1

Circuit Theory for Power Transfer Networks

1.1 Introduction

In circuit theory, a power transfer network is known as a lossless two-port which matches a given voltage generator with internal impedance Z_G to a load Z_L. The lossless two-port consists of lossless circuit elements such as capacitors, inductors, coupled coils, transmission lines and transformers.

In practice, the complex impedances Z_G and Z_L are measured and modeled using idealized lossy and reactive circuit elements. In circuit theory, losses are associated with resistors. Reactive elements can be considered as capacitors, inductors, transmission lines or a combination of these.

It is well known that passive or lossy impedances consume energy. This is also known as power dissipation (i.e. energy consumption per unit time).

For given design specifications, such as the frequency band of operations and a desirable minimum flat gain level, the design problem of a power transfer network involves fundamental concepts of circuit theory. On the other hand, the fundamentals of circuit theory stem from electromagnetic fields. Especially at high frequencies, where the size of the circuit components is comparable to the wavelength of operational signals, use of electromagnetic field theory becomes inevitable for assessing the performance of the circuits. Therefore, at high frequencies, circuit design procedures must include electromagnetic field-dependent behavior of circuit components to produce actual reliable electrical performance.

In designing power transfer networks, we usually deal with mathematical functions employed in classical circuit theory.[1] These functions are determined directly from the given design specifications by means of optimization. Eventually, they are synthesized at the component level, yielding the desired power transfer network. Therefore, a formal understanding of circuit functions and their electromagnetic field assessments is essential for dealing with design problems.

[1] Circuit functions may be described as positive real driving point impedance or admittance functions or corresponding bounded real input reflection coefficients. The mathematical properties of these functions will be given in the following chapters.

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As mentioned above, power transfer networks are designed as lossless two-ports which may include only reactive lumped elements,[2] or only distributed elements, or a combination of both; that is, lumped and distributed elements. Usually, distributed elements are considered as ideal transmission lines.[3]

In Figure 1.1, a conceptual power transfer network is shown. The input port may be driven by an amplifier which is modeled as a Thévenin voltage source with complex internal impedance Z_G. The output port may be terminated by an antenna which is considered as a complex passive impedance Z_L.

At this point, it may be appropriate to give the formal definitions of ideal circuit components so that we can build some concrete properties of network functions.

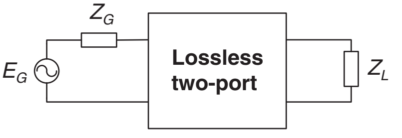

Figure 1.1 Conceptual power transfer network

1.2 Ideal Circuit Elements

In classical circuit theory, circuit elements may be described in terms of their terminal or port-related quantities such as voltage and current or incident and reflected wave relations.

In essence, descriptive port quantities are related to power delivered to that port. Referring to Figure 1.2, multiplication of port voltage $v(t)$ by port current $i(t)$ yields the power delivered to that port at any time t.

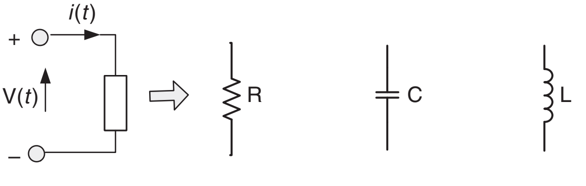

Figure 1.2 Ideal one-port circuit components

For a dissipative or lossy one-port the delivered power

$$\boxed{P(t) = v(t).i(t)} \tag{1.1}$$

[2] Reactive elements are also known as lossless circuit components such as capacitors and inductors.

[3] An ideal transmission line is lossless and propagates uniform transverse electromagnetic waves. These waves are called uniform plane waves.

must be positive. Consequently, the total energy consumed by that 'one-port' is given as the integral of the delivered power such that

$$\boxed{W = \int_{-\infty}^{+\infty} P(t)dt = \int_{-\infty}^{+\infty} v(t).i(t)dt < 0} \tag{1.2}$$

Specifically, for a lossless one-port, $W = 0$ since there is no power consumption on it.

Now let us elaborate the concept of power by means of the following examples.

Example 1.1: Let the applied voltage to a port be $v(t) = 3$ volts (or V) (DC) and the corresponding current response be $i(t) = 1$ ampere (or A) (DC) over the entire time domain. Find the power dissipation of the one-port under consideration.

Solution: Power delivered to the port is given by Equation (1.1). Thus, $P(t) = v(t)i(t) = 3\,\text{V} \times 1\,\text{A} = 3$ watts (or W).

Example 1.2: Let the applied voltage to a port be $v(t) = 3\sin(2\pi \times 50t)$ volts (50 Hz AC) and the corresponding current response be $i(t) = 1\sin(2\pi \times 50t)$ amps (50 Hz AC) over the time domain $t \geq 0$. Find the power dissipation of the one-port at time $t = 5$ milliseconds.

Solution: Instantaneous power dissipation at any time $t \geq 0$ is given by

$$\begin{aligned} P(t) = v(t) \times i(t) &= 3\sin(2\pi \times 50t)\text{volts} \times 1\sin(2\pi \times 50t)\text{amps} \\ &= 3\sin^2(2\pi \times 50t)\text{watts} \end{aligned}$$

Hence, $P(t = 50\,\text{ms}) = 3\,\text{W}$.

Note that, in this problem, the 'voltage and current' pair is sinusoidal with a frequency of $f = 50$ Hz; or equivalently with the time period of $T = {}^1\!/_{50} = 20$ ms. In practice, however, we are interested in average power dissipation over a period. Now let us define the average power dissipation as follows.

1.3 Average Power Dissipation and Effective Voltage and Current

For a one-port, let the port voltage and current pair be specified as

$$\boxed{\begin{aligned} v(t) &= V_m\sin(\omega_0 t - \varphi_v) \\ i(t) &= I_m\sin(\omega_0 t - \varphi_i) \end{aligned}} \tag{1.3}$$

where

$$\boxed{\omega_0 = 2\pi f_0 = \frac{2\pi}{T}} \tag{1.4}$$

is the angular frequency with frequency f_0 and the period

$$\boxed{T = \frac{1}{f_0}} \tag{1.5}$$

Then, for a periodic voltage and current pair, the average power dissipation over a period T is defined as

$$\boxed{\begin{aligned} P_{av} &= \frac{1}{T}\int_0^T v(t)i(t)dt \\ &= \frac{V_m I_m}{T}\int_0^T \sin\left(\frac{2\pi}{\mathrm{T}}t - \varphi_v\right) . \sin\left(\frac{2\pi}{\mathrm{T}}t - \varphi_i\right)dt \end{aligned}} \tag{1.6}$$

Note that

$$\sin(\alpha) . \sin(\beta) = \frac{1}{2}\cos(\alpha - \beta) - \frac{1}{2}\cos(\alpha + \beta)$$

Furthermore,

$$\cos(\boldsymbol{A}) = \cos(-\mathrm{A})$$

In the above trigonometric equalities, by replacing α by $(2\pi/\mathrm{T})t - \phi_v$ and β by $(2\pi/\mathrm{T})t - \phi_i$, one obtains

$$\boxed{P_{av} = \frac{V_m I_m}{2T}\cos(\varphi_i - \varphi_v)\int_0^T dt - \frac{V_m I_m}{2T}\int_0^T \cos\left(\frac{4\pi}{\mathrm{T}}t - \varphi_v - \varphi_i\right)dt} \tag{1.7}$$

Note that the second integral is zero since the area under the cosine function is zero over a full period T. Hence, we have

$$\boxed{\begin{aligned} &P_{av} = \frac{1}{2}V_m I_m \cos(\varphi_v - \varphi_i) \\ &\text{or} \\ &P_{av} = \left[\frac{V_m}{\sqrt{2}}\right]\left[\frac{I_m}{\sqrt{2}}\right]\cos(\varphi_v - \varphi_i) \end{aligned}} \tag{1.8}$$

In the above form, the quantities

$$\boxed{\begin{aligned} &V_{eff} = \frac{V_m}{\sqrt{2}} \\ &\text{and} \\ &I_{eff} = \frac{I_m}{\sqrt{2}} \end{aligned}} \tag{1.9}$$

are called the effective values of the peak voltage V_m and the peak current I_m respectively.

1.4 Definitions of Voltage and Current Phasors

In the classical circuit theory literature, complex quantities can be expressed in terms of the Euler formula. For example,

$$\boxed{e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)} \tag{1.10}$$

Furthermore, sinusoidal time domain signals can be expressed using the Euler formula such that

$$\boxed{v(t) = V_m\cos(\omega t - \varphi_v) = \text{real}\left\{e^{j\omega t}\left[V_m e^{-j\varphi_v}\right]\right\}} \tag{1.11}$$

In Equation (1.11) the quantity

$$\boxed{\mathbf{V} \triangleq \left[V_m e^{-j\varphi_v}\right]} \tag{1.12}$$

is called the voltage phasor. Similarly, the current phasor is defined as

$$\boxed{\begin{gathered}\boldsymbol{I} = [I_m e^{-j\varphi i}] \\ \text{In terms of the current phasor, the actual current is given by} \\ i(t) = \text{real}\{\boldsymbol{I}e^{j\omega t}\} = I_m\cos(\omega t - \varphi_i)\end{gathered}} \tag{1.13}$$

By means of voltage and current phasors, average power can be expressed as

$$P_{av} = \text{real}\{\boldsymbol{V}\boldsymbol{I}^*\} = \text{real}\{\boldsymbol{V}^*\boldsymbol{I}\} = \frac{1}{2}V_m I_m\cos(\varphi_v - \varphi_i) = V_{eff}I_{eff}\cos(\varphi_v - \varphi_i)$$

Example 1.3: Let $v(t) = 10\cos(\omega t - 10°)$ and $i(t) = 20\cos(\omega t - 40°)$.

(a) Find the voltage and current phasors.
(b) Find the average power dissipated over a period T.

Solution:
(a) By definition, voltage phasor is $\boldsymbol{V} = 10.e^{j10°}$. Similarly, the current phasor is given by $I = 20.e^{j40°}$.
(b) The average power is $P_{av} = \frac{1}{2}10 \times 20.\cos(10° - 40°) = 100.\cos(30°) = 86.6\,\text{W}$.

Example 1.4: Let the voltage phasor be $\boldsymbol{V} = 1.e^{j60°}$. Find the steady state time domain form of the voltage at $\omega = 10\,\text{rad/s}$.

