1
DELTA-FUNCTIONS AND DISTRIBUTIONS

The delta-function and its derivatives are frequently encountered in the technical literature. The function was first conceived as a tool which, if properly handled, could lead to useful results in a particularly concise way. Its popularity is now justified by solid mathematical arguments, developed over the years by authors such as Sobolev, Bochner, Mikusinski, and Schwartz. In the following pages we give the essentials of the Schwartz approach (distribution theory). The level of treatment is purely utilitarian. Rigorous exposés, together with descriptions of the historical evolution of the theory, may be found in the numerous texts quoted in the bibliography.

1.1 The δ-function

The idea of the δ-function is quite old, and dates back at least to the times of Kirchhoff and Heaviside (van der Pol *et al.* 1951). In the early days of quantum mechanics, Dirac put the accent on the following properties of the function:

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1, \qquad \delta(x) = 0 \quad \text{for } x \neq 0. \tag{1.1}$$

The notation $\delta(x)$ was inspired by δ_{ik}, the Kronecker delta, equal to 0 for $i \neq k$, and to 1 for $i = k$. Clearly, $\delta(x)$ must be 'infinite' at $x = 0$ if the integral in (1.1) is to be unity. Dirac recognized from the start that $\delta(x)$ was not a function of x in the usual mathematical sense, but something more general which he called an 'improper' function. Its use, therefore, had to be confined to certain simple expressions, and subjected to careful codification. One of the expressions put forward by Dirac was the 'sifting' property

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0). \tag{1.2}$$

This relationship can serve to define the delta function, not by its value at each point of the x axis, but by the ensemble of its scalar products with suitably chosen 'test' functions $f(x)$.

It is clear that the infinitely-peaked delta function can be interpreted intuitively as a strongly concentrated forcing function. The function may represent, for example, the force density produced by a unit force acting on a one-dimensional mechanical structure, e.g. a flexible string. This point of view leads to the concept of $\delta(x)$ being the limit of a function which becomes more and more concentrated in the vicinity of $x = 0$, whereas its integral from $-\infty$ to $+\infty$ remains equal to one. Some of the limit functions which behave in that manner are shown in Fig. 1.1. The first one is the rectangular pulse, which becomes 'needle-like' at high values of n (Fig. 1.1a). The other ones are (de Jager 1969; Bass 1971)

$$\lim_{n\to\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \qquad \text{(shown in Fig. 1.1b),} \tag{1.3}$$

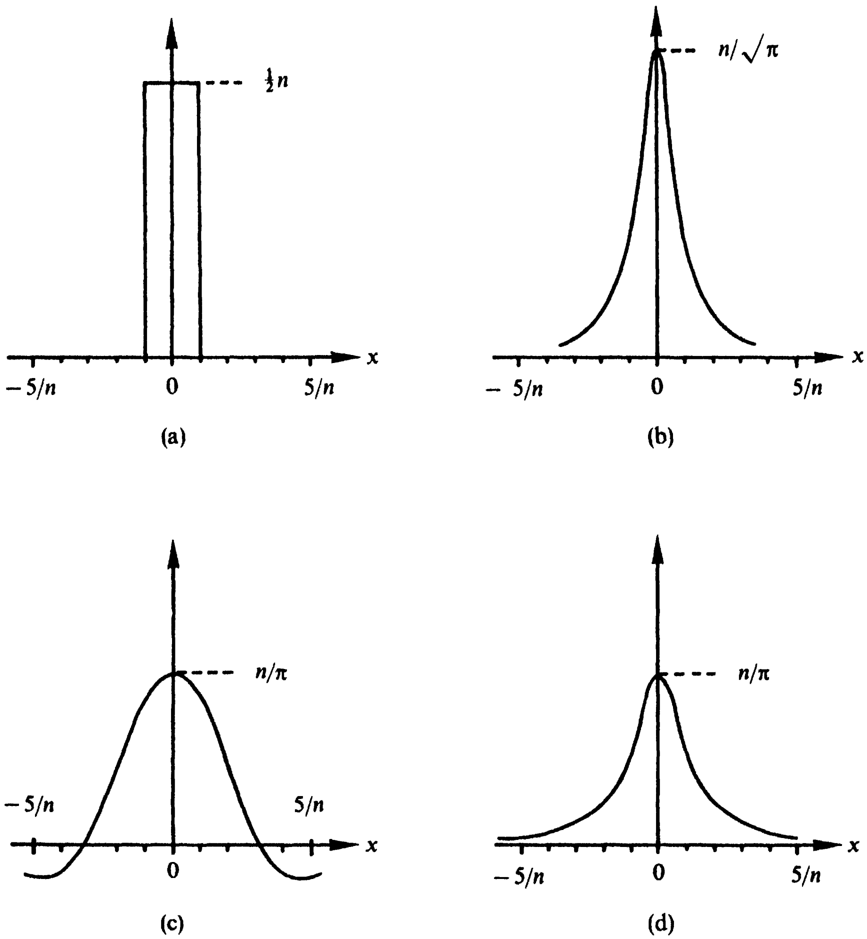


Fig. 1.1. Functions which represent Dirac's function in the limit $n \to \infty$.

$$\lim_{n\to\infty} \frac{\sin nx}{\pi x} \qquad \text{(shown in Fig. 1.1c)}, \tag{1.4}$$

$$\lim_{n\to\infty} \frac{n}{\pi(1 + n^2x^2)} = \lim_{n\to\infty} \left(-\frac{n}{\pi} \operatorname{Im} \frac{1}{nx + \mathrm{j}} \right) \qquad \text{(shown in Fig. 1.1d)}. \tag{1.5}$$

1.2 Test functions and distributions

The notion of distribution is obtained by generalizing the idea embodied in (1.2), namely that a function is defined by the totality of its scalar products with reference functions termed *test functions*. The test functions used in the Schwartz theory are complex continuous functions $\phi(\mathbf{r})$ endowed with continuous derivatives of all orders. Such functions are often termed 'infinitely smooth'. They must vanish outside some finite domain, which may be different for each ϕ. They form a space $\mathscr{D}$. The smallest closed set which contains the set of points for which $\phi(\mathbf{r}) \neq 0$ is the *support* of ϕ. A typical one-dimensional test function is

$$\phi(x) = \begin{cases} \exp \dfrac{|ab|}{(x-a)(x-b)} & \text{for } x \text{ in } (a, b), \\ 0 & \text{for } x \text{ outside } (a, b). \end{cases} \tag{1.6}$$

The support of this function is the interval $[a, b]$. At the points $x = a$ and $x = b$, all derivatives vanish, and the graph of the function has a contact of infinite order with the x axis. A particular case of (1.6) is

$$\phi(x) = \begin{cases} \exp \dfrac{-1}{1-x^2} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geqslant 1. \end{cases} \tag{1.7}$$

In n dimensions, with $R^2 = x_1^2 + \cdots + x_n^2$, we have

$$\phi(\mathbf{r}) = \begin{cases} \exp \dfrac{-1}{1-R^2} & \text{for } |\mathbf{r}| < 1, \\ 0 & \text{for } |\mathbf{r}| \geqslant 1. \end{cases} \tag{1.8}$$

A few counterexamples are worth mentioning: $\phi(x) = x^2$ (for all x) is *not* a test function because its support is not bounded. The same is true of $\phi(x) = \sin|x|$, which furthermore has no continuous derivative at the origin.

To introduce the concept of 'distribution', it is necessary to first define *convergence* in $\mathscr{D}$ (Schwartz 1965). A sequence of functions $\phi_m(x)$ belonging to $\mathscr{D}$ is said to converge to $\phi(x)$ for $m \to \infty$ if

(1) the supports of the ϕ_m are contained in the same closed domain, independently of m;

(2) the ϕ_m and their derivatives of all orders converge uniformly to ϕ and its corresponding derivatives.

The next step is to define a *linear functional* on $\mathscr{D}$. This is an operation which associates a complex number $t(\phi)$ with every ϕ belonging to $\mathscr{D}$, in such a way that

$$t(\phi_1 + \phi_2) = t(\phi_1) + t(\phi_2), \qquad t(\lambda\phi) = \lambda t(\phi), \tag{1.9}$$

where λ is a complex constant. The complex number $t(\phi)$ is often written in the form

$$t(\phi) = \langle t, \phi \rangle \tag{1.10}$$

The functional is *continuous* if, when ϕ_m converges to ϕ for $m \to \infty$, the complex numbers $t(\phi_m)$ converge to $t(\phi)$. *Distributions are continuous linear functionals* on $\mathscr{D}$. They form a vector space $\mathscr{D}'$. To clarify these concepts, assume that $\tau(x)$ is a locally integrable function (i.e. a function which is integrable over any compact set). Such a function generates a distribution by the operation (Schwartz 1965)

$$\tau(\phi) = \langle \tau, \phi \rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \tau(x)\phi(x)\mathrm{d}x. \tag{1.11}$$

Many distributions cannot be written as an integral of that form, except in a formal way. For such cases the 'generating function' $\tau(x)$ becomes a symbolic function, and (1.11) only means that the integral, whenever it is encountered in an analytical development, may be replaced by the value $\tau(\phi)$. It should be noted, in this respect, that experiments do not yield instantaneous, punctual values of quantities such as a force or an electric field. Instead, they generate *integrated* outputs, i.e. averages over some non-vanishing intervals of time and space. The description of a quantity by scalar products of the form (1.11) is therefore quite acceptable from a physical point of view.

