1.1 WHAT IS A SYMMETRY?

What is a symmetry? A symmetry is a transformation that leaves an object unchanged or “invariant.” For example, if we start with a basic equilateral triangle with the vertices labeled as 1, 2, and 3 (Figure 1.1), then a reflection through any one of three different bisection axes (Figure 1.2) or rotations through angles of \( \frac{2\pi}{3} \) and \( \frac{4\pi}{3} \) (Figure 1.3) leaves the triangle invariant.

Another example is the rotation of a disk through an angle \( \epsilon \). Consider the points \((x, y)\) and \((\bar{x}, \bar{y})\), on the circumference of a circle of radius \( r \) (Figure 1.4). We can write these in terms of the radius and the angles \( \theta \) (a reference angle) and \( \theta + \epsilon \), (after rotation), that is,

These then become:

\[
\begin{align*}
  x &= r \cos \theta, & \bar{x} &= r \cos(\theta + \epsilon), \\
  y &= r \sin \theta, & \bar{y} &= r \sin(\theta + \epsilon),
\end{align*}
\]

or, after eliminating \( \theta \)

\[
\begin{align*}
  \bar{x} &= x \cos \epsilon - y \sin \epsilon \quad & (1.2a) \\
  \bar{y} &= y \cos \epsilon + x \sin \epsilon. & (1.2b)
\end{align*}
\]
CHAPTER 1 An Introduction

![Diagram of an equilateral triangle](image1)

**FIGURE 1.1** An equilateral triangle.

![Reflections of an equilateral triangle](image2)

**FIGURE 1.2** Reflections of an equilateral triangle.

![Rotations of an equilateral triangle through \(\frac{2\pi}{3}\) and \(\frac{4\pi}{3}\)](image3)

**FIGURE 1.3** Rotations of an equilateral triangle through \(\frac{2\pi}{3}\) and \(\frac{4\pi}{3}\).

![Rotation of a circle](image4)

**FIGURE 1.4** Rotation of a circle.
1.1 What Is a Symmetry?

To show invariance of the circle under (1.2) is to show that $\overline{x}^2 + \overline{y}^2 = r^2$ if $x^2 + y^2 = r^2$. Therefore,

$$
\overline{x}^2 + \overline{y}^2 = (x \cos \epsilon - y \sin \epsilon)^2 + (y \cos \epsilon + x \sin \epsilon)^2 \\
= x^2 \cos^2 \epsilon - 2xy \sin \epsilon \cos \epsilon + y^2 \sin^2 \epsilon \\
+ x^2 \sin^2 \epsilon + 2xy \sin \epsilon \cos \epsilon + y^2 \cos^2 \epsilon \\
= x^2 + y^2 \\
= r^2.
$$

As a third example, consider the line $y = \frac{1}{2}x$ and the transformation

$$
\overline{x} = e^\epsilon x \quad \overline{y} = e^\epsilon y. \quad (1.3)
$$

The line is invariant under (1.3) as (Figure 1.5)

$$
\overline{y} = \frac{1}{2} \overline{x} \quad \text{then} \quad e^\epsilon y = \frac{1}{2} e^\epsilon x \quad \text{if} \quad y = \frac{1}{2} x.
$$

**EXAMPLE 1.1**

Show the equation

$$
x^2 y^2 - xy^2 + 2xy - y^2 - y + 1 = 0 \quad (1.4)
$$
is invariant under
\[ \bar{x} = x + \epsilon, \quad \bar{y} = \frac{y}{1 - \epsilon y}. \tag{1.5} \]
For this example, it is actually easier to rewrite (1.4) as
\[ \left( x + \frac{1}{y} \right)^2 - \left( x + \frac{1}{y} \right) - 1 = 0. \tag{1.6} \]
Under the transformation (1.5), the term \( x + \frac{1}{y} \) becomes
\[ \bar{x} + \frac{1}{\bar{y}} = x + \epsilon + \frac{1 - \epsilon y}{y} = x + \epsilon + \frac{1}{y} - \epsilon = x + \frac{1}{y} \tag{1.7} \]
and invariance of (1.6) readily follows.

