About the Authors

Robert E. Reys is Curators’ Professor Emeritus of Mathematics Education at the University of Missouri-Columbia, where he teaches courses for elementary, middle, and secondary mathematics teachers. He is a former mathematics teacher and district mathematics coordinator. His research interests are in the areas of calculators, mental computation, estimation, and number sense. Bob has authored over 220 articles in professional journals. He was General Editor for five yearbooks of the National Council of Teachers of Mathematics, and co-edited the 2010 NCTM Yearbook Mathematics Curriculum: Issues, Trends, and Future Directions. He was awarded the Lifetime Achievement Award for his service, leadership, and research by the Mathematics Education Trust of the National Council of Teachers of Mathematics.

Mary M. Lindquist is Fuller E. Callaway Professor of Mathematics Education, Emeritus at Columbus State University (Georgia). She taught undergraduate and graduate students in early childhood, middle grades, and secondary mathematics education. Mary was president of the National Council of Teachers of Mathematics and chair of the Commission on the Future of the Standards, the oversight committee for Principles and Standards for School Mathematics (NCTM, 2000). She has served or is serving on many national and international committees, including those involved with the National Assessment of Education Progress (NAEP) and the Trends in International Mathematics and Science Study (TIMSS). She was awarded the Lifetime Achievement Award for her service, leadership, and research by the Mathematics Education Trust.

Diana V. Lambdin is Armstrong Professor of Teacher Education Emerita and Professor of Mathematics Education Emerita at Indiana University in Bloomington, where she taught courses for prospective elementary teachers, supervised students’ field experience, worked with masters and doctoral students in mathematics education, and served as Associate Dean for Teacher Education. Prior to entering the field of teacher education, she was a mathematics teacher in Massachusetts, Michigan, and Iowa. Diana has been active as an author, editor, project evaluator, and leader in the National Council of Teachers of Mathematics (NCTM). She was a member of the writing team for Principles and Standards for School Mathematics (NCTM, 2000) and served as a member of NCTM’s Board of Directors from 2009 to 2012.

Nancy L. Smith has been an educator for over 30 years. She taught elementary school and middle school mathematics for ten years in Richmond, Missouri. She is currently Professor in the Department of Elementary Education/Early Childhood/Special Education at Emporia State University in Emporia, Kansas, where she teaches elementary mathematics education courses for preservice and inservice teachers, teaches general elementary education courses, and supervises student teachers in the Olathe Professional Development Schools.
Welcome to the 11th edition of Helping Children Learn Mathematics. Earlier editions have helped several generations of elementary teachers. We are grateful to the many instructors and students who have written us with ideas, stories, and suggestions about ways of helping children learn mathematics. This edition of Helping Children Learn Mathematics reflects the ever changing world of learning and teaching elementary school mathematics. You will experience many changes and have the opportunity to shape their direction.

Helping Children Learn Mathematics is built around three main themes:

- helping children make sense of mathematics
- incorporating practical experiences
- using research to guide teaching

Additionally, throughout the book you will also find connections and implications from the Common Core State Standards: Mathematics (CCSS-M).

Equity and diversity are key issues addressed throughout the book. Practical advice and suggestions about ways to help treat children equally and fairly are provided. Elementary teachers must strive to insure equity in their classrooms, and this becomes more challenging as the diversity within many classrooms grows.

This book is intended for those of you who are or who will be teachers of mathematics in elementary school. It is designed to help you help children develop understanding and proficiency with mathematics so they can solve problems. We hope our book will challenge your thinking and stimulate your interest in learning and teaching mathematics.

Helping Children Learn Mathematics consists of two main parts. The first part (Chapters 1–6) provides a basis for understanding the changing mathematics curriculum and how children learn mathematics. These chapters offer guidelines for planning, instructing, and evaluating that are aligned with the Mathematical Practices from the CCSS-M and include reference to the Principles to Action: Ensuring Mathematical Success for All (NCTM, 2014). Particular attention is given to problem solving and assessment, both of which have profound implications for mathematics teaching. Their importance is reflected by the integration of problem solving and assessment throughout the book.