Solution: By formal definition of phasor within this book, we can write $v(t) = \text{real}\left\{\boldsymbol{V}e^{j10t}\right\} = \cos(10t - 60°)$.

For the sake of completeness, it should be noted that the steady state voltage $v(t)$ may also be defined as the imaginary part of $\{\boldsymbol{V}e^{j\omega t}\}$ if the input drive is $v_{in}(t) = \sin(\omega t)$.

In general, usage of phasor notation facilitates the sinusoidal steady state analysis of a circuit in the time domain. In principle, network equations (more specifically, equations originating from Kirchhoff's voltage and current laws) are written using voltage and current phasors. Eventually, time domain expressions can easily be obtained by Equation (1.11), like mappings.[4]

[4] Here, what we mean is that any steady state time domain expression of a phasor $\dot{\boldsymbol{A}} = \dot{A}_m e^{J\phi_A}$ can be obtained as $A(t) = \text{real}\left\{\dot{\boldsymbol{A}}e^{j\omega t}\right\}$. In this representation $A(t)$ may designate any node or mesh voltage and current in a network.

1.5 Definitions of Active, Passive and Lossless One-ports

Referring to Figure 1.2, let $v(t)$ and $i(t)$ be the voltage and current pair with designated polarity and direction of an ideal circuit component. We assume that these quantities are given as a function of time t. Based on the given polarity and directions:

- A one-port is called passive if $W = \int_{-\infty}^{+\infty} P(t)dt = v(t).i(t) < 0$.
- A one port is called lossless if $W = \int_{-\infty}^{+\infty} P(t)dt = v(t).i(t) = 0$.
- On the other hand, if $W = \int_{-\infty}^{+\infty} P(t)dt = v(t).i(t) > 0$, then the one-port is called active. Obviously, a conventional voltage or current source is an active one-port.

In the following section, we will present elementary definitions of passive and lossless circuit components based on their port voltages and currents.

An ideal circuit component may be a resistor R, a capacitor C or an inductor L. Formal circuit theory definitions of these components are given next.

1.6 Definition of Resistor

A resistor R is a lumped one-port circuit element which is defined by means of Ohm's law:[5]

$$\boxed{\begin{array}{l} v_R(t) \triangleq R i_R(t) \\ \text{or} \\ i_R = \dfrac{v_R}{R} \triangleq G v_R, \; G = \dfrac{1}{R} \end{array}} \tag{1.14}$$

where R is called the resistance and it is measured by means of the ratio of port voltage to port current. The symbol '$\triangleq$' refers to equality by definition.

The units of voltage $v(t)$ and current $i(t)$ are volt (V) and ampere (A) respectively. The unit of resistance R is given by V/A, which is called the ohm and designated by Ω. G is called conductance and it is measured in siemens or Ω^{-1}.[6] The power dissipated on a resistor is given by multiplication of its port voltage and current such that

$$\boxed{P_R(t) = v_R(t) i_R(t) = R i_R^2(t) = V_R^2(t)/R \geq 0} \tag{1.15}$$

Dissipated power is always non-negative.[7] Therefore, the value of resistance must always be non-negative (i.e.$R \geq 0$).[8]

[5] A one-port circuit element is placed between two nodes and described in terms of its port quantities such as voltage and current pairs. These nodes are referred to as terminals of the one-port.

[6] Ω is a Greek letter read as omega.

[7] That is, $P_R(t) \geq 0, \forall t$.

[8] Here, it should be noted that for a real physical system, time is measured as a real number; voltage and current in the time domain are measured as real numbers with respect to selected references. Therefore, energy and power quantities are also measured as real numbers which in turn yield a non-negative real resistance value for the port under consideration.

The unit of power is volt × ampere which is called the watt and designated by W; 1 watt describes 1 joule of energy (1 J) dissipated per second (s).[9]

1.7 Definition of Capacitor

In electromagnetic field theory, we talk about energy stored both in electric and magnetic fields which produce actual work when applied to a moving electric charge. With this understanding, electric energy is stored on a circuit element called a capacitor and is usually designated by the letter C. As an ideal lumped circuit element, a capacitor C is described in terms of its port voltage v_C and port current i_C as

$$\boxed{i_C(t) \triangleq C\frac{dv_C(t)}{dt}} \tag{1.16}$$

where C is the capacitance and its unit is the farad (F).[10]

Total electric energy stored in a capacitor C is given in terms of the time integral of the power flow $P_C(t) = v_c(t).i_c(t)$ by

$$\boxed{\begin{array}{l} W_C = \int_{-\infty}^{t} v_C(\tau)i_C(\tau)d\tau = C\int_{-\infty}^{t} v_C(\tau)dv_C = \frac{1}{2}Cv_C^2 \\ \text{provided that initially the capacitor is empty,} \\ \text{i.e. } v(-\infty) = 0. \end{array}} \tag{1.17}$$

Since the stored electric energy W_C must be non-negative (or positive), then capacitance C must always be non-negative (or positive) (i.e.$C \geq 0$). At this point we should mention that this is potential electric energy. It is not dissipation. In other words, it is not consumed by the capacitor; rather it is stored. However, it may generate work or, equivalently, it can be transformed into kinetic energy when it is applied to a moving charge.

In practice, a capacitor is charged with a constant voltage source E_G, say a simple battery which has a series internal resistance R_G. When the charging process is completed within T_C seconds, the capacitor is said to be full and passes no current (i.e. $i_c(T_C) = 0$). The voltage $v_C(T_C)$ across its plates becomes constant and is equal to E_G. In this case, the total stored electric energy is given by $W_C = \frac{1}{2}CE_G^2$. However, consumed energy will be zero since $i_C(T_C) = 0$. In this explanation, any transient process is ignored and the charging time period $T_C = 0^+$ seconds is assumed. This means that the capacitor is immediately charged having $v_C(t < 0^+) = E_G$ and $i_C(t < 0^+) = 0$, yielding no power dissipation (i.e. $P(t) = 0$) or equivalently total energy consumption $W = 0$ (Figure 1.3).

[9] That is, 1 W = 1 J/s.

[10] In this book all the units are given in the International Standard Unit (ISU) system. Basic units of ISU are the meter, kilogram and second (MKS). Therefore, ISU is also known as the MKS unit system. In MKS, voltage and current units are given as volt (V) and ampere (A).

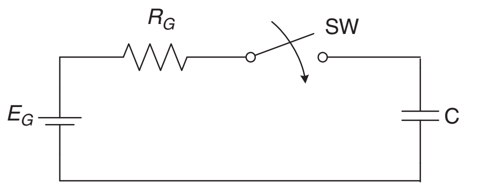

Figure 1.3 Electric energy storage element: capacitor (C)

1.8 Definition of Inductor

An inductor L is an ideal lumped circuit element. It stores magnetic energy. Its formal definition is given in terms of its port voltage $v_L(t)$ and port current $i_L(t)$ as

$$\boxed{v_L(t) \triangleq L\frac{di_L(t)}{dt}} \tag{1.18}$$

where L called inductance and its unit is is given by the henry (H).

Total magnetic energy W_L stored in an inductor L over an interval of time $(-\infty, t]$ is given by

$$\boxed{W_L = \int_{-\infty}^{t} v_L(\tau).i_L(\tau)d\tau = \int_{-\infty}^{t} Li_L(\tau)di_L = \frac{1}{2}Li_L^2} \tag{1.19}$$

Since the stored magnetic field energy must be non-negative (or positive), then inductance L must be non-negative (or positive) (i.e.)$L \geq 0$).[11]

In a similar manner to that of a capacitor, as an ideal lumped circuit element, an inductor L is lossless. This means that it does not dissipate power but rather holds magnetic energy over a specified period of time unless it is emptied. When an inductor is connected to an excitation, say to a constant current source I_G with an internal shunt resistance R_G, at time $t = 0$ seconds, a constant current $I_L = I_G = i_L(t < 0^+)$ immediately builds up over a very short period of time ending at $t = 0^+$ seconds. Then, this current circulates indefinitely within the circuit. Let the voltage drop on R_G be equal to $R_G I_G$ at time $t = 0^-$ seconds. Roughly speaking, when the inductor L is connected to the current source I_G, this voltage immediately appears on the inductor $v_L(t = 0) = R_G I_G$ and rapidly reduces to zero within $T_L = 0^+$ seconds while the inductor current i_L rises to the level of I_G, yielding zero power transfer. During this process, as i_L increases, current through the shunt resistance R_G goes to zero resulting in zero voltage across inductor L (see Figure 1.4).

We should emphasize that this is a macroscopic explanation. Details are skipped here.

Just to summarize the above discussions based on the definitions, as ideal circuit elements a capacitor or an inductor is a lossless one-port, and it can only store energy. On the other hand, a resistor is a lossy circuit element which dissipates or consumes energy by heating itself. In practice, however, there is no ideal circuit element; one can always associate a real dissipation perhaps in series with an ideal inductance L, say r_L, or in parallel with an ideal capacitance C which may be designated as conductance G_C as shown in Figure 1.5.

[11] It should be noted that in Equation (1.16) and (1.19), initially capacitor C and inductor L were assumed to be empty. Therefore, in these equations the integration constant is set to zero.

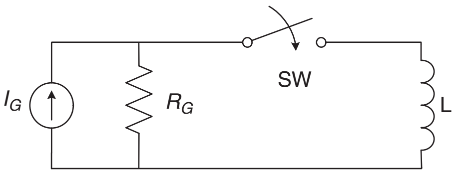

Figure 1.4 Magnetic energy storage element: inductor (L)

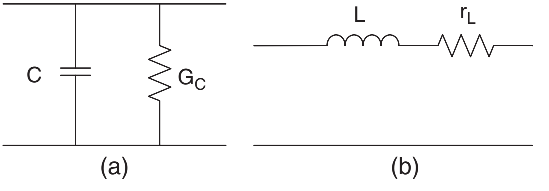

Figure 1.5 A practical capacitor and a practical inductor in shunt and series configurations respectively

Example 1.5: Let a resistive one-port with an input impedance of $10\,\Omega$ be derived by a voltage excitation of $v(t) = 10\cos(5t)$ V.

(a) Find the current through the resistive input.
(b) Find the average power dissipated on the resistive one-port.
(c) Find the energy consumed on the resistive one-port over a period.
(d) Find the total energy consumed on the one-port over a time interval of $[0, +\infty)$.