It is important to realize that not all equations are invariant under all transformations. Consider \( y - 1 = 3(x - 1) \) and the transformation (1.3) again. If this were invariant, then
\[ \bar{y} - 1 = 3(\bar{x} - 1) \quad \text{if} \quad y - 1 = 3(x - 1). \]
Upon substitution, we have \( e^\epsilon y - 1 = 3(e^\epsilon x - 1) \), which is clearly not the original line and hence not invariant under (1.3).

The transformations (1.2), (1.3), and (1.5) are very special and are referred to as \textit{Lie transformation groups} or just \textit{Lie groups}.

1.2 **LIE GROUPS**

In general, we consider transformations
\[ \bar{x}_i = f_i(x_j, \epsilon), \quad i, j = 1, 2, 3, \cdots n. \]
These are called a \textit{one-parameter Lie group}, where \( \epsilon \) is the parameter. First and foremost, they form a group. That is, they satisfy the following axioms, where \( G \) is the group and \( \phi(a, b) \) the law of composition.

1. \textit{Closure}. If \( a, b \in G \), then \( \phi(a, b) \in G \).
2. \textit{Associative}. If \( a, b, c \in G \), then \( \phi(a, \phi(b, c)) = \phi(\phi(a, b), c) \).
3. \textit{Identity}. If \( a \in G \), then there exists an \( e \in G \) such that \( \phi(a, e) = \phi(e, a) = a \).
4. **Inverse.** If \( a \in G \), then there exists a unique element \( a^{-1} \in G \) such that \( \phi(a, a^{-1}) = \phi(a^{-1}, a) = e \).

Second, they further satisfy the following properties:

1. \( f_i \) is a smooth function of the variables \( x_j \).
2. \( f_i \) is analytic function in the parameter \( \varepsilon \), that is, a function with a convergent Taylor series in \( \varepsilon \).
3. \( \varepsilon = 0 \) can always be chosen to correspond with the identity element \( e \).
4. the law of composition can be taken as \( \phi(a, b) = a + b \).

Our focus is on transformation groups, so our discussion is confined to these types of groups.

---

**EXAMPLE 1.2**

Consider (Figure 1.6)

\[
\bar{x} = ax, \quad a \in \mathbb{R} \setminus \{0\}.
\]

(1.8)

1. **Closure.** If \( \bar{x} = ax \) and \( \bar{x} = b\bar{x} \), then \( \bar{x} = abx \). In this example, the law of composition is \( \phi(a, b) = ab \).
2. **Associative.** As \( \phi(a, b) = ab \), then \( \phi(a, \phi(b, c)) = \phi(ab, c) = (ab)c = \phi(a, b, c) \).
3. **Identity.** In this case, \( e = 1 \) as \( \phi(a, 1) = a1 = a \).
4. **Inverse.** Here \( a^{-1} = \frac{1}{a} \) as \( \phi(a, \frac{1}{a}) = 1 \).

![FIGURE 1.6 Scaling group \( \bar{x} = ax \) and its composition.](image)

We note that if we reparameterize the group by letting \( a = e^\varepsilon \), then the group becomes a Lie group. \( \blacksquare \)
**EXAMPLE 1.3**

Consider

\[
\tilde{x} = \frac{xy}{y - \epsilon}, \quad \tilde{y} = y - \epsilon, \quad \epsilon \in \mathbb{R}.
\]  

(1.9)

1. **Closure.** If

\[
\bar{x} = \frac{xy}{y - \alpha} \quad \text{and} \quad \bar{y} = y - \alpha
\]

and if

\[
\tilde{x} = \frac{xy}{y - \beta} \quad \text{and} \quad \tilde{y} = y - \beta
\]

then

\[
\check{x} = \frac{xy}{y - \alpha} \cdot \left(y - \alpha\right) = \frac{xy}{y - \left(\alpha + \beta\right)}
\]

and

\[
\check{y} = y - \left(\alpha + \beta\right).
\]

In this example, the law of composition is \(\phi(a, b) = a + b\).

2. **Associative.** As \(\phi(a, b) = a + b\), then \(\phi(a, \phi(b, c)) = a + (b + c) = (a + b) + c = \phi(\phi(a, b), c)\).

3. **Identity.** In this case, \(e = 0\).

4. **Inverse.** Here \(a^{-1} = -\epsilon\) as \(\phi(\epsilon, -\epsilon) = 0\).

