

# CHAPTER ONE

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## FUNDAMENTALS OF STABILITY THEORY

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### 1.1 INTRODUCTION

It is not necessary to be a structural engineer to have a sense of what it means for a structure to be stable. Most of us have an inherent understanding of the definition of instability—that a small change in load will cause a large change in displacement. If this change in displacement is large enough, or is in a critical member of a structure, a local or member instability may cause collapse of the entire structure. An understanding of stability theory, or the mechanics of why structures or structural members become unstable, is a particular subset of engineering mechanics of importance to engineers whose job is to design safe structures.

The focus of this text is not to provide in-depth coverage of all stability theory, but rather to demonstrate how knowledge of structural stability theory assists the engineer in the design of safe steel structures. Structural engineers are tasked by society to design and construct buildings, bridges, and a multitude of other structures. These structures provide a load-bearing skeleton that will sustain the ability of the constructed artifact to perform its intended functions, such as providing shelter or allowing vehicles to travel over obstacles. The structure of the facility is needed to maintain its shape and to keep the facility from falling down under the forces of nature or those made by humans. These important characteristics of the structure are known as *stiffness* and *strength*.

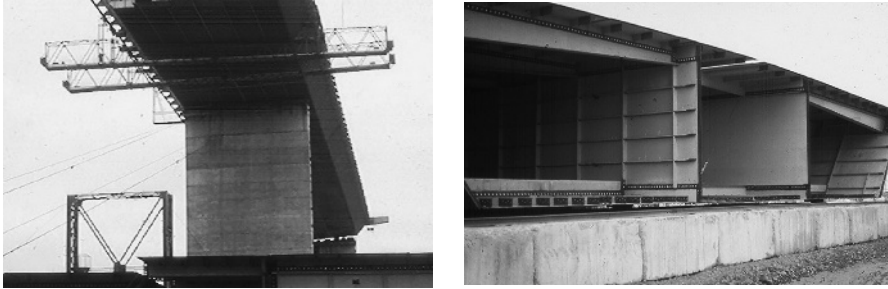
This book is concerned with one aspect of the strength of structures, namely their stability. More precisely, it will examine how and under what loading condition the structure will pass from a stable state to an unstable one. The reason for this interest is that the structural engineer, knowing the circumstances of the limit of stability, can then proportion a structural scheme that will stay well clear of the zone of danger and will have an adequate margin of safety against collapse due to instability. In a well-designed structure, the user or occupant will never have to even think of the structure's existence. Safety should always be a given to the public.

Absolute safety, of course, is not an achievable goal, as is well known to structural engineers. The recent tragedy of the World Trade Center collapse provides understanding of how a design may be safe under any expected circumstances, but may become unstable under extreme and unforeseeable circumstances. There is always a small chance of failure of the structure.

The term *failure* has many shades of meaning. Failure can be as obvious and catastrophic as a total collapse, or more subtle, such as a beam that suffers excessive deflection, causing floors to crack and doors to not open or close. In the context of this book, *failure* is defined as the behavior of the structure when it crosses a *limit state*—that is, when it is at the limit of its structural usefulness. There are many such limit states the structural design engineer has to consider, such as excessive deflection, large rotations at joints, cracking of metal or concrete, corrosion, or excessive vibration under dynamic loads, to name a few. The one limit state that we will consider here is the limit state where the structure passes from a stable to an unstable condition.

Instability failures are often catastrophic and occur most often during erection. For example, during the late 1960s and early 1970s, a number of major steel box-girder bridges collapsed, causing many deaths among erection personnel. The two photographs in Figure 1.1 were taken by author Galambos in August 1970 on the site two months before the collapse of a portion of the Yarra River Crossing in Melbourne, Australia. The left picture in Figure 1.1 shows two halves of the multi-cell box girder before they were jacked into place on top of the piers (see right photo), where they were connected with high-strength bolts. One of the 367.5 ft. spans collapsed while the ironworkers attempted to smooth the local buckles that had formed on the top surface of the box. Thirty-five workers and engineers perished in the disaster.

There were a number of causes for the collapse, including inexperience and carelessness, but the Royal Commission (1971), in its report pinpointed the main problem: “We find that [the design organization] made assumptions about the behavior of box girders which extended beyond the range of engineering knowledge.” The Royal Commission concluded “. . . that the design firm “failed altogether to give proper and careful regard to the



**Fig. 1.1** Stability-related failures.

process of structural design.” Subsequent extensive research in Belgium, England, the United States, and Australia proved that the conclusions of the Royal Commission were correct. New theories were discovered, and improved methods of design were implemented. (See Chapter 7 in the *Stability Design Criteria for Metal Structures* (Galambos 1998)).

Structural instability is generally associated with the presence of compressive axial force or axial strain in a plate element that is part of a cross-section of a beam or a column. *Local instability* occurs in a single portion of a member, such as local web buckling of a steel beam. *Member instability* occurs when an isolated member becomes unstable, such as the buckling of a diagonal brace. However, member instability may precipitate a *system instability*. System instabilities are often catastrophic.

This text examines the stability of some of these systems. The topics include the behavior of columns, beams, and beam-columns, as well as the stability of frames and trusses. Plate and shell stability are beyond the scope of the book. The presentation of the material concentrates on steel structures, and for each type of structural member or system, the recommended design rules will be derived and discussed. The first chapter focuses on basic stability theory and solution methods.

## **1.2 BASICS OF STABILITY BEHAVIOR: THE SPRING-BAR SYSTEM**

A *stable* elastic structure will have displacements that are proportional to the loads placed on it. A *small* increase in the load will result in a *small* increase of displacement.

As previously mentioned, it is intuitive that the basic idea of instability is that a *small* increase in load will result in a *large* change in the displacement. It is also useful to note that, in the case of axially loaded members,

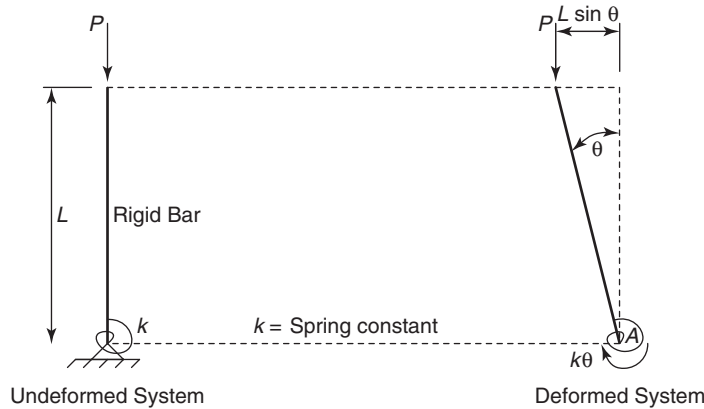


Fig. 1.2 Simple spring-bar system.

the large displacement related to the instability is not in the same direction as the load causing the instability.

In order to examine the most basic concepts of stability, we will consider the behavior of a spring-bar system, shown in Figure 1.2. The left side in Figure 1.2 shows a straight vertical rigid bar of length  $L$  that is restrained at its bottom by an elastic spring with a spring constant  $k$ . At the top of the bar there is applied a force  $P$  acting along its longitudinal axis. The right side shows the system in a deformed configuration. The moment caused by the axial load acting through the displacement is resisted by the spring reaction  $k\theta$ . The symbol  $\theta$  represents the angular rotation of the bar in radians.