Solution:

(a) The current of the resistive input is given by $i(t) = v(t)/R = 1.\cos(5t)$A.
(b) $P_{av} = \frac{1}{2}V_m I_m = \frac{1}{2}10 \times 1 = 5$W.
(c) $T = 2\pi/\omega = 2\pi/5$ and $W_T = \int_0^T R.i^2(t)dt = 10\int_0^{2\pi/5}\cos^2(5t)dt$. Note that $\cos^2 5t = \frac{1}{2} + \cos(10t)$. Then $W_T = 10\left[\frac{1}{2}\frac{2\pi}{5} + \frac{1}{10}\sin\left(10.\frac{2\pi}{5}\right)\right] = \frac{10\pi}{5} = 2\pi \cong 6.28$ J
(d) $W_{[0,+\infty)} = \int_0^{+\infty} R\cos^2(5t)dt = \infty$J. This means that the resistive input consumes energy as time approaches to infinity.

Obviously, a resistor must consume energy as long as it is connected to a power supply. Therefore, one must be very careful to save power when an iron type of house appliance is operated and not let it stand if not in use.

Example 1.6: Let a one-port consist of a single capacitor of C = 10 F which is driven by an excitation of $v(t) = 1.\sin(5t)$ V.

(a) Find the current through the capacitor C.
(b) Find the average power dissipated on the capacitor over a period T.
(c) Find the maximum electric field energy stored on the capacitor.

Solution:

(a) By definition, the current of a capacitor is given by $i(t) = C\,dv/dt = 10 \times 5\cos(5t) = 50\cos(5t)$A.

(b) In this problem, the period T is specified as in the above example. Hence, $T = 2\pi/5$ The average power dissipated on the capacitor over a period is specified as

$$\begin{aligned} P_{av} &= \frac{1}{T}\int_0^T v(t)i(t)dt \\ &= \frac{50}{T}\int_0^T \sin(5t)\cos(5t)dt \\ &= \frac{50}{T}\frac{1}{2}\int_0^T \sin(10t)dt = \frac{25}{4\pi}[-\cos(4\pi)+\cos(0)] = \frac{25}{4\pi}[-1+1] \\ &= 0W^{12} \end{aligned}$$

as it should be, since a capacitor is a lossless lumped circuit element.[13] However, one should keep in mind that instantaneous power may not be zero. It depends on the instantaneous values of the applied voltage and current.

(c) Obviously, a capacitor charges and discharges as the applied voltage across it varies. Then, the maximum stored energy of the capacitor is found at the peak value of the applied voltage. Hence,

$$\begin{aligned} W_{\max} &= \max\left\{\frac{1}{2}Cv^2(t)\right\} = \frac{1}{2}C \times \max\{v^2(t)\} \\ &= \frac{1}{2}CV_m^2 = \frac{1}{2} \times 10 \times 1 = 5\,\mathrm{J} \end{aligned}$$

Example 1.7: Let a one-port consist of a single inductor of L = 1 H which is driven by an excitation of $i(t) = 1.\sin(5t)$ A.

(a) Find the voltage through the inductor capacitor L.
(b) Find the average power dissipated on the inductor over a period T.
(c) Find the maximum magnetic energy stored on the inductor.

Solution:

(a) By definition, the voltage of an inductor is given by $v(t) = L\,di/dt = 1 \times 5\cos(5t) = 5\cos(5t)$V.
(b) As in the previous problem, it is straightforward to show that

$$P_{av} = \frac{1}{T}\int_0^T v(t)\ i(t)dt = \frac{5}{T}\int_0^T \sin(5t)\ \cos(5t)\ dt = 0\ W$$

as it should be, since an inductor is a lossless lumped circuit element. However, one should keep in mind that instantaneous power may not be zero. It depends on the instantaneous values of voltage and current.

(c) Obviously, an inductor charges and discharges as the current through it varies. Then, the maximum stored magnetic energy is found at the peak value of the current. Hence,

$$W_{\max} = \max\left\{\frac{1}{2}\ Li^2(t)\right\} = \frac{1}{2}\ L\ \times\ \max\ \{i^2(t)\} = \frac{1}{2}LI_m^2 = \frac{1}{2}\ \times\ 1\ \times\ 1 = 0.5\,\mathrm{J}$$

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[12] Here, $T = 2\pi/\omega = 2\pi/5$.

[13] Note that $\sin(\alpha).\cos(\alpha) = \frac{1}{2}\sin(2\alpha)$ Furthermore,

$$\int_0^T \sin(10t)dt = -\frac{1}{10}\cos(10t)\big|_0^{2\pi/5} = \frac{1}{10} \times 0 = 0$$

1.9 Definition of an Ideal Transformer

A transformer is an ideal two-port circuit element which consists of two magnetically perfectly coupled coils as shown in Figure 1.6. The coil on the left is called the primary coil with n_1 turns and constitutes port 1 with port voltage v_1(t) and port current i_1(t). The coil on the right is called the secondary coil with n_2 turns and constitutes port 2 with port voltage $v_2(t)$ and port current $i_2(t)$ satisfying the following relations:

$$\boxed{\begin{array}{l} v_2(t) \triangleq \dfrac{n_2}{n_1} v_1(t) = n v_1(t) \\ \text{and} \\ \qquad i_2(t) \triangleq -\dfrac{1}{n} i_1(t) \end{array}} \tag{1.20}$$

where $n = n_2/n_1$ and it is called the turns ratio of the transformer.

From the above equations, it is clear that total power delivered to the transformer is given by

$$\boxed{P_T = v_1(t).\ i_1(t) + v_2(t).\ i_2(t) = 0} \tag{1.21}$$

This means that no power is dissipated on the transformer. Hence, by definition, a transformer is a lossless two-port. The above equations are only valid for time-varying voltage and current. Otherwise, no transformation can occur.[14]

Referring to Figure 1.6, let the impedance seen from port 2 be Z_2. Let port 1 be terminated in impedance Z_1.

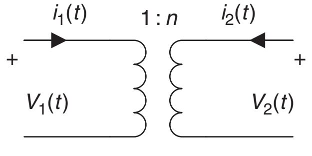

Figure 1.6 An ideal transformer

Then, Equation (1.20) reveals that

$$\boxed{Z_2 = \frac{v_2}{i_2} = -n^2 \frac{v_1}{i_1} = n^2 \left[\frac{-v_1}{i_1} \right] = n^2 Z_1} \tag{1.22}$$

where $Z_1 = v_{Z1}/i_{Z1} = -v_1/i_1$. Thus, we see that an ideal transformer also transforms the terminating impedances[15] in a flat manner[16] over the entire frequency band.

[14] That is, transformation neither from the primary to the secondary nor from the secondary to the primary coils can occur.

[15] Loosely speaking, the voltage to current ratio is defined as the impedance and its inverse is called the admittance. However, their formal definitions will be given in the following sections.

[16] Here, we use the term 'flat' to indicate the entire frequency band $(-\infty < \omega < \infty)$.

1.10 Coupled Coils

Referring to Figure 1.7, inductors L_1 and L_2 are said to constitute a coupled structure with a coupling coefficient M if the following equations are satisfied:

$$\boxed{\begin{aligned} v_1 &\triangleq L\frac{di_1}{dt} + M\frac{di_2}{dt} \\ v_2 &\triangleq M\frac{di_1}{dt} + L_2\frac{di_2}{dt} \end{aligned}} \tag{1.23}$$

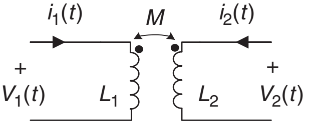

Figure 1.7 Coupled coils

Here, we should note that the position of the dots in Figure 1.7 defines the sign of the coupling coefficient M. This coefficient is positive if the dots are on the same side (i.e. they are either at the top or at the bottom) and negative if the dots are on opposite sites (i.e. one of the dots is at the top and the other is at the bottom).[17]

1.11 Definitions: Laplace and Fourier Transformations of a Time Domain Function *f*(*t*)

Two-sided or bilateral Laplace transformation of a continuous function $f(t)$ over the entire time domain is defined as

$$\boxed{F(p) \triangleq \mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty} f(t).e^{-pt}dt} \tag{1.24}$$

where $p = \sigma + j\omega$ is a complex variable and known as the complex frequency.

We may define a similar transformation for complex variable $p = j\omega$ by setting $\sigma = 0$. In this case, Equation (1.24) can be written as

$$\boxed{F(p) \triangleq \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt} \tag{1.25}$$

[17] It should be noted that one may also define coupled capacitors in a similar manner such that $i_1(t) = C_1\,dv_1/dt + M_C\,dv_2/dt$ and $i_2(t) = M_C\,dv_1/dt + C_2\,dv_2/dt$ where C_1 and C_2 are coupled capacitors and M_c is the capacitive coupling coefficient. However, for most practical applications M_c becomes negligibly small and therefore is set to zero.

Equation (1.25) is called the Fourier transformation of a continuous time domain function $f(t)$.[18] In this equation, ω is called the real angular frequency or, in short, the real frequency, and it is given by

$$\boxed{\omega = 2\pi f} \tag{1.26}$$

where f is called the frequency or the actual frequency.[19]

In everyday problems, we usually deal with real time functions (or signals) which are bounded and, perhaps, we know 'how and when' they are initiated. The initiation point of these functions can be fixed at $t = 0$ seconds. In this case, we can say that these functions do not exist before they are initiated and therefore they are set to zero before their initiation time. Hence, mathematically, we can state that

$$\boxed{f(t) = \begin{cases} 0 \text{ for } -\infty < t < 0 \\ \neq 0 \text{ and bounded for } 0 \leq t < +\infty \end{cases}} \tag{1.27}$$

Those functions which satisfy Equation (1.27) are called causal functions.

Strictly speaking, for the existence of Laplace and Fourier transforms, $f(t)$ must be a member of the class of functions which are integrable. For example, a class of functions is called L^1 if $f(t)$, as a member of the class, satisfies the inequality of $\int_0^{+\infty} |f(t)|\, dt < M < \infty$. $f(t)$ is called L^2 if it belongs to a class of functions which satisfies the inequality $\int_0^{+\infty} |f(t)|^2 dt < M < \infty$.

