11.1 Ring Ising model \((N = 3, 4)\). Calculate \(\langle \sigma \rangle\) and \(\langle \sigma_i \sigma_{i+1} \rangle\) for (1) an \(N = 3\) ring Ising model in Section 11.2.3.2 and (2) an \(N = 4\) ring Ising model in Section 11.2.3.3. Then, find \(\langle \sigma \rangle\) and \(\langle \sigma_i \sigma_{i+1} \rangle\) in the limits of \(\beta J \to 0\) and \(\beta J \to \infty\). You should be able to tell whether your answers are correct or not, following the same discussion as the one we had for the \(N = 3\) linear Ising model in Section 11.2.3.1.

\[
\langle \sigma \rangle = \frac{1}{3\beta} \frac{\partial \ln Z}{\partial H} = \frac{e^{3\beta J} \sinh 3\beta H + e^{-\beta J} \sinh \beta H}{e^{3\beta J} \cosh 3\beta H + 3e^{-\beta J} \cosh \beta H}
\]

\[
\langle \sigma_i \sigma_{i+1} \rangle = \frac{1}{3\beta} \frac{\partial \ln Z}{\partial J} = \frac{e^{3\beta J} \cosh 3\beta H - e^{-\beta J} \cosh \beta H}{e^{3\beta J} \cosh 3\beta H + 3e^{-\beta J} \cosh \beta H}
\]

In the limit of \(\beta J \to 0\),

\[
\langle \sigma \rangle \approx \frac{\sinh 3\beta H + \sinh \beta H}{\cosh 3\beta H + 3 \cosh \beta H} = \tanh \beta H
\]

\[
\langle \sigma_i \sigma_{i+1} \rangle \approx \frac{\cosh 3\beta H - \cosh \beta H}{\cosh 3\beta H + 3 \cosh \beta H} = \tanh^2 \beta H
\]

Spins are independent of each other.

When \(\beta J \gg 1\),

\[
\langle \sigma \rangle \approx \tanh 3\beta H
\]

\[
\langle \sigma_i \sigma_{i+1} \rangle \approx 1
\]

The three spins are in unison.

(2)

\[
\langle \sigma \rangle = \frac{1}{4\beta} \frac{\partial \ln Z}{\partial H} = \frac{e^{4\beta J} \sinh 4\beta H + 2 \sinh 2\beta H}{e^{4\beta J} \cosh 4\beta H + 4 \cosh 2\beta H + e^{-4\beta J} + 2}
\]

\[
\langle \sigma_i \sigma_{i+1} \rangle = \frac{1}{4\beta} \frac{\partial \ln Z}{\partial J} = \frac{e^{4\beta J} \cosh 4\beta H - e^{-4\beta J}}{e^{4\beta J} \cosh 4\beta H + 4 \cosh 2\beta H + e^{-4\beta J} + 2}
\]
In the limit of $\beta J \to 0$, 
\[
\langle \sigma \rangle \approx \frac{\sinh 4\beta H + 2 \sinh 2\beta H}{\cosh 4\beta H + 4 \cosh 2\beta H + 3} = \tanh \beta H
\]
\[
\langle \sigma_i \sigma_{i+1} \rangle \approx \frac{\cosh 4\beta H - 1}{\cosh 4\beta H + 4 \cosh 2\beta H + 3} = \tanh^2 \beta H
\]

When $\beta J \gg 1$,
\[
\langle \sigma \rangle \equiv \tanh 4\beta H
\]
\[
\langle \sigma_i \sigma_{i+1} \rangle \equiv 1
\]

**11.2 Ring Ising model ($N = 6$).** Six spins $\sigma_i$ through $\sigma_6$ are arranged in a hexagon, as shown below.

There is no external magnetic field, and the interaction exists only between an adjacent pair on the ring. The energy of spin system is given as

\[
E = -J \sum_{i=1}^{6} \sigma_i \sigma_{i+1}
\]

where $J > 0$ and $\sigma_7 = \sigma_1$.

(1) Complete the table below. The typical spin arrangements should show all possibilities, eliminating those that are identical by symmetry operations (rotation by 60°, reflection by a plane through the center that divides the ring into two equal parts (sans spins), inversion).

<table>
<thead>
<tr>
<th>Energy level $6J$</th>
<th>Typical spin arrangements</th>
<th>Number of states</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6J$</td>
<td>&lt;image&gt;</td>
<td>2</td>
</tr>
</tbody>
</table>

(2) What is $\langle \sigma \rangle$?

(3) What is $\langle \sigma_i \sigma_{i+1} \rangle = \langle \Sigma \sigma_i \sigma_{i+1} \rangle / 6$?
(4) What is the low-temperature limit of $<\sigma_i\sigma_{i+1}>$?

(5) What is the high-temperature asymptote of $<\sigma_i\sigma_{i+1}>$?

<table>
<thead>
<tr>
<th>Energy level</th>
<th>Typical spin arrangements</th>
<th>Number of states</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6J$</td>
<td><img src="image1.png" alt="Spin Arrangements" /></td>
<td>2</td>
</tr>
<tr>
<td>$2J$</td>
<td><img src="image2.png" alt="Spin Arrangements" /></td>
<td>30</td>
</tr>
<tr>
<td>$-2J$</td>
<td><img src="image3.png" alt="Spin Arrangements" /></td>
<td>30</td>
</tr>
<tr>
<td>$-6J$</td>
<td><img src="image4.png" alt="Spin Arrangements" /></td>
<td>2</td>
</tr>
</tbody>
</table>

(2) 0

(3)

$$Z = \sum_{\sigma_i=\pm 1} \cdots \sum_{\sigma_{i+1}=\pm 1} \exp \left( \beta J \sum_{i=1}^{6} \sigma_i\sigma_{i+1} \right)$$

$$= 2e^{6\beta J} + 30e^{2\beta J} + 30e^{-2\beta J} + 2e^{-6\beta J} = 4\cosh(6\beta J) + 60\cosh(2\beta J)$$

$$\langle \sigma_i\sigma_{i+1} \rangle = \frac{1}{6\beta} \frac{\partial}{\partial J} \ln Z = \frac{\sinh(6\beta J) + 5\sinh(2\beta J)}{\cosh(6\beta J) + 15\cosh(2\beta J)}$$

