands for the $\Gamma$–X-direction, we could use the same cell as before. However, an iteration is required to enforce the condition $k_{x_1} = k_{x_2}$ in this case. Therefore, we choose the elementary cell shown on the right side of Fig. 5.23. The further steps occur analogously to those for the $\Gamma$–X band. We discretise in $\xi_2$-direction and introduce the condition $k_{\xi_2} = 0$ into...
The finite difference operator. Then we determine the Floquet modes for the $\xi_1$-direction.

The band diagram for the 2D square lattice and the TM modes is given in Fig. 5.24. We follow the labelling used by Joannopoulos [11] here. In this case, the electric field is parallel to the rods. We have approximated the round dielectric rods with squares of the same area. Nevertheless, our results agree very well with those in [11]. Only at the M point is there a small step in the highest curve. This is because the squares as approximations of the circles are rotated by 45 degrees (see Fig. 5.23). More accurate approximations of the round rods can be obtained by staircase approximations (simply by using a cross) or by discretisation lines of varying lengths [5].

Fig. 5.25 shows examples of PCs with hexagonal lattice. In Fig. 5.25a we again have dielectric rods in air, whereas Fig. 5.25b shows air holes in a dielectric substrate. The first structure has band gaps for the TM modes, whereas the band gaps appear in the second one for TE modes [11]. The irreducible Brillouin zone of the hexagonal lattice is presented in Fig. 5.26. To compute the band diagram, we use rectangular cells as shown in Fig. 5.25. To compute the $\Gamma$–M band we discretise in $x_2$-direction and determine the Floquet modes in $\xi_1$-direction, and for the M–K band we just do this the opposite way around. For the PBCs we must introduce $k_{x_2} = 0$ ($\Gamma$–M) and $\beta a \sqrt{3} = 2\pi$ (M–K). Both conditions require that the fields on the boundaries perpendicular to the direction of propagation are identical.

For the $\Gamma$–K-direction we have to use the right rectangular, where we discretise in $\zeta_2$-direction and determine the Floquet modes for the $\xi_1$-direction. Here, the fields at the boundaries in $\xi_2$-direction must be equal. A closer inspection shows that we therefore have the same conditions as for the M–K band and both parts are obtained in one program run.

Fig. 5.27 shows the band diagram for the hexagonal lattice shown in Fig. 5.25a for the TM modes. The dielectric rods have the diameter $2r = 0.4a$,
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Fig. 5.24 Band diagram for the square lattice of Fig. 5.6 and TM modes: dielectric rods with $2r = 0.4a$ and $\varepsilon_r = 8.9$. --- results from [11]

and the permittivity is given by $\varepsilon_r = 12.0$. The results were compared with those in [13]. Again, an excellent agreement was found even though we used a quadratic approximation of the rods.

Fig. 5.28 shows the band diagram for the hexagonal lattice of Fig. 5.25b and TE modes. The air holes have the diameter $2r = 0.6a$ and the permittivity of the substrate is given by $\varepsilon_r = 12.0$. A comparison with the results of [13] shows a larger deviation than in the previous cases. This is caused by the quadratic approximation that we used for the rods. The corners produce a singularity for the E-field, whereas circular structures do not have such singularities. To improve the results we could use a finer staircase...
Fig. 5.26 Irreducible Brillouin zone of the hexagonal lattices sketched in the reciprocal lattice space [11]

Fig. 5.27 Band diagram for the hexagonal lattice of Fig. 5.25a and TM modes: rods with $2r = 0.4a$ and $\varepsilon_r = 12.0$. - - - results from [13]

approximation or use discretisation lines of varying lengths (see Section 6.1 and [5]).

5.5.2 Waveguide circuits in photonic crystals
By introducing defects into PCs, we can design waveguides and other devices. Examples are given in Fig. 5.29. In the first row we have a straight waveguide and a 60 degree sharp bend. The second row shows a 90 degree sharp bend and a general junction. Such sharp bends cannot be realised with e.g. rib or channel waveguides. Therefore, PCs present new possibilities in miniaturising optical circuits.
To analyse the wave propagation in the two-dimensional structures only one-dimensional discretisation is necessary. The field decays very fast in the transverse direction. Therefore, ABCs should be introduced at the side walls of the computation window and only a very few rods on each side are required. For the three-dimensional structures, the discretisation is done as shown in the example presented in Fig. 5.30. The analysis of the wave propagation in the periodic waveguides (or waveguides in photonic crystals) is now straightforward. In the $z$-direction we should choose (if possible, symmetric) periods and use the algorithms described before.