1.3 Simple examples

A first simple example is the integral of ϕ from 0 to ∞. This integral is a distribution, which may be written as

$$\langle \mathrm{Y}, \phi \rangle \stackrel{\text{def}}{=} \int_0^{\infty} \phi(x)\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{Y}(x)\phi(x)\mathrm{d}x. \tag{1.12}$$

The generating function is the Heaviside unit function $\mathrm{Y}(x)$, defined by the values

$$\mathrm{Y}(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geqslant 0. \end{cases} \tag{1.13}$$

As a second example, we consider a function $f(x)$, possibly undefined at c and unbounded near c, but integrable in the intervals $(a, c - \varepsilon)$ and $(c + \eta, b)$, where ε and η are positive. If

$$I = \lim_{\substack{\varepsilon \to 0 \\ \eta \to 0}} \left(\int_a^{c-\varepsilon} f(x)\mathrm{d}x + \int_{c+\eta}^{b} f(x)\mathrm{d}x \right) \tag{1.14}$$

exists for ε and η approaching zero independently of each other, this limit is termed the integral of $f(x)$ from a to b. Sometimes the limit exists only for $\varepsilon = \eta$. In such a case, its value is the principal value of Cauchy, and one writes

$$\mathrm{PV} \int_a^b f(x)\mathrm{d}x = \lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f(x)\mathrm{d}x + \int_{c+\varepsilon}^{b} f(x)\mathrm{d}x \right). \tag{1.15}$$

An example of such an integral is

$$\begin{aligned} \mathrm{PV} \int_{-a}^{a} \frac{\mathrm{d}x}{x} &= \lim_{\varepsilon \to 0} \left(\int_{-a}^{-\varepsilon} \frac{\mathrm{d}x}{x} + \int_{\varepsilon}^{a} \frac{\mathrm{d}x}{x} \right) \\ &= \lim_{\varepsilon \to 0} (\log \varepsilon - \log a + \log a - \log \varepsilon) = 0. \end{aligned} \tag{1.16}$$

The function $1/x$ does not define a distribution since it is not integrable in the vicinity of $x = 0$. But a well-defined meaning may be attached to $\mathrm{PV}(1/x)$ by introducing the functional:

$$\langle \mathrm{PV}(1/x), \phi \rangle \overset{\text{def}}{=} \mathrm{PV} \int_{-\infty}^{\infty} \frac{\phi(x)}{x} \mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{PV}(1/x)\, \phi(x)\mathrm{d}x. \tag{1.17}$$

The third example, of great importance in mathematical physics, is the original Dirac distribution, which associates the value $\phi(0)$ with any test function $\phi(x)$. Thus,

$$\langle \delta_0, \phi \rangle \overset{\text{def}}{=} \phi(0) = \int_{-\infty}^{\infty} \delta(x)\phi(x)\mathrm{d}x. \tag{1.18}$$

Similarly,

$$\langle \delta_{x_0}, \phi \rangle \overset{\text{def}}{=} \phi(x_0) = \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)\mathrm{d}x. \tag{1.19}$$

As an application

$$\int_{-\infty}^{\infty} x^m\, \delta(x)\phi(x)\mathrm{d}x = \int_{-\infty}^{\infty} \delta(x)\, [x^m \phi(x)]\, \mathrm{d}x = 0 \tag{1.20}$$

holds when m is a positive integer, in which case $x^m\phi(x)$ is a test function. Property (1.20) may therefore be written in symbolic form as

$$x^m\delta(x) = 0. \tag{1.21}$$

As mentioned above, $\delta(x)$ has no 'values' on the x axis, but the statement that the delta function $\delta(x)$ is zero in the vicinity of a point such as $x_0 = 1$ can be given a well-defined meaning by introducing the concept 'support of a distribution'. A distribution $\langle t, \phi\rangle$ is said to vanish in an interval Δ if, for every $\phi(x)$ which has its support in that interval, $\langle t, \phi\rangle = 0$. This clearly holds, in the case of $\delta(x)$, for intervals Δ which do not contain the origin. The support of t is what remains of the x axis when all the Δ intervals have been excluded. The support of δ_0 is therefore the point $x = 0$ (Schwartz 1965).

1.4 Three-dimensional delta-functions

The three-dimensional δ-function is defined by the sifting property

$$\langle\delta_0, \phi\rangle \overset{\text{def}}{=} \phi(0) = \iiint \delta(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V; \tag{1.22}$$

here and in the future, the omission of the integration limits means that the integral is extended over all space.

In Cartesian coordinates, the volume element is $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, and $\delta(\boldsymbol{r})$ can be written explicitly as

$$\delta(\boldsymbol{r}) = \delta(x)\,\delta(y)\,\delta(z). \tag{1.23}$$

In a more general coordinate system, the form of $\mathrm{d}V$ determines that of $\delta(\boldsymbol{r})$. Let (u, v, w) be a set of curvilinear coordinates. The volume element at a regular point is $J\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$, where J denotes the Jacobian of the transformation from the (x, y, z) coordinates into the (u, v, w) coordinates. More explicitly:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{1.24}$$

The three-dimensional delta function can be expressed in terms of one-dimensional functions by the relationship

$$\delta(u-u_0, v-v_0, w-w_0) = \frac{\delta(u-u_0)\delta(v-v_0)\delta(w-w_0)}{J(x_0, y_0, z_0)}. \tag{1.25}$$

The singular points of the coordinate system are those at which the Jacobian vanishes. At such points, the transformation from (x, y, z) into (u, v, w) is no longer of the one-to-one type, and some of the (u, v, w) coordinates become ignorable, i.e. they need not be known to find the corresponding (x, y, z). Let J_k be the integral of J over the ignorable coordinates. Then δ is the product of the δ's relative to the nonignorable coordinates, divided by J_k. In cylindrical coordinates, for example, J is equal to r, and

$$\begin{aligned}\delta(\mathbf{r} - \mathbf{r}_0) &= \delta(r - r_0, \varphi - \varphi_0, z - z_0) \\ &= (1/r_0)\delta(r - r_0)\delta(\varphi - \varphi_0)\delta(z - z_0).\end{aligned} \tag{1.26}$$

Points on the z axis are singular, and φ is ignorable there. We therefore write

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(r)\delta(z - z_0)\Big/\int_0^{2\pi} r\,\mathrm{d}\varphi = \frac{1}{2\pi r}\delta(r)\delta(z - z_0). \tag{1.27}$$

This representation is valid with the convention

$$\int_0^\infty \delta(r)\mathrm{d}r = 1. \tag{1.28}$$

If one chooses

$$\int_0^\infty \delta(r)\mathrm{d}r = \tfrac{1}{2}, \tag{1.29}$$

then the $1/2\pi$ factor in (1.27) should be replaced by $1/\pi$.

In spherical coordinates, J is equal to $R^2 \sin\theta$, and

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(R - R_0)\delta(\varphi - \varphi_0)\delta(\theta - \theta_0)/R^2 \sin\theta\,. \tag{1.30}$$

On the polar axis (where θ_0 is zero or π), the azimuth φ is ignorable, and

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(R - R_0)\delta(\theta - \theta_0)/2\pi R^2 \sin\theta\,. \tag{1.31}$$

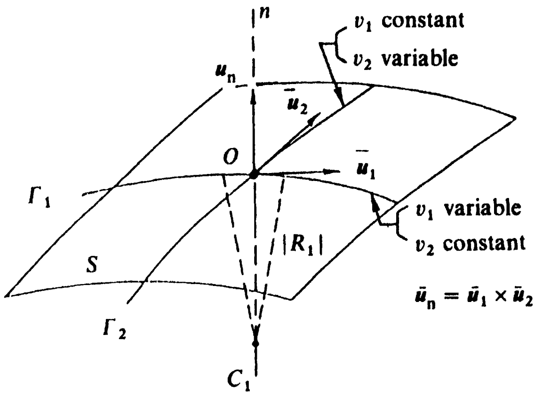
At the origin, both φ and θ are ignorable, and

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(R)/4\pi R^2\,. \tag{1.32}$$

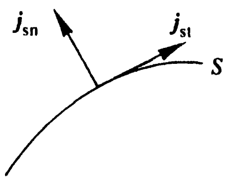
This formula holds when $\delta(R)$ satisfies (1.28), with r replaced by R. If $\delta(R)$ is assumed to satisfy (1.29), then the factor $1/4\pi$ in (1.32) must be replaced by $1/2\pi$.

