

# 1

## Basic Concepts in the Theory of Errors and Uncertainties

### 1.1 Systematic and Random Errors

In modern measurement systems we can observe, along with growth of complexity, the evolution of measurement methods to estimate accuracy. On one hand, this is a consequence of the increasing complexity of measurement models: the number of input quantities increases and dependencies between inputs and outputs become more and more complicated. It makes it difficult to estimate accuracy with the use of classical methods that employ analytical descriptions. On the other hand, technical progress enables better insight into physical reality, which, among other things, involves changes to definitions of units of measure, which are the basis of each metric system. For example, consider how the definition of the meter has evolved over the last two centuries ([www.gum.gov.pl](http://www.gum.gov.pl)):

**1793:** The meter is 1/10 000 000 of the distance from the equator to the Earth's North Pole (i.e. the Earth's circumference is equal to 40 million meters).

**1899:** The meter is the distance, measured at 0°C, between two engraved lines on the top surface of the international prototype meter standard, made of a platinum–iridium bar (102 cm in length) with an H-shaped cross-section.

**1960:** The meter is equal to 1 650 763.73 wavelengths of the orange–red radiation of the krypton-86 isotope.

**1983:** The meter is the distance traveled by light in vacuum in 1/299 792 458 seconds.

For the evaluation of measurement accuracy, it is necessary to define basic theoretical concepts of error and uncertainty. Below we present definitions of the measurement error for a single value of a measured quantity.

The **absolute error** of a measurement is the difference between measured value  $\hat{y}$  and actual value  $y$ :

$$\Delta y = \hat{y} - y. \quad (1.1)$$

The **relative error** of a measurement is the ratio of the absolute error to the actual value:

$$\delta y = \frac{\Delta y}{y} = \frac{\hat{y} - y}{y}. \quad (1.2)$$

In practice, the unknown actual value  $y$  in formula (1.1) is substituted by the true conventional value. Since the exact value of the absolute error is unknown, it is very important to evaluate a range in which the actual value is located. Such reasoning leads to the definition of the limiting error.

The **limiting error** is the smallest range around the measured value  $\hat{y}$  containing actual value  $y$  (Guide 2004):

$$\hat{y} - \Delta y_{\min} \geq y \geq \hat{y} + \Delta y_{\max}. \quad (1.3)$$

In the analysis of measurement errors occurring in repeatable experiments, a division into systematic and random errors is made. Investigating the results of repeated measurements of the same quantity leads to the observation that one component of the error does not change its sign or value, or evolves with changes in the reference conditions according to a specific law (function). This component was named the systematic error or bias (Taylor 1997, Guide 1995). It is defined as follows: the **systematic error (bias)** is the difference between the mean value calculated for an infinite number of measurements of a quantity – carried out under the same conditions – and its actual value.

The second component of the error is commonly called the random error (Guide 2004). It can be reduced by repeating the measurement. In VIM (1993) the **random error** is defined as the difference between the result of the individual measurement and the mean value calculated for an infinite number of measurements of a quantity, carried out under the same conditions.

The above definitions of measurement errors refer to the results of individual measurements. When a measurement model is given in the form of a function of input quantities, it is called an indirect measurement. The error of an indirect measurement is determined on the basis of the law of error propagation (Taylor 1997). According to this law, to determine an output quantity error on the basis of known errors of input quantities, one of two methods can be used: the expansion of the model function in a Taylor series up to first-order terms (method of total differential); or the method of increments.

The **method of increments** (exact method) consists of determining the increment of a measurement model function for the known increments of input quantities (i.e. absolute errors). Let us consider a measurement model in the form of a function of several variables:

$$y = f(x_1, x_2, \dots, x_n), \quad (1.4)$$

where  $x_1, x_2, \dots, x_n$  are inputs and  $y$  the measurement result. Let us also denote  $\Delta x_1, \Delta x_2, \dots, \Delta x_n$  as absolute errors of the inputs (increments of  $f$  arguments). We can then write the increment of the function as:

$$\Delta y = y + \Delta y - y. \quad (1.5)$$

The first two components on the right-hand side of (1.5) can be expressed as:

$$y + \Delta y = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_n + \Delta x_n). \quad (1.6)$$

Finally:

$$\Delta y = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_n + \Delta x_n) - f(x_1, x_2, \dots, x_n). \quad (1.7)$$

Hence, from (1.7), the relative error of  $y$  is:

$$\delta y = \frac{f(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_n + \Delta x_n) - f(x_1, x_2, \dots, x_n)}{f(x_1, x_2, \dots, x_n)}. \quad (1.8)$$

The above method was used in this work in computer simulations of the method error in infrared thermography presented in Chapter 4.