In this context, the causal functions that we deal with must belong either to L^1 or L^2. Therefore, their Laplace and Fourier transform integrals start from $t = 0$ seconds. In practice, however, we employ L^2 functions. Thus, for a bounded causal (perhaps L^2) real time domain signal $f(t)$, the 'one-sided Laplace transform'

$$\boxed{F(p) = \int_0^{+\infty} f(t)e^{-pt} dt = \int_0^{+\infty} [f(t)e^{-\sigma t}]\, e^{-j\omega t} dt} \tag{1.28}$$

exists if and only if the integrand $f(t)e^{-\sigma t}$ is bounded, i.e. $|f(t)e^{-\sigma t}| < \infty$.

This is possible if σ is non-negative. Therefore, for a bounded causal signal, the 'one-sided Laplace transformation' exists for the values of $\sigma \geq 0$. So, roughly speaking, we say that $F(p)$ is analytic (or differentiable everywhere) in the right half plane of $p = \sigma + j\omega$ (or simply in the RHP).[20]

By setting $\sigma = 0$, the Laplace transform of Equation (1.27) becomes the one-sided Fourier transform and for bounded causal signals it is given by

$$\boxed{F(j\omega) = \int_0^{+\infty} f(t)e^{-j\omega t} dt} \tag{1.29}$$

[18] In circuit theory, $f(t)$ may refer to a time domain signal such as voltage or current. Then, $F(j\omega)$ is called the Fourier transform of a time domain signal $f(t)$.

[19] In the classical literature of power transfer networks or broadband matching theory, for the sake of simplicity, angular frequency ω and the actual frequency f are both referred to as the *real frequency* since they carry the same information within a constant 2π.

[20] A rigorous discussion of Fourier and Laplace transforms of causal signals can be found in Chapter 3 of the book '*Wideband Circuit Design*' by H. J. Carlin and P. C. Civalleri, CRC Press, 1998.

For bounded causal signals, if the Fourier transform $F(j\omega)$ is known, the original time domain signal $f(t)$ can be uniquely determined from $F(j\omega)$ by means of the inverse Fourier transform as

$$\boxed{f(t) = \mathcal{F}^{-1}\{F(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}d\omega} \tag{1.30}$$

Or, replacing $j\omega$ by p, the inverse Laplace transform is given by

$$\boxed{f(t) = \mathcal{L}^{-1}\{F(p)\} = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} F(p)e^{pt}dp} \tag{1.31}$$

Fourier and Laplace transformations are handy tools for solving integral and differential equations, as demonstrated below.

In Equation (1.31), using the 'integration by parts' rule, it is straightforward to show that

$$\boxed{\begin{array}{l} \mathcal{L}\left\{\dfrac{df}{dt}\right\} = pF(p) \\ \text{and} \\ \mathcal{L}\left\{\displaystyle\int f(t)dt\right\} = \dfrac{1}{p}F(p) \end{array}} \tag{1.32}$$

1.12 Useful Mathematical Properties of Laplace and Fourier Transforms of a Causal Function[21]

In the following subsection useful properties of the Laplace and Fourier transforms are introduced from the design perspective of power transfer networks.

Even and Odd Parts

Assume that complex function $F(p)$ is defined as the one-sided Laplace transform of a causal, continuous, bounded (CCB) function. As was discussed earlier, $F(p)$ exists for all values of $t \geq 0$ if and only if $\text{real}\{p\} = \sigma \geq 0$. This condition immediately yields that $F(p)$ must be analytic in the closed RHP. Let us comment on this statement for some practical situations.

For example, let $v(t)$ and $i(t)$ be the CCB voltage and current response of a one-port which consists of various combinations of connections of R, L and C elements in a sequential manner.[22] In this case, it can be shown that Laplace transforms $V(p) = \mathcal{L}\{v(t)\}$ and $I(p) = \mathcal{L}\{i(t)\}$ are rational functions. Obviously, rational functions should have their singularities at the poles. Therefore, $V(p)$ and $I(p)$ must be free of closed RHP poles.

Generically speaking, a complex variable function $F(p)$ which is free of open RHP zeros and closed RHP poles is called a minimum function.[23]

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[21] Formal discussions of the subject can be found in Chapter 3 of the book '*Wideband Circuit Design*' by H. J. Carlin and P. P. Civalleri, CRC Press, 1998, and also in Chapters 1–4 of the book '*Signal Processing and Linear Systems*' by B. P. Lathi, Cambridge University Press, 1998.

[22] Such as a cascaded connection of parallel and series circuit elements.

[23] Open RHP excludes the entire $j\omega$ axis whereas closed RHP includes the $j\omega$ axis. It should be noted that the origin $(\omega = 0)$ is also included on the $j\omega$ axis.

The Fourier transform of a minimum function can be uniquely determined from its Laplace transform $F(p)$ by setting $\sigma = 0$ in $p = \sigma + j\omega$ or equivalently by setting $p = \sigma + j\omega$. Let $F(p)$ be a rational function. Then,

$$\boxed{F_e(p^2) = \frac{1}{2}[F(p) + F(-p)]} \tag{1.33}$$

and

$$\boxed{F_o(p) = \frac{1}{2}[F(p) - F(-p)]} \tag{1.34}$$

represent its even and the odd parts respectively. In this case, Fourier transform $F(j\omega)$ can be written as

$$\boxed{F(j\omega) = A(\omega^2) + jB(\omega)} \tag{1.35}$$

where $A(\omega^2) = F_e(p^2 = -\omega^2)$ is an even function in ω^2 and $jB(\omega) = F_0(j\omega)$ or $B(\omega) = -jF_0(j\omega)$ is an odd function of ω.

Similarly,

$$\boxed{F(j\omega) = \rho(\omega)\, e^{j\varphi(\omega)}} \tag{1.36}$$

where $\rho^2(\omega) = A^2 + B^2$ and $\phi(\omega) = \tan^{-1}(B/A)$ are even and odd functions of ω respectively.

Example 1.8: Let $F(p) = 1/(1+p)$.

(a) Find the even part of $F(p)$.
(b) Find the odd part of $F(p)$.
(c) Find the frequency representation of even and odd parts.

Solution:

(a) By Equation(1.33),

$$F_e(p) = \frac{1}{2}[F(p) + F(-p)] = \frac{1}{2}\left[\frac{1}{1+p} + \frac{1}{1-p}\right] = \frac{1}{1-p^2}$$

(b) By Equation (1.34),

$$F_o(p) = \frac{1}{2}\left[\frac{1}{1+p} - \frac{1}{1-p}\right] = -\frac{p}{1-p^2}$$

(c) Replacing p by $j\omega$, we have

$$F_e(j\omega) = A(\omega) = \frac{1}{1+\omega^2}$$

$$F_o(j\omega) = -j\,\frac{\omega}{1+\omega^2}$$

which yields

$$B(\omega) = -\frac{\omega}{1+\omega^2}$$

Example 1.9: Let

$$F(p) = \frac{2p^2 + 2p + 1}{1 + p}$$

(a) Find the even part of $F(p)$.
(b) Find the odd part of $F(p)$.
(c) Find the frequency domain representation of the even part.
(d) Find the frequency domain representation of the odd parts.

Solution:

(a) By Equation (1.33),

$$F_e(p) = \frac{1}{2}[F(p) + F(-p)] = \frac{1}{2}\left[\frac{2p^2 + 2p + 1}{1 + p} + \frac{2p^2 - p + 1}{1 - p}\right] = \frac{1}{1 - p^2}$$

(b) By Equation (1.34),

$$F_o(p) = \frac{1}{2}[F(p) - F(-p)] = \frac{1}{2}\left[\frac{2p^2 + 2p + 1}{1 + p} - \frac{2p^2 - p + 1}{1 - p}\right] = \frac{-2p^3 + p}{1 - p^2}$$

(c) In this case, $F_e(j\omega) = A(\omega) = \frac{1}{\omega^2 + 1}$
(d) Here,

$$F_o(j\omega) = \frac{j\omega^3 + j\omega}{\omega^2 + 1} = j\frac{\omega^3 + \omega}{\omega^2 + 1} = jB(\omega)$$

which yields[24]

$$B(\omega) = \frac{2\omega^3 + \omega}{\omega^2 + 1} = \frac{\omega}{\omega^2 + 1} + \frac{2\omega^3}{\omega^2 + 1}$$

It is interesting to note that in this example:

- The real part $A(\omega)$ of $F(j\omega)$ is the same as the one determined in Example 1.8 even though $F(j\omega)$ is different.
- On the other hand, the odd term $B(\omega)$ differs from that of Example 1.8 by a single additive term of $2\omega^3/(\omega^2 + 1)$.
- The above points can easily be verified by writing

$$F(p) = \frac{2p^2 + 2p + 1}{p + 1} = 2p + \frac{1}{p + 1}$$

  which only differs from that of $F(p)$ of Example 1.8 by a *simple* term $2p$.
- It is straightforward to show that $F(p)$ of Example 1.8 is a minimum function whereas $F(p)$ of Example 1.9 is not, since it has a pole on the $j\omega$ axis at $\omega = \infty$ (or $p = \infty$).

Analytic Continuation of F(jω)

Let $f(t)$ be a 'CCB' real-valued function in real-time variable $t \geq 0$. Let $F(j\omega)$ be its Fourier transform. Then, its one-sided Laplace transform $F(p)$ can be uniquely determined from $F(j\omega)$, replacing $j\omega$ by

[24] Note that $(j\omega)^3 = -j\omega^3$. Therefore, $-p^3$ yields $+j\omega^3$. Then, the above results follow.

$p = \sigma + j\omega$ for all $\sigma \geq 0$ since $F(p)$ is analytic in the closed RHP. This process is called the analytic continuation of $F(j\omega)$ in the closed RHP.

Hilbert Transformation

Let $F(p) = N(p)/D(p)$ be a minimum rational function which is free of closed RHP poles as defined above. Let $F(j\omega) = A(\omega) + jB(\omega)$ be its Fourier transform. Then, $F(j\omega)$ can uniquely be determined from the given real part $A(\omega)$ by

$$B(\omega) = A_\infty + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{A(\Omega)}{\omega - \Omega} d\Omega$$

where the constant term A_∞ is the value of $A(\omega)$ at infinity.