This is an example of a Lie group. It is an easy matter to show that \(xy = 1\) is invariant under (1.9). Figure 1.7 illustrates the composition of two successive transformations.

**1.3 INVARINANCE OF DIFFERENTIAL EQUATIONS**

We are starting to discover that equations can be invariant under a Lie group. This leads us to the following question: Can differential equations be invariant under Lie groups? The following examples illustrate an answer to that question. Consider the simple differential equation

\[
\frac{dy}{dx} = xy^3,
\]

(1.10)
1.3 Invariance of Differential Equations

and the Lie group

\[ \bar{x} = e^\varepsilon x, \quad \bar{y} = e^{-\varepsilon} y. \]  

(1.11)

Is the ODE (1.10) invariant under (1.11)? It is an easy matter to calculate

\[ \frac{d\bar{y}}{d\bar{x}} = e^{-2\varepsilon} \frac{dy}{dx} \quad \text{and} \quad \bar{xy}^3 = e^{-2\varepsilon} xy^3 \]  

(1.12a)

and clearly, under (1.11), (1.10) is invariant, as from (1.12) we see that

\[ \frac{d\bar{y}}{d\bar{x}} = \bar{xy}^3 \quad \text{since} \quad \frac{dy}{dx} = xy^3. \]

EXAMPLE 1.4

Show

\[ \frac{dy}{dx} = \frac{(xy + 1)^3}{x^5} + \frac{1}{x^2} \]  

(1.13)

is invariant under

\[ \bar{x} = \frac{x}{1 + \varepsilon x}, \quad \bar{y} = y - \varepsilon. \]  

(1.14)
We first calculate \( \frac{dy}{dx} \) by the chain rule

\[
\frac{dy}{dx} = \frac{dy}{d\bar{x}} \frac{d\bar{x}}{dx} = \frac{dy}{d\bar{x}} (1 + \epsilon x)^2. \tag{1.15}
\]

Next, we focus on the first term on the right-hand side of (1.13). So

\[
\frac{(\bar{x}y + 1)^3}{\bar{x}^5} = \left( \frac{x}{1 + \epsilon x} \cdot (y - \epsilon) + 1 \right)^3 \frac{x}{(1 + \epsilon x)^3} = \left( \frac{x}{1 + \epsilon x} \right)^3 \frac{x}{(1 + \epsilon x)^5} = \frac{(xy + 1)^3}{x^5} (1 + \epsilon x)^2.
\]

Thus, the entire right-hand side of (1.13) becomes

\[
\frac{(\bar{x}y + 1)^3}{\bar{x}^5} + \frac{1}{x^2} = \frac{(xy + 1)^3}{x^5} (1 + \epsilon x)^2 + \frac{1}{x^2}. \tag{1.16}
\]

To show invariance is to show that

\[
\frac{dy}{dx} = \frac{(\bar{x}y + 1)^3}{\bar{x}^5} + \frac{1}{x^2} \quad \text{if} \quad \frac{dy}{dx} = \frac{(xy + 1)^3}{x^5} + \frac{1}{x^2}. \tag{1.17}
\]

Using (1.15) and (1.16) in (1.17) shows (1.17) to be true.

We will now turn our attention to solving some differential equations.

### 1.4 SOME ORDINARY DIFFERENTIAL EQUATIONS

Consider the Riccati equation

\[
\frac{dy}{dx} = y^2 - \frac{y}{x} - \frac{1}{x^2}. \tag{1.18}
\]
1.4 Some Ordinary Differential Equations

Typically, to solve this ODE, we need one solution. It is an easy matter to show that if we guess a solution of the form \( y = \frac{k}{x} \), then

\[ k = \pm 1. \]

If we choose

\[ y_1 = \frac{1}{x} \]

and let

\[ y = \frac{1}{u} + \frac{1}{x}, \tag{1.19} \]

where \( u = u(x) \), then substituting into equation (1.18) and simplifying gives

\[ u' + \frac{u}{x} = -1, \]

which is linear. The integrating factor \( \mu \) is

\[ \mu = e^{\int \frac{du}{x}} = x, \]

and so the linear equation is easily integrated, giving

\[ u = \frac{c - x^2}{2x}, \]

where \( c \) is an arbitrary constant of integration. Substituting this into (1.19) gives

\[ y = \frac{2x}{c - x^2} + \frac{1}{x}. \tag{1.20} \]