1.5 Delta-functions on lines and surfaces

The electric charge density ρ_s on a surface is a concentrated source; hence it should be possible to express its value in terms of some appropriate δ-function. A first method to achieve this goal is based on partial separation of variables in the (v_1, v_2, n) coordinate system (Fig. 1.2a). The n coordinate is



(a)



(b)

Fig. 1.2. Coordinates and currents on a surface.

measured along the normal, whereas the (v_1, v_2) coordinates fix the position of a point on S. The lines of constant v_1 (or v_2) are the (orthogonal) lines of curvature Γ_2 (or Γ_1). On a line such as Γ_1, the normals at consecutive points intersect at a common point C_1, termed the centre of curvature. Similar considerations hold for Γ_2. The distances from O to C_1 and C_2, counted positive in the direction of $\boldsymbol{u}_n$, are the two principal radii of curvature: R_1 and R_2. With this sign convention, they are negative at a point where the surface is convex. On a sphere of radius a, for example, $R_1 = R_2 = -a$.

An increase $(\mathrm{d}v_1, \mathrm{d}v_2, \mathrm{d}n)$ in the coordinates results in a displacement $\mathrm{d}l$ given by

$$\mathrm{d}l^2 = h_1^2 \mathrm{d}v_1^2 + h_2^2 \mathrm{d}v_2^2 + \mathrm{d}n^2. \tag{1.33}$$

The quantities h_1 and h_2 are the metrical coefficients.

To determine the distributional form of the surface charge density, we start from a strongly concentrated volume density ρ straddling the surface S. Let us assume that the law of variation of ρ with n is the same for all points of S. Under those circumstances, we write $\rho = g(v_1, v_2)f(n)$. When the concentration increases without limit, the volume charge goes over into a surface charge, and the 'profile function' $f(n)$ becomes a $\delta(n)$ function. We write

$$\rho = \rho_s(v_1, v_2)\delta(n). \tag{1.34}$$

Integrated over space, this expression yields

$$\iiint \rho \, dV = \iiint \rho_s(v_1, v_2)\delta(n)dS \, dn = \iint_S \rho_s(v_1, v_2)dS. \tag{1.35}$$

where $dS = h_1 h_2 \, dv_1 \, dv_2$. A representation such as (1.34) is particularly useful when separation of variables is applicable, and n is one of the coordinates. The correct distributional representation of ρ_s, however, is not based on $\delta(n)$, but on a symbolic (or generalized) function δ_s, defined by the functional

$$\langle \delta_s, \phi \rangle \stackrel{\text{def}}{=} \iint_S \phi(\mathbf{r})dS = \iiint \delta_s \phi(\mathbf{r})dV. \tag{1.36}$$

The meaning of this relationship is the usual one, that is whenever the volume integral is encountered in an analytical development, it may be replaced by the surface integral. The support of δ_s is the surface S. A distribution of surface charge density ρ_s may now be represented by the volume density

$$\rho = \rho_s(v_1, v_2)\delta_s. \tag{1.37}$$

The corresponding functional is (de Jager 1969, 1970)

$$\langle \rho_s \delta_s, \phi \rangle \stackrel{\text{def}}{=} \iint_S \rho_s \phi \, dS = \iiint \rho_s \delta_s \phi \, dV. \tag{1.38}$$

Similarly, a surface electric current may be written in the form (Fig. 1.2b)

$$\mathbf{j} = \mathbf{j}_s(v_1, v_2)\delta_s = \mathbf{j}_{st}(v_1, v_2)\delta_s + \mathbf{j}_{sn}(v_1, v_2)\delta_s. \tag{1.39}$$

The normal component of this current represents a surface distribution of elementary currents, oriented along the normal.

Finally, a distribution of linear charge density ρ_c on a curve C may be represented by the volume density

$$\rho = \rho_c \delta_c, \tag{1.40}$$

where

$$\langle \rho_c \delta_c, \phi \rangle \stackrel{\text{def}}{=} \int_C \rho_c \phi \, dC = \iiint \rho_c \delta_c \phi \, dV. \tag{1.41}$$

1.6 Multiplication of distributions

There is no natural way to define the product of two distributions. A locally integrable function, for example, generates a distribution; but the product of two such functions might not be locally integrable, and hence it might not generate a distribution. To illustrate the point, consider the function $f(x) = 1/\sqrt{|x|}$, which is locally integrable. Its square $f^2(x) = 1/|x|$, however, is not, and does *not* define a distribution. In general, the more f is irregular, the more g must be regular if the product fg is to have a meaning. Multiplication by an infinitely differentiable function $\alpha(x)$, however, is always meaningful. More precisely:

$$\langle \alpha t, \phi \rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \alpha(x) t(x) \phi(x) \mathrm{d}x = \int_{-\infty}^{\infty} t(x) [\alpha(x)\phi(x)] \mathrm{d}x = \langle t, \alpha\phi \rangle. \tag{1.42}$$

This result is based on the fact that $\alpha\phi$ is a test function. An example of application of (1.42) is relationship (1.21). Another one is

$$\alpha(x)\delta(x - x_0) = \alpha(x_0)\delta(x - x_0). \tag{1.43}$$

The restriction to infinitely differentiable $\alpha(x)$ is not always necessary. The function $\alpha(x)\delta(x)$, for example, has a meaning, namely $\alpha(0)\delta(x)$, once $\alpha(x)$ is continuous at the origin.

It should be noted that multiplication of distributions, even when defined, is not necessarily associative. For example:

$$\left(\frac{1}{x} x\right)\delta(x) = \delta(x), \qquad \frac{1}{x}[x\delta(x)] = \frac{1}{x} 0 = 0. \tag{1.44}$$

1.7 Change of variables

The operation 'change of variables' starts from a generating function $t(x)$, and introduces $t[f(x)]$ by means of the formula (Friedman 1969)

$$\int_{-\infty}^{\infty} t[f(x)]\phi(x)\mathrm{d}x = \int_{-\infty}^{\infty} t(y)\left(\frac{\mathrm{d}}{\mathrm{d}y}\int_{f(u)<y} \phi(u)\mathrm{d}u\right)\mathrm{d}y. \tag{1.45}$$

The right-hand side has a meaning, provided that the term in big parentheses is a test function. Let us investigate, for example, the properties of $\delta(\alpha x - \beta)$. From (1.45):

$$\int_{-\infty}^{\infty} \delta(\alpha x - \beta)\phi(x)\mathrm{d}x = \int_{-\infty}^{\infty} \delta(y)\left(\frac{\mathrm{d}}{\mathrm{d}y}\int_{\alpha u - \beta < y} \phi(u)\mathrm{d}u\right)\mathrm{d}y. \tag{1.46}$$

Assume first that $\alpha > 0$. The integral becomes

$$\int_{-\infty}^{\infty} \delta(y)\left(\frac{\mathrm{d}}{\mathrm{d}y}\int_{-\infty}^{(1/\alpha)(y+\beta)} \phi(u)\mathrm{d}u\right)\mathrm{d}y = \frac{1}{\alpha}\phi\left(\frac{\beta}{\alpha}\right); \tag{1.47}$$

hence

$$\delta(\alpha x - \beta) = (1/\alpha)\delta(x - \beta/\alpha). \tag{1.48}$$

Consider now the case $\alpha < 0$. The integral takes the form

$$\int_{-\infty}^{\infty} \delta(y)\left(\frac{\mathrm{d}}{\mathrm{d}y}\int_{(1/\alpha)(y+\beta)}^{\infty} \phi(u)\mathrm{d}u\right)\mathrm{d}y = \int_{-\infty}^{\infty} \delta(y)\left[-\frac{1}{\alpha}\phi\left(\frac{y+\beta}{\alpha}\right)\right]\mathrm{d}y$$

$$= -\frac{1}{\alpha}\phi\left(\frac{\beta}{\alpha}\right); \tag{1.49}$$

hence

$$\delta(\alpha x - \beta) = -(1/\alpha)\delta(x - \beta/\alpha). \tag{1.50}$$

The two cases may be combined into a single formula:

$$\delta(\alpha x - \beta) = (1/|\alpha|)\delta(x - \beta/\alpha). \tag{1.51}$$

Similarly

$$\delta[(x-a)(x-b)] = (1/|b-a|)[\delta(x-a) + \delta(x-b)] \quad (a \neq b). \tag{1.52}$$

A particular application of (1.51) is

$$\delta(x) = \delta(-x). \tag{1.53}$$

The δ-function is therefore an 'even' function, a property which is in harmony with the profile of the curves shown in Fig. 1.1.

Additional useful formulas may be obtained from (1.45). For example:

$$\delta(x^2 - a^2) = (1/2|a|)[\delta(x-a) + \delta(x+a)]. \tag{1.54}$$

As a particular case:

$$|x|\delta(x^2) = \delta(x). \tag{1.55}$$

Consider further a function $f(x)$ which varies monotonically, vanishes at $x = x_0$, and satisfies $f'(x_0) \neq 0$. For such a function,

$$\delta[f(x)] = (1/|f'(x_0)|)\delta(x - x_0). \tag{1.56}$$

Result (1.51) follows directly from this formula. Finally, (1.45) also leads to

$$\langle t(x/a), \phi(x)\rangle = |a|\langle t(x), \phi(ax)\rangle. \tag{1.57}$$

1.8 The derivative of a distribution

The derivative of a distribution t is a new distribution t', defined by the functional

$$\langle t', \phi \rangle \stackrel{\text{def}}{=} -\left\langle t, \frac{d\phi}{dx} \right\rangle = -\int_{-\infty}^{\infty} t \frac{d\phi}{dx} dx = \int_{-\infty}^{\infty} \frac{dt}{dx} \phi \, dx. \tag{1.58}$$

Every distribution, therefore, has a derivative: a property which obviously has no analogue in the classical theory of functions.