We will begin with the most basic solution of this problem. That is, we will find the *critical load* of the structure. The critical load is the load that, when placed on the structure, causes it to pass from a stable state to an unstable state. In order to solve for the critical load, we must consider a deformed shape, shown on the right in Figure 1.2. Note that the system is slightly perturbed with a rotation  $\theta$ . We will impose equilibrium on the deformed state. Summing moments about point  $A$  we obtain

$$\sum M_A = 0 = PL \sin \theta - k\theta \quad (1.1)$$

Solving for  $P$  at equilibrium, we obtain

$$P_{cr} = \frac{k\theta}{L \sin \theta} \quad (1.2)$$

If we consider that the deformations are very small, we can utilize *small displacement theory* (this is also referred to in mechanics texts as small strain theory). Small displacement theory allows us to simplify the math by

recognizing that for very small values of the angle,  $\theta$ , we can use the simplifications that

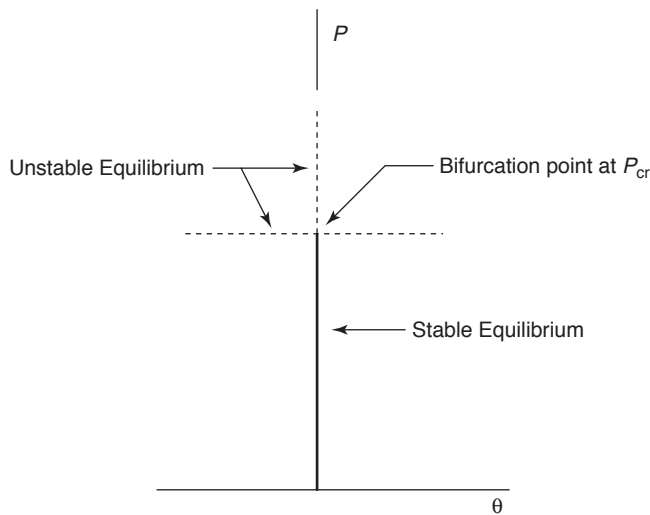
$$\begin{aligned}\sin \theta &= \theta \\ \tan \theta &= \theta \\ \cos \theta &= 1\end{aligned}$$

Substituting  $\sin \theta = \theta$ , we determine the critical load  $P_{cr}$  of the spring-bar model to be:

$$P_{cr} = \frac{k\theta}{L\theta} = \frac{k}{L} \quad (1.3)$$

The equilibrium is in a neutral position: it can exist both in the undeformed and the deformed position of the bar. The small displacement response of the system is shown in Figure 1.3. The load ratio  $PL/k = 1$  is variously referred in the literature as the *critical load*, the *buckling load*, or the *load at the bifurcation of the equilibrium*. The bifurcation point is a branch point; there are two equilibrium paths after  $P_{cr}$  is reached, both of which are unstable. The upper path has an increase in  $P$  with no displacement. This equilibrium path can only exist on a perfect system with no perturbation and is therefore not a practical solution, only a theoretical one.

Another means of solving for the critical load is through use of the *principle of virtual work*. Energy methods can be very powerful in describing structural behavior, and have been described in many structural analysis



**Fig. 1.3** Small displacement behavior of spring-bar system.

and structural mechanics texts. Only a brief explanation of the method will be given here. The total potential  $\Pi$  of an elastic system is defined by equation 1.4 as

$$\Pi = U + V_p \quad (1.4)$$

1.  $U$  is the elastic strain energy of a conservative system. In a conservative system the work performed by both the internal and the external forces is independent of the path traveled by these forces, and it depends only on the initial and the final positions.  $U$  is the *internal work* performed by the internal forces;  $U = W_i$
2.  $V_p$  is the *potential of the external forces*, using the original deflected position as a reference.  $V_p$  is the *external work*;  $V_p = -W_e$ .

Figure 1.4 shows the same spring-bar system we have considered, including the distance through which the load  $P$  will move when the bar displaces.

The strain energy is the work done by the spring,

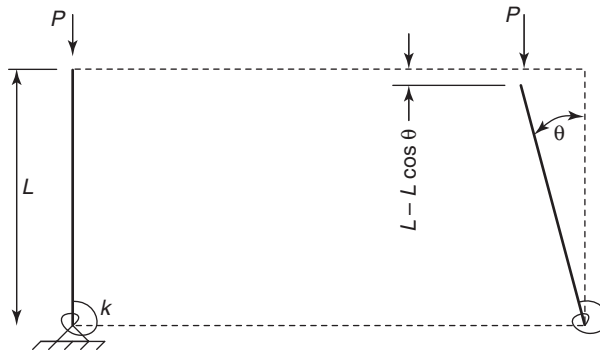
$$U = W_i = \frac{1}{2}k\theta^2. \quad (1.5)$$

The potential of the external forces is equal to

$$V_p = -W_e = -PL(1 - \cos \theta) \quad (1.6)$$

The total potential in the system is then given by:

$$\Pi = U + V_p = \frac{1}{2}k\theta^2 - PL(1 - \cos \theta) \quad (1.7)$$



**Fig. 1.4** Simple spring-bar system used in energy approach.

According to the principle of virtual work the maxima and minima are equilibrium positions, because if there is a small change in  $\theta$ , there is no change in the total potential. In the terminology of structural mechanics, the total potential is *stationary*. It is defined by the derivative

$$\frac{d\Pi}{d\theta} = 0 \quad (1.8)$$

For the spring bar system, equilibrium is obtained when

$$\frac{d\Pi}{d\theta} = 0 = k\theta - PL \sin \theta \quad (1.9)$$

To find  $P_{cr}$ , we once again apply small displacement theory ( $\sin \theta = \theta$ ) and obtain

$$P_{cr} = k/L$$

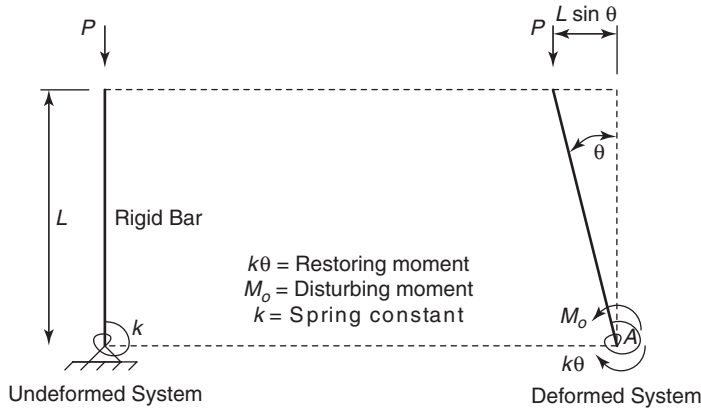
as before.

#### Summary of Important Points

- Instability occurs when a small change in load causes a large change in displacement. This can occur on a local, member or system level.
- The critical load, or buckling load, is the load at which the system passes from a stable to an unstable state.
- The critical load is obtained by considering *equilibrium* or *potential energy* of the system in a deformed configuration.
- Small displacement theory may be used to simplify the calculations if only the critical load is of interest.

### 1.3 FUNDAMENTALS OF POST-BUCKLING BEHAVIOR

In section 1.2, we used a simple example to answer a fundamental question in the study of structural stability: At what load does the system become unstable, and how do we determine that load? In this section, we will consider some basic principles of stable and unstable behavior. We begin by reconsidering the simple spring-bar model in Figure 1.2, but we introduce a disturbing moment,  $M_o$  at the base of the structure. The new system is shown in Figure 1.5.