We find that the procedure is long and we do need one solution to find the general solution of (1.18). However, if we let

\[ x = e^s, \quad y = re^{-s}, \tag{1.21} \]

where \( s = s(r) \), then substituting into (1.18) gives

\[ \frac{e^{-s} - re^{-s} s'}{e^s s'} = r^2 e^{-2s} - re^{-2s} - e^{-2s} \]
and solving for \( s' \) gives

\[
\frac{dx}{dr} = \frac{1}{r^2 - 1},
\]

(1.22)
an equation which is separable and independent of \( s' \). This is easily integrated and using (1.21) gives rise to the solution (1.20).

Consider

\[
\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{x + y}.
\]

(1.23)

Unfortunately, there is no simple way to solve this ODE. However, if we let

\[
x = s, \quad y = rs,
\]

(1.24)

where again \( s = s(r) \), then (1.23) becomes

\[
\frac{ds}{dr} = r + 1,
\]

(1.25)
an equation which is also separable and independent of \( s' \). Again, it is easily integrated, giving

\[
s = \frac{1}{2} r^2 + r + c,
\]

(1.26)

and using (1.24) gives rise to the solution of (1.23)

\[
x = \frac{1}{2} \frac{r^2}{x^2} + \frac{y}{x} + c.
\]

(1.27)

Finally, we consider

\[
\frac{dy}{dx} = \frac{2y^2(x - y - xy)}{x(x - y)^2}.
\]

(1.28)

This is a complicated ODE without a standard way of solving it. However, under the change of variables

\[
x = \frac{1}{r + s}, \quad y = \frac{1}{s},
\]

(1.29)

(1.28) becomes

\[
\frac{ds}{dr} = -\frac{2(r + 1)}{r^2 + 2r + 2}.
\]

(1.30)
again, an equation which is separable and independent of \( s \)!

Integrating (1.30) gives

\[
s = -\ln |r^2 + 2r + 2| + c
\]  

(1.31)

and via (1.29) gives

\[
\frac{1}{y} = -\ln \left( \left( \frac{1}{x} - \frac{1}{y} \right)^2 + 2 \left( \frac{1}{x} - \frac{1}{y} \right) + 2 \right) + c,
\]  

(1.32)

the exact solution of (1.28).

In summary, we have considered three different ODEs and have shown that by introducing new variables, these ODEs can be reduced to new ODEs (Table 1.1) that are separable and independent of \( s \).

We are naturally led to the following questions:

1. What do these three seemingly different ODEs have in common?

2. How did I know to pick the new coordinates \((r, s)\) (if they even exist) so that the original equation reduces to one that is separable and independent of \( s \)?

The answer to the first question is that all of the ODEs are invariant under some Lie group. The first ODE

\[
\frac{dy}{dx} = y^2 - \frac{y}{x} - \frac{1}{x^2}
\]

(1.33)

**TABLE 1.1**

Equation (1.18), (1.23) and (1.28) and their separability

<table>
<thead>
<tr>
<th>Equation</th>
<th>Transformation</th>
<th>New Equation</th>
</tr>
</thead>
</table>
| \[
\frac{dy}{dx} = y^2 - \frac{y}{x} - \frac{1}{x^2}
\]
| \[
x = e^t, \quad y = re^{-t}
\]
| \[
\frac{ds}{dr} = \frac{1}{r^2 - 1}
\]| |
| \[
\frac{dy}{dx} = \frac{y + x^2}{x + y}
\]
| \[
x = s, \quad y = rs
\]
| \[
\frac{ds}{dr} = r + 1
\]| |
| \[
\frac{dy}{dx} = \frac{2y^2(x - y - xy)}{x(x - y)^2}
\]
| \[
x = \frac{1}{r + s}, \quad y = \frac{1}{s}
\]
| \[
\frac{ds}{dr} = \frac{-2(r + 1)}{r^2 + 2r + 2}
\]| |
is invariant under
\[ \bar{x} = e^\varepsilon x, \quad \bar{y} = e^{-\varepsilon} y, \]  
(1.34)
the second ODE
\[ \frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{x+y}, \]  
(1.35)
is invariant under
\[ \bar{x} = x + \varepsilon, \quad \bar{y} = \frac{(x + \varepsilon) y}{x}, \]  
(1.36)
and the third ODE
\[ \frac{dy}{dx} = \frac{2y^3(x - y - xy)}{x(x-y)^2} \]  
(1.37)
is invariant under
\[ \bar{x} = \frac{x}{1 + \varepsilon x}, \quad \bar{y} = \frac{y}{1 + \varepsilon y}. \]  
(1.38)
The answer to the second question will be revealed in Chapter 2!