(4) $\beta J \gg 1$. 


\[
\langle \sigma_i \sigma_{i+1} \rangle \approx \frac{e^{6\beta J} + 5e^{2\beta J}}{e^{6\beta J} + 15e^{2\beta J}} \approx 1
\]

(5) $\beta J \ll 1$.

\[
\langle \sigma_i \sigma_{i+1} \rangle \approx \frac{6\beta J + 5 \times 2\beta J}{1 + 15} = \beta J
\]

11.3 Coarse graining. In the one-dimensional Ising model of $N$ spins, we change the way the states are counted and adopt a course-grained view. In the modified counting method, a unit is a pair of adjacent spins, and the state of each unit, $\tau_j$, is $+2$, $0$, and $-2$, where $j = 1, 2, \ldots, \frac{1}{2}N$. Their degeneracies are $1$, $2$, and $1$, respectively. The interaction exists between adjacent units, and is expressed as $-\gamma f \tau_j \tau_{j+1}$, where a constant $\gamma$ may not be equal to one. The energy is then expressed as

\[
E = -H \sum_{j=1}^{N/2} \tau_j - \gamma J \sum_{j=1}^{N/2} \tau_j \tau_{j+1}
\]

Employ the Bragg-Williams approximation to show that $\gamma = 1/2$ gives the same partition function as the one given by Eq. (11.27).

\[
Z = \sum_{\tau_1} \sum_{\tau_2} \cdots \sum_{\tau_{N/2}} \exp \left( \beta H \sum_{j=1}^{N/2} \tau_j + \beta \gamma J \sum_{j=1}^{N/2} \tau_j \tau_{j+1} \right)
\]

With the B-W approximation,

\[
Z = \sum_{\tau_1} \sum_{\tau_2} \cdots \sum_{\tau_{N/2}} \exp \left( \beta \sum_{j=1}^{N/2} \left( H + \gamma J \langle \tau \rangle \right) \tau_j \right)
\]

It is rewritten to

\[
Z = \left[ \sum_{\tau = -2,0,2} e^{\beta(H + \gamma J \langle \tau \rangle) \tau} \right]^{N/2}
= \left[ e^{2\beta(H + \gamma J \langle \tau \rangle)} + 2 + e^{-2\beta(H + \gamma J \langle \tau \rangle)} \right]^{N/2}
= [2 \cosh(\beta(H + \gamma J \langle \tau \rangle))]^N
\]

\[
\ln Z = N \ln(2 \cosh(\beta(H + \gamma J \langle \tau \rangle)))
\]

\[
\langle \tau \rangle = \frac{2}{\beta N} \frac{\partial \ln Z}{\partial H} = \frac{2}{\beta N} \frac{N \beta \sinh(\beta(H + \gamma J \langle \tau \rangle))}{\cosh(\beta(H + \gamma J \langle \tau \rangle))} = 2 \tanh(\beta(H + \gamma J \langle \tau \rangle))
\]
This equation is equivalent to Eq. (11.27) if

\[ \frac{\langle \tau \rangle}{2} = \langle \sigma \rangle \]

and

\[ \gamma J \langle \tau \rangle = J \langle \sigma \rangle \]

The first relationship just accounts for the difference in the counting method. From the two equations, we have \( \gamma = 1/2 \).

### 11.4 Three-state model.

In the 1D Ising model, we change the number of states for each spin from 2 to 3:

\[
\sigma_i = \begin{cases} 
1 & \text{(up)} \\
-1 & \text{(down)} \\
0 & \text{(otherwise)}
\end{cases}
\]

The degeneracy is 1 for each of the three states, and the energy in the presence of magnetic field \( H \) and the interaction are identical to those in Section 11.2. The expression of the energy \( E \) is identical to Eq. (11.2). Apply a B-W approximation and discuss how the mean spin changes from the one discussed in that section.

\[
Z = \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \exp \left( \beta H \sum_{i=1}^{N} \sigma_i + \beta J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} \right)
\]

With the B-W approximation,

\[
Z = \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \exp \left( \beta \sum_{i=1}^{N} (H + J \langle \sigma \rangle) \sigma_i \right) = \left[ \sum_{\sigma=1,0,-1} e^{\beta(H+J\langle \sigma \rangle)\sigma} \right]^N
\]

which is calculated as

\[
Z = \left[ 1 + 2 \cosh(\beta(H + J \langle \sigma \rangle)) \right]^N
\]
Therefore,

\[ \ln Z = N \ln(1 + 2 \cosh(\beta(H + J\langle\sigma\rangle))) \]

The mean magnetization \( \langle\sigma\rangle \) is then obtained as

\[ \langle\sigma\rangle = \frac{1}{N\beta} \frac{\partial \ln Z}{\partial H} = \frac{2 \sinh(\beta(H + J\langle\sigma\rangle))}{1 + 2 \cosh(\beta(H + J\langle\sigma\rangle))} \]

When \( H = 0 \), the plot of

\[ f(x) \equiv \frac{2 \sinh \beta J x}{1 + 2 \cosh \beta J x} \]

is shown below.

If \( \beta J < 3/2 \), the solution of the self-consistent equation is \( \langle\sigma\rangle = 0 \). If \( \beta J > 3/2 \), the self-consistent equation has two stable solutions \( x_\pm \). An approximate expression for \( x_+ \) is shown below.