**Fig. 1.5** Spring-bar system with disturbing moment.

Similar to Figure 1.1, the left side of Figure 1.5 shows a straight, vertical rigid bar of length  $L$  that is restrained at its bottom by an infinitely elastic spring with a spring constant  $k$ . At the top of the bar there is applied a force  $P$  acting along its longitudinal axis. The right sketch shows the deformation of the bar if a disturbing moment  $M_o$  is acting at its base. This moment is resisted by the spring reaction  $k\theta$ , and it is augmented by the moment caused by the product of the axial force times the horizontal displacement of the top of the bar. The symbol  $\theta$  represents the angular rotation of the bar (in radians).

### 1.3.1 Equilibrium Solution

Taking moments about the base of the bar (point  $A$ ) we obtain the following equilibrium equation for the displaced system:

$$\sum M_A = 0 = PL \sin \theta + M_o - k\theta$$

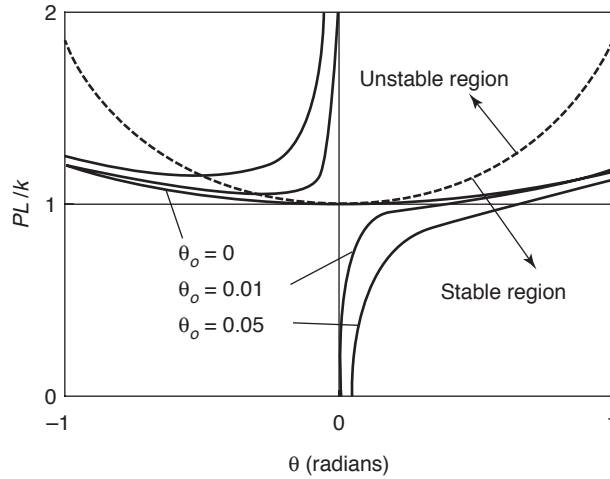
Letting  $\theta_o = M_o/k$  and rearranging, we can write the following equation:

$$\frac{PL}{k} = \frac{\theta - \theta_o}{\sin \theta} \tag{1.10}$$

This expression is displayed graphically in various contexts in Figure 1.6.

The coordinates in the graph are the load ratio  $PL/k$  as the abscissa and the angular rotation  $\theta$  (radians) as the ordinate. Graphs are shown for three values of the disturbing action

$$\theta_o = 0, \theta_o = 0.01, \text{ and } \theta_o = 0.05.$$



**Fig. 1.6** Load-deflection relations for spring-bar system with disturbing moment.

When  $\theta_o = 0$ , that is  $\frac{PL}{k} = \frac{\theta}{\sin \theta}$ , there is no possible value of  $PL/k$  less than unity since  $\theta$  is always larger than  $\sin \theta$ . Thus no deflection is possible if  $PL/k < 1.0$ . At  $PL/k > 1.0$  deflection is possible in either the positive or the negative direction of the bar rotation. As  $\theta$  increases or decreases the force ratio required to maintain equilibrium becomes larger than unity. However, at relatively small values of  $\theta$ , say, below 0.1 radians, or about  $5^\circ$ , the load-deformation curve is flat for all practical purposes. Approximately, it can be said that equilibrium is possible at  $\theta = 0$  and at a small adjacent deformed location, say  $\theta < 0.1$  or so. The load  $PL/k = 1.0$  is thus a special type of load, when the system can experience two adjacent equilibrium positions: one straight and one deformed. The equilibrium is thus in a neutral position: It can exist both in the undeformed and the deformed position of the bar. The load ratio  $PL/k = 1$  is variously referred in the literature as the *critical load*, the *buckling load*, or the *load at the bifurcation of the equilibrium*. We will come back to discuss the significance of this load after additional features of behavior are presented next.

The other two sets of solid curves in Figure 1.6 are for specific small values of the disturbing action  $\theta_o$  of 0.01 and 0.05 radians. These curves each have two regions: When  $\theta$  is positive, that is, in the right half of the domain, the curves start at  $\theta = \theta_o$  when  $PL/k = 0$  and then gradually exhibit an increasing rotation that becomes larger and larger as  $PL/k = 1.0$  is approached, finally becoming affine to the curve for  $\theta_o = 0$  as  $\theta$  becomes very large. While this is not shown in Figure 1.6, the curve for smaller and smaller values of  $\theta_o$  will approach the curve of the bifurcated equilibrium. The other branches of the two curves are for negative values of  $\theta$ . They are

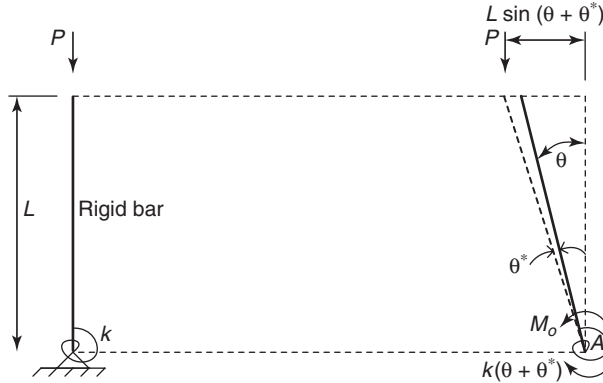


Fig. 1.7 Disturbed equilibrium configuration.

in the left half of the deformation domain and they lie above the curve for  $\theta_o = 0$ . They are in the unstable region for smaller values of  $-\theta$ , that is, they are above the dashed line defining the region between stable and unstable behavior, and they are in the stable region for larger values of  $-\theta$ . (Note: The stability limit will be derived later.) The curves for  $-\theta$  are of little practical consequence for our further discussion.

The nature of the equilibrium, that is, its stability, is examined by disturbing the already deformed system by an additional small rotation  $\theta^*$ , as shown in Figure 1.7.

The equilibrium equation of the disturbed geometry is

$$\sum M_A = 0 = PL \sin(\theta + \theta^*) + M_o - k(\theta + \theta^*)$$

After rearranging we get, noting that  $\theta_o = \frac{M_o}{k}$

$$\frac{PL}{k} = \frac{\theta + \theta^* - \theta_o}{\sin(\theta + \theta^*)} \tag{1.11}$$

From trigonometry we know that  $\sin(\theta + \theta^*) = \sin \theta \cos \theta^* + \cos \theta \sin \theta^*$ . For small values of  $\theta^*$  we can use  $\cos \theta^* \approx 1.0$ ;  $\sin \theta^* \approx \theta^*$ , and therefore

$$\frac{PL}{k} = \frac{\theta + \theta^* - \theta_o}{\sin \theta + \theta^* \cos \theta} \tag{1.12}$$

This equation can be rearranged to the following form:  $\frac{PL}{k} \sin \theta - \theta + \theta_o + \theta^* (\frac{PL}{k} \cos \theta - 1) = 0$ . However,  $\frac{PL}{k} \sin \theta - \theta + \theta_o = 0$  as per equation 1.10,  $\theta^* \neq 0$ , and thus

$$\frac{PL}{k} \cos \theta - 1 = 0 \tag{1.13}$$

Equation 1.13 is the locus of points for which  $\theta^* \neq 0$  while equilibrium is just maintained, that is the equilibrium is *neutral*. The same result could have been obtained by setting the derivative of  $F = \frac{PL}{k} \sin \theta - \theta + \theta_0$  with respect to  $\theta$  equal to zero:

$$\frac{dF}{d\theta} = \frac{PL}{k} \cos \theta - 1.$$

The meaning of the previous derivation is that when

1.  $\cos \theta < \frac{1}{PL/k}$ , the equilibrium is *stable*—that is, the bar returns to its original position when  $q^*$  is removed; energy must be added.
2.  $\cos \theta = \frac{1}{PL/k}$ , the equilibrium is *neutral*—that is, no force is required to move the bar a small rotation  $\theta^*$ .
3.  $\cos \theta > \frac{1}{PL/k}$ , the equilibrium is *unstable*—that is, the configuration will snap from an unstable to a stable shape; energy is released.