The second part (Chapters 7–18) discusses teaching strategies and techniques, as well as learning activities related to specific mathematical topics. The emphasis is on making mathematics make sense for our diverse population of students.

**FEATURES OF THIS TEXT**

Helping Children Learn Mathematics reflects current recommendations from professional associations, as well as recent research findings relevant to teaching mathematics. We have maintained the characteristics and features that have made this book a popular choice of instructors for many years. A userable text for instructors, it is also readable and understandable by students who are being introduced to teaching elementary mathematics. Yet, its depth also makes it appropriate for teachers to use as they continue to learn about teaching.

**NEW Snapshots of a Lesson** reflect video clips. These videos are new and drawn from the Teaching Channel. Each snapshot is aligned with the CCSS-M and is easily accessed online. Each chapter-opening snapshot is an excerpt from an actual mathematics lesson or mathematical situation. These videos demonstrate many effective classroom practices and provide a smooth, practical segue into the body of the chapter. You’ll see the snapshots called out with this camera icon:

**NEW Reflecting on the Snapshot Lesson** provides questions to stimulate discussion about the Snapshot of a Lesson. Answering these questions will be a good preparation for licensure tests.

**NEW Common Errors** in Chapters 7–18 illustrate errors that children or teachers often make. Both research and experiences in working with children reveal frequent errors and misconceptions. We have identified errors that are often predictable and highlighted them in Common Error boxes. This feature is designed to stimulate your thinking as you strive to help children make sense of mathematics.

**NEW Tech Connect.** The Tech Connect boxes highlight a variety of technology resources, including apps for tablets and smartphones, and URLs that link to information aligned with the content being discussed.

**NEW Learning Outcomes** are identified early in each chapter and address major issues within each chapter. While these learning outcomes are not intended to be exhaustive, they do provide some key questions to guide reading and promote discussions.

**Chapter on Fractions.** Research has shown the importance of children having a conceptual understanding of fractions as well as fluency with computation. A chapter in this edition is
devoted entirely to common fractions. Decimals have been moved to the place value chapter.

In the Classroom activities are found throughout the text in special boxes. These activities provide a wealth of ideas and strategies for helping children explore, learn, and in some cases practice skills related to specific mathematical topics. Lesson plans for most of these activities can be easily found in the companion book, Teaching Elementary Mathematics: A Resource for Field Experiences.

Children’s Literature. Literature can be a powerful ally in helping children learn. We have made a concerted effort to identify books that you can use effectively to stimulate children’s interest in mathematics and to promote mathematics learning. We cite and discuss specific books at various points within the text, and we provide an updated annotated list of useful books in the Book Nook section at the end of each chapter (except Chapter 1).

Updated and Streamlined Research. Updated references to important research are found throughout the text. As a teacher, you are often called on to provide a rationale for curricular or instructional decisions, and we think you will find these references to relevant research useful; much of the research we cite has important implications for learning and teaching. Complete reference information for each citation is available on the book companion site at www.wiley.com/college/reys.

Things to Do sections found near the end of each chapter embody our active learning and teaching approach to mathematics. Divided into two parts, the Things to Do sections are designed to engage you in inquiring and thinking about mathematics—to offer you experiences and introduce you to investigations that will help you achieve the understanding and insight you need to be a successful teacher. The first part, “From What You’ve Read,” relates directly to this book. The second part, “Going Beyond This Book,” generally offers four types of activities: In the field, In your journal, With additional resources, and With technology. These include several activities linked to Teaching Elementary Mathematics: A Resource for Field Experiences.

WILEY E-TEXT

Powered by VitalSource.
The Wiley E-Text: Powered by VitalSource gives students anytime, anywhere, access to the best math methods teaching content when and where they study: on their desktop, laptop, tablet, or smartphone. Students can search across content, highlight, and take notes that they can share with teachers and classmates.