At high temperatures (including \( H = 0 \)),

\[ \langle\sigma\rangle \approx \frac{2\beta(H + J\langle\sigma\rangle)}{3} \]

which leads to

\[ \langle\sigma\rangle \approx \frac{2\beta H}{3 - 2\beta J} \]

At low temperatures (including \( H = 0 \)),

\[ \langle\sigma\rangle \approx \frac{e^{\beta(H + J\langle\sigma\rangle)}}{1 + e^{\beta(H + J\langle\sigma\rangle)}} = 1 - e^{-\beta(H + J\langle\sigma\rangle)} \]
11.5 $\langle \sigma \rangle$ in F-H approximation. Use the expression of $\ln Z$ given by Eq. (11.58) to show that it is consistent with the formula, $\langle \sigma \rangle = (N\beta)^{-1}\partial \ln Z/\partial H$. Keep in mind that $\langle \sigma \rangle$ is a function of $\beta$, $H$, and $J$.

\[
\frac{\partial \ln Z}{\partial H} \cong N\langle \sigma \rangle \left( \frac{1}{1 - \langle \sigma \rangle^2} - 2\beta J \right) \frac{\partial \langle \sigma \rangle}{\partial H}
\]

From Eq. (11.51),

\[
\frac{\partial \langle \sigma \rangle}{\partial H} = \text{sech}^2(\beta H + 2\beta J \langle \sigma \rangle) \left( \beta + 2\beta J \frac{\partial \langle \sigma \rangle}{\partial H} \right)
\]

\[
= \left[ 1 - \tanh^2(\beta H + 2\beta J \langle \sigma \rangle) \right] \beta + 2\beta J \frac{\partial \langle \sigma \rangle}{\partial H} = \left( 1 - \langle \sigma \rangle^2 \right) \left( \beta + 2\beta J \frac{\partial \langle \sigma \rangle}{\partial H} \right)
\]

which leads to

\[
\left( \frac{1}{1 - \langle \sigma \rangle^2} - 2\beta J \right) \frac{\partial \langle \sigma \rangle}{\partial H} = \beta
\]

Therefore,

\[
\frac{1}{N\beta} \frac{\partial \ln Z}{\partial H} \cong \langle \sigma \rangle
\]

11.6 $\langle \sigma_i \sigma_{i+1} \rangle$ in F-H approximation. Apply Eq. (11.6) to Eq. (11.58) to find $\langle \sigma_i \sigma_{i+1} \rangle$ in the F-H approximation of the 1D Ising model.

\[
\langle \sigma_i \sigma_{i+1} \rangle = \frac{1}{N\beta} \frac{\partial}{\partial J} N \left[ \ln 2 - \frac{1}{2} \ln(1 - \langle \sigma \rangle^2) - \beta J \langle \sigma \rangle^2 \right]
\]

\[
= \frac{1}{\beta} \left[ \langle \sigma \rangle \left( \frac{1}{1 - \langle \sigma \rangle^2} - 2\beta J \right) \frac{\partial \langle \sigma \rangle}{\partial J} - \beta \langle \sigma \rangle^2 \right]
\]

(*)

Differentiating Eq. (11.51) by $J$ gives

\[
\frac{\partial \langle \sigma \rangle}{\partial J} = 2\beta \left( 1 - \langle \sigma \rangle^2 \right) \left( \langle \sigma \rangle + J \frac{\partial \langle \sigma \rangle}{\partial J} \right)
\]
Then,
\[
\frac{\partial \langle \sigma \rangle}{\partial J} = \frac{2\beta \langle \sigma \rangle (1-\langle \sigma \rangle^2)}{1-2\beta J (1-\langle \sigma \rangle^2)}
\]
Inserting this equation into eq * gives
\[
\langle \sigma_i \sigma_{i+1} \rangle = \langle \sigma \rangle^2
\]

**1.7 2D Ising model.** The figure below illustrates spins in a 2D Ising model on a square lattice. The interaction is between nearest neighbors. For example, the spin indicated by an outlined ellipse has four nearest neighbors indicated by gray ellipses. The system has a total $N$ spins ($N \gg 1$).

The energy of the system is given as
\[
E = -H \sum_{i=1}^{N} \sigma_i - J \sum_{i,j \in NN} \sigma_i \sigma_j
\]
The second series is calculated for all nearest neighbor pairs of $i$ and $j$. Apply the three methods of mean-field approximation to the 2D Ising model to find the equation for $\langle \sigma \rangle$ that, when solved self-consistently, would give $\langle \sigma \rangle$.

The system has a total $2N$ interactions, 2 neighbors per spin.

(1) B-W approximation. The energy is approximated as
\[
E = -H \sum_{i=1}^{N} \sigma_i - 2J \sum_{i=1}^{N} \sigma_i \langle \sigma \rangle = - \sum_{i=1}^{N} \sigma_i (H + 2J \langle \sigma \rangle)
\]
The partition function is
\[
Z = \left[ \sum_{\sigma = \pm 1} e^{\beta (H + 2J \langle \sigma \rangle)} \right]^N = \left[ 2 \cosh(\beta (H + 2J \langle \sigma \rangle)) \right]^N
\]
Then, $\langle \sigma \rangle$ is obtained as
\[ \langle \sigma \rangle = \frac{1}{\beta} \frac{\partial}{\partial H} \ln \left( 2 \cosh \left( \beta \left( H + 2J \langle \sigma \rangle \right) \right) \right) = \tanh(\beta(H + 2J\langle \sigma \rangle)) \]

(2) F-H approximation

For random distribution of spins under \( \Sigma \sigma = M \), the interaction is

\[
\sum_{i,j \in NN} \sigma_i \sigma_j = 2N \left[ \left( \frac{N + M}{2N} \right)^2 + \left( \frac{N - M}{2N} \right)^2 \right] \left( +1 \right)^2 + 2 \left( \frac{N + M}{2N} \right) \left( \frac{N - M}{2N} \right) \left( \left( +1 \right)(-1) \right) = 2 \frac{M^2}{N} \]

The partition function is

\[ Z = \sum_{M=-N}^{N} f(M) \]

where

\[ f(M) = \binom{N}{(N + M) / 2} \exp \left( \beta H M + 2 \beta J \frac{M^2}{N} \right) \]

The difference from the 1D Ising model in the F-H approximation is that the interaction term is “2J” in place of “J”. Therefore, the equation for \( \langle \sigma \rangle \) is

\[ \langle \sigma \rangle = \tanh(\beta H + 4 \beta J \langle \sigma \rangle) \]

(3) Mean-field theory. The interaction is expressed as

\[
\sum_{i,j \in NN} \sigma_i \sigma_j \approx 2(N \langle \sigma \rangle^2 + 2 \langle \sigma \rangle \sum_{i=1}^{N} \delta \sigma_i) \]

This expression is different from the one in the 1D Ising model only by the factor of 2. Therefore,

\[ \langle \sigma \rangle = \tanh \beta(H + 4J \langle \sigma \rangle) \]

11.8 Polycation. A polycation of \( N \) repeat units \((N >> 1)\) is dissolved in water. Each repeat unit has one cationic site. Small anions (acid HX) are added to the solution and some of them may pair with the cations. The state of each cation may be expressed by \( \sigma \), as
The energy of pairing may be expressed as

\[ E = \varepsilon \sum_{i=1}^{N} \sigma_i + \xi \sum_{i=1}^{N} \sigma_i \sigma_{i+1} \]

where \( \varepsilon \) (< 0) represents the energy for a single cation to pair with \( X^- \), and \( \xi \) the increase in the energy when two adjacent sites are paired. The latter may be positive or negative.