General waveguide devices consist of concatenations of waveguide sections and junctions. The analysis is in principle analogous to that of planar structures (see Section 2.4.1). To analyse the junction in Fig. 5.29d, we should take into account the fact that the fields are propagating in perpendicular directions, usually in the direction of the discretisation lines. Therefore, we must use crossed discretisation lines in the junction region for the analysis.

The details are demonstrated by the example given in Fig. 5.31a. The junction region is bounded by four ports (A to D). In general, all of these four ports have connecting waveguides, in our case the waveguides $W_A$ to $W_D$. These connecting waveguides may be different to the waveguides inside the junction. For sharp bends (which may be considered as a special case of junction) we have only two ports with connecting waveguides (e.g. the ports A and D with $W_A$ and $W_D$). To derive relations for the ports, consider Fig. 5.31b, where only the junction region is sketched. Going from port A to port B, we assume $K$ concatenated waveguide sections. This number is generally

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**Fig. 5.28** Band diagram for the hexagonal lattice of Fig. 5.25b and TE modes: air holes with $2r = 0.6a$. Permittivity $\varepsilon_r = 12.0$. - - - results from [13]
Fig. 5.29 Waveguide circuits in photonic crystals (Top: Reproduced by permission of CRC Press; bottom: R. Pregla, ‘Analysis of waveguide junctions and sharp bends with general anisotropic material by using orthogonal propagating waves’, in ICTON Conf. © 2003 Institute of Electrical and Electronics Engineers (IEEE))

much greater than that in ‘normal junctions’ (see e.g. Section 5.4). The two ports of the section numbered $k$ are labelled $A^k$ and $B^k$. At the side walls of the connecting waveguides we assume absorbing boundary conditions (ABC). These are placed where we would otherwise introduce Dirichlet walls. Under these conditions, we can use the formulas described in Section 6.4 or in Section 8.4 in the case of arbitrary anisotropy.

5.5.3 Numerical results for photonic crystal circuits

5.5.3.1 Waveguides

Fig. 5.32 shows the dispersion diagram of the TM mode in the waveguide of Fig. 5.33. For $d/a = 0.25$ $a = 0.6 \, \mu$m and $n_s = 3.4$ $n_0 = 1.0$ the band gap is between $\lambda = 1.26 \, \mu$m and $\lambda = 1.7 \, \mu$m.
Fig. 5.30 Discretisation in the cross-section of a three-dimensional waveguide in photonic crystals

Fig. 5.31 Junction of waveguides in photonic crystals (a) and inner junction (enlarged) (b) (R. Pregla, ‘Analysis of waveguide junctions and sharp bends with general anisotropic material by using orthogonal propagating waves’, in ICTON Conf. © 2003 Institute of Electrical and Electronics Engineers (IEEE))

Fig. 5.33 shows the lateral distribution of the field component \( E_y \), determined for an infinite number of lateral periods, compared with a structure that was cut after seven rods. We can clearly see how the field decreases (alternating) to zero. As mentioned before, only very few periods are required. The waveguides were designed for the frequency region in which we have a band gap for the lateral direction. Therefore, for the phase in lateral direction, we have \( \Gamma d = a + j\pi \) (see Fig. 5.12). For this reason, we can introduce
Fig. 5.32 Dispersion diagram of the TM mode in a two-dimensional photonic crystal such as in Fig. 5.7; comparison between MoL and FDTD [14] (Reproduced by permission of CRC Press)

Fig. 5.33 Lateral field distribution for the TM mode in a two-dimensional photonic crystal of Fig. 5.7 with \( n_0 = 1.0 \) and \( n_s = 3.4 \)

periodic conditions as shown in Fig. 5.34 into the discretisation scheme. The relation between the fields in a distance of one period can be given as:

\[
F_{N+P} = F_N e^{-\Gamma} \quad F_{N+i} = F_{N-P+i} e^{-\Gamma} \quad 0 \leq i \leq P \quad s = e^{-\Gamma} \quad (5.90)
\]
This $s$ can be introduced into the difference operator in the following way.