Since $A(\omega)$ is an even function,

$$B(\omega) = A_\infty + \frac{2\omega}{\pi} \int_{0}^{+\infty} \frac{A(\Omega)}{\Omega^2 - \omega^2} d\Omega$$

Using integration by parts, it is straightforward to show that

$$B(\omega) = A_\infty + \frac{1}{\pi} \int_{0}^{+\infty} \left[\frac{dA(\Omega)}{d\Omega}\right] \ln\left|\frac{\Omega + \omega}{\Omega - \omega}\right| d\Omega \tag{1.37}$$

or in compact form

$$F(p) = A_\infty + \frac{2p}{\pi} \int_{0}^{+\infty} \frac{A(\Omega)}{p^2 + \Omega^2} d\Omega$$

Similarly, $A(\omega)$ is given in terms of $B(\omega)$ as

$$A(\omega) = A_\infty - \frac{2\omega}{\pi} \int_{0}^{+\infty} \frac{B(\Omega)}{\Omega^2 - \omega^2} d\Omega$$

Equation (1.37) is known as the Hilbert transformation relations.[25]

The above $\{A(\omega)$ and $B(\omega)\}$ Hilbert transform pairs are useful tools for generating minimum functions from the real part $A(\omega)$. On the other hand, depending on the form of the even part, evaluation of these integrals may be troublesome. Therefore, we prefer to work with various numerical or algebraic techniques to generate the minimum functions from their even parts without evaluating the integral. Especially in designing ultra wideband power transfer networks, 'non-integral techniques' are employed. In the following chapters details of these techniques will be presented. Nevertheless, for the sake of completeness, let us look at a simple example to show how complicated it is to evaluate the Hilbert integrals.

Example 1.10: Let $A(\omega^2) = 1/(1 + \omega^2)$ be the real part of a minimum function $F(p)$ on the real frequency axis $j\omega$ as in Example 1.8. Find the analytic form of the $F(p)$ using the Hilbert transformation.

Solution: From Example 1.8 we know that $F(p) = 1/(p + 1)$. However, in a more formal way, by Equation (1.37),

[25] In many practical situations, the constant term A_∞ approaches zero. Therefore, it may be neglected.

$$F(p) = A_\infty + \frac{2p}{\pi}\int_0^{+\infty} \frac{A(\Omega)}{p^2+\Omega^2}d\Omega = A_\infty + \frac{p}{\pi}\int_{-\infty}^{+\infty} \frac{A(\Omega)}{p^2+\Omega^2}d\Omega$$

Close examination of $A(\omega^2) = 1/(1+\omega^2)$ reveals that $A_\infty = 0$.

This integral may be evaluated by means of a complex contour integral technique employing the Cauchy residue formula. In short, it is known that for any function $\alpha(\xi)$ which is analytic inside a closed region C,

$$(2\pi j).\alpha(p) = \oint \frac{\alpha(\xi)}{\xi - p}d\xi = \int_{-\infty}^{+\infty} \frac{\alpha(\xi)}{\xi - p}d\xi$$

provided that $\lim_{R\to\infty} \alpha(\xi)/(\xi - p) = 0$ where R is the radius of the closed contour C.[26] Now, to go back to our problem, let $\xi = j\Omega$ and $\alpha(\xi) = 1/(1+\xi)(p+\xi)$. Then,

$$\Omega^2 = -\xi^2 \quad d\Omega = \frac{1}{j}d\xi \quad A(\xi) = \frac{1}{1-\xi^2} = \frac{1}{(1-\xi)(1-\xi)}$$

and $p^2 + \Omega^2 = p^2 - \xi^2 = (p+\xi)(p-\xi)$.

Thus, the original Hilbert transformation becomes

$$\begin{aligned}\frac{(j\pi)F(p)}{p} &= \int\limits_{-\infty}^{+\infty} \frac{A(\Omega)}{p^2+\Omega^2}d\Omega = \int\limits_{-\infty}^{+\infty} \frac{\alpha(\xi)}{(1-\xi)(p-\xi)}d\xi \\ &= \int_{-\infty}^{+\infty}\left[\frac{\mu(1)}{1-\xi} + \frac{\eta(p)}{p-\xi}\right]d\xi = (2\pi j)[\mu(1) + \eta(p)]\end{aligned}$$

where

$$\mu(\xi) = \frac{\alpha(\xi)}{p - \xi}$$

and

$$\eta(\xi) = \frac{\alpha(\xi)}{1-\xi}$$

Note that

$$\alpha(1) = \frac{1}{2(p+1)}$$

$$\alpha(p) = \frac{1}{2p(1+p)}$$

[26] For this problem it is assumed that the closed countour is the closed right half of the complex p plane.

Then,

$$[\mu(1)+\eta(p)] = \frac{1}{2p(p+1)}$$

Thus,

$$F(p) = \frac{1}{1+p}$$

confirms the original function specified by Example 1.8.

There are some other methods for evaluating the above Hilbert integral but they are all tedious. In designing power transfer networks, that is why we wish to work with numerical or algebraic methods to

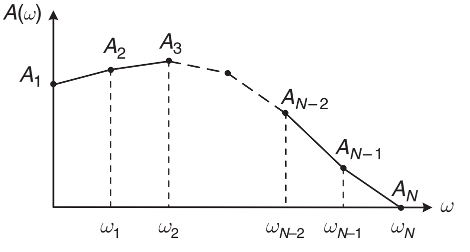

Figure 1.8 Piecewise linearization of real part $A(\omega)$

generate a minimum function from its even part.

As will be shown in the following section, $A(\omega)$ is always non-negative, and a smooth function of angular frequency or real frequency ω in many practical situations eventually reaches zero as frequency goes to infinity. Therefore, its piece wise linear approximation is possible as shown in Figure 1.8.

Let us consider that $A(\omega)$ as expressed in terms of N sampled point pairs $\{(A_k, \omega_k); k = 1, 2, \ldots, N\}$ or equivalently total $(N-1)$ pieces of line segment such that

$$A(\omega) = \left\{ \begin{array}{lll} a_j\omega + b_j & \text{for} \quad \omega_j \le \omega \le \omega_{j+1}, & j = 1, 2, \ldots, (N-1) \\ 0 & \omega \ge \omega_N & \end{array} \right\}$$

where

$$a_j = \frac{A_j - A_{j+1}}{\omega_j - \omega_{j+1}} = \frac{\Delta A_j}{\omega_{j+1} - \omega_j} \tag{1.38}$$

and

$$b_j = \frac{(A_{j+1})\omega_j - (A_j)\omega_{j+1}}{\omega_{j-\omega_{j+1}}}$$

where $\Delta A_j = A_{j+1} - A_j$

The above form of $A(\omega)$ is suitable for numerical evaluation of $B(\omega)$ by employing the Hilbert transformation as illustrated in the following section.

1.13 Numerical Evaluation of Hilbert Transform

Referring to Equation (1.38), let

$$\boxed{\begin{aligned}\hat{B}_j(\omega) &= \frac{1}{\pi}\int_{\omega_j}^{\omega_{j+1}}\left[\frac{dA(\Omega)}{d\Omega}\right]\ln\left|\frac{\Omega+\omega}{\Omega-\omega}\right|d\Omega\\ &= \frac{a_j}{\pi}\int_{\omega_j}^{\omega_{j+1}}\ln\left|\frac{\Omega+\omega}{\Omega-\omega}\right|d\Omega\\ &\text{or}\\ \hat{B}_j(\omega) &= \frac{A_j-A_{j+1}}{\pi(\omega_j-\omega_{j+1})}\int_{\omega_j}^{\omega_{j+1}}\ln\left[\frac{\Omega+\omega}{\Omega-\omega}\right]d\Omega\\ &\text{Then,}\\ B(\omega) &= \sum_{k=1}^{N}\hat{B}_k\end{aligned}} \tag{1.39}$$

For numerical reasons, it may be appropriate to express Equation (1.39) in terms of excursions of $\Delta A_j = A_{j+1} - A_j$ and its multiplying coefficients

$$\boxed{B_j(\omega) = \frac{1}{\pi(\omega_j-\omega_{j+1})}\int_{\omega_j}^{\omega_{j+1}}\ln\left|\frac{\Omega+\omega}{\Omega-\omega}\right|d\Omega} \tag{1.40}$$

In this regard, let

$$\boxed{F_j(\omega) = (\omega+\omega_j)\ln(|\omega+\omega_j|) + (\omega-\omega_j)\ln(|\omega-\omega_j|)} \tag{1.41}$$

Then, it is straightforward to show that

$$\boxed{B_j(\omega) = \frac{1}{\pi(\omega_j-\omega_{j+1})}\left[F_{j+1}(\omega)-F_j(\omega)\right]} \tag{1.42}$$

Thus,

$$\boxed{B(\omega) = \sum_{j=1}^{N-1} B_j(\omega)\Delta A_j} \tag{1.43}$$

This equation clearly indicates that the imaginary part $B(\omega)$ of a minimum function $F(j\omega)$ can also be expressed in terms of a linear combination of the same break points $\{A_k, \omega_k;\ K = 1, 2, \ldots, N\}$.

Details will be presented in Chapter 11.

1.14 Convolution

Let $f(t)$ and $h(t)$ be CCB signals. Their convolution is defined as

$$\boxed{\begin{aligned} y(t) &= f(t) * h(t) \\ &= \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} f(t-\tau)h(\tau)d\tau, t \geq 0 \end{aligned}} \tag{1.44}$$

Let $Y(j\omega) = \mathcal{F}\{y(t)\}$, $f(j\omega) = \mathcal{F}\{f(t)\}$ and $H(j\omega) = \mathcal{F}\{h(t)\}$. Then, it can be shown that

$$\boxed{\begin{aligned} &Y(j\omega) = F(j\omega)\ H(j\omega) \\ &\text{or, replacing } j\omega \text{ by } p, \text{ we can write} \\ &Y(p) = F(p)H(p) \end{aligned}} \tag{1.45}$$

1.15 Signal Energy

Let $v(t)$ and $i(t)$ be the port voltage and current of a circuit component. Let $f(t)$ designate either $v(t)$ or $i(t)$. In general these quantities can be complex. In this case, it can be shown that total energy W delivered to a component or to a one-port is proportional to $W \sim \int_{-\infty}^{+\infty} |f(t)|^2 dt$.

It must be clear that for a passive one port W must be non-negative. For example, for a resistor R[27]

$$\boxed{\begin{aligned} W_R &= \int_{-\infty}^{+\infty} v(t)i^*(t)dt \\ &= \int_{-\infty}^{+\infty} R|i(t)|^2 dt = \int_{-\infty}^{+\infty} \frac{1}{G}|v(t)|^2 dt \\ &\geq 0 \\ &\text{For real signals we can simply write} \\ &\quad w = \int_{-\infty}^{+\infty} v(t)i(t)dt \geq 0 \\ &\text{where the superscript } * \text{ designates the complex conjugate of the signal.} \end{aligned}} \tag{1.46}$$

[27] Strictly speaking, the resistor R may be a complex quantity. However, for a real passive physical system, R must be real.