Unfortunately, for complicated measurement models, evaluation of the error by means of (1.7) and (1.8) is very tedious. Therefore, the error is often evaluated by using an approximated method – the method of the total differential.

The **method of the total differential** (approximated method) is based on the expansion of the function  $f(x_1, x_2, \dots, x_n)$  as a Taylor series around the point defined by the actual (true conventional) values of the inputs. Assuming that function (1.4) is continuous and, for simplicity, that only input  $x_1$  is burdened with error  $\Delta x_1$ , the expansion in a Taylor series has the following form:

$$\begin{aligned} f(x_1 + \Delta x_1, x_2, \dots, x_n) &= f(x_1, x_2, \dots, x_n) + \frac{\Delta x_1}{1!} f'(x_1, x_2, \dots, x_n) \\ &+ \frac{(\Delta x_1)^2}{2!} f''(x_1, x_2, \dots, x_n) + \frac{(\Delta x_1)^3}{3!} f'''(x_1, x_2, \dots, x_n) + \dots \end{aligned} \quad (1.9)$$

The terms of order higher than one can be omitted in the above expansion, assuming that their influence on the result is negligible. On the basis of (1.6), we can write:

$$\Delta y_1 = \Delta x_1 \cdot f'(x_1, x_2, \dots, x_n) = \Delta x_1 \cdot \frac{\partial y}{\partial x_1}, \quad (1.10)$$

where  $\Delta y_1$  is called the component of the output error associated with  $x_1$ . Partial derivative  $\partial y / \partial x_1$ , calculated at point  $(x_1, x_2, \dots, x_n)$ , is called the sensitivity index to input  $x_1$ . If we take into account errors from all inputs  $x_1, x_2 \dots x_n$ , the total error of the indirect measurement can be written as the sum:

$$\Delta y = \sum_{i=1}^n \Delta x_i \frac{\partial y}{\partial x_i}, \quad (1.11)$$

where partial derivatives  $\partial y / \partial x_i$  are calculated at  $(x_1, x_2, \dots, x_n)$ . Since in (1.11) all the increments of input variables  $x_1, x_2, \dots, x_n$  are taken with the same sign, the total error is overestimated. In real experiments, the probability that all input measurements are burdened with positive (or negative) errors is small and decreases with an increasing number of inputs (Fuller 1987). Therefore, a more realistic estimate of the indirect measurement absolute error is commonly used in practice – the mean square error:

$$\Delta y = \sqrt{\left(\Delta x_1 \frac{\partial y}{\partial x_1}\right)^2 + \left(\Delta x_2 \frac{\partial y}{\partial x_2}\right)^2 + \dots + \left(\Delta x_n \frac{\partial y}{\partial x_n}\right)^2}. \quad (1.12)$$

From the point of view of temperature measurement using an infrared system, analysis of error can be useful only for strictly defined reference conditions. Such an analysis can also be helpful in the sensible estimation of measurement accuracy in situations where there is no information on these conditions. An additional aim of the analysis is the comparison of various measurement models used in contemporary infrared cameras. Lastly, analysis of error can be a starting point for the investigation of sensitivity, when thermography is applied to validate numerical models (e.g. when temperature measured at different points is used in finite element method (FEM) computations).

## 1.2 Uncertainties in Indirect Measurements

In accurate comparative measurements (such as standard measurements) it is necessary to describe the reference conditions in the form of random variables with assumed probability distributions. In such situations it is more convenient to use the concept of measurement uncertainty. In general, the measurement uncertainty characterizes the doubt about a measurement result. With this meaning, the uncertainty does not determine any specific quantitative measure. It only expresses the lack of accurate knowledge about a value of the measured quantity. Therefore, a measurement result is always an estimate of the measured quantity. More specifically, the measurement uncertainty is defined in the following way (VIM 1993): the **uncertainty of a measurement** is a parameter characterizing the spread of measurement values that can be assigned to the measured quantity in a justified way.

Unfortunately, the above definition does not determine how this assignment can be made. Therefore, to characterize the measurement accuracy precisely, the following definition of the standard uncertainty was introduced as a quantitative measure of spread (Guide 1995): the **standard uncertainty of a measurement** is the uncertainty of measurement values expressed in the form of the standard deviation.