One expects the generating function $t'(x)$ to coincide with the usual derivative when both t and dt/dx are continuous. That this is so may be shown by the following elementary integration:

$$-\int_{-\infty}^{\infty} t \frac{d\phi}{dx} dx = -[t(x)\phi(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{dt}{dx} \phi dx = \int_{-\infty}^{\infty} \frac{dt}{dx} \phi dx. \tag{1.59}$$

To obtain this result, we took into account that $t(x)$ is bounded, and that $\phi(x)$ vanishes at $x = \infty$ and $x = -\infty$.

As a second application, consider the automatic introduction of a δ-function into the derivative of a function $t(x)$ which suffers a jump discontinuity A at x_0 (Fig. 1.3). From (1.58), since t remains bounded in x_0,

$$-\int_{-\infty}^{\infty} t \frac{d\phi}{dx} dx = -\int_{-\infty}^{x_0^-} t \frac{d\phi}{dx} dx - \int_{x_0^+}^{\infty} t \frac{d\phi}{dx} dx \tag{1.60}$$

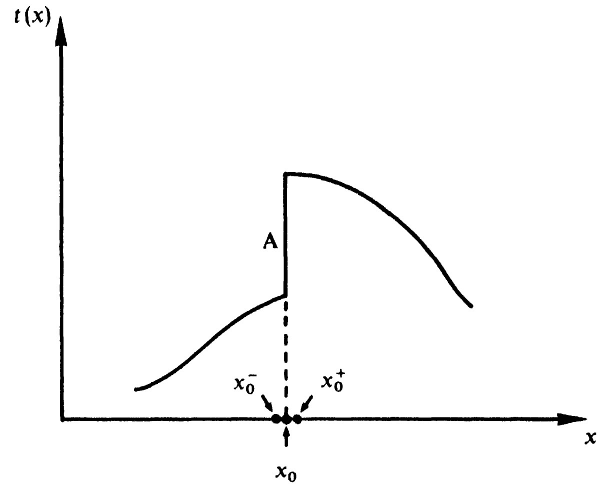


Fig. 1.3. A piecewise continuous function.

An integration by parts yields

$$\begin{aligned}\langle t', \phi\rangle &= A\phi(x_0) + \int_{-\infty}^{x_0^-} \frac{dt}{dx}\phi\,dx + \int_{x_0^+}^{\infty} \frac{dt}{dx}\phi\,dx \\ &= \int_{-\infty}^{\infty} \left(A\delta(x - x_0) + \left\{\frac{dt}{dx}\right\}\right)\phi\,dx.\end{aligned} \tag{1.61}$$

The notation $\{dt/dx\}$, used frequently in the sequel, represents a function which is equal to the usual derivative anywhere but at x_0, where it remains undefined. The generating function of t' is therefore

$$\frac{dt}{dx} = A\delta(x - x_0) + \left\{\frac{dt}{dx}\right\}. \tag{1.62}$$

The derivative of the Heaviside step function, in particular, is

$$\frac{dY}{dx} = \delta(x). \tag{1.63}$$

The definition (1.58), applied to the Naperian logarithm, yields (Dirac, 1958, Schwartz, 1965, Lützen, 1982)

$$\frac{d}{dx}\log_e|x| = \mathrm{PV}\frac{1}{x}, \tag{1.64}$$

$$\frac{d}{dx}\log_e x = \frac{1}{x} - j\pi\delta(x) \quad \text{(on one branch)}. \tag{1.65}$$

As an illustration of these concepts, consider the example of a particle of mass m which moves under the influence of a continuous force f, and experiences a sudden momentum increase $m\mathbf{v}_0$ (a kick) at $t = 0$. In the spirit of (1.62) we write the equation of motion of the particle as

$$m\frac{d\mathbf{v}}{dt} = \{f\} + m\mathbf{v}_0\delta(t). \tag{1.66}$$

The right-hand member is the generalized force.

1.9 Properties of the derivative

We will now list some of the important properties of the derivative in the distributional sense:

(1) A distribution has derivatives of all orders. Further, the ordering of differentiation in a partial derivative may always be permuted.

(2) A series of distributions which converges in the sense discussed in Section 1.2 may be differentiated term by term. This holds, for example, for

the Fourier series (Dirac 1958)

$$\sum_{k=-\infty}^{\infty} e^{j2\pi kx} = \sum_{k=-\infty}^{\infty} \delta(x-k). \tag{1.67}$$

The left-hand member is divergent in the classical sense but, in the sense of distributions, it yields periodic 'sharp' spectral lines at $x = 0, \pm 1, \pm 2, \ldots$. Differentiation yields

$$j2\pi \sum_{k=-\infty}^{\infty} k\, e^{j2\pi kx} = \sum_{k=-\infty}^{\infty} \delta'(x-k). \tag{1.68}$$

(3) The usual differentiation formulas are valid, for instance:

$$\frac{d}{dx}(\alpha t) = \frac{d\alpha}{dx} t + \alpha \frac{dt}{dx}. \tag{1.69}$$

Also, $dt/dx = ds/dx$ implies that t and s differ by a constant.

(4) The operations of differentiation and passing to the limit may always be interchanged. Specifically, if f_m converges to f as $m \to \infty$, then

$$\lim_{m\to\infty} \left\langle \frac{df_m}{dx}, \phi \right\rangle = \left\langle \frac{df}{dx}, \phi \right\rangle. \tag{1.70}$$

(5) The chain rule for differentiation remains valid. Thus,

$$\frac{d}{dx} t[f(x)] = t'[f(x)]\, f'(x). \tag{1.71}$$

Let us apply this formula to $t(x) = \mathrm{Y}(x)$ and $f(x) = x^2 - a^2$. From (1.54):

$$\frac{d}{dx}\mathrm{Y}(x^2 - a^2) = \delta(x^2 - a^2)2x = \frac{x}{|a|}[\delta(x-a) + \delta(x+a)]$$
$$= \delta(x-a) - \delta(x+a). \tag{1.72}$$

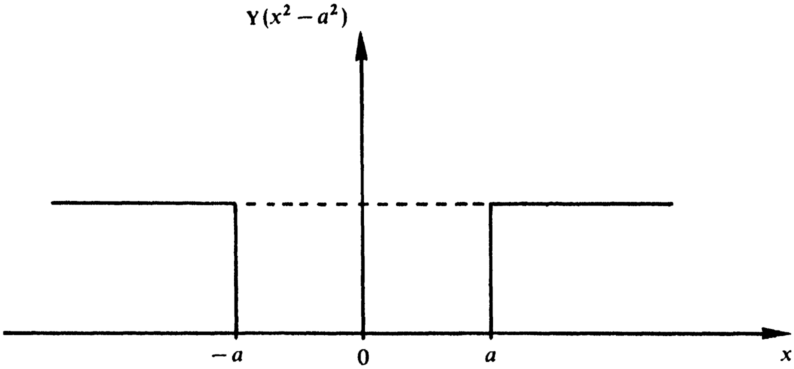


Fig. 1.4. An example of a piecewise continuous function.

This result could have been predicted from the graphical representation of $Y(x^2 - a^2)$, given in Fig. 1.4.

1.10 Partial derivatives of scalar functions

In three dimensions the partial derivative $\partial t/\partial x_i$ is defined by the functional

$$\left\langle \frac{\partial t}{\partial x_i}, \phi \right\rangle \stackrel{\text{def}}{=} -\iiint t \frac{\partial \phi}{\partial x_i} \mathrm{d}V = \iiint \frac{\partial t}{\partial x_i} \phi \, \mathrm{d}V. \tag{1.73}$$

This definition can serve to express $\operatorname{grad} t$ in the sense of distributions. Applying (1.73) successively to $\partial t/\partial x$, $\partial t/\partial y$, and $\partial t/\partial z$ yields

$$\langle \operatorname{grad} t, \phi \rangle \stackrel{\text{def}}{=} -\iiint t \operatorname{grad} \phi \, \mathrm{d}V = \iiint \phi \operatorname{grad} t \, \mathrm{d}V. \tag{1.74}$$

Let us apply this formula to the three-dimensional Heaviside function Y_s, equal to one in V_1, and to zero in V_2 (Fig. 1.5). From (1.74) (Bouix 1964):

$$\begin{aligned}\iiint \phi \operatorname{grad} Y_s \, \mathrm{d}V &= -\iiint_{V_1} Y_s \operatorname{grad} \phi \, \mathrm{d}V \\ &= -\iiint_{V_1} \operatorname{grad}(\phi \, Y_s) \mathrm{d}V + \iiint_{V_1} \phi \operatorname{grad} Y_s \mathrm{d}V \\ &= \iint_S \phi \, Y_s \boldsymbol{u}_{n1} \, \mathrm{d}S = \iint_S \phi \boldsymbol{u}_{n1} \, \mathrm{d}S.\end{aligned} \tag{1.75}$$