In general,

$$\boxed{W_f \triangleq \int_{-\infty}^{+\infty} |f(t)|^2 dt \geq 0} \tag{1.47}$$

is called the signal energy.

Obviously, for physical passive one-ports which are defined in terms of real parameters, W_f must be non-negative, finite and L^2. Therefore, we can comfortably state that for any real, passive one-port, the port descriptive functions $v(t)$ and $i(t)$ must be real, CCB and L^2. Let $V(j\omega)$ and $I(j\omega)$ be the Fourier transforms of $v(t)$ and $i(t)$ respectively. Using the inverse Fourier transform, we can write $v(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V(j\omega)e^{j\omega t} d\omega$.

Then, for any real, passive one port, Equation (1.46) becomes

$$\boxed{W = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V(j\omega)\left[\int_{-\infty}^{+\infty} i(t)e^{-j\omega t}dt\right]^* d\omega} \tag{1.48}$$

or

$$\boxed{W = \int_{-\infty}^{+\infty} v(t)i\,(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V(j\omega)I^*(j\omega)d\omega \geq 0} \tag{1.49}$$

1.16 Definition of Impedance and Admittance

Equation (1.49) is very useful for expressing definitions of circuit components in terms of Laplace transforms since the derivatives of voltage and current are removed from the mathematical expressions. Thus, employing the Laplace-transformed pairs for voltage and current as $v \leftrightarrow V(p)$ and $i \leftrightarrow I(p)$; passive ideal resistor R, capacitor C and inductor L definitions can be given in a straightforward manner as

$$\boxed{\begin{aligned} &V_R(p) \triangleq RI_R(p) \\ &I_c(p) \triangleq [Cp]V_c(p) = Y_c(p)V_c(p) \\ &V_L(p) \triangleq [Lp]I_L(p) = Z_L(p)I_L(p) \end{aligned}} \tag{1.50}$$

where $Z_L(p) = pL = V_L(p)/I_L(p)$ is called the impedance of an inductor and $Y_L(p) = pC = I_C(p)/V_C(p)$ is called the admittance of a capacitor.

Similarly the coupled coil definition is given by

$$\boxed{\begin{aligned} V_1(p) &= pL_1 I_1(p) + pM\, I_2(p) \\ V_2(p) &= pM_1(p) + pL_2\, I_2(p) \end{aligned}} \tag{1.51}$$

or

$$\boxed{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} pL_1 & pM \\ pM & pL_2 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \boldsymbol{Z_M}(p)\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}} \tag{1.52}$$

where $\boldsymbol{Z_M}(p)$ is called the impedance matrix of the coupled coils.

1.17 Immittance of One-port Networks

Referring to Figure 1.9, consider a one-port network which consists of passive lumped R, L and C elements and is driven by a bounded causal excitation.[28]

Let $v_{in} \leftrightarrow V_{in}(p)$ and $i_{in} \leftrightarrow I_{in}(p)$ be the time and Laplace transform pairs of the one-port. Then, the driving point impedance function $Z_{in}(p)$ of the one-port is defined as

$$\boxed{Z_{in}(p) \triangleq \frac{V_{in}(p)}{I_{in}(p)}} \tag{1.53}$$

Similarly,

$$\boxed{Y_{in}(p) \triangleq \frac{I_{in}(p)}{V_{in}(p)}} \tag{1.54}$$

is called the driving point admittance function. In short, if we drop the subscripts *in* then, for any one port network, $Z(p) = V(p)/I(p)$ and $Y(p) = I(p)/V(p)$ are called the impedance and admittance functions respectively. By nature, impedance and admittance functions possess the same mathematical properties and are referred to as immittance functions. It is straightforward to show that, for a linear passive lumped element circuit, the 'immittance' function is rational with real numerator $N(p)$ and real denominator $D(p)$ polynomials. Hence,

$$\boxed{F(p) = \frac{N(p)}{D(p)} = \frac{a_n p^n + a_{n-1}p^{n-1} + \ldots + a_1 p + a_0}{b_m p^m + b_{m-1}p^{m-1} + \ldots + b_1 p + b_0}} \tag{1.55}$$

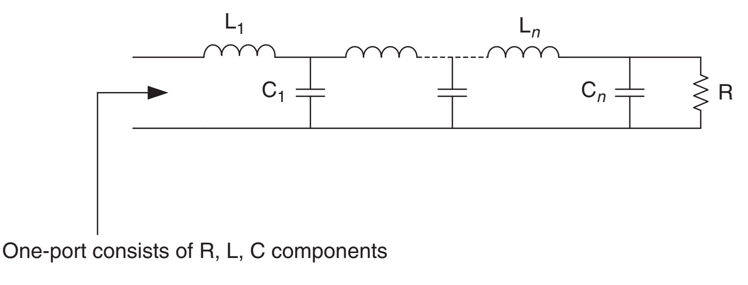

Figure 1.9 One-port network consisting of passive elements

[28] That means port voltage $v(t)$ and port current $i(t)$ are bounded and causal signals.

Clearly, an immittance function $F(p)$ is a complex function in $p = \sigma + j\omega$. In addition, voltage $V(p)$ and current $I(p)$ can be expressed in terms of impedance and admittance functions as

$$\boxed{\begin{aligned} V(p) &= Z(p)I(I) \\ I(p) &= Y(p)V(p) \end{aligned}} \tag{1.56}$$

Close examination of Equation (1.55) reveals that $Z(p)$ and $Y(p)$ must be regular or pole free in the RHP to make $v(t)$ and $i(t)$ causal.[29] Furthermore, if they have poles on $j\omega$ the axis, these poles must be simple to make $v(t)$ and $i(t)$ bounded.[30]

Using classical circuit nomenclature, the transformed functions in $p = j\omega$[31] can be expressed as follows:

$$\boxed{\begin{aligned} V(j\omega) &= |V(j\omega)|.e^{j\varphi_v(\omega)} \\ I(j\omega) &= |I(j\omega)|.e^{j\varphi_i(\omega)} \\ Z(j\omega) &= R(\omega) + jX(\omega) \\ Y(j\omega) &= G(\omega) + jB(\omega) \\ F(j\omega) &= A(\omega) + jB(\omega) \end{aligned}} \tag{1.57}$$

Based on the description of immittance functions, Equations (1.48–1.49) can be expressed as

$$\boxed{\begin{aligned} W &= \int_{-\infty}^{+\infty} v(t)i(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} V(j\omega)I(-j\omega)d\omega \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} R\left(-\omega^2\right)|I(j\omega)|^2 d\omega + j\int_{-\infty}^{+\infty} X(\omega)|I(j\omega)|^2 d\omega\right] \geq 0 \end{aligned}} \tag{1.58}$$

In Equation (1.57) $\int_{-\infty}^{+\infty} X(\omega)|I(j\omega)|^2 d\omega = 0$ since $X(\omega)$ is an odd function of ω and integral $\int_{-\infty}^{+\infty} R\left(-\omega^2\right)|I(j\omega)|^2 d\omega = 2\int_0^{+\infty} R(\omega)|I(j\omega)|^2 d\omega \geq 0$ since $R(-\omega^2)$ is an even function of ω. Thus, the total energy delivered to the one-port is given by

$$\boxed{W_p = \frac{1}{\pi}\int_0^{+\infty} R(\omega)|I(j\omega)|^2 d\omega \geq 0} \tag{1.59}$$

[29] A complex function is said to be regular in a domain D if it is analytic (differentiable everywhere) in D. Note that inverse Laplace transformation of a single pole term $K_\sigma/(p - \sigma_k)$ is equal to $K_\sigma.e^{\sigma_k t}u(t)$, which is not bounded for $\sigma_k < 0$. Therefore, an immittance function $F(p)$ must be free of RHP poles. In this representation, $u(t)$ designates the unit step function.

[30] Suppose that $F(p)$ has simple complex conjugate poles as in $1/(p^2 + \omega_r^2)$. Then, considering the inverse Laplace transformation of $1/(p^2 + \omega_r^2)$, a term $\mathcal{L}^{-1}\{F(p)\} = (1/\omega_r)\sin(\omega_r t)\ u(t)$ must appear either on $v(t)$ or on $i(t)$, or on both, which is causal. Therefore, simple $j\omega$ poles are acceptable in $F(p)$. On the other hand, multiple $j\omega$ poles are not acceptable since they yield non-dissipative immittance functions. Details of this issue are skipped here.

[31] That is, by choosing $\sigma = 0$, we set $p = j\omega$.

yielding $R(\omega) \geq 0$ for all ω; equivalently, $R(p^2) \geq 0$ for all real $\{p\}$ ≥ 0. Similarly, one can write that

$$\boxed{W_p = \frac{1}{\pi}\int_0^{+\infty} G(-\omega^2)|V(j\omega)|^2 d\omega \geq 0} \tag{1.60}$$

yielding $G(-\omega^2) \geq 0$ for all ω; equivalently, $G(p^2) \geq 0$ for all real $\{p\} \geq 0$.

1.18 Definition: 'Positive Real Functions'

The above derivations lead us to define a positive real function.

A complex variable function F(p) which designates either the impedance or the admittance function of a one-port is said to be 'positive real' (or in short PR) if:

1. $F(p)$ is real when p is real.
2. $A(p^2) = \text{real}\{F(p)\} \geq 0$ and bounded when $\sigma = \text{real}\ \{p\} \geq 0$.

Condition 1 implies that all the parameters in $F(p)$ other than p (such as coefficients, exponents, etc.) must be real. In condition 2, the two equality signs do not hold simultaneously unless they hold identically.

Clearly, for a passive one-port network described by Equations (1.58–1.59), the driving point immittance function $F(p)$ must be positive real. Furthermore, if the one-port consists of lumped circuit elements such as resistors, inductors, capacitors and transformers, then $F(p)$ must be a rational PR function.

As verified from the above discussions, a rational function

$$F(p) = \frac{N(p)}{D(p)} = \frac{a_m p^m + a_{m-1}p^{m-1} + \ldots + a_1 p + a_0}{b_n p^n + b_{n-1}p^{n-1} + \ldots + b_1 p + b_0}$$

is PR if and only if:

- Coefficients $\{a_i, b_j\}$ are real.
- It is free of RHP zeros and poles. In other words, both itself and its inverse are analytic in RHP.
- Its $j\omega$ axis zeros and poles are simple, which implies that
- $|m - n| \leq 1$, meaning either $m = n$ or $|m - n| = 1$.