To estimate the quantitative accuracy of a measurement, a description of the measurement model inputs in the form of random variables is introduced. These variables are characterized by specific probability distribution functions. For estimating the measurement accuracy, the most important statistics of a random variables are the expected value and the standard deviation.

The **expected value**  $E(X)$  of a discrete random variable  $X$ , whose values  $x_i$  appear with probabilities  $p_i$ , is:

$$E(X) = \sum p_i x_i, \quad (1.13)$$

where the sum is taken over all possible values  $x_i$  of variable  $X$ .

In practice, the set of measured values  $x_i$  is a finite  $N$ -element set. Therefore, the expected value is substituted by its estimator – the arithmetic mean from  $N$  independent observations (Söderström and Stoica 1994):

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i. \quad (1.14)$$

The **standard deviation**  $\sigma(X)$  of a random variable is the positive square root of the variance:

$$\sigma(X) = \sqrt{E[X - E(X)]^2}. \quad (1.15)$$

In practical problems an estimator of the standard deviation, called the experimental standard deviation, is used. It is calculated from  $N$  independent observations  $x_i$ :

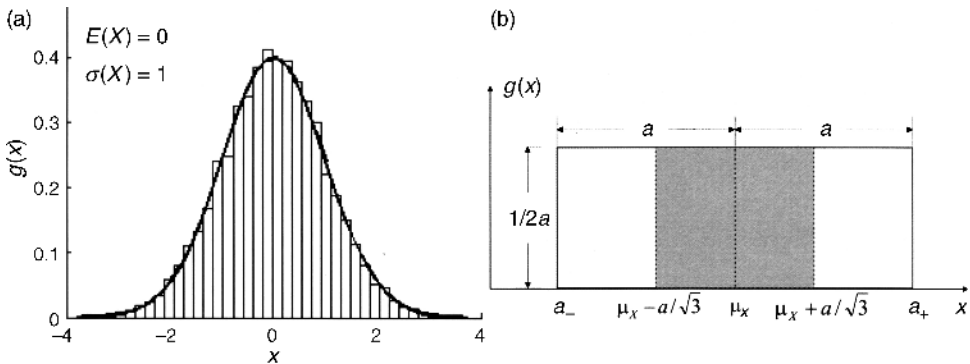
$$s(x_i) = \sqrt{\frac{1}{N-1} \sum (x_i - \bar{x})^2}. \tag{1.16}$$

In accordance with Recommendation INC-1 (1980), the components of a measurement uncertainty can be grouped into two categories (CIPM 1981, CIPM 1986):

**Type A standard uncertainty** determined on the basis of the observed frequency distribution.  
**Type B standard uncertainty** determined on the basis of a frequency distribution assumed a priori.

**Example 1.1** *Estimation of the parameters of a probability density function from a series of measurements – Type A uncertainty*

In this example we simulated an experiment consisting of multiple measurements of quantity  $X$  under repeated conditions. The uncertainty analysis was conducted on the basis of a series of realizations generated to simulate the results of real measurements. The estimation of parameters of the density function was conducted assuming that the measured series was subject to a Gaussian distribution (this distribution was determined on the basis of the shape of the histogram). The results of numerical calculations are shown in Figure 1.1a. The solid line denotes the probability density function obtained for the estimated parameters. To evaluate the standard uncertainty, we used the arithmetic mean (1.14) and the experimental standard deviation 1.16 as the best estimators of the expected value and the standard uncertainty respectively. Hence, we could use MATLAB’s functions `mean()` and `std()` to determine these basic statistics of the obtained distribution. This experiment illustrates how to evaluate Type A standard uncertainty.



**Figure 1.1** Evaluation of standard uncertainty of: (a) Type A, simulation in MATLAB; and (b) Type B, for uniform probability distribution (Guide 1995)

**Example 1.2** *Evaluation of Type B standard uncertainty from the parameters of a uniform distribution density function*

Figure 1.1b shows how Type B standard uncertainty can be determined assuming the uniform probability distribution of variable  $X$ . The distribution density function is:

$$\begin{aligned} g(x) &= 1/2a & \text{for } a_- \leq x \leq a_+ \\ g(x) &= 0 & \text{for other } x. \end{aligned} \quad (1.17)$$

In this example, to determine the Type B standard uncertainty, we used information on the permissible interval of the measured values. The assumption of a uniform distribution is the worst case, for which the standard uncertainty is  $a/\sqrt{3}$ , where  $a$  is one-half of the interval length.