\[
D = \begin{bmatrix}
1 & -1 \\
-1 & 1 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-1 & 1 & -1 & 1 \\
\end{bmatrix} \quad \rightarrow \quad D = \begin{bmatrix}
1 & -s \\
-1 & 1 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & 1 & -1 & 1 \\
\end{bmatrix}
\]

\begin{equation}
(5.91)
\end{equation}
We applied the derived formulas to a photonic crystal waveguide. Fig. 5.35 shows the electric field distribution of the fundamental Floquet mode. The curve labelled ‘discretised’ was computed with the formulas derived above with ABC. In the curve ‘analytical’, we used analytic expressions in the horizontal direction (see Fig. 5.34). We can recognise a good agreement between these two curves.

Fig. 5.37 Connection of a slab waveguide with a PC waveguide (a); loss as function of waveguide width $d$, comparison of the MoL with the FDTD [14], wavelength $\lambda = 1.5 \, \mu\text{m}$ (b) (Reproduced by permission of CRC Press)
5.5.3.2 Numerical results for PC circuits

Based on the waveguide presented in Fig. 5.33, we examined various circuits, for which we will show numerical results here. One of the interesting applications of PCs is the potential possibility to design sharp bends and concatenations of them with low losses. Fig. 5.36a shows an S bend made of photonic crystals. In Fig. 5.36b we see the determined transmission curve. For some wavelength regions, nearly all of the power is transmitted. Such high values would not be possible with ‘usual’ structures. The dips are caused by reflection and not by radiation. We should also state that no optimisation was done at all.

![Resonator structures with photonic crystals](image)

**Fig. 5.38** Resonator structures with photonic crystals

![Stop-band characteristic of the filter structure](image)

**Fig. 5.39** Stop-band characteristic of the filter structure shown in Fig. 5.38

In integrated circuits we will need to connect PC waveguides to regular ones. This coupling should occur with a high efficiency. An example is shown in Fig. 5.37a. We computed the power loss as a function of the waveguide
width $d$ in Fig. 5.37b. A minimum was found for $d \approx 0.65 \ \mu m$ (wavelength $\lambda = 1.5 \ \mu m$). The same optimum was found by the FDTD [14]. However, the total loss obtained with the latter method was slightly lower.

5.5.3.3 Numerical results for filters
Resonator structures (filters) can be made very easily from PCs, e.g. simply by removing one of the rods. Two examples are shown in Fig. 5.38a and b.
Fig. 5.42 Holey fibre (a) and model with discretisation lines (b) (A. Barcz, S. F. Helfert and R. Pregla, ‘The method of lines applied to numerical simulation of 2D and 3D bandgap structures’, in ICTON Conf. © 2003 Institute of Electrical and Electronics Engineers (IEEE))

Fig. 5.43 Holey fibre – determined effective index (A. Barcz, S. F. Helfert and R. Pregla, ‘The method of lines applied to numerical simulation of 2D and 3D bandgap structures’, in ICTON Conf. © 2003 Institute of Electrical and Electronics Engineers (IEEE))

A comparison of the determined filter characteristics is presented in Fig. 5.39. As we can see, we have a sharper resonance for configuration a.

The structures we have shown so far are two-dimensional. The filter sketched in Fig. 5.40 is an example of a 3D device. The transmission characteristic computed with the MoL was compared with that of the
FDTD [15] in Fig. 5.41a. The round holes were replaced by square ones for the MoL computations, whereas a rounder structure was examined with the FDTD. This might explain the slight shift of the determined curves. Fig. 5.41b shows the influence of the distance between the holes (in the middle of the structure) on the transmission. We see a shift of the curve to the left when we increase the distance.

5.5.3.4 Numerical results for holey fibres
Another PC structure is the holey fibre plotted in Fig. 5.42a. Here the waves propagate in the direction of the holes (and not perpendicular to them as in the previous examples). For the analysis, we used the discretisation scheme shown in Fig. 5.42b. We determined the propagation constant (effective index) of the fundamental mode. The results are shown in Fig. 5.43.
References


Further Reading


