Supplemental Document 2 - Two-Asset P&L Distribution

Sections / Topics Covered

- Focus on how distributions combine
- Build P&L distribution for two assets
- Again, based around Figure 8.7 from chapter 8 (p. 209)
- Builds P&L distribution by variance-covariance (VCV) or delta-normal method, by combining security DV01 and Risk Factor distribution
- Focus on graphical explanation, also go over formula for normal distribution. Make sure the graphical approach fails gracefully with 3+ RFs
- Also use triangle addition

Parameters, Initialization, Functions

Estimating Portfolio P&L Distribution

In Document 1 we focused on the process of estimating the P&L distribution for a single security (using the linear, delta-normal, or VCV approach). The process for multiple securities follows the same general steps:

1. Asset / Risk Factor Mapping - Calculate transformation from individual assets to risk factors
2. Risk Factor Distributions - Estimate the range of possible levels and changes in market risk factors
3. Generate P&L Distribution - Generate risk factor P&L and sum to produce the portfolio P&L distribution
4. Calculate Risk Measures - Estimate the VaR, volatility, or other desired characteristics of the P&L distribution

In this document we will examine the process for two securities and specifically focus on step 3 and combining multiple risk factor P&L distributions into a single overall or portfolio distribution. For a single security we often have only a single risk factor and the issue of combining separate distributions into a portfolio distribution just does not arise. For multiple securities and in any practical situation we have to combine separate distributions into a single portfolio distribution.

Graphical Representation of Building P&L Distribution

We use the same graphical representation, based on Figure 8.7 of section 8.3, to outline how we build the portfolio P&L distribution. In this case we start with two securities, $20 million of a 10-year and €7 million of a CAC equity index futures. Figure 1 shows parametric (delta-normal) estimation of the portfolio P&L distribution.

Figure 1: Methodology for Estimating Characteristics of the P&L Distribution
1. ASSET / RISK FACTOR MAPPING

<table>
<thead>
<tr>
<th>SECURITY</th>
<th>10yr UST $20mn</th>
<th>CAC Index Futures €7mn</th>
</tr>
</thead>
</table>

Security Risk Factors

- USDYld10yr
- Delta: 18291.9
- CACEqIndex
- Delta: 18291.9

2. RISK FACTOR DIST’Ns

<table>
<thead>
<tr>
<th>Portfolio Risk Factors</th>
<th>USDYld10yr</th>
<th>CACEqIndex</th>
</tr>
</thead>
<tbody>
<tr>
<td>RF Volatility</td>
<td>7.148</td>
<td>0.0253651</td>
</tr>
</tbody>
</table>

3. GENERATE P&L DIST’Ns and COMBINE THEM

- Risk Factor Delta: 18291.9
- Pos’n Volatility: 130,800
- 9.1 x 10^6
- 230,800

4. CALCULATE RISK MEASURES

OVERAL PORTFOLIO 5%/95% VaR: 479,200
Volatility: 291,300

Combining Distributions

Most of the steps in Figure 1 are easy to represent or conceptualize graphically. The one step that is difficult to represent graphically, the step that is not clearly laid out in Figure 1, is step 3, combining the disparate distributions into a single overall portfolio P&L distribution. This is not easy to represent with a simple graphic, although the process itself is very simple.

To see why it is conceptually simple, think about any particular day. Table 1 shows the P&L for five particular days. For the first day the 10-year US yield goes up and so there is a negative P&L. The CAC index goes down so there is also a negative P&L. Adding the individual risk factor P&Ls together gives the total portfolio P&L (negative for this particular day since both the 10-year bond and the CAC futures lose money). This happens every day - the risk factor P&Ls add to give the portfolio P&L. This is very easy when we consider each day on its own - the P&L for each security or risk factor simply adds to that for the others to give the overall portfolio P&L. The overall P&L distribution is built up, day-by-day. When we estimate the P&L using Monte Carlo that is exactly what we do: simulate a P&L for each security for the first simulated trading day and then add the P&Ls. Do the same for a second simulated day and repeat for as many simulated trading days as we think necessary to get a good estimate of the portfolio P&L distribution.
<table>
<thead>
<tr>
<th>Day</th>
<th>10-yr US Yld</th>
<th>CAC Index</th>
<th>Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>yld change</td>
<td>P&amp;L</td>
<td>% change</td>
</tr>
<tr>
<td>Day 1</td>
<td>10.87</td>
<td>-198,900</td>
<td>-1.41</td>
</tr>
<tr>
<td>Day 2</td>
<td>3.76</td>
<td>-68,860</td>
<td>2.00</td>
</tr>
<tr>
<td>Day 3</td>
<td>1.92</td>
<td>-35,150</td>
<td>0.29</td>
</tr>
<tr>
<td>Day 4</td>
<td>-7.27</td>
<td>133,000</td>
<td>2.06</td>
</tr>
<tr>
<td>Day 5</td>
<td>-7.09</td>
<td>129,600</td>
<td>0.09</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

This process is simple to conceptualize when we think of each day and then look over multiple days. It is simple to implement when we use Monte Carlo. But it is not easy to conceptualize or to represent graphically when we think of P&L densities and distributions such as in Figure 2. These are the distributions for the 10-year Treasury and the CAC futures separately. From these diagrams alone we do not have any way of combining them into an overall portfolio P&L.