These derivations are very simple, yet they give us a lot of information:

1. The load-deflection path of the system that sustains an applied action  $\theta_0$  from the start of loading. This will be henceforth designated as an *imperfect* system, because it has some form of deviation in either loading or geometry from the ideally *perfect* structure that is straight or unloaded before the axial force is applied.
2. It provides the critical, or buckling, load at which the equilibrium become neutral.
3. It identifies the character of the equilibrium path, whether it is neutral, stable, or unstable.

It is good to have all this information, but for more complex actual structures it is often either difficult or very time-consuming to get it. We may not even need all the data in order to design the system. Most of the time it is sufficient to know the buckling load. For most practical structures, the determination of this critical value requires only a reasonably modest effort, as shown in section 1.2.

In the discussion so far we have derived three hierarchies of results, each requiring more effort than the previous one:

1. Buckling load of a perfect system (Figure 1.2)
2. The post-buckling history of the perfect system (Figure 1.5)
3. The deformation history of the “imperfect” system (Figure 1.7)

In the previous derivations the equilibrium condition was established by utilizing the statical approach. Equilibrium can, however, be determined by using the theorem of virtual work. It is sometimes more convenient to use this method, and the following derivation will feature the development of this approach for the spring-bar problem.

**1.3.2 Virtual Work Solution**

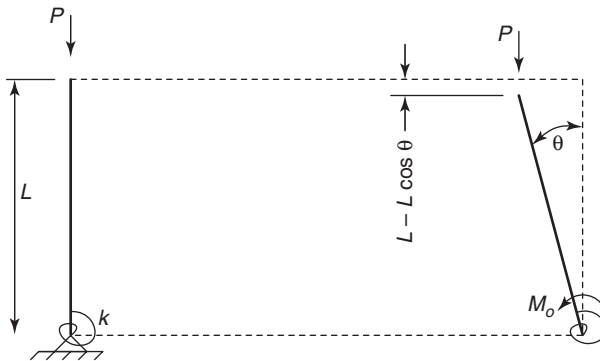
We also examine the large displacement behavior of the system using the energy approach described in section 1.2. The geometry of the system is shown in Figure 1.8

For the spring-bar system the strain energy is the work done by the spring,  $U = W_i = \frac{1}{2}k\theta^2$ . The potential of the external forces is equal to  $V_p = W_e = -PL(1 - \cos \theta) - M_o\theta$ . With  $\theta_o = \frac{M_o}{k}$  the total potential becomes

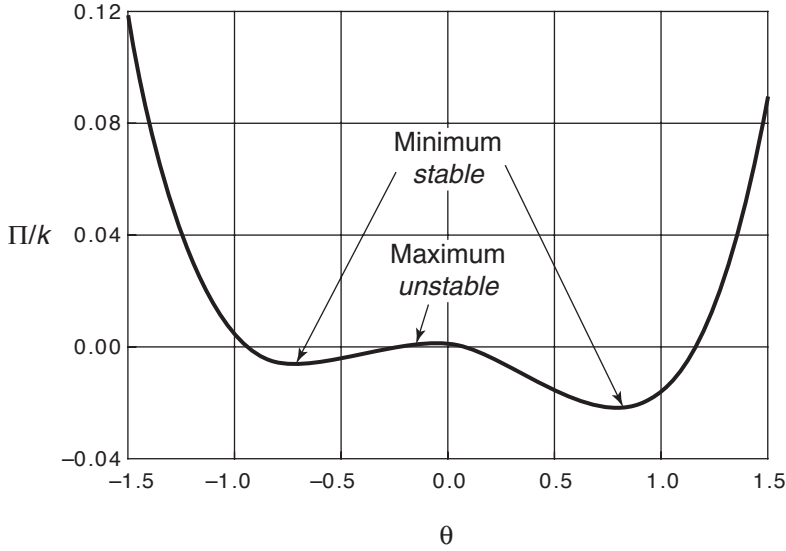
$$\frac{\Pi}{k} = \frac{\theta^2}{2} - \frac{PL}{k}(1 - \cos \theta) - \theta_o\theta \tag{1.14}$$

The total potential is plotted against the bar rotation in Figure 1.9 for the case of  $\theta_o = 0.01$  and  $PL/k = 1.10$ . In the range  $-1.5 \leq \theta \leq 1.5$  the total potential has two minima (at approximately  $\theta = 0.8$  and  $-0.7$ ) and one maximum (at approximately  $\theta = -0.1$ ). According to the *Principle of Virtual Work*, the maxima and minima are equilibrium positions, because if there is a small change in  $\theta$ , there is no change in the total potential. In the terminology of structural mechanics, the total potential is *stationary*. It is defined by the derivative  $\frac{d\Pi}{d\theta} = 0$ . From equation 1.6,  $\frac{d\Pi}{d\theta} = 0 = \frac{2\theta}{2} - \frac{PL}{k} \sin \theta - \theta_o$ , or

$$\frac{PL}{k} = \frac{\theta - \theta_o}{\sin \theta} \tag{1.15}$$



**Fig. 1.8** Geometry for the total potential determination.



**Fig. 1.9** Total potential for  $\theta_o = 0.01$  and  $PL/k = 1.10$ .

This equation is identical to equation 1.10. The status of stability is illustrated in Figure 1.10 using the analogy of the ball in the cup (stable equilibrium), the ball on the top of the upside-down cup (unstable equilibrium), and the ball on the flat surface.


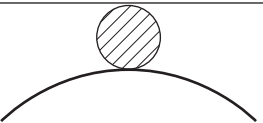
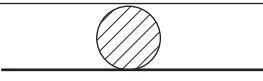
The following summarizes the problem of the spring-bar model's energy characteristics:

$$\begin{aligned} \frac{\Pi}{k} &= \frac{\theta^2}{2} - \frac{PL}{k}(1 - \cos \theta) - \theta_o \theta \rightarrow \text{Total potential} \\ \frac{d(\Pi/k)}{d\theta} &= \theta - \theta_o - \frac{PL}{k} \sin \theta = 0 \rightarrow \frac{PL}{k} = \frac{\theta - \theta_o}{\sin \theta} \rightarrow \text{Equilibrium} \quad (1.16) \\ \frac{d^2(\Pi/k)}{d\theta^2} &= 1 - \frac{PL}{k} \cos \theta = 0 \rightarrow \frac{PL}{k} = \frac{1}{\cos \theta} \rightarrow \text{Stability} \end{aligned}$$

These equations represent the energy approach to the large deflection solution of this problem.