This exciting new learning model brings textbook pages to life—no longer just a static e-book, the E-Text enriches the study experience with dynamic features:

- Pre- and Post-chapter quizzes
- Teaching Channel video introductions to each chapter
- Integrated Teaching Elementary Mathematics: A Resource for Field Experiences activities
- Access to student study tools, including Lecture PowerPoints and Blackline Masters

TEACHING AND LEARNING RESOURCES


Featuring a wealth of materials to use in schools, this resource is designed for students to use when they participate in field experiences such as practicums, observations, and professional development school (PDS) experiences in K–8 classrooms. It is aligned with Helping Children Learn Mathematics and provides:

- In the School activities focus on observing and gathering information about the school and its resources. Resources for teacher and student interviews are included.
- Helping Children Learn activities include mini lessons for small groups or whole class, that include games, technology, and In the Activities in this Resource provide opportunities in collecting information about the school and school resources, observing and interviewing teachers and children, and doing mathematics with children in the form of mini-lessons, some involving games and technology. Pages from this Resource may be downloaded and printed for classroom use.

NEW Appendix and References

The Appendix and References are available on the book companion site at www.wiley.com/college/reys.

SUPPLEMENTS

The eleventh edition of Helping Children Learn Mathematics is accompanied by the following instructor and student supplements:

NEW Instructor’s Manual

The Instructor’s Manual is a useful resource for both veteran and new elementary methods instructors. It is available on the Wiley Web site at www.wiley.com/college/reys. Each chapter includes:

- Chapter overview, student objectives, and key vocabulary
- Supplemental lecture ideas, textbook extension ideas, and class and field activity suggestions
- Extensive resource list (both print and media)

Pre- & Post-Chapter Test

Objective questions are available online. These short quizzes may be helpful for focusing learning and reflecting on the content of each chapter. (More in-depth questions are available at the end of each chapter.)
We thank Marilyn Suydam for her hard work and insights during the early years of this text’s development. Marilyn is now retired, but her legacy of significant contributions remains a vital part of this book.

In preparation for this edition, the authors interviewed several faculty members who were using the book to solicit ideas and suggestions for making it better. In that regard, we thank the following people for taking the time to talk with us and offer valuable feedback:

Julie Bauman, Waynesburg University
Brenda Doll, McKendree University
Patricia Emmons, Bridgewater State University
Janine Fendrich, St. Josephs University
Diane Sehe and Cary Tuckey, Fontbonne University
Mercedes Tichenor, Stetson University

We also welcome any ideas you have for improving and strengthening this 11th edition of Helping Children Learn Mathematics.

Finally, we give a special thanks to Joe Champion, Boise State University, and Erin Krupa, Montclair State University, for their contributions to this book. They did a careful search of links and apps that would be useful for helping elementary school teachers. You will find many of their recommendations reflected in the Tech Connect boxes that appear throughout the book.

Reflecting on the many people who have helped shape this book, we say THANKS for your help and contributions.

We also wish to acknowledge the many colleagues, friends, and students who have contributed in various ways to the development of this book over the years. In particular, we thank Barbara Reys, of the University of Missouri, Frank Lester, of Indiana University, and Paul Lindquist for their help and support.

We also wish to recognize the help of many reviewers and contributors of ideas and suggestions for prior editions, including:

Roda Amaria, Salem State College
Kimberly Arp, Cabrini College
Peter Appelbaum, Arcadia University
Tom Bassarear, Keene State College Jennifer Bay-Williams, University of Louisville
Martha Boecker, Northwestern Oklahoma State University
Carol Bonilla Bowman, Ramapo College
Daniel Brahier, Bowling Green State University
Christine Browning, Western Michigan University
Lecretia Buckley, Purdue University
Grace Burton, University of North Carolina, Wilmington
Rick Callan, Franklin College
Richard Caulfield, Indiana University
Delta Cavner, Southwest Baptist University
Astrida Cirulis, National-Louis University
Sandi Cooper, Baylor University
Bob Drake, University of Cincinnati
Martha Eggers, McKendree University
Dianne Erickson, Oregon State University
Skip Fennell, McDaniel College
Marvel Froemming, Moorhead State University
Jeff Frykholm, University of Colorado
K. Gaddis, Lewis & Clark College Lowell
Gadberry, Southwestern Oklahoma State University
Enrique Galindo, Indiana University Madeleine Gregg, University of Alabama
Elsa L. Geskus, Kutztown
Yvelyne Ger-main-McCarty, The University of New Orleans
Claire Graham, Framingham State College
Andrea Guillaume, California State University
Janet Handler, Mount Mercy College
Kim Harris, University of North Carolina-Charlotte
Kim Hartweg, Western Illinois University
Edith Hays, Texas Woman’s University
Ruth M. Heaton, University of Nebraska, Lincoln
Karen Higgins, Oregon State University
Ellen Hines, Northern Illinois University
Robert Jackson, University of Minnesota
Gae Johnson, Northern Arizona University
Susan Johnson, Northwestern College
Todd Johnson, Eastern Washington University
Mary Kabiri, Lincoln University
Henry S. Kepner, University of Wisconsin-Milwaukee
Diane H. Klein, Indiana University of Pennsylvania
Michele Koomen, Gustavus Adolphus College
Rick Kruschinsky, University of St. Thomas
William Lacefield, Mercer University
Vena Long, University of Tennessee
Margie Mason, College of William and Mary
David Martin, Florida Atlantic University
Robert Matulis, Millersville University
Sueanne McKinney, Old Dominion University
William Merrill, Central Michigan University
Alice Mills, Quincy University
Jean Mitchell, California State University-Monterey Bay
Eula Ewing Monroe, Brigham Young University
Sarah Murray, Centre College
Margaret (Maggie) Niess, Oregon State University
Jamar Pickreign, Rhode Island College
Don Ploger, Florida Atlantic University
Sara Powell, University of Charleston (SC)
Frank Powers, University of Idaho
Jacelyn Marie Rees, McNeese State University
Gay Ragan, Southwest Missouri State University
Denise M. Reboli, King’s College (PA)
Andy Reeves, University of South Florida-Tampa
Barbara Ridener, Florida Atlantic University
Candice L. Ridlon, Towson University
Tom Romberg, University of Wisconsin
Thomas E. Rowan, University of Maryland, College Park
Mary Ellen Schmidt, Ohio State University-Mansfield
Linda Sheeran, Oklahoma State University
Jason Silverman, Drexel University
Tina Sloan, Athens State University
Marian Smith, Florida A&M University
Marilyn Soucie, University of Missouri
Frances Stern, New York University

David L. Stout, University of West Florida
Jane Strawhecker, University of Nebraska at Kearney
Gertrude R. Toher, Hofstra University
Frederick L. Uy, California State University, Los Angeles
Juan Vazquez, Missouri Southern State College
Elsa Villa, The University of Texas at El Paso
Beth Vinson, Athens State
Kay Wall, University of Central Oklahoma
Pat Wall, Northern Arizona University
Judy Wells, Indiana State University
Tad Watanabe, Kennesaw State University
Dorothy Y. White, University of Georgia
Margaret Wyckoff, University of Maine-Farmington
John Yang, Lakeland College
Bernard Yvon, University of Maine
Preface iv
Acknowledgments vi

CHAPTER 1
School Mathematics in a Changing World 1
   Snapshot of a Lesson 1
   Chapter Learning Outcomes 1
   Introduction 1
What is Mathematics? 2
What Determines the Mathematics Being Taught? 2
   Historical Influences 2
   Recent Influences 4
Where Can You Turn? 6
   National Guidelines for School Mathematics 7
   State and Local Guidelines 7
   Research 7
   Cultural and International Resources 8
   Text Books and Other Materials 8
   Electronic Materials 8
   Professional Organizations 9
   Professional Development 9
   Other Teachers 9
   Yourself 9
What is Your Role Now? 9
Things to Do: From What You’ve Read 9
Things to Do: Going Beyond This Book 9