In the case of errors, the problem on how individual standard uncertainties of inputs of a complex analytical model affect the precision of an indirect measurement (i.e. evaluation of a measurement uncertainty) often occurs in practice. In such a situation it is necessary to evaluate the combined standard uncertainty. Depending on whether the model inputs are correlated or not, a covariance factor appears in the definition of the combined uncertainty. Provided that the input variables are uncorrelated, the combined standard uncertainty is defined according to VIM (1993): the **combined standard uncertainty**  $u_c(y)$  is the positive square root of the combined variance  $u_c^2(y)$ , defined as:

$$u_c^2(y) = \sum_{i=1}^N \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i), \quad (1.18)$$

where  $y = f(x_1, x_2, \dots, x_n)$  is the measurement model function (1.4) and  $u^2(x_i)$  is the variance of the  $i$ th input of the model.

When the input variables are correlated, the expression describing the uncertainty is more complicated because it includes estimates of the covariance of the inputs. The combined uncertainty of measurement  $y$  is determined as (Taylor 1997):

$$u_c^2(y) = \sum \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j), \quad (1.19)$$

where  $u(x_i, x_j)$  is the estimate of covariance between  $x_i$  and  $x_j$ .

Because in the evaluation of indirect measurement uncertainty the inputs of the model are considered as random variables, the determined estimators (expected values, standard deviations) are also random variables. That is why we need to define the determined parameters using concepts of probability. These concepts are (VIM 1993) discussed below.

The **one-sided coverage interval** is as follows. If  $T$  is a function of observed values, such that for estimated parameter of population  $\theta$ , probability  $Pr(T \geq \theta)$  or  $Pr(T \leq \theta)$  is at least equal to  $(1 - \alpha)$  (where  $(1 - \alpha)$  is a fixed number, positive and smaller than one), then the interval from the smallest possible value of  $\theta$  to  $T$  (or the interval from  $T$  to the biggest possible value of  $\theta$ ) is the one-sided coverage interval  $\theta$  with confidence level  $(1 - \alpha)$ .

The **confidence level** is the value  $(1 - \alpha)$  of probability associated with a confidence interval or statistical coverage interval.

Estimation of the combined standard uncertainty is usually associated with the simultaneous evaluation of probability, with which a measurement result lies inside the interval determined

by this uncertainty. For the strict determination of this probability, the concept of so-called expanded uncertainty is introduced: the **expanded uncertainty**  $U$  is the uncertainty obtained by multiplying the combined standard uncertainty  $u_c(y)$  by expansion factor  $k$ :

$$U = ku_c(y). \quad (1.20)$$

The expanded uncertainty specifies the limits of the uncertainty interval for a given confidence level. The value of the expansion factor depends on the probability distribution of the model output variable. For example, if a random variable has a Gaussian distribution at the model output, the probability that a measurement result will fall in the interval from  $y - u_c(y)$  to  $y + u_c(y)$ , that is for  $k = 1$ , is about 68%; from  $y - 2u_c(y)$  to  $y + 2u_c(y)$ , that is for  $k = 2$ , about 95%; and from  $y - 3u_c(y)$  to  $y + 3u_c(y)$ , that is for  $k = 3$ , about 99%. The requirement for exact knowledge of the type of the output variable distribution is an inconvenience in determining the expansion factor. For models with a large number of inputs it can be assumed that the central limit theorem applies. In such a case it is assumed a priori that the output variable has a Gaussian distribution. However, measurement practice reveals significant differences in evaluated expanded uncertainties obtained under the assumption of the Gaussian distribution, especially when the measurement model exhibits strong nonlinearities and the true distribution of the output variable is asymmetric.

When the probability distribution of the model output variable is not known, we need to determine the relationship between the expansion factor and the coverage interval (confidence level). This problem is usually solved by determining the resultant number of degrees of freedom  $\nu$  from the Welch–Satterthwaite equation (Welch 1936, Satterthwaite 1941):

$$\nu_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^N \frac{u_i^4(y)}{\nu_i}}, \quad (1.21)$$

and calculating the expanded uncertainty as (Guide 1995):

$$U_p = k_p u_c(y) = t_p(\nu_{eff}) u_c(y), \quad (1.22)$$

where coefficient  $t_p(\nu_{eff})$  is the value of Student's  $t$  distribution calculated on the basis of the number of degrees of freedom approximated by formula (1.21).