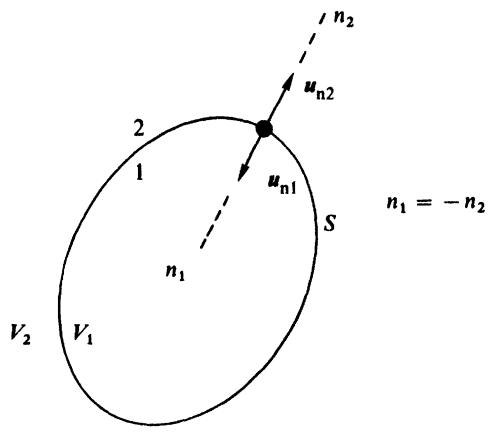


Fig. 1.5. A surface across which discontinuities may occur

This manipulation shows that, in the $\delta(n)$ formalism,

$$\operatorname{grad} Y_s = \delta(n_1)\boldsymbol{u}_{n1} \tag{1.76}$$

Applied to a test-function triple $\boldsymbol{\phi} = (\phi_x, \phi_y, \phi_z)$, (1.76) leads to the relationship

$$\langle \operatorname{grad} Y_s, \boldsymbol{\phi}\rangle = \iiint \boldsymbol{\phi}\cdot \operatorname{grad} Y_s \,\mathrm{d}V = \iint \boldsymbol{\phi}\cdot \boldsymbol{u}_{n1}\,\mathrm{d}S. \tag{1.77}$$

The extension of the concept 'derivative' to higher orders is immediate. In one-dimensional space, for example,

$$\left\langle \frac{\mathrm{d}^m t}{\mathrm{d}x^m}, \phi\right\rangle \overset{\text{def}}{=} (-1)^m \left\langle t, \frac{\mathrm{d}^m \phi}{\mathrm{d}x^m}\right\rangle. \tag{1.78}$$

Let us apply this formula to the second derivative of $|x|$. The steps are elementary:

$$\begin{aligned}\left\langle \frac{\mathrm{d}^2|x|}{\mathrm{d}x^2}, \phi\right\rangle &= -\int_{-\infty}^{0^-} x\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\mathrm{d}x + \int_{0^+}^{\infty} x\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\mathrm{d}x \\ &= -\int_{-\infty}^{0^-}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) - \frac{\mathrm{d}\phi}{\mathrm{d}x}\right]\mathrm{d}x + \int_{0^+}^{\infty}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) - \frac{\mathrm{d}\phi}{\mathrm{d}x}\right]\mathrm{d}x \\ &= \phi(0^-) + \phi(0^+) = 2\,\phi(0).\end{aligned} \tag{1.79}$$

The generating function of the second derivative is therefore

$$\frac{\mathrm{d}^2|x|}{\mathrm{d}x^2} = 2\delta(x). \tag{1.80}$$

Higher derivatives in n dimensions are defined along analogous lines. A linear differential operator in n dimensions is typically a summation of the form

$$\mathscr{L} = \sum_{\boldsymbol{p}} A_{\boldsymbol{p}}\left(\frac{\partial}{\partial x_1}\right)^{p_1}\left(\frac{\partial}{\partial x_2}\right)^{p_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{p_n}, \tag{1.81}$$

where $\boldsymbol{p} = (p_1, \ldots, p_n)$. The adjoint of $\mathscr{L}$ is

$$\mathscr{L}^\dagger = \sum_{\boldsymbol{p}} A_{\boldsymbol{p}}(-1)^{p_1+\cdots+p_n}\left(\frac{\partial}{\partial x_1}\right)^{p_1}\left(\frac{\partial}{\partial x_2}\right)^{p_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{p_n}, \tag{1.82}$$

and the meaning of $\mathscr{L}t$ follows from

$$\langle \mathscr{L}t, \phi\rangle \overset{\text{def}}{=} \langle t, \mathscr{L}^\dagger \phi\rangle. \tag{1.83}$$

Applied to the Laplacian, this gives

$$\langle \nabla^2 t, \phi\rangle \overset{\text{def}}{=} \langle t, \nabla^2 \phi\rangle \tag{1.84}$$

In particular (Schwartz 1965; Bass 1971; Petit 1987):

$$\begin{aligned} \nabla^2 \log_e 1/|\boldsymbol{r}-\boldsymbol{r}'| &= -2\pi\delta(\boldsymbol{r}-\boldsymbol{r}') \quad \text{in 2 dimensions,} \\ \nabla^2 (1/|\boldsymbol{r}-\boldsymbol{r}'|) &= -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}') \quad \text{in 3 dimensions.} \end{aligned} \tag{1.85}$$

The 'weak' definition of the derivative given above allows recasting a differential equation such as

$$\nabla^2 f = g \tag{1.86}$$

in the form

$$\langle \nabla^2 f, \phi \rangle = \langle g, \phi \rangle = \langle f, \nabla^2 \phi \rangle. \tag{1.87}$$

This formulation transfers the operator ∇^2 from the unknown f to the test function ϕ. It further avoids the difficulties which arise with a classical differential equation such as

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = 0. \tag{1.88}$$

This equation is satisfied by every function of x alone, whereas

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0. \tag{1.89}$$

need not have a sense for such a function. One method to avoid this difficulty is to follow Schwartz' example, and supplement 'usual' functions with new objects: the distributions. These always allow differentiation, and in particular the exchange of the *order* of differentiation.

1.11 Derivatives of δ(x)

According to the general definition (1.58) $\delta'(x)$ is the generating function of

$$\langle \delta', \phi \rangle = -\int_{-\infty}^{\infty} \delta(x) \frac{\mathrm{d}\phi}{\mathrm{d}x} \mathrm{d}x = -\phi'(0). \tag{1.90}$$

One can visualize $\delta'(x)$ by considering the derivatives of the functions shown in Fig. 1.1, which represent the δ-function in the limit $n \to \infty$. A typical graph of the first derivative is sketched in Fig. 1.6a. In the case of the 'rectangular pulse' (Fig. 1.1a), the graph reduces to two delta functions: a positive one supported at $x = 1/n$, and a negative one at $x = -1/n$. Such a 'doublet' can be physically realized by two point charges separated by a short distance 2ε (Fig. 1.6b). If ε approaches zero while $2\varepsilon q$ keeps a fixed value p_e, the linear charge density of the doublet takes the form

$$\rho = q\delta[x - (x_0 + \varepsilon)] - q\delta[x - (x_0 - \varepsilon)]. \tag{1.91}$$

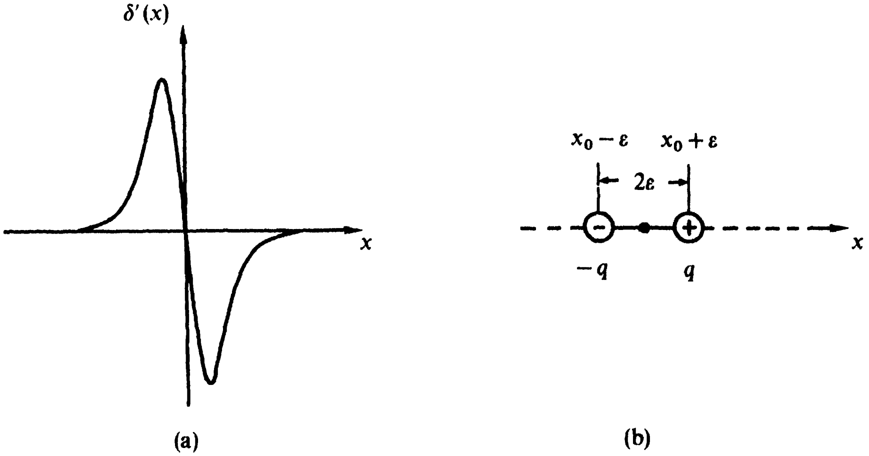


Fig. 1.6. Illustrations relating to the derivative of the delta-function.

Following Dirac's example, we treat the δ-function as a usual function and write

$$\rho = -q\{\delta[(x-x_0)+\varepsilon] - \delta[(x-x_0)-\varepsilon]\} = -q2\varepsilon\,\delta'(x-x_0)$$
$$= -p_e\,\delta'(x-x_0). \tag{1.92}$$

This is the volume density of an x-directed dipole. Schwartz (1965) remarked that the mathematical distributions constitute a correct description of the distributions encountered in physics (monopoles, dipoles, quadrupoles, etc.). This point of view is further discussed in Chapter 2.

Higher derivatives of $\delta(x)$ are defined by the relationship

$$\langle \delta^{(m)}, \phi \rangle \stackrel{\text{def}}{=} (-1)^m \phi^{(m)}(0). \tag{1.93}$$

The first derivative has the following useful property:

$$\alpha(x)\delta'(x) = \alpha(0)\delta'(x) - \alpha'(0)\delta(x), \tag{1.94}$$

where $\alpha(x)$ is infinitely differentiable. As an example:

$$x\,\delta'(x) = -\delta(x), \qquad x^2\,\delta'(x) = 0. \tag{1.95}$$

Other properties of interest are:

$$x\,\delta^{(m)}(x) = -m\,\delta^{(m-1)}(x), \qquad x^n\,\delta^{(m)}(x) = 0 \quad \text{when } n > m+1, \tag{1.96}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta[g(x)] = \delta'[g(x)]\,g'(x).$$

If g is infinitely differentiable for $x < x_0$ and $x > x_0$, and if g and all its derivatives have left-hand and right-hand limits at $x = x_0$, then

$$g' = \{g'\} + \sigma_0\delta(x - x_0), \qquad g'' = \{g''\} + \sigma_0\delta'(x - x_0) + \sigma_1\delta(x - x_0),$$
$$g^{(m)} = \{g^{(m)}\} + \sigma_0\delta^{(m-1)}(x - x_0) + \cdots + \sigma_{m-1}\delta(x - x_0). \tag{1.97}$$

Here σ_k denotes the difference between the right-hand and the left-hand limit of the kth derivative, while $\{g^k\}$ denotes the distribution generated by a function equal to the usual kth derivative for $x \neq x_0$, but not defined at $x = x_0$. Equation (1.80) may be obtained directly from the value of g'' given above.