For the small deflection problem we set  $\theta_o = 0$  and note that  $1 - \cos \theta \approx \frac{\theta^2}{2}$ . The total potential is then equal to  $\Pi = \frac{k\theta^2}{2} - \frac{PL\theta^2}{2}$ . The derivative with respect to  $\theta$  gives the critical load:

$$\frac{d\Pi}{d\theta} = 0 = \theta(k - PL) \rightarrow P_{cr} = k/L \quad (1.17)$$

<ul style="list-style-type: none"> <li>• Minimum of <math>\Pi</math></li> <li>• <b>Stable equilibrium</b></li> <li>• Energy must be added to change configuration.</li> </ul>	$\frac{d^2\Pi}{d\theta^2} > 0$		<p><i>Ball in cup can be disturbed, but it will return to the center.</i></p>
<ul style="list-style-type: none"> <li>• Maximum of <math>\Pi</math></li> <li>• <b>Unstable equilibrium</b></li> <li>• Energy is released as configuration is changed.</li> </ul>	$\frac{d^2\Pi}{d\theta^2} < 0$		<p><i>Ball will roll down if disturbed.</i></p>
<ul style="list-style-type: none"> <li>• Transition from minimum to maximum</li> <li>• <b>Neutral equilibrium</b></li> <li>• There is no change in energy.</li> </ul>	$\frac{d^2\Pi}{d\theta^2} = 0$		<p><i>Ball is free to roll.</i></p>

**Fig. 1.10** Table illustrating status of stability.

Thus far, we have considered three methods of stability evaluation:

1. The small deflection method, giving only the buckling load.
2. The large deflection method for the perfect structure, giving information about post-buckling behavior.
3. The large deflection method for the imperfect system, giving the complete deformation history, including the reduction of stiffness in the vicinity of the critical load.

Two methods of solution have been presented:

1. Static equilibrium method
2. Energy method

Such stability-checking procedures are applied to analytically exact and approximate methods for real structures in the subsequent portions of this book.

The spring-bar system of Figure 1.5 exhibited a particular post-buckling characteristic: The post-buckling deflections increased as the load was raised above the bifurcation point, as seen in Figure 1.6. Such *hardening* behavior is obviously desirable from the standpoint of safety. However, there are structural systems where the post-buckling exhibits a *softening* character. Such a spring-bar structure will be considered next for the system of Figure 1.11.

Equilibrium is obtained by taking moments about the pinned base of the rigid bar that is restrained by a horizontal spring a distance  $a$  above its base and is disturbed by a moment  $M_o$ :

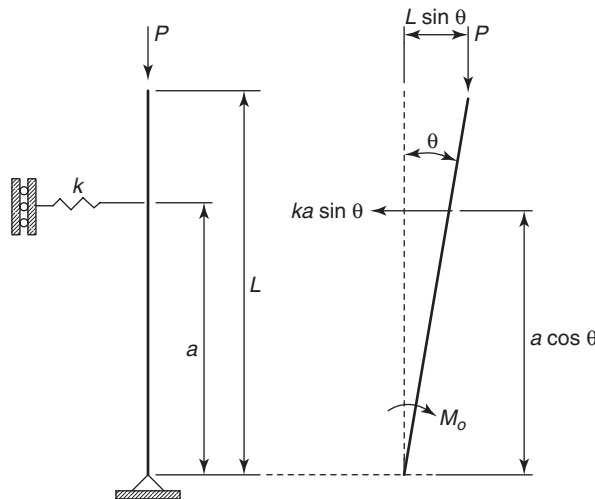
$$(ka \sin \theta) a \cos \theta - M_o - PL \sin \theta = 0$$

Rearrangement and introduction of the imperfection parameter  $\theta_o = \frac{M_o}{ka^2}$  gives the following equation:

$$\frac{PL}{ka^2} = \frac{\sin \theta \cos \theta - \theta_o}{\sin \theta} \tag{1.18}$$

The *small deflection ideal geometry* assumption ( $\theta_o = 0$ ;  $\sin \theta = \theta$ ;  $\cos \theta = 1$ ) leads to the buckling load

$$P_{cr} = \frac{ka^2}{L} \tag{1.19}$$



**Fig. 1.11** Softening spring-bar structure.

From the *large deflection-ideal geometry* assumption ( $\theta_o = 0$ ) we get the post-buckling strength:

$$P_{cr} = \frac{ka^2}{L} \cos \theta \tag{1.20}$$

The load-rotation curves from equations 1.18 and 1.20 are shown in Figure 1.12 for the perfect ( $\theta_o = 0$ ) and the imperfect ( $\theta_o = 0.01$ ) system. The post-buckling behavior is *softening*—that is, the load is decreased as the rotation increases. The deflection of the imperfect system approaches that of the perfect system for large bar rotations. However, the strength of the imperfect member will never attain the value of the ideal critical load. Since in actual structures there will always be imperfections, the theoretical buckling load is upper bound.

The nature of stability is determined from applying a virtual rotation to the deformed system. The resulting equilibrium equation then becomes equal to

$$[ka \sin(\theta + \theta^*)]a \cos(\theta + \theta^*) - M_o - PL \sin(\theta + \theta^*) = 0$$

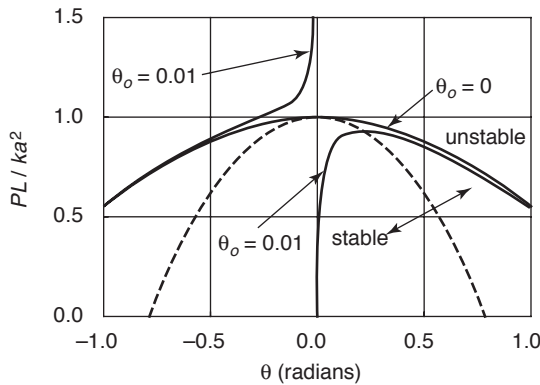
Noting that  $\theta^*$  is small, and so  $\sin \theta^* = \theta^*$ ;  $\cos \theta^* = 1$ . Also making use of the trigonometric relationships

$$\sin(\theta + \theta^*) = \sin \theta \cos \theta^* + \cos \theta \sin \theta^* = \sin \theta + \theta^* \cos \theta$$

$$\cos(\theta + \theta^*) = \cos \theta \cos \theta^* - \sin \theta \sin \theta^* = \cos \theta - \theta^* \sin \theta$$

we can arrive at the following equation:

$$\begin{aligned} & [ka^2 \sin \theta \cos \theta - M_o - PL \sin \theta] \\ & + \theta^* [ka^2 (\cos^2 \theta - \sin^2 \theta) - PL \cos \theta] \\ & - \theta^* [ka^2 \cos \theta \sin \theta] = 0 \end{aligned}$$



**Fig. 1.12** Load-rotation curves for a softening system.

The first line is the equilibrium equation, and it equals zero, as demonstrated above. The bracket in the third line is multiplied by the square of a small quantity ( $\theta^* \gg \theta^2$ ) and so it can be neglected. From the second line we obtain the stability condition that is shown in Figure 1.12 as a dashed line:

$$\begin{aligned} \frac{PL}{ka^2} &= \frac{\cos^2\theta - \sin^2\theta}{\cos\theta} \\ &= \frac{2\cos^2\theta - 1}{\cos\theta} \end{aligned} \quad (1.21)$$

This problem is solved also by the energy method, as follows:

$$\text{Total potential: } \Pi = \frac{k(a \sin \theta)^2}{2} - M_o\theta - PL(1 - \cos \theta)$$

$$\begin{aligned} \text{Equilibrium: } \frac{\partial \Pi}{\partial \theta} &= ka^2 \sin \theta \cos \theta - M_o - PL \sin \theta = 0 \rightarrow \frac{PL}{ka^2} \\ &= \frac{\sin \theta \cos \theta}{\sin \theta} \end{aligned}$$

$$\text{Stability: } \frac{\partial^2 \Pi}{\partial \theta^2} = ka^2[\cos^2\theta - \sin^2\theta] - PL \cos \theta = 0 \rightarrow \frac{PL}{ka^2} = \frac{2\cos^2\theta - 1}{\cos \theta}$$

The two spring-bar problems just discussed illustrate three post-buckling situations that occur in real structures: hardening post-buckling behavior, softening post-buckling behavior, and the transitional case where the post-buckling curve is flat for all practical purposes. These cases are discussed in various contexts in subsequent chapters of this book. The drawings in Figure 1.13 summarize the different post-buckling relationships, and indicate the applicable real structural problems. Plates are insensitive to initial imperfections, exhibiting reliable additional strength beyond the buckling load. Shells and columns that buckle after some parts of their cross section have yielded are imperfection sensitive. Elastic buckling of columns, beams, and frames have little post-buckling strength, but they are not softening, nor are they hardening after buckling.