Figure 2: Separate (Marginal) P&L Distributions for 10-year Treasury Yields and CAC Index Percent Changes

To combine into an overall P&L we need to consider how the risk factors move together - we need to examine the joint distribution. The joint distribution is shown in Figure 3.

Figure 3: Joint P&L Distributions for 10-year US Treasury Yields and CAC Futures Changes
Figure 4 tries to explain this a little better. The upper left is a contour plot of the joint distribution - in other words the joint distribution viewed from above. The colors represent height of the joint density. The 3D plot of the joint density is shown in the upper right, reproduced from above.

The marginal density for the 10-year yield is shown below the contour plot, in two different graphical representations. The marginal density for the 10-year is the projection of the joint density onto the horizontal axis. The first graphic for the marginal density is just this - the 3D joint density but viewed from the perspective of the horizontal axis. The second graphic for the marginal density is more conventional graphic - a 2-dimensional line drawing of the density.

Turning to the mathematical definition, the marginal density is due to changes in the CAC index \textit{without regard to how 10-year yields change}. When we start with the joint density, the marginal density is obtained from the joint density by integrating out the 10-year yield:

\[
CAC \text{ index density } = P[CAC \& L = x] = \int [P[CAC \& L = x \& 10 \text{ yr } P \& L = y] \ dy]
\]  

(1)

The CAC marginal density is calculated from the joint density by integrating along the verticals of the joint density, represented by the dotted lines in the contour plot. The CAC marginal density is the projection of the joint density onto the horizontal, in other words along the vertical dotted lines. Similarly the 10-year marginal plot would be calculated by integrating along horizontal lines or projecting to the vertical (not shown in Figure 4).

\textbf{Figure 4: Joint Density for 10-year US Treasury Yields and CAC Index, and Marginal Density for CAC Index}
Now let us consider the density of the portfolio P&L, which is the sum of the 10-year and CAC P&Ls. If the P&L due to the 10-year is $y$ and due to the CAC Index is $x$ then the sum is $(x+y)$:

\[ y = \text{P&L due to 10 yr} \quad x = \text{P&L due to CAC Index} \quad \text{Overall P&L} = t = x + y \]

The probability density for the overall P&L is obtained by integrating the joint density under the condition that $x+y=t$:

\[
\text{Overall density} = P[\text{Overall P&L} = t] = \int P[\text{CAC P&L} = x \& 10 \text{ yr P&L} = t - x] \, dx
\]

This looks exactly like Equation 1 except that now we are integrating along the line.
The joint density that is highly elongated more-or-less along the diagonal, as in Figure 5, the overall portfolio dispersion will be relatively high. If, instead, the distribution is not elongated, or is aligned along the anti-diagonal, then the portfolio dispersion will be much reduced. The correlation is the number that controls the elongation and alignment.
Figure 6 is an interactive version of Figure 5, where the user can alter the correlation between -0.9 and +0.9. The historical correlation is close to +0.2 (it is 0.24). Changing the correlation to +0.9 aligns the joint distribution more-or-less along the diagonal and produces a high portfolio volatility (roughly $353,000 using the original volatilities of $130,751 and $230,822). Changing the correlation to -0.9 aligns the joint distribution more-or-less along the anti-diagonal and produces a low portfolio volatility (roughly $126,700 using the original volatilities of $130,800 and $230,800). A correlation of 0.0 produces a more-nearly circular joint density and a portfolio volatility between the extremes produced by correlation -0.9 and +0.9.

Figure 6: Joint and Marginal P&L Densities for 10-year US Treasury Yields and CAC Index Changes, Together with Density for Sum

| vol10yr | 130,751. |
| volCAC  | 230,822. |
| correlation | 0.24 |

10yr UST  CAC Index Futures  
130,800  230,800

(portfolio: 130,800 + 230,800 = 291,600)
This way of thinking about how marginals combine to the portfolio density - as projecting onto the diagonal - is perfectly general. The figures illustrate the idea with normal distributions but the idea is the same for any type of distribution.

When we work with normal densities, however, the mathematics is much simplified. A jointly-normal density is fully characterized by the correlation coefficient (and the marginal volatilities). The separate volatilities combine according to the formula:

\[ \sigma_p^2 = \sigma_x^2 + 2 \rho \sigma_x \sigma_y + \sigma_y^2 \]

\[ \sigma_p = \sqrt{\sigma_x^2 + 2 \rho \sigma_x \sigma_y + \sigma_y^2} \]

(3)

In section 10.1 we discussed triangle addition. Volatilities combine as the sides of a triangle:

\[ \sigma_p^2 = \sigma_x^2 + 2 \rho \sigma_x \sigma_y + \sigma_y^2 \iff A^2 = B^2 - 2 \cos \theta \times B \times C + C^2 \]

for

\[ \cos \theta = -\rho \]

This triangle addition is included in the bottom of the interactive Figure 6