Summing up, the steps in evaluating the combined standard uncertainty can be presented as follows (Guide 2004):

1. Evaluation of the expected values and standard deviations  $u(x) = (u(x_1) \dots u(x_n))^T$  of probability distributions of random variables  $X_1 \dots X_n$  representing inputs  $x = (x_1 \dots x_n)^T$  of the measurement model. If the inputs are correlated with each other, we should use joint probability distributions of the variables.
2. Evaluation of covariances (reciprocal uncertainties)  $u(x_i, x_j)$  as  $Cov(X_i, X_j)$ .
3. Evaluation of partial derivatives of measurement model (1.4) with respect to its inputs.
4. Calculation of estimate of the output  $y$  on the basis of the measurement model function  $f$ .
5. Evaluation of sensitivity indices of the model using the partial derivatives determined in step 3, calculated at point  $\mathbf{x} = (x_1 \dots x_n)$ .
6. Evaluation of combined standard uncertainty  $u_c(y)$  on the basis  $u(\mathbf{x})$ ,  $u(x_i, x_j)$  and the model sensitivity indices, using formula 1.18 or 1.19.

7. Evaluation of the number of degrees of freedom using, for example, formula (1.21).
8. Calculation of the expanded uncertainty from (1.22).

These steps lead to correct evaluation of the expanded uncertainty only when the three following conditions are satisfied:

1. Nonlinearity of the model is negligible. Because it is difficult to indicate an objective measure of nonlinearity, we mean here that neglecting the higher order terms in the Taylor series expansion of the model function does not greatly affect the results of the estimation.
2. The assumptions of the central limit theorem are satisfied. In particular, the distribution of the model output is Gaussian with good accuracy.
3. The approximation of the resultant number of degrees of freedom from the Welch–Satterthwaite equation is accurate enough.

In practical measurements it is often not possible to fulfill the above conditions. Consequently, Working Group No. 1 of the Common Committee for Basic Problems in Metrology produced Supplement No. 1 to Guide (1995) entitled ‘Numerical methods for the propagation of distributions’. In this supplement, an idea of the numerical evaluation of the coverage interval is presented. With such an approach, knowledge of an analytical form of the probability distribution function is not required. In further parts of this book we present the basic guidelines and aims of this method.

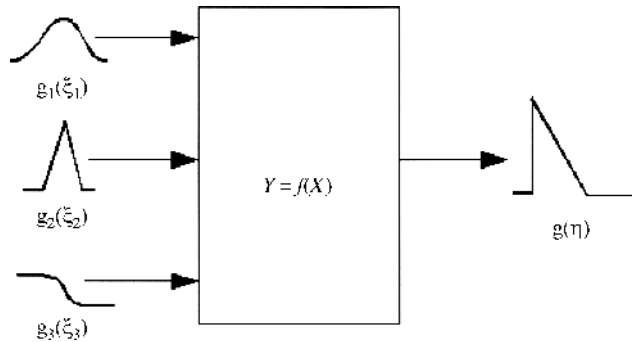
### 1.3 Method for the Propagation of Distributions

In the previous points we presented the basic concepts associated with the evaluation of precision in indirect measurements making use of complex mathematical models. We described the basics of the theory of errors and uncertainties and indicated the problems arising in the evaluation of the expanded uncertainty. These problems are consequences of the fact that in practice the types of probability distributions of input random variables (i.e. measured quantities) are unknown.

The Common Committee for Basic Problems in Metrology took this into consideration when producing the supplement mentioned immediately above (Guide 2004). This supplement deals with the evaluation of precision in indirect measurements, with particular emphasis on strongly nonlinear and/or complicated measurement models, such as the processing algorithm of an infrared camera measurement path. The propagation of distributions method allows for the correct estimation of measurement precision, in particular in the following cases (Dudzic and Minkina 2007):

- partial derivatives are unavailable;
- the distribution of the output variable is not Gaussian;
- the distributions of the input variables exhibit asymmetry;
- the measurement model is a strongly nonlinear function of input quantities;
- uncertainty ranges of individual input quantities are incomparable.

The idea of the propagation of distributions is illustrated in Figure 1.2.



**Figure 1.2** Illustration of the propagation of distributions

The symbols in Figure 1.2 denote:  $g_i(\xi_i)$ , the probability density functions of permissible values  $\xi_i$  of the  $i$ th input quantity  $X_i$ ; and  $g(\eta)$ , the probability density function of permissible values  $\eta$  of output quantity  $Y$  of measurement model  $Y = f(X)$ .