Before leaving the topic of spring-bar stability, we will consider two more topics: the snap-through buckling and the multidegree of freedom column.

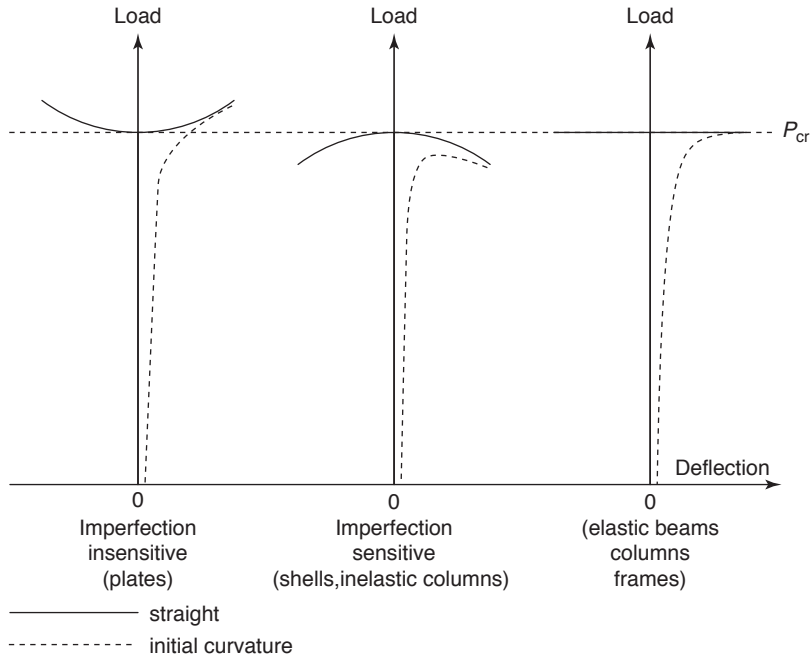


Fig. 1.13 Illustration of post-buckling behavior.

### 1.4 SNAP-THROUGH BUCKLING

Figure 1.14 shows a two-bar structure where the two rigid bars are at an angle to each other. One end of the right bar is on rollers that are restrained by an elastic spring. The top Figure 1.14 shows the loading and geometry, and the bottom features the deformed shape after the load is applied. Equilibrium is determined by taking moments of the right half of the deformed structure about point *A*.

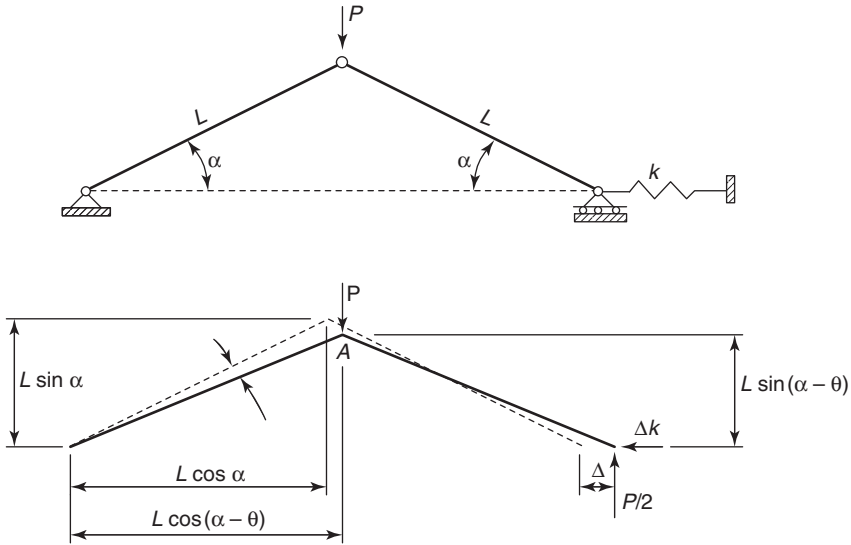
$$\sum M_A = 0 = \frac{P}{2} [L \cos (\alpha - \theta)] - \Delta kL \sin (\alpha - \theta)$$

From the deformed geometry of Figure 1.14 it can be shown that

$$\Delta = 2L \cos (\alpha - \theta) - 2L \cos \theta$$

The equilibrium equation thus is determined to be

$$\frac{P}{kL} = 4[\sin (\alpha - \theta) - \tan (\alpha - \theta) \cos \alpha] \tag{1.22}$$



**Fig. 1.14** The snap-through structure.

The state of the equilibrium is established by disturbing the deflected structure by an infinitesimally small virtual rotation  $\theta^*$ . After performing trigonometric and algebraic manipulations it can be shown that the curve separating stable and unstable equilibrium is

$$\frac{P}{kL} = 4 \left[ \frac{1 - 2 \cos^2(\alpha - \theta) + \cos(\alpha - \theta) \cos \alpha}{\sin(\alpha - \theta)} \right] \quad (1.23)$$

If we substitute  $PL/k$  from equation 1.22 into equation 1.23, we get, after some elementary operations, the following equation that defines the angle  $\theta$  at the limit of stable equilibrium:

$$\cos^3(\alpha - \theta) - \cos \alpha = 0 \quad (1.24)$$

The curve shown in Figure 1.15 represents equilibrium for the case of  $\alpha = 30^\circ$ . Bar rotation commences immediately as load is increased from zero. The load-rotation response is nonlinear from the start. The slope of the curve increases until a peak is reached at  $P/kl = 0.1106$  and  $\theta = 0.216$  radians. This is also the point of passing from stable to unstable equilibrium as defined by equations 1.23 and 1.24. The deformation path according to equation 1.22 continues first with a negative slope until minimum is reached, and then it moves up with a positive slope. However, the actual path of the deflection of the structure does not follow this unstable path, but the structure snaps through to  $\theta = 1.12$  radians. Such behavior is typical of shell-type structures, as can be easily demonstrated by standing on the top of

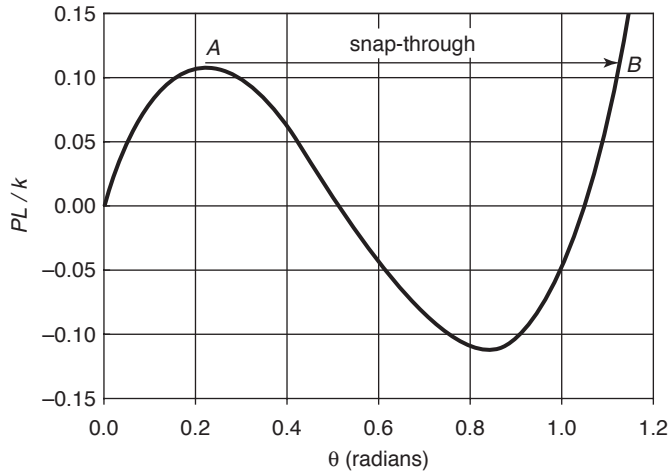


Fig. 1.15 Load-rotation curve for snap-through structure for  $\alpha = 30^\circ$ .

an empty aluminum beverage can and having someone touch the side of the can. A similar event takes place any time a keyboard of a computer is pushed. Snap-through is sudden, and in a large shell structure it can have catastrophic consequences.