**EXERCISES**

1. Show that the following are Lie groups:

   (i) \[ \bar{x} = e^\varepsilon x, \]
   (ii) \[ \bar{x} = \sqrt{x^2 + \varepsilon}, \]
   (iii) \[ \bar{x} = \frac{(y + \varepsilon)x}{y}, \quad \bar{y} = y + \varepsilon, \]
   (iv) \[ \bar{x} = \frac{x}{1 + \varepsilon x}, \quad \bar{y} = \frac{y}{1 + \varepsilon y}. \]

2. Show that the following equations are invariant under the given Lie group

   (i) \[ x^2 y^2 + e^{xy} = 1 + xy, \quad \bar{x} = e^\varepsilon x, \quad \bar{y} = e^{-\varepsilon} y, \]
   (ii) \[ y^4 + 2xy^2 + x^2 + 2y^2 + 2x = 0, \quad \bar{x} = x - \varepsilon, \quad \bar{y} = \sqrt{y^2 + \varepsilon}, \]
   (iii) \[ x^2 - y^2 - 2xy \sin \frac{y}{x} = 0, \quad \bar{x} = \frac{x}{1 + \varepsilon x}, \quad \bar{y} = \frac{y}{1 + \varepsilon y}. \]
3. Find functions $a(\epsilon)$, $b(\epsilon)$, $c(\epsilon)$, and $d(\epsilon)$ such that

$$y - y_0 = m(x - x_0)$$

is invariant under

$$\bar{x} = a(\epsilon)x + b(\epsilon), \quad \bar{y} = c(\epsilon)y + d(\epsilon).$$

Does this form a Lie group? If not, find the form of (1.40) that not only leaves (1.39) invariant but also forms a Lie group.

4. Show that the following ODEs are invariant under the given Lie groups

(i) $\frac{dy}{dx} = 2y^2 + xy^3$, \quad $\bar{x} = e^{a(\epsilon)}x$, \quad $\bar{y} = e^{b(\epsilon)}y$,

(ii) $\frac{dy}{dx} = \frac{x^2y}{x^3 + xy + y^2}$, \quad $\bar{x} = \frac{x}{1 + \epsilon y}$, \quad $\bar{y} = \frac{y}{1 + \epsilon y}$,

(iii) $\frac{dy}{dx} = \frac{y^2}{x^2} F\left(\frac{1}{x} - \frac{1}{y}\right)$, \quad $\bar{x} = \frac{x}{1 + \epsilon x}$, \quad $\bar{y} = \frac{y}{1 + \epsilon y}$.

5. Find the constants $a$ and $b$ such that the ODE

$$\frac{dy}{dx} = \frac{3xy + 2y^3}{x^2 + 3xy^2}$$

is invariant under the Lie group of transformations

$$\bar{x} = e^{a(\epsilon)} x, \quad \bar{y} = e^{b(\epsilon)} y.$$
7. (i) Show that
\[
\frac{dy}{dx} = \frac{y^2 + (x - 2x^2)y - x^3}{x(x + y)}
\]
is invariant under
\[
\bar{x} = x + \epsilon, \quad \bar{y} = \frac{(x + \epsilon)y}{x}.
\]
(ii) Show that under the change of variables
\[
r = \frac{y}{x}, \quad s = x,
\]
the original equation becomes
\[
\frac{ds}{dr} = -\frac{r + 1}{2r + 1}.
\]

8. (i) Show that
\[
\frac{dy}{dx} = F(x)
\]
is invariant under
\[
\bar{x} = x, \quad \bar{y} = y + \epsilon.
\]
(ii) Prove that the only ordinary differential equation of the form
\[
\frac{dy}{dx} = F(x, y)
\]
that is invariant under (1.42) is of the form of (1.41).