(1) What is the partition function? Use a B-W approximation.

(2) Find a self-consistent solution up to the linear order of \( \beta \xi \).

(3) Draw a sketch for a plot of \( \langle \sigma \rangle \) as a function of \( \beta \xi \), and discuss the plot.

With a B-W approximation,

\[ Z = \sum_{\sigma_i=0,1} \sum_{\sigma_j=0,1} \cdots \sum_{\sigma_N=0,1} \exp(-\beta \varepsilon \sum_{i=1}^{N} \sigma_i - \beta \xi \sum_{i=1}^{N} \sigma_i \sigma_{i+1}) \]

\[ Z = \sum_{\sigma_i=0,1} \sum_{\sigma_j=0,1} \cdots \sum_{\sigma_N=0,1} \exp\left(-\beta \sum_{i=1}^{N} (\varepsilon + \xi \langle \sigma \rangle) \sigma_i \right) = \left[ 1 + e^{-\beta(\varepsilon + \xi \langle \sigma \rangle)} \right]^N \]

(2)

\[ \ln Z = N \ln(1 + e^{-\beta(\varepsilon + \xi \langle \sigma \rangle)}) \]

\[ \langle \sigma \rangle = -\frac{1}{N \beta} \frac{\partial \ln Z}{\partial \varepsilon} = \frac{1}{1 + e^{\beta(\varepsilon + \xi \langle \sigma \rangle)}} \]

When \( \beta \xi \ll 1 \),

\[ \langle \sigma \rangle \approx \frac{1}{1 + e^{\beta \xi (1 + \beta \xi \langle \sigma \rangle)}} \]

which is a quadratic equation of \( \langle \sigma \rangle \):
\[ \beta \xi \langle \sigma \rangle^2 + (1 + e^{-\beta \xi}) \langle \sigma \rangle - e^{-\beta \xi} = 0 \]

Since \( \langle \sigma \rangle \) is positive,

\[ \langle \sigma \rangle = \frac{-(1 + e^{-\beta \xi}) + \left[ (1 + e^{-\beta \xi})^2 + 4 \beta \xi e^{-\beta \xi} \right]^{1/2}}{2 \beta \xi} = \frac{e^{-\beta \xi/2}}{\cosh \frac{\beta \xi}{2} + (\cosh^2 \frac{\beta \xi}{2} + \beta \xi)^{1/2}} \]

\( \langle \sigma \rangle \)

\( \langle \sigma \rangle \) decreases with an increasing \( \beta \xi \), since that does not favor adjacent repeat units having counterions.

### 11.9 Alternative listing of states, mean-field

The state of a one-dimensional Ising model of \( N \) spins, \( \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_N \), can alternatively be specified by \( \sigma_1 \) and \( \sigma_1 \sigma_2, \sigma_2 \sigma_3, \ldots, \sigma_{N-1} \sigma_N \). The product of an adjacent pair of spins is either +1 or −1. The energy of the \( N \) spins in magnetic field \( H \) is given as

\[ E = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} - H \sum_{i=1}^{N} \sigma_i \]

(11.155)

and the partition function is expressed as

\[ Z = \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \sum_{\sigma_3 = \pm 1} \cdots \sum_{\sigma_{N-1} \sigma_N = \pm 1} \exp \left( \beta J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} + \beta H \sum_{i=1}^{N} \sigma_i \right) \]

(11.156)

We note that

\[ \sigma_2 = \sigma_1 \sigma_1 \sigma_2 \]

(11.157)

\[ \sigma_3 = \sigma_1 \sigma_1 \sigma_2 \sigma_2 \sigma_3 \]

(11.158)

\[ \ldots \]
We adopt a mean-field approximation for calculating the partition function. Our method is different from the Bragg-Williams approximation that replaces $\sigma_{i+1}$ in $\sigma_i \sigma_{i+1}$ with $\langle \sigma \rangle$. We replace $\sigma_1 \sigma_2$, etc in Eqs. (11.157)–(11.159) with $\langle \sigma_1 \sigma_2 \rangle = \eta$. Thus,

$$
\sigma_2 = \sigma_1 \eta, \quad \sigma_3 = \sigma_1 \eta^2, \quad \cdots \quad \sigma_N = \sigma_1 \eta^{N-1}
$$

(11.160)

Then,

$$
\sum_{i=1}^{N} \sigma_i = \sigma_1 \sum_{j=0}^{N-1} \eta^j = \sigma_1 \frac{1-\eta^N}{1-\eta}
$$

(11.161)

for $\eta < 1$. When $\eta = 1$, the series equals $N$. The figure below shows a plot of $(1-\eta^N)/(1-\eta)$ for $N = 10, 30, \text{and} 100$. Assume that $N$ is even in this problem.

With this approximation, the partition function is expressed as

$$
Z = \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1, \sigma_2 \sigma_3=\pm 1} \cdots \sum_{\sigma_{N-1} \sigma_N=\pm 1} \exp \left( \beta J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} + \beta H \sigma_1 \frac{1-\eta^N}{1-\eta} \right)
$$

(11.162)

(1) Use Eq. (11.162) to calculate $Z$.

(2) Calculate $\langle \sigma_1 \sigma_2 \rangle$.

(3) Calculate $\langle \sum_{i=1}^{N} \sigma_i \rangle$.