In the method for the propagation of distributions the uncertainties are evaluated using the Monte Carlo method. The principal aim of the computational procedure is to evaluate the statistical coverage interval at a specified confidence level. It is worth emphasizing that the procedure gives correct results even for strongly nonlinear functional relationships of measurement models as well as for asymmetric probability density functions of input random variables. The following steps can be distinguished in the evaluation of the uncertainty:

1. Definition of the output quantity of the considered measurement model (indirectly measured quantity).
2. Definition of input quantities of the model.
3. Design of the measurement model on the basis of available (experimental or theoretical) knowledge of the measured quantity.
4. Determination of shapes of probability density functions of the model inputs (based on analysis of a series of input measurements or on a single experiment).
5. Evaluation of the probability distribution of the output using the measurement model and the determined distributions of the inputs. The calculations can be carried out using the Monte Carlo method.
6. Estimation of parameters of the resulting probability density function; that is, the standard uncertainty and the corresponding expected value of the output as well as the coverage (confidence) interval that includes the measurement results with probability determined by an assumed confidence level.

The Monte Carlo method makes possible the numerical approximation of cumulative distribution  $G(\eta)$  of the output quantity. The simulation is based on the assumption that any value of an input quantity chosen at random from all permissible values of this input is as justified as any other. In other words, no value is preferred. Hence, drawing values of each input quantity according to the probability distribution function assigned to this input validates the

set of its values. The value of the measurement model output corresponding to the drawn values of the inputs is a representative output. Consequently, a big enough set of output values obtained from the model in this way can approximate, to the required accuracy, the probability density distribution of permissible values of the output (measured quantity). The Monte Carlo simulation is performed in the following steps (Guide 2004):

1. Generation of a set of  $N$  values by independent sampling of the probability density function of each input variable  $X_i, i = 1, \dots, N$ . In the case of statistically dependent variables the samples must be generated with the use of the joint density function of the variables. The sampling is repeated  $M$  times, where  $M$  is a large number. As a result, we obtain  $M$  independent sets of  $N$  values of the inputs.
2. Simulation of the model for each set of values. As a result, we obtain a set of  $M$  values (realizations) of the model output variable  $Y$ . This set is in fact a numerical approximation of the probability density distribution of the output variable.
3. Determination of approximation  $\hat{G}(\eta)$  of cumulative density function  $G(\eta)$  of  $Y$ , based on the generated set of values.
4. Evaluation of statistical parameters of the output variable distribution on the basis of  $\hat{G}(\eta)$ . In particular, this determines: the measured value  $y$  as the expected value of  $\hat{G}(\eta)$ ; the estimate of standard uncertainty  $u(y)$  as the standard deviation of  $\hat{G}(\eta)$ ; and the end points of coverage interval  $I_p(y)$  for the assumed coverage probability, as two quantiles of  $\hat{G}(\eta)$ .

One very important aspect of the propagation of distributions is the approximation of the cumulative density function of the output quantity. The steps for the approximation procedure are as follows:

1. Sorting of values  $y_r, r = 1, \dots, M$ , of the output variable (obtained from the Monte Carlo simulation) in non-decreasing order. The sorted values are further denoted as  $y_{(r)}, r = 1, \dots, M$ .
2. Assignment of equidistant cumulative probabilities to the sorted values according to the formula (Cox *et al.* 2001):

$$p_r = \frac{r - 0.5}{M}, \quad r = 1, \dots, M. \quad (1.23)$$

3. Forming of piecewise linear function  $\hat{G}(\eta)$  by joining  $M$  points of coordinates  $(y_{(r)}, p_r)$ :

$$\hat{G}(\eta) = p_r + \frac{\eta - y_{(r)}}{M(y_{(r+1)} - y_{(r)}), \quad y_{(r)} \leq \eta \leq y_{(r+1)}, \quad r = 1, \dots, M - 1. \quad (1.24)$$

Having determined approximation  $\hat{G}(\eta)$  of the output probability distribution, it is possible to calculate its expected value  $\hat{y}$ , which is an estimate of measured quantity  $Y$ , and its standard deviation, which is an estimate of the standard uncertainty. Estimates of the expected value and the variance can be calculated as:

$$\hat{y} = \frac{1}{M} \sum_{r=1}^M y_r \quad (1.25)$$

and:

$$u_c^2(\hat{y}) = \frac{1}{M-1} \sum_{r=1}^M (y_r - \hat{y})^2. \quad (1.26)$$

The last step in the propagation of distributions algorithm is the evaluation of the coverage interval resulting from the assumed confidence level (coverage probability). Usually, the 95% confidence level is adopted.