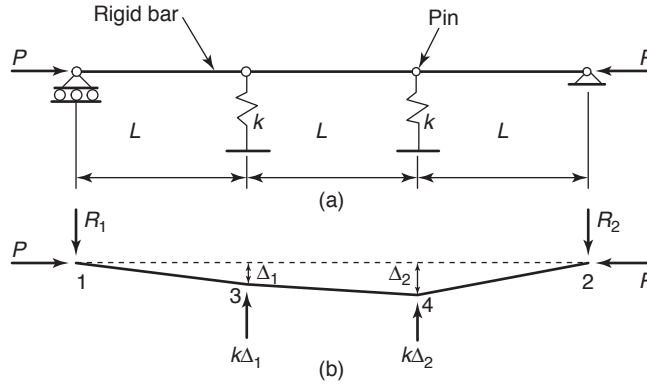
Similarly to the problems in the previous section, the energy approach can be also used to arrive at the equilibrium equation of equation 1.22 and the stability limit of equation 1.23 by taking, respectively, the first and second derivative of the total potential with respect to  $\theta$ . The total potential of this system is

$$\Pi = \frac{1}{2}k\{2L[\cos(\alpha - \theta) - \cos\alpha]\}^2 - PL[\sin\alpha - \sin(\alpha - \theta)] \quad (1.25)$$

The reader can complete to differentiations to verify the results.

### 1.5 MULTI-DEGREE-OF-FREEDOM SYSTEMS

The last problem to be considered in this chapter is a structure made up of three rigid bars placed between a roller at one end and a pin at the other end. The center bar is connected to the two edge bars with pins. Each interior pinned joint is restrained laterally by an elastic spring with a spring constant  $k$ . The structure is shown in Figure 1.16a. The deflected shape at buckling is presented as Figure 1.16b. The following buckling analysis is performed by assuming small deflections and an initially perfect geometry. Thus, the only information to be gained is the critical load at which a straight and a buckled configuration are possible under the same force.



**Fig. 1.16** Three-bar structure with intermediate spring supports.

Equilibrium equations for this system are obtained as follows:

$$\text{Sum of moments about Point 1: } \sum M_1 = 0 = k\Delta_1 L + k\Delta_2(2L) - R_2(3L)$$

$$\text{Sum of vertical forces: } \sum F_y = 0 = R_1 + R_2 - k\Delta_1 - k\Delta_2$$

$$\text{Sum of moments about point 3, to the left: } \sum M_3 = 0 = P\Delta_1 - R_1 L$$

$$\text{Sum of moments about point 4, to the right: } \sum M_4 = 0 = P\Delta_2 - R_2 L$$

Elimination of  $R_1$  and  $R_2$  from these four equations leads to the following two homogeneous simultaneous equations:

$$\begin{bmatrix} P - \frac{2kL}{3} & -\frac{kL}{3} \\ -\frac{kL}{3} & P - \frac{2kL}{3} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = 0 \quad (1.26)$$

The deflections  $\Delta_1$  and  $\Delta_2$  can have a nonzero value only if the determinant of their coefficients becomes zero:

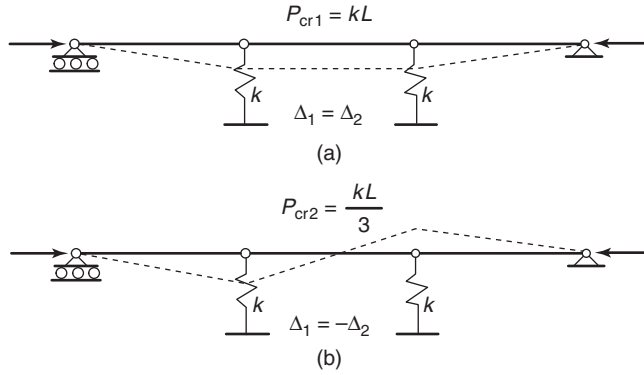
$$\begin{vmatrix} P - \frac{2kL}{3} & -\frac{kL}{3} \\ -\frac{kL}{3} & P - \frac{2kL}{3} \end{vmatrix} = 0 \quad (1.27)$$

Decomposition of the determinant leads to the following quadratic equation:

$$3\left(\frac{P}{kL}\right)^2 - 4\frac{P}{kL} + 1 = 0 \quad (1.28)$$

This equation has two roots, giving the following two critical loads:

$$\begin{aligned} P_{\text{cr1}} &= kL \\ P_{\text{cr2}} &= \frac{kL}{3} \end{aligned} \quad (1.29)$$



**Fig. 1.17** Shapes of the buckled modes.

The smaller of the two critical loads is then the buckling load of interest to the structural engineer. Substitution each of the critical loads into equation 1.26 results in the mode shapes of the buckled configurations, as illustrated in Figure 1.17.

Finally then,  $P_{cr} = \frac{kL}{3}$  is the governing buckling load, based on the small deflection approach.

The energy method can also be used for arriving at a solution to this problem. The necessary geometric relationships are illustrated in Figure 1.18, and the small-deflection angular and linear deformations are given as follows:

$$\Delta_1 = \psi L \quad \text{and} \quad \Delta_2 = \theta L$$

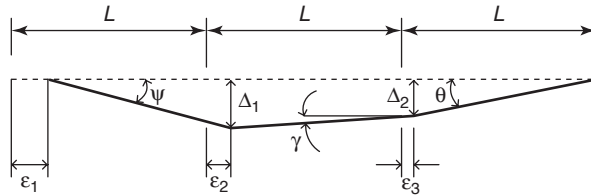
$$\frac{\Delta_1 - \Delta_2}{L} = \gamma = \psi - \theta$$

$$\varepsilon_3 = L - L \cos \theta \approx \frac{L\theta^2}{2}$$

$$\varepsilon_2 = \varepsilon_3 + L[1 - \cos(\psi - \theta)] = \frac{L}{2}(2\theta^2 + \psi^2 - 2\psi\theta)$$

$$\varepsilon_3 = \varepsilon_2 + \frac{L\psi^2}{2} = L(\theta^2 + \psi^2 - \psi\theta)$$

The strain energy equals  $U_P = \frac{k}{2}(\Delta_1^2 + \Delta_2^2) = \frac{kL^2}{2}(\psi^2 + \theta^2)$ .



**Fig. 1.18** Deflections for determining the energy solution.

The potential of the external forces equals  $V_P = -P\varepsilon_1 = -PL(\theta^2 + \psi^2 - \psi\theta)$

The total potential is then

$$\Pi = U + V_P = \frac{kL^2}{2}(\psi^2 + \theta^2) - PL(\theta^2 + \psi^2 - \psi\theta) \quad (1.30)$$

For equilibrium, we take the derivatives with respect to the two angular rotations:

$$\begin{aligned} \frac{\partial \Pi}{\partial \psi} = 0 &= \frac{kL^2}{2}(2\psi) - 2PL\psi + PL\theta \\ \frac{\partial \Pi}{\partial \theta} = 0 &= \frac{kL^2}{2}(2\theta) - 2PL\theta + PL\psi \end{aligned}$$

Rearranging, we get

$$\begin{bmatrix} (kL^2 - 2PL) & PL \\ PL & (kL^2 - 2PL) \end{bmatrix} \begin{bmatrix} \theta \\ \psi \end{bmatrix} = 0$$

Setting the determinant of the coefficients equal to zero results in the same two critical loads that were already obtained.