(4) When $\eta = 1$, all the spins are aligned. Then, you can calculate $Z$ exactly. Show that your answer of part (3) for $\eta = 1$ is identical to the one obtained from the exact $Z$.

(5) When $\eta = 0$, what is your answer of part (3) equal to? Is the result reasonable? Explain the result.

(6) When $\eta = -1$, what is your answer of part (3) equal to? Is the result reasonable? Explain the result.
\[
Z = \sum_{\sigma = \pm 1} \exp(\beta H \sigma_1 \frac{1-\eta^N}{1-\eta}) \left[ \sum_{\sigma_2 = \pm 1} \exp(\beta J \sigma_1 \sigma_2) \right]^{N-1} \\
= 2 \cosh(\beta H \frac{1-\eta^N}{1-\eta}) (2 \cosh \beta J)^{N-1}
\]

\[(2)\]

\[
\ln Z = \ln 2 + \ln \left( \cosh \left( \beta H \frac{1-\eta^N}{1-\eta} \right) \right) + (N-1)[\ln 2 + \ln(cosh \beta J)]
\]

\[
\langle \sigma_1 \sigma_2 \rangle = \frac{1}{N-1} \frac{1}{\beta} \frac{\partial \ln Z}{\partial J} = \tanh \beta J
\]

\[(3)\]

\[
\left\langle \sum_{i=1}^{N} \sigma_i \right\rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial H} = \frac{1-\eta^N}{1-\eta} \tanh \left( \beta H \frac{1-\eta^N}{1-\eta} \right)
\]

\[(4)\] The exact \(Z\) is

\[
Z = \sum_{\sigma_i = \pm 1} \exp(\beta J (N-1) + \beta H N \sigma_i) = 2 \cosh(\beta H N)e^{\beta J (N-1)}
\]

\[
\ln Z = \ln(2 \cosh(\beta H N)) + \beta J (N-1)
\]

\[
\left\langle \sum_{i=1}^{N} \sigma_i \right\rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial H} = N \tanh(\beta H N)
\]

In the answer of part (3), \((1-\eta^N)/(1-\eta) = N\). They agree with each other.

\[(5)\]

\[
\left\langle \sum_{i=1}^{N} \sigma_i \right\rangle = \tanh(\beta H)
\]

This result is identical to a single spin in field \(H\). \(\eta = \langle \sigma_1 \sigma_2 \rangle = 0\) forces \(\sigma_2\) to be +1 and -1 with equal probabilities. Therefore, \(\langle \sigma_2 \rangle = 0\), and the effect of \(H\) remains only in the first spin.
\[ \left\langle \sum_{i=1}^{N} \sigma_i \right\rangle = 0 \]

\eta = -1 means alternate spin orientations. Since \( N \) is even, the mean of the spins is zero.

11.10 Exact linear Ising model. Confirm that the partition function in the exact 1D linear Ising model agrees with the one to be obtained from discrete listing of all the states for (1) \( N = 2 \) and (2) \( N = 3 \) spins.

(1) The table below lists the states of the 1D linear Ising model with \( N = 2 \).

<table>
<thead>
<tr>
<th>( \sigma_1 )</th>
<th>( \sigma_2 )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(-2H - J)</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>( J )</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>( J )</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>( 2H - J )</td>
</tr>
</tbody>
</table>

The partition function is

\[ Z = e^{\beta(2H + J)} + e^{\beta(-2H + J)} + 2e^{-\beta J} = 2(e^{\beta J} \cosh 2\beta H + e^{-\beta J}) \]

\((*)\)

Equation (11.116) for \( N = 2 \) is

\[ Z_2 = \left( \frac{h + \frac{1}{h}}{h} \right) (\lambda_1 + \lambda_2) - 2 \left( j - \frac{1}{j} \right) = j \left( h^2 + \frac{1}{h^3} \right) + \frac{2}{j} \]

which is identical to (*)..

(2) Equation (11.116) for \( N = 3 \) is

\[ Z_3 = \left( \frac{h + \frac{1}{h}}{h} \right) (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) - 2 \left( j - \frac{1}{j} \right) (\lambda_1 + \lambda_2) = j^2 \left( \frac{1}{h^3} + h^3 \right) + \left( 2 + \frac{1}{j^2} \right) \left( h + \frac{1}{h} \right) \]

which is identical to Eq. (11.15).
11.11 Exact ring Ising model. Confirm that the partition function in the exact 1D ring Ising model agrees with the one to be obtained from discrete listing of all the states for (1) $N = 2$ and (2) $N = 3$ spins.

(1) The table below lists the states in the 1D ring Ising model of $N = 2$.

<table>
<thead>
<tr>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$-2H - 2J$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>$2J$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>$2J$</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>$2H - 2J$</td>
</tr>
</tbody>
</table>

The partition function is

$$Z = e^{\beta(2H+2J)} + e^{\beta(-2H+2J)} + 2e^{-\beta J} = 2(e^{\beta J} \cosh 2\beta H + e^{-\beta J})$$

Equation (11.126) for $N = 2$ is

$$\lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = j^2\left(h + \frac{1}{h}\right)^2 - 2\left(j^2 - \frac{1}{j^2}\right) = j^2\left(h^2 + \frac{1}{h^2}\right) + \frac{2}{j^2}$$

These two expressions are identical.

(2)

$$\lambda_1^3 + \lambda_2^3 = (\lambda_1 + \lambda_2)[(\lambda_1 + \lambda_2)^2 - 3\lambda_1\lambda_2] = j\left(h + \frac{1}{h}\right)\left[j^2\left(h + \frac{1}{h}\right)^2 - 3\left(j^2 - \frac{1}{j^2}\right)^2\right]$$

$$= j\left(h + \frac{1}{h}\right)\left[j^2\left(h^2 - 1 + \frac{1}{h^2}\right) + \frac{3}{j^2}\right] = j^3\left(h^3 + \frac{1}{h^3}\right) + \frac{3}{j}\left(h + \frac{1}{h}\right)$$

This result is identical to Eq. (11.23).

11.12 Exact Ising model, large $N$. Figure 11.17 shows a plot of $<\sigma_1\sigma_{i+1}>$ as a function of $\beta J$ for different values of $\beta H$. Draw a sketch of $<\sigma_1\sigma_{i+1}>$ as a function of $\beta H$ for different values of $\beta J$. The range of $\beta H$ should include positive and negative values of $\beta H$. 