The **quantile of order  $\beta$**  of a probability distribution described by cumulative distribution function  $G(\eta)$  is such that a value  $\eta$  of the random variable satisfies the equality  $G(\eta) = \beta$ . This means that the probability of occurrence of this value is equal to  $\beta$ .

If we denote by  $\alpha$  a value for the interval from 0 to  $1 - p$ , where  $p$  is the required coverage probability, then the ends of the coverage interval  $I_p(y)$  can be determined as quantiles of order  $\alpha$  and  $\alpha + p$  of the distribution defined by  $G(\eta)$ . For example, if we adopt  $\alpha = 0.025$ , the ends of the 95% coverage interval will be quantiles of order 0.025 and 0.0975. As a result, we obtain the probabilistically symmetric coverage interval  $I_{0.95}(y)$ .

In general, if a probability distribution is symmetric, the shortest coverage interval is associated with quantile:

$$\alpha = \frac{1-p}{2}. \quad (1.27)$$

As we can see, for the 95% coverage interval and provided that the distribution is symmetric and satisfies (1.27), we obtain  $\alpha = 0.025$ ; that is, the value used in the above example.

The Monte Carlo simulation may reveal that the probability distribution density of the output is not symmetric with respect to its expected value (it is not the centered expected value). In this case there are many intervals satisfying equality:

$$g(G^{-1}(\alpha)) = g(G^{-1}(\alpha + p)), \quad (1.28)$$

and we should choose such a value of  $\alpha$  that determines the shortest possible coverage interval associated with assumed probability  $p$ . The value of  $\alpha$  chosen in this way satisfies the condition:

$$\hat{G}^{-1}(\alpha + p) - \hat{G}^{-1}(\alpha) = \min. \quad (1.29)$$

Below we present an example of uncertainty analysis using the propagation of distributions. (The procedure described in this example corresponds in principle to a much more complicated case study: the simulation evaluation of uncertainty of an infrared camera processing path algorithm. The methodology and results of such an evaluation are presented in Chapter 5.)

**Example 1.3** *Application of the propagation of distributions for the determination of the 95% coverage interval of a simple nonlinear model*

Let us consider a simple measurement model with two input and one output random variables. The relationship between the inputs and the output is:

$$Y = X_1^2 + 2X_2. \quad (1.30)$$

**Table 1.1** Input data for the Monte Carlo simulation of model (1.30)

Model input quantity	$\hat{x}$	$u^2(x)$	$a$	$b$
$X_1$	1.0	2.0	7.550	12.45
$X_2$	100	200	92.25	107.7

Input variables  $X_1$  and  $X_2$  are subject to uniform probability distributions defined as:

$$g(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{for other } x. \end{cases} \quad (1.31)$$

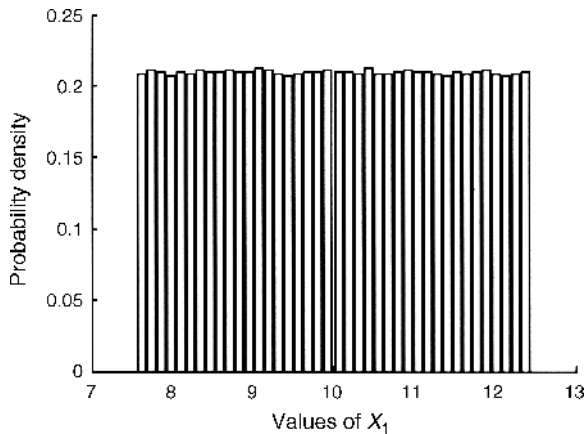
Parameters  $a$  and  $b$  of these distributions can be calculated from given statistics of the inputs, namely the expected values and the variances (squared standard deviations):

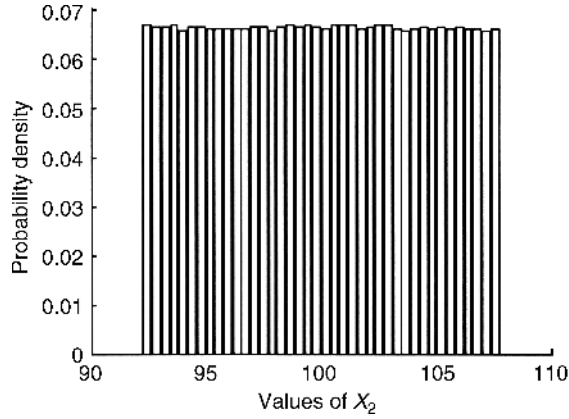
$$\begin{aligned} a &= \hat{x} - \sqrt{3u^2(x)} \\ b &= \hat{x} + \sqrt{3u^2(x)}. \end{aligned} \quad (1.32)$$