## 1.6 SUMMARY

This chapter presented an introduction to the subject of structural stability. Structural engineers are tasked with designing and building structures that are safe under the expected loads throughout their intended life. Stability is particularly important during the erection phase in the life of the structure, before it is fully braced by its final cladding. The engineer is especially interested in that critical load magnitude where the structure passes from a stable to an unstable configuration. The structure must be proportioned so that the expected loads are smaller than this critical value by a safe margin.

The following basic concepts of stability analysis are illustrated in this chapter by several simple spring-bar mechanisms:

- The critical, or buckling load, of geometrically perfect systems
- The behavior of structures with initial geometric or static imperfections
- The amount of information obtained by small deflection and large deflection analyses
- The equivalence of the geometrical and energy approach to stability analysis

- The meaning of the results obtained by a *bifurcation* analysis, a computation of the *post-buckling* behavior, and by a *snap-through* investigation
- The hardening and the softening post-buckling deformations
- The stability analysis of multi-degree-of-freedom systems

We encounter each of these concepts in the subsequent parts of this text, as much more complex structures such as columns, beams, beam-columns, and frames are studied.

**PROBLEMS**

- 1.1. Derive an expression for the small deflection bifurcation load in terms of  $\frac{EI}{L^2}$ .

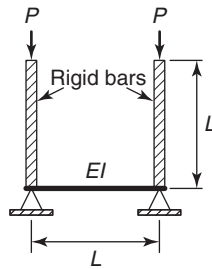


Fig. p1.1

- 1.2. Determine the critical load of this planar structural system if

$$a = L, L_1 = L \text{ and } L_2 = 3L.$$

*Hint:* The flexible beam provides a rotational and translational spring to the rigid bar compression member.

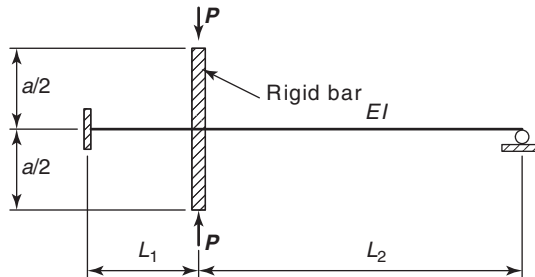
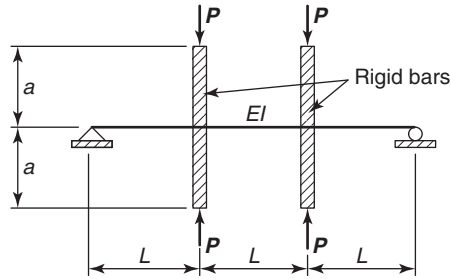


Fig. p1.2

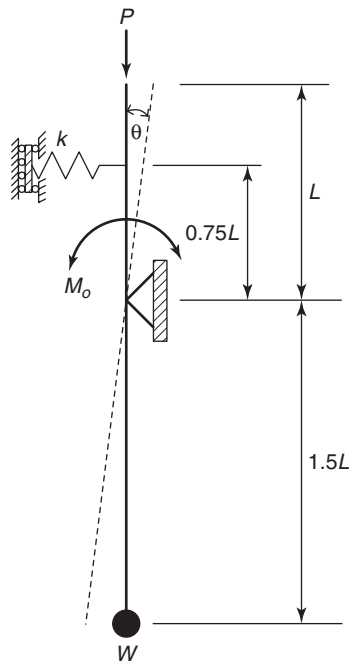
- 1.3. Determine the critical load of this planar structural system.  
*Hint:* The flexible beam provides a rotational and translational spring to the rigid bar compression member.



**Fig. p1.3**

- 1.4.** In the mechanism a weightless infinitely stiff bar is pinned at the point shown. The load  $P$  remains vertical during deformation. The weight  $W$  does not change during buckling. The spring is unstretched when the bar is vertical. The system is disturbed by a moment  $M_o$  at the pin.
- Determine the critical load  $P$  according to small deflection theory.
  - Calculate and plot the equilibrium path  $p - \theta$  for  $0 \leq \theta \leq \frac{\pi}{2}$  when  $\theta_o = 0$  and

$$\theta_o = 0.01, \quad p = \frac{PL - Wb}{ka^2} \quad \text{and} \quad \theta_o = \frac{M_o}{ka^2}, \quad a = 0.75L \quad \text{and} \quad b = 1.5L.$$



**Fig. p1.4**

- c. Investigate the stability of the equilibrium path.
- d. Discuss the problem.

*Note:* This problem was adapted from Chapter 2, Simitses “An introduction to the elastic stability of structures” (see end of Chapter 2 for reference details).

- 1.5. Develop an expression for the critical load using the small-deflection assumption. Employ both the equilibrium and the energy method.  
*Note:* that the units of  $K_1$  are inch-kip/radian, and the units of  $K_2$  are kip/inch

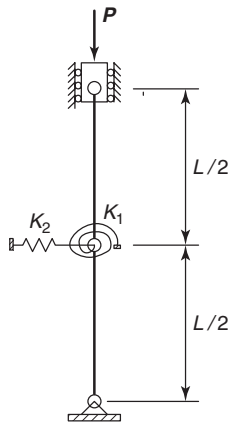


Fig. p1.5

- 1.6. Develop an expression for the critical load using the small-deflection assumption. The structure is made up of rigid bars and elastic springs. Employ both the equilibrium and the energy method.

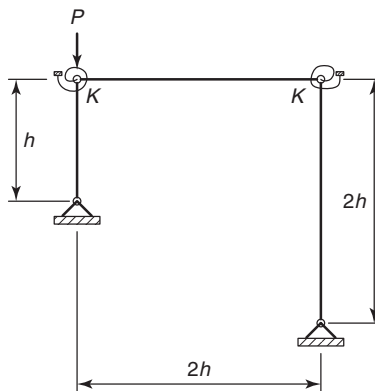
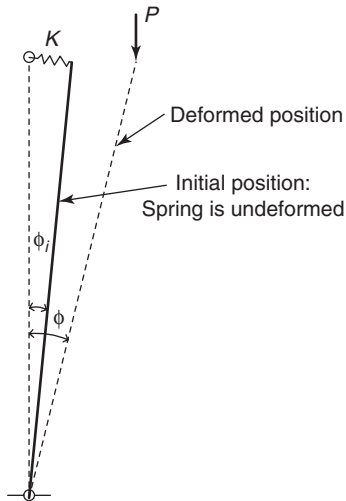


Fig. p1.6

- 1.7. The length of the bar is  $L$ , and it is in an initially rotated condition  $\phi_i$  from the vertical. The spring is undistorted in this initial configuration. A vertical load  $P$  is applied to the system, causing it to deflect an angle  $\phi$  from the vertical. The load  $P$  remains vertical at all times. Derive equations for equilibrium and stability, using the equilibrium and the energy methods. Plot  $P$  versus  $\phi$  for  $\phi_i = 0.05$  radians.



**Fig. p1.7**