It can be shown that any rational PR impedance function $Z(p)$ can be represented as

$$\boxed{F(p) = \left[K_\infty p + \frac{k_0}{p} + \sum_{j=1}^{N\omega} \frac{k_{\omega j} p}{p^2 + \omega_j^2}\right] + \left[K + \sum_{j=1}^{n} \frac{k_j}{p - p_j}\right]} \tag{1.61}$$

or

$$\boxed{Z(p) = Z_F(p) + Z_{min}(p)} \tag{1.62}$$

where

$$\boxed{Z_F(p) = \left[k_\infty p + \frac{k_0}{p} + \sum_{j=1}^{N_\omega} \frac{k_{\omega j} p}{p^2 + \omega_j^2}\right]} \tag{1.63}$$

is an odd function with all its poles on the $j\omega$ axis; more specifically, at infinity, at zero and at finite frequencies ω_j with real non-negative residues k_∞, k_0 and $k_{\omega j}$ respectively. In classical circuit theory it is known as the Foster function. Also,

$$\boxed{Z_{min}(p) = \left[K + \sum_{i=1}^{N_p} \frac{k_i}{p - p_i}\right]} \tag{1.64}$$

is the $j\omega$ pole-free part of the impedance $Z(p)$ and is called the minimum reactance function. The complex residue k_i of the poles p_i must have a negative real part. Obviously, a pole $p_i = -\alpha_i + j\beta_i$ must be placed in the open left hand plane (LHP) and therefore α_i must be positive. Furthermore, if $\beta \neq 0$, then it must be accompanied with its conjugate to make the impedance function real.

It is interesting to note that the even part of $Z(p)$ is equal to the even part of $Z_{min}(p)$ and it is given by

$$\boxed{\begin{aligned} R(p^2) &= \text{even}\{Z(p)\} = \text{even}\{Z_{min}(p)\} \\ &= \frac{1}{2}\left[2K + \sum_{i=1}^{n}\left(\frac{k_i}{p - p_i} + \frac{k_i}{-p - p_i}\right)\right] \\ &= \left[K + \sum_{i=1}^{n} \frac{k_i p_i}{(p^2 - p_i^2)}\right] \end{aligned}} \tag{1.65}$$

In this case, for a pole p_j, the complex residues k_j can be directly computed from Equation (1.65) as

$$\boxed{\begin{aligned} & k_j = (p^2 - p_j^2)\frac{R(p^2)}{p_j}\Big|_{p = p_j} \\ & \text{and} \\ & K = \lim_{p \to \infty} R(p^2) \end{aligned}} \tag{1.66}$$

Equation (1.66) indicates that a minimum reactance function can be uniquely generated from its real part in a similar manner to that of Equation (1.39). This greatly facilitates the generation of PR functions to construct lossless power transfer networks. By Darlington's theorem it is also well known that any PR function can be represented as a lossless two-port in resistive termination. In this case, The PR function can be utilized to describe lossless power transfer networks. Furthermore, if we deal with minimum reactance or minimum susceptance functions, then the lossless power transfer network can be uniquely described in terms of the real part of the driving point input immittance of the power transfer network when it is terminated in a resistance. Details of this issue will be covered in the following chapters.

Based on Darlington's theorem, we can also consider the generation of PR functions by means of lossless two-ports when they are terminated in a resistance R (or equivalently a conductance $G = 1/R$).

In practice, lossless filters and matching networks or, in short, power transfer networks, are constructed using ladder networks. Therefore, the following chapters are devoted to generating PR functions from ladder structures.

Below we give an example to test a rational function if it is PR.

Example 1.11: Let

$$F(p) = \frac{N(p)}{D(p)} = \frac{30p^5 + 30p^4 + 23p^3 + 17p^2 + 4p + 1}{24p^6 + 24p^5 + 68p^4 + 44p^3 + 23p^2 + 3p + 1}$$

Find:

(a) If $F(p)$ is PR.
(b) If it is a minimum function.
(c) The analytic form of the even part of $F(p)$.
(d) The real part $R(\omega) = \text{real}\{F(j\omega)\}$ and $X(\omega) = \text{im}\{F(j\omega)\}$ and plot them vs. angular frequency ω.
(e) The imaginary part $X(\omega) = \text{im}\{F(j\omega)\}$ using the numerical Hilbert transformation of Equations (1.40–1.43).

Solution:

(a) The above function is rational. If it is PR, then it must have all its zeros and poles in the LHP of the complex p domain. Furthermore, if there exist $j\omega$ poles, they must be simple.

Using MATLAB®, we can easily compute the zeros and the poles and verify if $F(p)$ is PR.

Let $\boldsymbol{N} = [30\ 30\ 23\ 17\ 4\ 1]$ be the vector which contains all the coefficients of the numerator polynomial $N(p)$ in descending order. Then, using MATLAB command '>>roots(N)', we can compute the zeros of $F(p)$.

Thus, we have the following zeros (Table 1.1):

Table 1.1 Zeros of $F(p)$

Zero 1	−0.7654
Zero 2	−0.0000 + 0.7071i
Zero 3	−0.0000 – 0.7071i
Zero 4	−0.1173 + 0.2708i
Zero 5	−0.1173 – 0.2708i

As can be seen from Table 1.1, all the zeros of $F(p)$ are in the LHP, which is acceptable.

Similarly, we can compute the poles of $F(p)$. Let $\boldsymbol{D} = [24\ 24\ 68\ 44\ 23\ 3\ 1]$ be the vector which includes all the coefficients of the denominator polynomial in descending order. Then, the poles are found using the MATLAB command '>>roots(D)' as in Table 1.2:

These poles are also in the LHP. Thus $F(p)$ is PR.

Table 1.2 Poles of $F(p)$

Pole 1	−0.1244 + 1.4886i
Pole 2	−0.1244 – 1.4886i
Pole 3	−0.3548 + 0.4505i
Pole 4	−0.3548 – 0.4505i
Pole 5	−0.0207 + 0.2374i
Pole 6	−0.0207 – 0.2374i

(b) Obviously, $F(p)$ is minimum since it has all its zeros and poles in the LHP. That is, it is free of $j\omega$ poles.

(c) The even part of $F(p)$ is given by

$$R(p) = \frac{1}{2}\left[F(p) + F(-p)\right]$$
$$= \frac{576p^8 + 672p^6 + 244p^4 + 28p^2 + 1}{576p^{12} + 2688p^{10} + 3616p^8 + 1096p^6 + 401p^4 + 37p^2 + 1}$$

(d) Setting $p = j\omega$, we can plot the real part $R(\omega) = \text{real}\{F(j\omega)\}$ as well as $X(\omega) = \text{im}\{F(j\omega)\}$ as shown in Figure 1.10 and 1.11 using MATLAB.

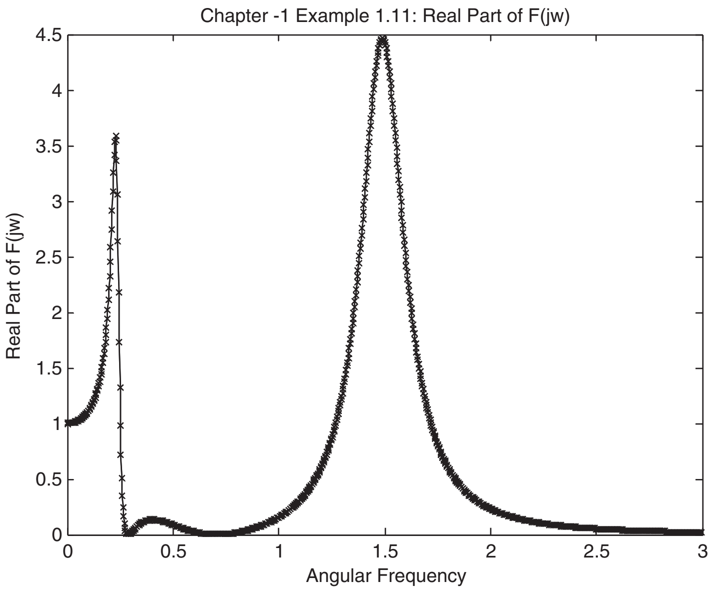

Figure 1.10 Plot of the real part of $F(j\omega)$

(e) In order to evaluate $X(\omega)$ by means of the numerical Hilbert transformation, firstly, we have to sample the real part $R(\omega)$ to specify the break points $\{R_k, k = 1, 2, \ldots, NB\}$ and the break frequencies $\omega_k, k = 1, 2, \ldots, NB$. Obviously, the quality of the computations depends on the total number of sampling points. The more the number of break points, the better the quality in computations. For our computational purposes, we developed a MATLAB program 'Chapter 1, Example 1.11, which is listed at the end of this chapter.

Close examination of Figure 1.10 reveals that, after $\omega \geq 3$, $R(\omega)$ practically approaches zero. Therefore, break frequencies can be uniformly distributed over the angular frequency interval $0 \leq \omega \leq 3$. In this case, choosing $NB = 10$ break point (Table 1.3), we end up with the break point pairs shown in Figure 1.12.

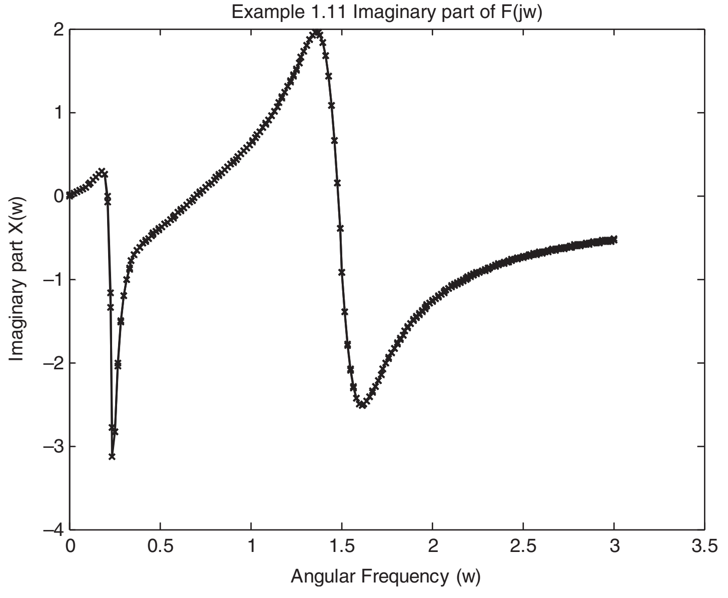

Figure 1.11 Plot of $X = \text{im}\{F(j\omega)\}$

Table 1.3 Selection of break points $NB = 10$ uniform samples for Example 1.11

Angular break frequencies, ω_k	Corresponding break points, R_k
0	1.0000
0.3333	0.0798
0.6667	0.0035
1.0000	0.1565
1.3333	1.6720
1.6667	1.4607
2.0000	0.2310
2.3333	0.0785
2.6667	0.0361
3.0000	0.0194

Employing Equation (1.43) we have

$$\boxed{X_H(\omega) = \sum_{k=1}^{NB-1} B_k(\omega)\Delta R_k} \tag{1.67}$$

where the excursions ΔR_k and the coefficients B_k and $X_H(\omega_k)$ are given in Table 1.4.