In Eq. (11.104), when $\beta H >> 1$,

$$\langle \sigma_i \sigma_{i+1} \rangle \approx \frac{1}{\tanh 2\beta J} - \frac{1}{e^{2\beta J} \sinh 2\beta J \tanh \beta H} \approx \frac{1}{\tanh 2\beta J} - \frac{e^{-2\beta J}}{\sinh 2\beta J} = 1$$

When $-\beta H >> 1$, the identical limit.

When $|\beta H| << 1$,

$$\langle \sigma_i \sigma_{i+1} \rangle \approx \frac{1}{\tanh 2\beta J} - \frac{1 + \frac{1}{2}(\beta H)^2}{\sinh 2\beta J \left[ 1 + \frac{1}{2} e^{4\beta J} (\beta H)^2 \right]} \approx \frac{1}{\tanh 2\beta J} - \frac{1 + \frac{1}{2} (1 - e^{4\beta J}) (\beta H)^2}{\sinh 2\beta J}$$

$$= \frac{1}{\tanh 2\beta J} - \frac{1}{\sinh 2\beta J} - \frac{1 - e^{4\beta J}}{2 \sinh 2\beta J} (\beta H)^2 = \frac{1}{\tanh \beta J} + e^{2\beta J} (\beta H)^2$$

![Graph showing the behavior of $\langle \sigma_i \sigma_{i+1} \rangle$ as a function of $\beta H$.](image)

**11.13 Chiral polymer.** Consider a one-dimensional chain of spins, each capable of being in three directions labeled as $-1$, $0$, and $1$. The state of the chain consisting of $N$ spins is specified by $\sigma_1, \sigma_2, \ldots, \sigma_N$, where $\sigma_i = -1, 0, \text{ or } 1$ ($i = 1, \ldots, N$). An example is shown below.

![Image of a chiral polymer chain.](image)

There is an interaction between an adjacent pair of spins, and the interaction $\xi$ between $\sigma_i$ and $\sigma_{i+1}$ is listed in the table below.
The partition function is expressed as

\[ Z = \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \exp \left( -\beta \sum_{i=1}^{N-1} \xi (\sigma_i, \sigma_{i+1}) \right) \]

(1) Find the transfer matrix \( M \) for calculating \( Z \) and the eigenvalues of \( M \).

(2) Assume \( N \gg 1 \) to obtain an approximate expression of \( \ln Z \).

(3) Calculate \( (N\beta)^{-1} \partial \ln Z / \partial J \). What quantity does it represent?

(1) Use a method of transfer matrix.

\[ Z_n = Z_n(1) + Z_n(0) + Z_n(-1) \]

\[
\begin{bmatrix}
Z_n(1) & Z_n(0) & Z_n(-1)
\end{bmatrix} = \begin{bmatrix}
Z_{n-1}(1) & Z_{n-1}(0) & Z_{n-1}(-1)
\end{bmatrix} \begin{bmatrix}
1 & e^{\beta J} & e^{-\beta J} \\
e^{-\beta J} & 1 & e^{\beta J} \\
e^{\beta J} & e^{-\beta J} & 1
\end{bmatrix}
\]

Let \( j \equiv e^{\beta J} \). Then, the transfer matrix \( M \) is

\[
M = \begin{bmatrix}
1 & j & j^{-1} \\
j^{-1} & 1 & j \\
j & j^{-1} & 1
\end{bmatrix}
\]

The eigenvalues are the zeros of
\[ |M - \lambda I| = \begin{vmatrix} 1 - \lambda & j & j^{-1} \\ j^{-1} & 1 - \lambda & j \\ j & j^{-1} & 1 - \lambda \end{vmatrix} \]

\[(1 - \lambda)^3 + j^3 + j^{-3} - 3(1 - \lambda) = 0\]
\[(1 - \lambda + j + j^{-1})[(1 - \lambda)^2 - (j + j^{-1})(1 - \lambda) + j^2 - 1 + j^{-2}] = 0\]

The three zeros are
\[\lambda_0 = 1 + j + j^{-1}\]
\[\lambda_{\pm} = 1 - \frac{(j + j^{-1}) \pm 3^{1/2} |j - j^{-1}|}{2}\]

(2) Compare the magnitudes of the three zero’s. Since
\[|\lambda_\pm|^2 = \left(1 - \frac{j + j^{-1}}{2}\right)^2 + \frac{3}{4} (j - j^{-1})^2 = (j + j^{-1} + 1)(j + j^{-1} - 2)\]
we get
\[|\lambda_\pm|^2 - \lambda_0^2 = -3(j + j^{-1} + 1) < 0\]
Therefore, \(\lambda_0 > |\lambda_\pm|\). When \(N \gg 1\),
\[\ln Z \cong N \ln \lambda_0 = N \ln(1 + j + j^{-1}) = N \ln(1 + 2 \cosh \beta J)\]

(3)
\[\frac{1}{N} \frac{\partial \ln Z}{\partial J} \cong \frac{2 \sinh \beta J}{1 + 2 \cosh \beta J}\]
\((N\beta)^{-1} \partial \ln Z/\partial J\) represents the number of \((1, 0), (0, -1),\) and \((-1,1)\) pairs in excess of the number of \((1, -1), (0, 1),\) and \((-1,0)\) pairs.

11.14 Quenched local fields, \(N = 3\). We consider a 1D Ising model (linear) of 3 spins with a quenched local field in which the energy \(E\) of the system is given as
Let us assume that $h_1 = 0, h_2 = h, h_3 = 0$ with $h > 0, J > 0$.

(1) What is the partition function $Z$?

(2) Calculate $\langle \sigma_1 \sigma_2 + \sigma_2 \sigma_3 \rangle$.

(3) What is $\langle \sigma_2 \rangle$?

(4) What is $\langle \sigma_1 \rangle$?

(5) Calculate the average energy $\langle E \rangle$.

(6) What is the low-temperature limit of $\langle E \rangle$? What is the state of the system in the low-temperature limit?