1.12 Partial derivatives of δ-functions

The first partial derivatives of $\delta(\boldsymbol{r} - \boldsymbol{r}_0)$ can be defined directly from (1.73) and (1.74). Thus,

$$\left\langle \frac{\partial \delta}{\partial x_i}, \phi \right\rangle = \iiint \frac{\partial \delta(\boldsymbol{r} - \boldsymbol{r}_0)}{\partial x_i} \phi \, \mathrm{d}V \overset{\text{def}}{=} - \iiint \delta(\boldsymbol{r} - \boldsymbol{r}_0) \frac{\partial \phi}{\partial x_i} \mathrm{d}V$$
$$= - \left[\frac{\partial \phi}{\partial x_i} \right]_{\boldsymbol{r}_0}, \tag{1.98}$$

$$\langle \operatorname{grad} \delta, \phi \rangle = \iiint \operatorname{grad} \delta(\boldsymbol{r} - \boldsymbol{r}_0) \mathrm{d}V \overset{\text{def}}{=} - \iiint \delta(\boldsymbol{r} - \boldsymbol{r}_0) \operatorname{grad} \phi \, \mathrm{d}V$$
$$= - [\operatorname{grad} \phi]_{\boldsymbol{r}_0}.$$

These relationships can serve to express the volume density of a concentrated dipole $\boldsymbol{p}_e$ as

$$\rho = - \boldsymbol{p}_e \cdot \operatorname{grad} \delta_{\boldsymbol{r}_0}. \tag{1.99}$$

The value (1.92), derived for an x-oriented dipole, is a particular case of this formula. Derivatives of δ_s may also be defined in accordance with (1.73). In a direction $\boldsymbol{a}$, for example,

$$\left\langle \frac{\partial \delta_s}{\partial a}, \phi \right\rangle \overset{\text{def}}{=} - \iint_S \frac{\partial \phi}{\partial a} \mathrm{d}S = \iiint \phi \frac{\partial \delta_s}{\partial a} \mathrm{d}V. \tag{1.100}$$

It follows that

$$\langle \operatorname{grad} \delta_s, \phi \rangle \overset{\text{def}}{=} - \iint_S \operatorname{grad} \phi \, \mathrm{d}S = \iiint \phi \operatorname{grad} \delta_s \mathrm{d}V. \tag{1.101}$$

When $\boldsymbol{a}$ coincides with the normal to S, and τ is a function of the surface coordinates v_1 and v_2 only,

$$\left\langle \tau \frac{\partial \delta_s}{\partial n}, \phi \right\rangle \overset{\text{def}}{=} - \iint_S \tau \frac{\partial \phi}{\partial n} \mathrm{d}S = \iiint \tau \frac{\partial \delta_s}{\partial n} \phi \, \mathrm{d}V. \tag{1.102}$$

Since the potential generated by a dipole layer of density τ is

$$\phi(\mathbf{r}_0) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\mathbf{r})}{|\mathbf{r}_0 - \mathbf{r}|}\,\mathrm{d}V = \frac{1}{4\pi\varepsilon_0}\iint_S \tau(\mathbf{r})\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}_0 - \mathbf{r}|}\,\mathrm{d}S, \tag{1.103}$$

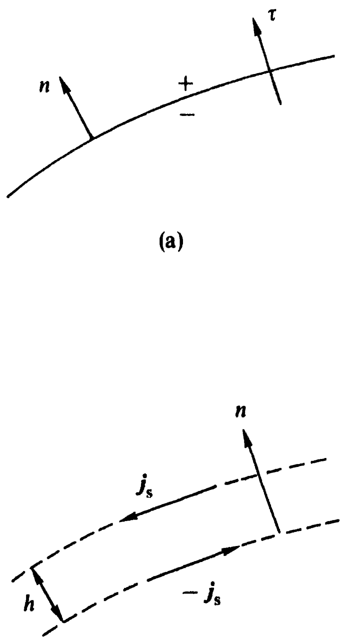
it becomes clear that the volume density of a double layer can be represented as

$$\rho = -\,\tau(v_1, v_2)\frac{\partial \delta_s}{\partial n}. \tag{1.104}$$

In this formula, n is counted positive in the direction of the dipoles (Fig. 1.7a). In Appendix A, we show that the generalized function $\partial\delta_s/\partial n$ is not equivalent to $\delta'(n)$ when the surface is curved.

The distributional representation of a double layer of surface *currents* can be obtained by similar steps. Using the sign convention on Fig. 1.7b, we write

$$\mathbf{j} = -\,\mathbf{c}_s(v_1, v_2)\frac{\partial \delta_s}{\partial n}. \tag{1.105}$$



(a)

(b)

Fig. 1.7. Double layers on a surface.

In this equation, we have assumed that $c_s = j_s h$ approaches a well-defined (nonzero) limit when the distance h between the two layers approaches zero. The detailed nature of the limit process yielding (1.105) is discussed in Appendix A.

Analogous considerations hold for the distribution δ_c defined in (1.41). The gradient is now

$$\langle \operatorname{grad} \delta_c, \phi \rangle \stackrel{\text{def}}{=} -\int_C \operatorname{grad} \phi \, dC = \iiint \phi \operatorname{grad} \delta_c \, dV. \tag{1.106}$$

1.13 Piecewise continuous scalar functions

The derivatives of a one-dimensional piecewise continuous function have been discussed in Sections 1.8 and 1.11. We extend these concepts to three-dimensions by considering a function t which is continuous and has continuous derivatives in regions 1 and 2 (Fig. 1.5). Both t and its derivatives are assumed to approach well-defined limits on sides 1 and 2 of S. The gradient of such a function is

$$\operatorname{grad} t = \{\operatorname{grad} t\} + (t_2 - t_1)\delta(n_2)\mathbf{u}_{n2}. \tag{1.107}$$

This relationship is obtained by applying to V_1 and V_2 the formula for the integral of the gradient over a volume. In harmony with previous notation, the term between brackets represents the value of the gradient anywhere but on S.

Since the effect of $\delta(n)$ is to reduce the volume integral to a surface integral, (1.107) may be rewritten more elegantly as

$$\operatorname{grad} t = \{\operatorname{grad} t\} + (\mathbf{u}_{n1} t_1 + \mathbf{u}_{n2} t_2)\delta_s. \tag{1.108}$$

In particular,

$$\operatorname{grad} Y_s = \delta_s \mathbf{u}_{n1}, \tag{1.109}$$

from which (1.76) immediately follows. By similar arguments, combined with an application of Green's theorem to the regions V_1 and V_2, the distributional form of the Laplacian of t is found to be (Schwartz 1965)

$$\nabla^2 t = \{\nabla^2 t\} + \left(\frac{\partial t}{\partial n_1} + \frac{\partial t}{\partial n_2}\right)\delta_s + t_1 \frac{\partial \delta_s}{\partial n_1} + t_2 \frac{\partial \delta_s}{\partial n_2}. \tag{1.110}$$

In Electrostatics, t is the potential ϕ, in which case, (1.110) implies that a discontinuity of ϕ may be represented by a double layer of charge, and a discontinuity of $\partial\phi/\partial n$ (i.e. of the normal component of $\mathbf{e}$) by a single layer. Conversely, the boundary conditions of ϕ on S can be derived directly from (1.110). The proof is elementary. Assume that S carries a charge density ρ_s (a single layer) and a dipole density τ (a dipole layer). The corresponding volume density is, from (1.37) and (1.104),

$$\rho = \rho_s \delta_s - \tau \frac{\partial \delta_s}{\partial n_2}. \tag{1.111}$$

In the philosophy of distribution theory, Poisson's equation

$$\nabla^2 \phi = -\rho/\varepsilon_0 \tag{1.112}$$

is valid throughout space. Comparing (1.110), (1.111), and (1.112) shows that

$$\left(-\varepsilon_0 \frac{\partial \phi_1}{\partial n_1}\right) + \left(-\varepsilon_0 \frac{\partial \phi_2}{\partial n_2}\right) = \rho_s \quad \text{and} \quad \phi_2 - \phi_1 = \frac{\tau}{\varepsilon_0} \quad \text{on } S. \tag{1.113}$$

These are the classical boundary conditions.