Table 1.4 is illustrated in Figure 1.13.

As can be seen from Figure 1.13 $NB = 10$ points results in poor estimation of the imaginary part of $F(j\omega)$. Nevertheless, resolution can be improved by increasing sample points on $R(\omega)$.

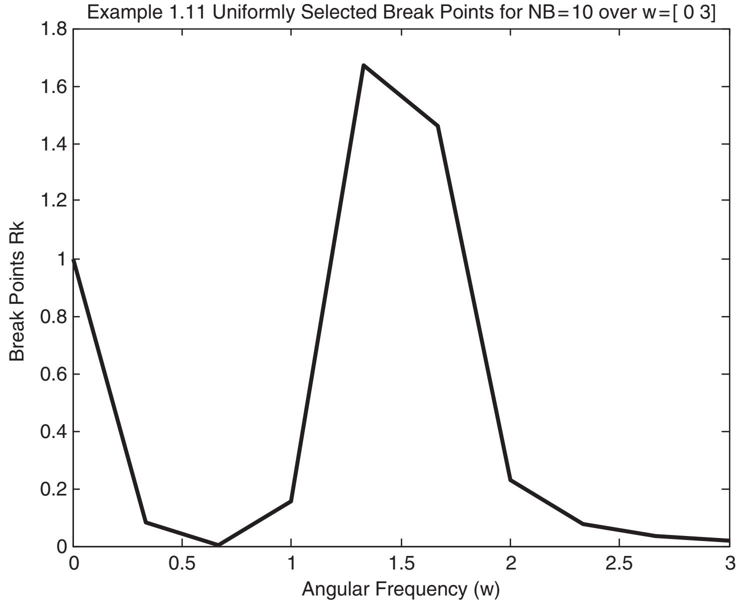

Figure 1.12 Break point distribution of Example 1.11

Table 1.4 Evaluation of numerical Hilbert transform of $R(\omega)$ for $NB = 10$ break points

	Break frequencies, ω_k	$\Delta\omega_k = \omega_k - \omega_{k-1}$	ΔR_k	B_k	Numerically computed $X_H(\omega_k)$
1	0	—	—	—	−0.0000
2	0.3333	0.3333	−0.9202	0.0354	−0.3194
3	0.6667	0.3333	−0.0763	0.1072	0.1444
4	1.0000	0.3333	0.1530	0.1818	0.7350
5	1.3333	0.3333	1.5155	0.2617	0.4005
6	1.6667	0.3333	−0.2113	0.3503	−0.8337
7	2.0000	0.3333	−1.2297	0.4535	−1.0609
8	2.3333	0.3333	−0.1525	0.5829	−0.6620
9	2.6667	0.3333	−0.0424	0.7693	−0.4911
10	3.0000	0.3333	−0.0166	1.2293	−0.3900

For example, $NB = 30$ yields uniformly distributed sample points and one can obtain much better estimation of $X_H(\omega)$ as shown in Figure 1.14.

Finally, note that an almost original approximation can be obtained with $NB = 50$ break points. This is left as an excercise to the reader.

Details of the Hilbert transformation will be given in the following chapters.

Finally, we present the MATLAB programs developed for this chapter to enable the reader to verify the above results.

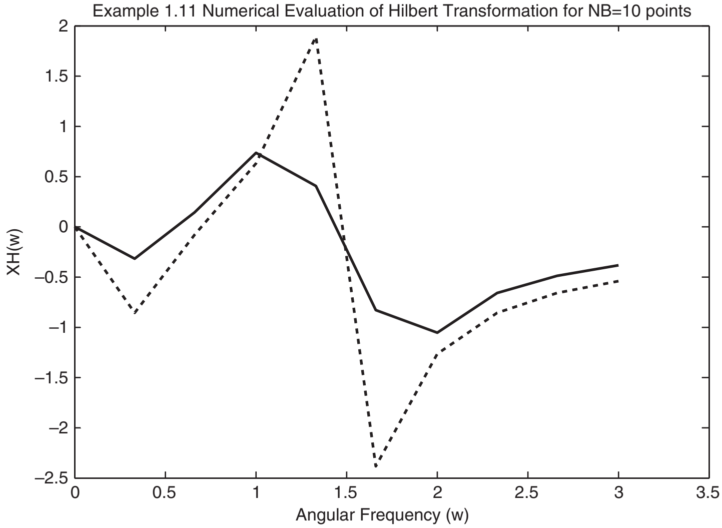

Figure 1.13 Approximation of the imaginary part via Hilbert transformation

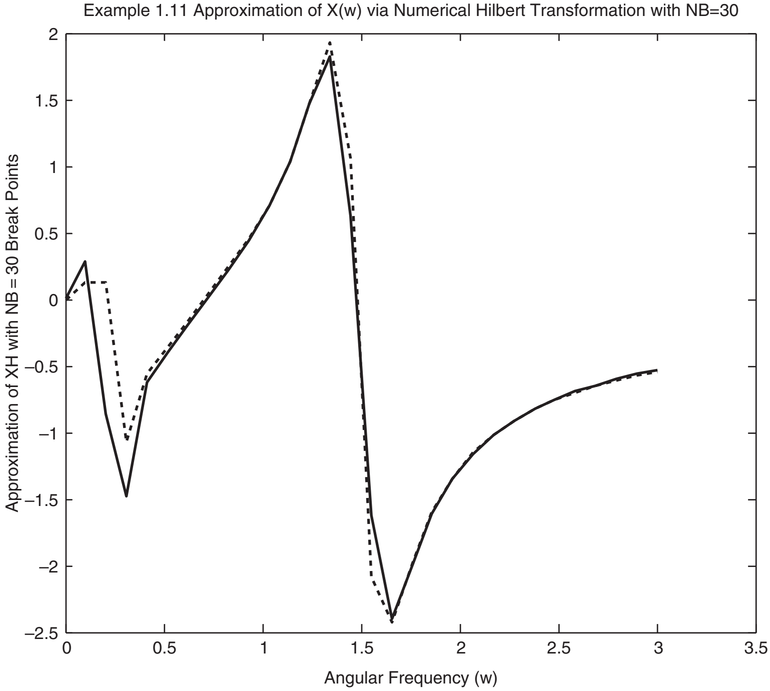

Figure 1.14 Better approximation of $X(j\omega)$ with more beak points

Program Listing 1.1 Chapter 1, Example 1.11, main program listing

```
% Program: Chapter 1; Example 1.11B
clear
% Given F(p) = N/D or F(p) = a(p)/b(p)
% Given N = [30 30 23 17 4 1] in descending order
% Given D = [24 24 68 44 23 3 1]in descending order
% Computation of R(w) and X(w)
N = [30 30 23 17 4 1];% Input
D = [24 24 68 44 23 3 1]; %Input

wstart = 0.0; % Beginning of angular frequency
wend = 3.0;   % End of angular frequency
NB = 30;      % Total number of sampling frequencies
dw = (wend-wstart)/(NB-1);% sweeping step size
%
%Computation of Break frequencies WB and Break Points RB
w = wstart;
j = sqrt(-1);
for i = 1:NB
    p = j*w;                % Complex variable on jw axis
    Num = polyval(N,p);     % Complex value of the numerator at w
    Denom = polyval(D,p);   % Complex value of the denominator at w
    F = Num/Denom;          % Complex value of the given function at w
    RB(i) = real(F);        % Break Points
    X(i) = imag(F);         % Original Imaginary part
    WB(i) = w;              % Break Frequencies
    w = w + dw;             % Sweeping frequency
end

% Numerical Hilbert Transformation
wstart = 0.0 + 1e-6; % Beginning of angular frequency
w = wstart;
%
for i = 1:NB
    [DW,DR,B,XA] = Num_HilbertB(w,WB,RB);
    XH(i) = XA;
    W(i) = w;
    w = w + dw;
end
figure (1)
plot (WB,RB)
xlabel('Angular Frequency (w)')
ylabel(' Break Points Rk')
title(' Example 1.11 Break Points Rk NB points')
figure (2)
plot(W,XH,WB,X)
xlabel('Angular Frequency (w)')
```

```
ylabel (' Approximation of XH with NB points')
title (' Example 1.11 Approximation of X(w) via Numerical Hilbert
Transformation NB points')

% Numerical Hilbert Transformation
wstart=0.0+1e-6; % Beginning of angular frequency
w=wstart;
%
for i=1:NB
    [DW,DR,B,XA]=Num_HilbertB(w,WB,RB);
    XH(i)=XA;
    W(i)=w;
    w=w+dw;
end
figure
plot(WB,RB)
figure
plot(W,XH,WB,X)
```

Program Listing 1.2 Evaluation of numerical Hilbert transformation for Example 1.11

```
function [DW,DR,B,XA]=Num_HilbertB(w,W,R)
%Numerical Hilbert Transformation
N=length(W);
%
for k=2:N
    DW(k)=W(k)-W(k-1);
    DR(k)=R(k)-R(k-1);
    M(k-1)=DR(k)/DW(k);
end
    for k=1:N
    F(k)=(w+W(k))*log(abs(w+W(k)))+(w-W(k))*log(abs(w-W(k)));
    end
for k=2:N
    DF=(1/pi)*(F(k)-F(k-1));
    B(k)=(1/pi)*(F(k)-F(k-1))/DW(k);
    X(k)=B(k)*DR(k);
  end
  %
      XA=0;
      for k=2:N
          XA=XA+X(k);
      end
```