(7) What is the high-temperature asymptote of $\langle E \rangle$?

\[
E = -\sum_{i=1}^{3} h_i \sigma_i - J \sum_{i=1}^{2} \sigma_i \sigma_{i+1}
\]

\[
Z = e^{\beta h + 2 \beta J} + 2e^{\beta h} + e^{-\beta h - 2 \beta J} + 2e^{-\beta h} + e^{\beta h - 2 \beta J} + e^{-\beta h + 2 \beta J} = 8 \cosh(\beta h) \cosh^2(\beta J)
\]

\[
\ln Z = \ln 8 + \ln(\cosh(\beta h)) + 2 \ln(\cosh(\beta J))
\]

\[
\left\langle \sum_{i=1}^{2} \sigma_i \sigma_{i+1} \right\rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial J} = 2 \tanh(\beta J)
\]
\[ \langle \sigma_2 \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial h} = \tanh(\beta h) \]

(4)
\[ Z \langle \sigma_1 \rangle = e^{\beta h + 2 \beta J} + e^{\beta h} + e^{-\beta h - 2 \beta J} + e^{-\beta h} - e^{\beta h - 2 \beta J} - e^{-\beta h - 2 \beta J} \]
\[ = 4 \sinh(\beta h) \sinh(2 \beta J) \]
\[ \langle \sigma_1 \rangle = \frac{4 \sinh(\beta h) \sinh(2 \beta J)}{8 \cosh(\beta h) \cosh^2(\beta J)} = \tanh(\beta h) \tanh(\beta J) \]

(5)
\[ \langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} = -h \tanh(\beta h) - 2J \tanh(\beta J) \]

(6) \( \beta h >> 1 \) and \( \beta J >> 1 \)
\[ \langle E \rangle \approx -h - 2J \]
All spins are up.

(7) \( \beta h << 1 \) and \( \beta J << 1 \)
\[ \langle E \rangle \approx -\beta(h^2 + 2J^2) \]

**11.15 Quenched random local field, short chain.** In Section 11.5, we considered a 1D Ising model of \( L \) spins in a quenched random field of \(+h\) and \(-h\) to obtain \( \langle \sigma_{rf} \rangle \) for \( \beta L^{1/2}h >> 1 \). The latter condition makes the transition of \( \tanh(\beta h_{\text{tot}}) \) take place in a range of \( h_{\text{tot}} \) much narrower compared with the distribution \( P(h_{\text{tot}}) \). This problem considers the other situation, namely, \( \beta L^{1/2}h << 1 \), i.e., the distribution of \( P(h_{\text{tot}}) \) is narrower compared with the transition in \( \tanh(\beta h_{\text{tot}}) \). \( P(h_{\text{tot}}) \) is distributed around \( h_{\text{tot}} = h_c \), where
\[ h_c \equiv 2hL(p - 1/2) \] (*)
(1) Expand \( \tanh(\beta h_{\text{tot}}) \) around \( h_{\text{tot}} = h_c \) into a Taylor series up to the second order of \( h_{\text{tot}} - h_c \) and calculate \( \langle \sigma \rangle_{\text{rf}} \).

(2) Draw a sketch for the plot of \( \langle \sigma \rangle_{\text{rf}} \) as a function of \( p \).

(3) Now combine the two situations, \( \beta L^{1/2} h \gg 1 \) and \( \beta L^{1/2} h \ll 1 \) to draw a sketch for the plot of \( \langle \sigma \rangle_{\text{rf}} \) as a function of \( \beta L^{1/2} h \). Assume that \( p > \frac{1}{2} \).

\[
\tanh(\beta h_{\text{tot}}) \approx \tanh(\beta h_c) + \text{sech}^2(\beta h_c) \beta (h_{\text{tot}} - h_c)
- \text{sech}^2(\beta h_c) \tanh(\beta h_c) \beta^2 (h_{\text{tot}} - h_c)^2
\]

Then,

\[
\langle \sigma \rangle_{\text{rf}} \approx \tanh(\beta h_c) \int_{-\infty}^{\infty} P(h_{\text{tot}}) dh_{\text{tot}} 
+ \text{sech}^2(\beta h_c) \int_{-\infty}^{\infty} P(h_{\text{tot}}) \beta (h_{\text{tot}} - h_c) dh_{\text{tot}}
- \text{sech}^2(\beta h_c) \tanh(\beta h_c) \int_{-\infty}^{\infty} P(h_{\text{tot}}) \beta^2 (h_{\text{tot}} - h_c)^2 dh_{\text{tot}}
\]

which is calculated as

\[
\langle \sigma \rangle_{\text{rf}} \approx \tanh(\beta h_c) \left[ 1 - \beta^2 h^2 L \text{sech}^2(\beta h_c) \right]
\]

With eq (*), it is rewritten to

\[
\langle \sigma \rangle_{\text{rf}} \approx \tanh(2\beta h L(p - 1/2)) \left[ 1 - \beta^2 h^2 L \text{sech}^2(2\beta h L(p - 1/2)) \right]
\]

Since \( \beta L^{1/2} h \ll 1 \), the above expression is further simplified to

\[
\langle \sigma \rangle_{\text{rf}} \approx \tanh(2\beta h L(p - 1/2))
\]
11.16 Quenched dilute local field, chain length. We learned that $\langle \sigma_d \rangle \equiv \tanh(\beta h r L)$ for a 1D Ising model with dilute quenched local fields. The $L$ is the length of a mono-domain, which is given as

$$\frac{1}{L} = \frac{1}{L_{th}} + \frac{1}{N}$$

When $N \ll L_{th}, L \equiv N$, and $\langle \sigma_d \rangle \equiv \tanh(\beta h r N)$. When $N \gg L_{th}, L \equiv L_{th}$, and $\langle \sigma_d \rangle \equiv \tanh(\beta h r L_{th})$. 

Draw a sketch for a plot of $\langle \sigma_d \rangle$ vs $N$. 