CHAPTER 2
Helping All Children Learn Mathematics with Understanding 11
   Snapshot of a Lesson 11
   Chapter Learning Outcomes 11
   Introduction 12
What do We Know About Learning Mathematics? 12
How Can We Support the Diverse Learners in Our Classroom? 12
   Creating a Positive Learning Environment 13
   Avoiding Negative Experiences that Increase Anxiety 14
   Establishing Clear Expectations 15
   Treating all Students as Equally Likely to Have Aptitude for Mathematics 15
   Helping Students Retain Mathematical Knowledge and Skills 16
Helping Children Acquire Both Procedural and Conceptual Knowledge 17
How do Children Learn Mathematics? 18
   Building Behavior 19
   Constructing Understanding 20
   Learning Trajectories 22
How Can We Help Children Make Sense of Mathematics? 22
   Recommendation 1: Teach to the Developmental Characteristics of Students 22
   Recommendation 2: Actively Involve Students 23
   Recommendation 3: Move Learning from Concrete to Abstract 25
   Recommendation 4: Use Communication to Encourage Understanding 27
Cultural Connections 28
A Glance at Where We’ve Been 28
   Things to Do: From What You’ve Read 29
   Things to Do: Going Beyond This Book 29
   Book Nook for Children 30

CHAPTER 3
Planning and Teaching 31
   Snapshot of a Lesson 31
   Chapter Learning Outcomes 31
   Introduction 31
Preparing to Teach: Questions to Ask 32
   Do I Understand the Mathematics I Am Teaching? 32
   What Do My Students Know? 33
   Where are My Students Developmentally? 34
   What Kinds of Tasks Will I Give My Students? 34
   How Will I Encourage My Students to Communicate? 35
   What Materials Will We Use? 37
Planning for Effective Teaching 43
   Levels of Planning 44
   Planning Different Types of Lessons 44
   Assessment and Analysis in Planning 51
Meeting the Needs of all Students 52
Cultural Connections 55
A Glance at Where We’ve Been 56
   Things to Do: From What You’ve Read 56
   Things to Do: Going Beyond This Book 57
   Book Nook for Children 57

CHAPTER 4
Using Assessment to Enhance Learning and Teaching 58
   Snapshot of a Lesson 58
   Chapter Learning Outcomes 58
   Introduction 59
Summative and Formative Assessments 59
Four Phases of Formative Assessment 59

Purposes for Formative Assessment 61
  Making Instructional Decisions 62
  Monitoring Student Progress 62
  Evaluating Student Achievement 62

Ways to Assess Students’ Abilities and Dispositions 62
  Observation 63
  Questioning 65
  Interviewing 65
  Performance Tasks 67
  Self-Assessment and Peer Assessment 68
  Work Samples 70
  Portfolios 70
  Writing 72
  Teacher-Designed Written Tests 72
  Standardized Achievement Tests 74

Keeping Records and Communicating About Assessments 74
  Recording the Information 74
  Communicating the Information 77

Cultural Connections 77

A Glance at Where We’ve Been 78
  Things to Do: From What You’ve Read 79
  Things to Do: Going Beyond This Book 79
  Book Nook for Children 80

CHAPTER 5
Practices and Processes of Doing Mathematics 81

  Snapshot of a Lesson 81
  Chapter Learning Outcomes 81
  Introduction 81

Mathematical Practices from the Common Core State Standards for Mathematics—CCSSM 83
  Practice 1: Problem Solving 83
  Practice 2: Reasoning 83
  Practice 3: Argumentation 83
  Practice 4: Modeling 85
  Practice 5: Using Tools 86
  Practice 6: Precision 86
  Practice 7: Structure 86
  Practice 8: Regularity 86

Mathematical Processes from NCTM’s Principles and Standards for School Mathematics 87
  Problem Solving 87
  Reasoning and Proof 90
  Communication 92
  Connections 93
  Representations 94

Cultural Connections 96

A Glance at Where We’ve Been 97

CHAPTER 6
Helping Children with Problem Solving 100

  Snapshot of a Lesson 100
  Chapter Learning Outcomes 100
  Introduction 101

What is a Problem and What is Problem Solving? 101

Teaching Mathematics Through Problem Solving 103
  Factors for Success in Problem Solving 104
  Choosing Appropriate Problems 105
  Finding Problems 109
  Having Students Pose Problems 110
  Using Calculators and Computers 111