1.14 Vector operators

Let $\boldsymbol{t}$ be a vector distribution, i.e. a triple of scalar distributions t_x, t_y, t_z. The operator $\operatorname{div} \boldsymbol{t}$ is defined, in classical vector analysis, by the expression

$$\operatorname{div} \boldsymbol{t} = \frac{\partial t_x}{\partial x} + \frac{\partial t_y}{\partial y} + \frac{\partial t_z}{\partial z}. \tag{1.114}$$

The distributional definition of $\operatorname{div} \boldsymbol{t}$ follows by applying (1.73) to the three derivatives shown above. More specifically:

$$\langle \operatorname{div} \boldsymbol{t}, \phi \rangle \stackrel{\text{def}}{=} -\iiint \boldsymbol{t} \cdot \operatorname{grad} \phi \, \mathrm{d}V = \iiint \phi \operatorname{div} \boldsymbol{t} \, \mathrm{d}V. \tag{1.115}$$

Such a definition gives a well-defined meaning to the equation

$$\operatorname{div} \boldsymbol{d} = \rho; \tag{1.116}$$

according to (1.115) it is

$$-\iiint \boldsymbol{d} \cdot \operatorname{grad} \phi \, \mathrm{d}V = \iiint \rho \, \phi \, \mathrm{d}V. \tag{1.117}$$

This relationship, which must hold for all test functions ϕ, remains valid when $\boldsymbol{d}$ does not possess everywhere the derivatives shown in (1.114). In consequence, Maxwell's equation

$$\operatorname{div} \boldsymbol{b} = 0 \tag{1.118}$$

now is interpreted as requiring

$$\iiint \boldsymbol{b} \cdot \operatorname{grad} \phi \, \mathrm{d}V = 0 \tag{1.119}$$

to hold for all ϕ.

As mentioned previously, integral formulations such as (1.116) and (1.118) make physical sense because macroscopic experimental evidence is obtained on an 'average' basis, rather than at a point. In addition, switching the differential operator to the test function has the advantage of broadening the class of admissible solutions to those which do not have a divergence in the classical sense of the word. This holds, for example, for the fields which exist at the leading front of a pulsed disturbance.

The distributional definition of curl $\boldsymbol{t}$ follows analogously from the classical value

$$\operatorname{curl} \boldsymbol{t} = \left(\frac{\partial t_z}{\partial y} - \frac{\partial t_y}{\partial z}, \frac{\partial t_x}{\partial z} - \frac{\partial t_z}{\partial x}, \frac{\partial t_y}{\partial x} - \frac{\partial t_x}{\partial y} \right). \tag{1.120}$$

The corresponding functionals are

$$\begin{aligned} \langle \operatorname{curl} \boldsymbol{t}, \phi \rangle &\overset{\text{def}}{=} \iiint \boldsymbol{t} \times \operatorname{grad} \phi \, \mathrm{d}V = \iiint \phi \operatorname{curl} \boldsymbol{t} \, \mathrm{d}V, \\ \langle \operatorname{curl} \boldsymbol{t}, \boldsymbol{\phi} \rangle &\overset{\text{def}}{=} \iiint \boldsymbol{t} \cdot \operatorname{curl} \boldsymbol{\phi} \, \mathrm{d}V = \iiint \boldsymbol{\phi} \cdot \operatorname{curl} \boldsymbol{t} \, \mathrm{d}V. \end{aligned} \tag{1.121}$$

A relationship such as

$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j} \tag{1.122}$$

now means, in a distributional sense, that

$$\iiint \phi \boldsymbol{j} \, \mathrm{d}V = \iiint \boldsymbol{h} \times \operatorname{grad} \phi \, \mathrm{d}V, \qquad \iiint \boldsymbol{\phi} \cdot \boldsymbol{j} \, \mathrm{d}V = \iiint \boldsymbol{h} \cdot \operatorname{curl} \boldsymbol{\phi} \, \mathrm{d}V. \tag{1.123}$$

An irrotational vector is therefore characterized by the properties

$$\iiint \boldsymbol{t} \times \operatorname{grad} \phi \, \mathrm{d}V = 0, \qquad \iiint \boldsymbol{t} \cdot \operatorname{curl} \boldsymbol{\phi} \, \mathrm{d}V = 0. \tag{1.124}$$

Extension to vector operators involving higher derivatives than the first proceeds in an analogous fashion. For example:

$$\begin{aligned} \langle \operatorname{curl} \operatorname{curl} \boldsymbol{t}, \phi \rangle &\overset{\text{def}}{=} -\iiint \boldsymbol{t} \, \nabla^2 \phi \, \mathrm{d}V + \iiint \boldsymbol{t} \cdot \operatorname{grad} \operatorname{grad} \phi \, \mathrm{d}V \\ &= \iiint \phi \operatorname{curl} \operatorname{curl} \boldsymbol{t} \, \mathrm{d}V, \\ \langle \operatorname{curl} \operatorname{curl} \boldsymbol{t}, \boldsymbol{\phi} \rangle &\overset{\text{def}}{=} \iiint \boldsymbol{t} \cdot \operatorname{curl} \operatorname{curl} \boldsymbol{\phi} \, \mathrm{d}V \\ &= \iiint \boldsymbol{\phi} \cdot \operatorname{curl} \operatorname{curl} \boldsymbol{t} \, \mathrm{d}V. \end{aligned} \tag{1.125}$$

The meaning of the symbol grad grad is discussed in Appendix A. Similarly the operator grad div is defined by the functionals

$$\begin{aligned}\langle \operatorname{grad}\operatorname{div} \boldsymbol{t}, \phi\rangle &\overset{\text{def}}{=} \iiint \boldsymbol{t}\cdot \operatorname{grad}\operatorname{grad}\phi \, \mathrm{d}V \\ &= \iiint \phi \operatorname{grad}\operatorname{div}\boldsymbol{t}\, \mathrm{d}V, \\ \langle \operatorname{grad}\operatorname{div} \boldsymbol{t}, \boldsymbol{\phi}\rangle &\overset{\text{def}}{=} \iiint \boldsymbol{t}\cdot \operatorname{grad}\operatorname{div}\boldsymbol{\phi}\, \mathrm{d}V \\ &= \iiint \boldsymbol{\phi}\cdot \operatorname{grad}\operatorname{div}\boldsymbol{t}\, \mathrm{d}V.\end{aligned} \tag{1.126}$$

Combining (1.125) and (1.126) yields, for the vector Laplacian,

$$\begin{aligned}\langle \nabla^2 \boldsymbol{t}, \phi\rangle &\overset{\text{def}}{=} \iiint \boldsymbol{t}\; \nabla^2\phi \, \mathrm{d}V = \iiint \phi\; \nabla^2 \boldsymbol{t}\, \mathrm{d}V, \\ \langle \nabla^2 \boldsymbol{t}, \boldsymbol{\phi}\rangle &\overset{\text{def}}{=} \iiint \boldsymbol{t}\cdot \nabla^2 \boldsymbol{\phi}\, \mathrm{d}V = \iiint \boldsymbol{\phi}\cdot \nabla^2 \boldsymbol{t}\, \mathrm{d}V.\end{aligned} \tag{1.127}$$

1.15 Piecewise continuous vector functions

Let $\boldsymbol{t}$ be a continuous vector function which suffers jumps across a surface S. The distributional formula for its *divergence* is (Fig. 1.5)

$$\operatorname{div}\boldsymbol{t} = \{\operatorname{div}\boldsymbol{t}\} + (\boldsymbol{u}_{n1}\cdot\boldsymbol{t}_1 + \boldsymbol{u}_{n2}\cdot\boldsymbol{t}_2)\delta_s. \tag{1.128}$$

This equation is obtained by applying the divergence theorem to volumes V_1 and V_2. When used to interpret Maxwell's equation (1.116) in the sense of distributions, expression (1.128) implies that d_n suffers a jump of ρ_s across S. It also allows writing the equation of continuity of charge in the form (Idemen 1973)

$$\{\operatorname{div}\boldsymbol{j}\} - (\boldsymbol{u}_n\cdot\boldsymbol{j})\delta_s + \left\{\frac{\partial\rho}{\partial t}\right\} + \frac{\partial\rho_s}{\partial t}\delta_s = 0. \tag{1.129}$$

In consequence (Fig. 1.8):

$$\operatorname{div}\boldsymbol{j} + \frac{\partial\rho}{\partial t} = 0 \quad \text{in } V, \qquad \frac{\partial\rho_s}{\partial t} = \boldsymbol{u}_n\cdot\boldsymbol{j} \quad \text{on } S. \tag{1.130}$$

If the surface S carries surface currents of density $\boldsymbol{j}_s$, not flowing beyond a curve C, the equation of conservation of charge becomes (Foissac 1975)

$$\operatorname{div}\boldsymbol{j} + \frac{\partial\rho}{\partial t} = \{\operatorname{div}_s\boldsymbol{j}_s\}\delta_s - \boldsymbol{u}_m\cdot\boldsymbol{j}_s\delta_c + \left\{\frac{\partial\rho_s}{\partial t}\right\}\delta_s + \frac{\partial\rho_c}{\partial t}\delta_c = 0, \tag{1.131}$$