(3)
11.17 Quenched dilute local field, normal approximation. In Section 11.5, we used the discrete distribution (binomial distribution) to get an expression of $<\sigma_d>$. We can get a more exact expression using a continuous approximation of the distribution. When $L >> 1$ (but $Lr << 1$), the distribution of $h_{tot}$ is approximated as

$$P(h_{tot}) = (2\pi h^2 Lr)^{-1/2} \exp \left( -\frac{(h_{tot} - hLr)^2}{2h^2 Lr} \right)$$

since the mean and variance are $hLr$ and $h^2 Lr$, respectively. Calculate $<\sigma_d>$.

Since

$$\tanh(\beta h_{tot}) \approx \tanh(\beta hLr) + \text{sech}^2 (\beta hLr) \beta (h_{tot} - hLr)$$

$$-\text{sech}^2 (\beta hLr) \tanh(\beta hLr) \beta^2 (h_{tot} - hLr)^2$$

we get

$$<\sigma_d> = \int_{-\infty}^{\infty} \tanh(\beta h_{tot}) P(h_{tot}) dh_{tot}$$

$$\approx \tanh(\beta hLr) \int_{-\infty}^{\infty} P(h_{tot}) dh_{tot} + \text{sech}^2 (\beta hLr) \beta \int_{-\infty}^{\infty} (h_{tot} - hLr) P(h_{tot}) dh_{tot}$$

$$-\text{sech}^2 (\beta hLr) \tanh(\beta hLr) \beta^2 \int_{-\infty}^{\infty} (h_{tot} - hLr)^2 P(h_{tot}) dh_{tot}$$

$$= \tanh(\beta hLr)[1 - \text{sech}^2 (\beta hLr) \beta^2 h^2 Lr]$$

11.18 Adsorption. We consider adsorption of molecules onto a planar surface that has $N$ adsorption sites ($N >> 1$) arranged on the grid points of a square lattice. A site is specified by two integers, $i$ and $j$, for the two-dimensional lattice; $(i, j)$ is a coordinate, but takes only integral values. The total number of $(i, j)$ is $N$.

Each site can accommodate up to one molecule. The state $\sigma_{ij}$ of site $(i, j)$ is 1 when it has a molecule and 0 otherwise. When a site adsorbs a molecule of chemical potential $\mu$, the energy of the site is lowered by $\epsilon$ ($> 0$), and furthermore facilitates adsorption at the adjacent four sites. For this substrate, we can express the energy of the substrate as

$$E = -\epsilon \sum_{ij} \sigma_{ij} - \frac{J}{2} \sum_{ij} \sigma_{ij} \sum_{k,l=\pm 1} \sigma_{i+k,j+l}$$
where $J (> 0)$ represents the interaction between occupied states on adjacent sites, and division by 2 is to cancel counting the same pair for a second time.

(1) The mean-field approximation for the interaction replaces $\sigma_{i+k,j+l}$ in the interaction term by $<\sigma> \equiv <\sigma_{ij}>$. What is the grand partition function $Z$ under the mean-field approximation?

(2) Show that $<\sigma>$ satisfies the following in the mean-field approximation

$$
<\sigma> = \frac{p}{p + q e^{-\beta(\varepsilon + 2J<\sigma>)}}
$$

where $p$ is the pressure, and $q$ is related to $p$ and $\mu$ as $e^{\beta \mu} = p/q$.

(3) The above equation is a transcendental equation for $<\sigma>$, and cannot be solved in general. However, when the temperature is sufficiently low or high, we can find an explicit result.

(a) What is low-temperature asymptote of $<\sigma>$?

(b) What is required for $T$ to qualify as “sufficiently low”?

(c) What is high-temperature asymptote of $<\sigma>$?

(d) Draw a sketch for the plot of $<\sigma>$ as a function of $T$.

(4) Calculate $<\sigma_{ij}\sigma_{ij+1}>$, where $\sigma_{ij}\sigma_{ij+1}$ is the product of occupancy at neighboring sites. Show that $<\sigma_{ij}\sigma_{ij+1}> = <\sigma>^2$.

---

(1)

$$
E = - (\varepsilon + 2J<\sigma>) \sum_{ij} \sigma_{ij}
$$

$$
Z = \prod_{ij} \sum_{\sigma_{ij} = 0, 1} \exp \left( \beta \mu \sum_{ij} \sigma_{ij} \right) \exp \left[ \beta (\varepsilon + 2J<\sigma>) \sum_{ij} \sigma_{ij} \right]
$$

$$
= \prod_{ij} \left[ 1 + e^{\beta (\mu + \varepsilon + 2J<\sigma>)} \right]^N
$$
\[
\ln Z = N \ln(1 + e^{\beta(\mu + \varepsilon + 2J\langle \sigma \rangle)})
\]
\[
\langle \sigma \rangle = \frac{1}{N\beta} \frac{\partial}{\partial \varepsilon} \ln Z = \frac{1}{1 + e^{-\beta(\mu + \varepsilon + 2\langle \sigma \rangle)}} = \frac{1}{1 + (q/p)e^{-\beta(\varepsilon + 2J\langle \sigma \rangle)}}
\]

(3) (a) At low temperatures, \(e^{-\beta(\varepsilon + 2J\langle \sigma \rangle)} << 1\). Then \(\langle \sigma \rangle \approx 1\), and

\[
\langle \sigma \rangle \approx 1 - \frac{q}{p} e^{-\beta(\varepsilon + 2J\langle \sigma \rangle)} \approx 1 - \frac{q}{p} e^{-\beta(\varepsilon + 2J)}
\]

(b) \(T << (\varepsilon + 2J)/k_B\).

(c) At high temperatures, \(\beta(\varepsilon + 2J\langle \sigma \rangle) << 1\).

\[
\langle \sigma \rangle = \frac{p}{p + q[1 - \beta(\varepsilon + 2J\langle \sigma \rangle)]} \approx \frac{p}{p + q\left[1 - \beta\left(\varepsilon + 2J\frac{p}{p+q}\right)\right]}
\]

(d)

\[
\langle \sigma \rangle = \frac{1}{2N\beta} \frac{\partial}{\partial J} \ln Z = \frac{\langle \sigma \rangle}{1 + e^{-\beta(\mu + \varepsilon + 2J\langle \sigma \rangle)}} = \langle \sigma \rangle^2
\]

(7)