Strategies for Problem Solving 112
  Act It Out 113
  Make a Drawing or Diagram 113
  Look for a Pattern 113
  Construct a Table 114
  Guess and Check 116
  Work Backward 116
  Solve a Similar but Simpler Problem 116

The Importance of Looking Back 119
  Looking Back at the Problem 119
  Looking Back at the Answer 119
  Looking Back at the Solution Process 119
  Looking Back at One’s Own Thinking 119

Helping All Students with Problem Solving 120
  Managing Time 120
  Managing Classroom Routines 120
  Managing Student Needs 120

Cultural Connections 121

A Glance at Where We’ve Been 123
  Things to Do: From What You’ve Read 123
  Things to Do: Going Beyond This Book 123
  Book Nook for Children 124

CHAPTER 7
Developing Counting and Number Sense 125

  Snapshot of a Lesson 125
  Chapter Learning Outcomes 125
  Introduction 125

Number Sense 125

Prenumber Concepts 127
  Classification 127
  Patterns 130

Early Number Development 131
  Conservation 131
  Subitizing 132
  Comparing and Ordering 132
Contents

Three-Dimensional Shapes 303
Describing and Sorting 304
Constructing to Explore and Discover 306

Two-Dimensional Shapes 309
Sides and Angles 310
Symmetry 313
Parallel and Perpendicular 314
Other Attributes 314

Space 317

Transformations 319

Visualization and Spatial Reasoning 320
Using Geometric Physical and Pictorial Materials 321
Using Mental Images 322

Cultural Connections 323

A Glance at Where We’ve Been 323
Things to Do: From What You’ve Read 323
Things to Do: Going Beyond This Book 324
Book Nook for Children 324

CHAPTER 16

Measurement 325

CHAPTER LEARNING OUTCOMES 325
INTRODUCTION 325

The Measurement Process 326
Identify Attributes by Comparing 327
Choose a Unit and Compare the Unit to an Object 332
Find and Report the Number of Units 334

Other Aspects of Measuring 340
Estimating Measurements 340
Creating Objects Given the Measurement 342
Comparing Measurements 342
Conversions 343

Connecting Attributes 344
Area and Shape 344
Volume and Shape 344
Perimeter and Area 344
Volume and Surface Area 345

Cultural Connections 346

A Glance at Where We’ve Been 347
Things to Do: From What You’ve Read 347
Things to Do: Going Beyond This Book 347
Book Nook for Children 348

CHAPTER 17

Data Analysis, Statistics, and Probability 349

CHAPTER LEARNING OUTCOMES 349
INTRODUCTION 349

Formulating Questions 351
Collecting Data 353

Experiments 354
Simulations 354

Analyzing Data: Graphical Organization 354
Quick and Easy Graphing Methods 355
Plots 355
Picture Graphs 358
Bar Graphs and Histograms 358
Pie Graphs 359
Line Graphs 359
Graphical Roundup 360
Misleading Graphs 361

Analyzing Data: Descriptive Statistics 363
Measures of Central Tendency or Averages 363
Measures of Variation 366

Interpreting Results 368
Data Sense 368
Communicating Results 369

Probability 369
Probability of an Event 370

Randomness 373
Independence of Events 373

Cultural Connections 374

A Glance at Where We’ve Been 375
Things to Do: From What You’ve Read 375
Things to Do: Going Beyond This Book 376
Book Nook for Children 377

CHAPTER 18

Number Theory 378

CHAPTER LEARNING OUTCOMES 378
INTRODUCTION 378
Why Study Number Theory 379

Number Theory in Elementary School Mathematics 381
Odds and Evens 381
Factors and Multiples 383
Primes and Composites 385

Divisibility 387

Other Number Theory Topics 389
Relatively Prime Pairs of Number 389
Polygonal Numbers 390
Modular Arithmetic 390
Pascal’s Triangle 391
Pythagorean Triples 391
Fibonacci Sequence 392

Cultural Connections 393

A Glance at Where We’ve Been 394
Things to Do: From What You’ve Read 394
Things to Do: Going Beyond This Book 394
Book Nook for Children 394

References 395
Appendix 407
Index 409