In the above equations we assume that the expected value is also an estimate of the input, denoted as  $\hat{x}$ , and  $u^2(x)$ , denoted as an estimate of the input variance. The adopted values of the estimates and uncertainties for inputs  $X_1$  and  $X_2$ , as well as the corresponding parameters of the uniform distribution functions calculated from these estimates, are shown in Table 1.1. They are the input data for the Monte Carlo simulation.

The Monte Carlo simulation consisted of  $M=10^5$  runs and generated approximate distributions of the input variables  $X_1$  and  $X_2$  for the conditions determined by the parameters from Table 1.1. These distributions are presented in Figures 1.3 and 1.4.

In order to determine the probability density function of output variable  $Y$  we also performed  $M$  runs of the Monte Carlo simulation of model (1.30). The approximation of the cumulative

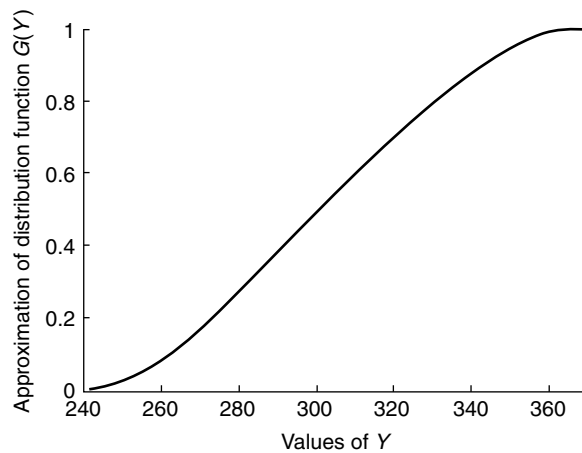
**Figure 1.3** Probability density distribution of input variable  $X_1$



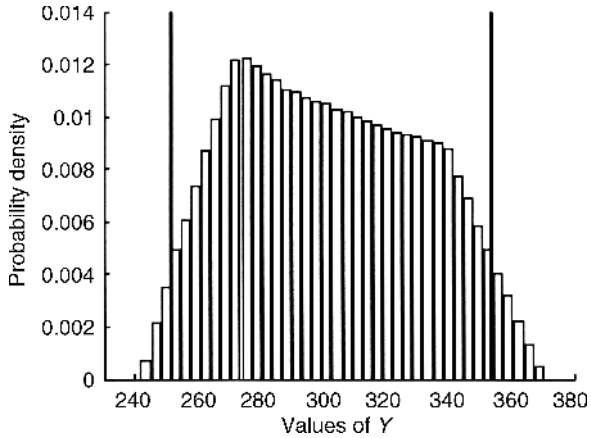
**Figure 1.4** Probability density distribution of input variable  $X_2$

distribution function of  $Y$  was determined using formulas (1.23) and (1.24). This approximation of the cumulative distribution is shown in Figure 1.5, and the corresponding probability density approximation is presented in Figure 1.6. The estimate of the measured quantity, calculated as the arithmetic mean of variable  $Y$ , is  $\hat{y} = 302$ , standard uncertainty  $u_c(\hat{y}) = 30$ , and the 95% coverage interval, marked by vertical lines in Figure 1.6, is  $I_{0.95}(Y) = [252, 354]$ .

The above example explains the propagation of distributions for a simple measurement model: a second-order polynomial. In a further part of this monograph we discuss the application of this method to much more complex models of measurement in infrared thermography (described in detail in Chapter 3). For such a complex model, the use of the propagation of distributions is justified more than an analytic approach. In addition, it yields



**Figure 1.5** Numerical approximation of the cumulative distribution of the model output variable



**Figure 1.6** Probability density function with marked 95% shortest coverage interval

more accurate results due to the model's nonlinearity. The methodology of the simulation and investigation of the infrared thermography measurement model is presented in Chapter 5.

Summing up the above considerations on the basic concepts of metrology, we want to emphasize that error analysis and uncertainty analysis do not exclude each other. One of our aims is to present the analysis of errors and uncertainties as complementary methods of evaluating accuracy in infrared thermography measurements.