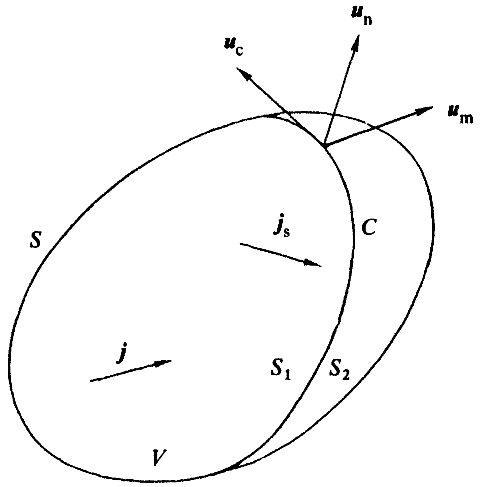


Fig. 1.8. Volume and surface electric currents.

where the brackets indicate the value at all points of S, contour C excluded. Unit vector $\boldsymbol{u}_m$ lies in the tangent plane, and is perpendicular to C. The symbol div_s denotes the surface divergence. In the (v_1, v_2) coordinates defined in Section 1.5, this operator is

$$\text{div}_s \boldsymbol{j}_s = \frac{1}{h_1 h_2}\left(\frac{\partial}{\partial v_1}(h_2 j_{s1}) + \frac{\partial}{\partial v_2}(h_1 j_{s2})\right). \tag{1.132}$$

Equating to zero the terms in δ_s and δ_c in (1.131) yields

$$\text{div}_s \boldsymbol{j}_s + \frac{\partial \rho_s}{\partial t} = 0 \quad \text{on } S, \qquad \frac{\partial \rho_c}{\partial t} = \boldsymbol{u}_m \cdot \boldsymbol{j}_s \quad \text{on } C. \tag{1.133}$$

These results are relevant for the 'physical optics' approximation, where S_1 is the illuminated part of a conductor, and S_2 the shadow region. They also apply to an open surface bounded by a rim (e.g. a parabolic reflector).

The distributional value of $\text{curl}\,\boldsymbol{t}$ is similarly obtained by applying the integral theorem

$$\iiint_V \text{curl}\,\boldsymbol{a}\,dV = \iint_S (\boldsymbol{u}_n \times \boldsymbol{a})\,dS \tag{1.134}$$

to volumes V_1 and V_2 (Fig. 1.5). The result is

$$\text{curl}\,\boldsymbol{t} = \{\text{curl}\,\boldsymbol{t}\} + (\boldsymbol{u}_{n1} \times \boldsymbol{t}_1 + \boldsymbol{u}_{n2} \times \boldsymbol{t}_2)\delta_s. \tag{1.135}$$

Applied to the magnetostatic equation (1.122), this formula gives, when the surface S carries a tangential current $\boldsymbol{j}_s$,

$$\{\operatorname{curl} \boldsymbol{h}\} + (\boldsymbol{u}_{n1} \times \boldsymbol{h}_1 + \boldsymbol{u}_{n2} \times \boldsymbol{h}_2)\delta_s = \boldsymbol{j}_s \delta_s . \tag{1.136}$$

Such a relationship implies that

$$(\boldsymbol{h}_2)_{\text{tang}} - (\boldsymbol{h}_1)_{\text{tang}} = \boldsymbol{j}_s \times \boldsymbol{u}_{n2} , \tag{1.137}$$

which is the traditional boundary condition.

Operators of a higher order acting on $\boldsymbol{t}$ may be defined by analogous methods. For example (Gagnon 1970):

$$\begin{aligned} \operatorname{curl}\operatorname{curl} \boldsymbol{t} &= \{\operatorname{curl}\operatorname{curl} \boldsymbol{t}\} + \{\boldsymbol{u}_{n1} \times \operatorname{curl} \boldsymbol{t}_1 + \boldsymbol{u}_{n2} \times \operatorname{curl} \boldsymbol{t}_2)\delta_s \\ &\quad + \operatorname{curl}[(\boldsymbol{u}_{n1} \times \boldsymbol{t}_1 + \boldsymbol{u}_{n2} \times \boldsymbol{t}_2)\delta_s] , \end{aligned} \tag{1.138}$$

$$\begin{aligned} \operatorname{grad}\operatorname{div} \boldsymbol{t} &= \{\operatorname{grad}\operatorname{div} \boldsymbol{t}\} + (\boldsymbol{u}_{n1} \operatorname{div} \boldsymbol{t}_1 + \boldsymbol{u}_{n2} \operatorname{div} \boldsymbol{t}_2)\delta_s \\ &\quad + \operatorname{grad}[(\boldsymbol{u}_{n1} \cdot \boldsymbol{t}_1 + \boldsymbol{u}_{n2} \cdot \boldsymbol{t}_2)\delta_s] . \end{aligned} \tag{1.139}$$

Equation (1.138) is obtained by two successive applications of (1.135), whereas (1.139) follows from a combination of (1.108) and (1.128). We delay until Chapter 2 a discussion of the meaning of a term such as $\operatorname{curl}(\boldsymbol{a}\delta_s)$.

References

Arsac, J. (1961). *Transformation de Fourier et théorie des distributions.* Dunod, Paris.

Bass, J. (1971). *Cours de Mathématiques, Tome III.* Masson, Paris.

Bouix, M. (1964). *Les fonctions généralisées ou distributions.* Masson, Paris.

Constantinescu, F. (1974). *Distributionen und ihre Anwendung in der Physik.* Teubner, Stuttgart.

Cristescu, R., and Marinescu, G. (1973). *Applications of the theory of distributions* (trans. S. Taleman). Wiley, London.

de Jager, E.M. (1969). *Applications of distributions in mathematical physics.* Math. Centrum, Amsterdam.

de Jager, E.M. (1970). Theory of distributions. Ch. 2 in *Mathematics applied to physics.* Springer, Berlin.

Dirac, P.A.M. (1958). *The principles of quantum mechanics,* 4th edn. Oxford University Press, Oxford.

Foissac, Y. (1975). L'application de l'algèbre extérieure et de la théorie des distributions à l'étude du rayonnement électromagnétique. *Comptes Rendus de l'Académie des Sciences de Paris,* **281 B**(13), 13–6.

Friedlander, F.G. (1982). *Introduction to the theory of distributions.* Cambridge University Press, Cambridge.

Friedman, B. (1969). *Lectures on applications-oriented mathematics.* Holden Day, San Francisco.

Gagnon, R.J. (1970). Distribution theory of vector fields, *American Journal of Physics,* **38**, 879–91.

GEL'FAND, I.M., and SHILOV, G.E. (1964). *Generalized functions.* Academic Press, New York.

HALPERIN, I. (1952). *Introduction to the theory of distributions.* University of Toronto Press.

HOSKINS, R.H. (1979). *Generalized functions.* Ellis Horwood, Chichester.

IDEMEN, M. (1973). The Maxwell's equations in the sense of distributions. *IEEE Transactions on Antennas and Propagation*, **21**, 736–8.

JONES, D.S. (1982). *The theory of generalised functions*, 2nd edn. Cambridge University Press, Cambridge.

KECS, W., and TEODORESCU, P.P. (1974). *Applications of the theory of distributions in mechanics.* Abacus Press, Tunbridge Wells.

KOREVAAR, J. (1968). *Mathematical Methods*, Vol. 1. Academic Press, New York.

LIGHTHILL, M.J. (1959). *An introduction to Fourier analysis and generalised functions.* Cambridge University Press.

LÜTZEN, J. (1982). *The prehistory of the theory of distributions.* Springer, New York.

MARCHAND, J.P. (1962). *Distributions: an outline.* North Holland, Amsterdam.

PETIT, R. (1987). *L'outil mathématique*, 2nd edn. Masson, Paris.

PREUSS, W., BLEYER, A., and PREUSS, H. (1985). *Distributionen und Operatoren: ihre Anwendung in Naturwissenschaft und Technik.* Springer, Wien.

RICHTMYER, R.D. (1978). *Principles of advanced mathematical physics*, Vol. 1. Springer, New York.

SCHWARTZ, L. (1948). Généralisation de la notion de fonction et de dérivation. Théorie des distributions. *Annales des Telecommunications*, **3**, 135–40.

SCHWARTZ, L. (1950). *Théorie des distributions.* Hermann et Cie, Paris.

SCHWARTZ, L. (1965). *Méthodes mathématiques pour les sciences physiques*, Hermann et Cie, Paris. An English translation has been published by Addison-Wesley in 1966.

SKINNER, R., and WEIL, J.A. (1989). An introduction to generalized functions and their application to static electromagnetic point dipoles, including hyperfine interactions. *American Journal of Physics*, **57**, 777–91.

VAN BLADEL, J. (1985). *Electromagnetic Fields.* Appendix 6. Hemisphere Publ. Co., Washington. Reprinted, with corrections, from a text published in 1964 by McGraw-Hill, New York.

VAN DER POL, B., and BREMMER, H. (1950). *Operational calculus based on the two-sided Laplace integral*, pp. 62–66. Cambridge University Press.

VLADIMIROV, V. (1979). *Distributions en physique mathématique.* Editions Mir, Moscou.

ZEMANIAN, A.H. (1987). *Distribution theory and transform analysis.* Dover Publications, New York. First published in 1965 by McGraw-Hill, New York.