Quatrer
ion
Algebr
a

This chapter introduces the quaternion algebra $\mathbb{H}$ and presents some properties that will be useful in later chapters.

1.1. Definitions

Quaternions are one of the four existing normed division algebras over the real numbers. Classically denoted by $\mathbb{H}$ in honor of Sir W.R. Hamilton who discovered them in 1843, they form a non-commutative algebra. A quaternion $q \in \mathbb{H}$ is a four-dimensional (4D) hypercomplex number and has a Cartesian form given by:

$$q = a + ib + jc + kd$$  \[1.1\]

where $a, b, c, d \in \mathbb{R}$ are called its components. The three imaginary units $i, j, k$ are square roots of $-1$ and are related through the famous\(^1\) relations:

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = -ji = k$$

$$ki = -ik = j$$

$$jk = -kj = i$$  \[1.2\]

A quaternion $q \in \mathbb{H}$ can be decomposed into a scalar part $S(q)$ and a vector part $V(q)$:

$$q = S(q) + V(q)$$  \[1.3\]

---

\(^1\) These relations defining the three imaginary units of an element of $\mathbb{H}$ were carved by Sir W.R. Hamilton on a stone of the Broome bridge in Dublin on 16 October 1843.
where $S(q) = a$ and $V(q) = q - S(q) = ib + jc + kd$. Obviously, $S(q) \in \mathbb{R}$ and we will also refer to it as the real part of $q$, i.e. $\mathbb{R}(q) = a$. Now, $q \in \mathbb{H}$ will be called a pure quaternion if its real part is null, i.e. if $S(q) = 0$. The set of pure quaternions will be denoted as $V(\mathbb{H})$, while the set of quaternions with a null vector part are trivially identified with elements of $\mathbb{R}$, i.e. $S(\mathbb{H}) \equiv \mathbb{R}$. If $q$ has a null vector part, $V(q) = 0$, then $q$ is simply an element of $\mathbb{R}$. To identify the different imaginary components of a quaternion $q = a + ib + jc + kd$, we will use the following notations:

$$\begin{align*}
\mathfrak{R}_i(q) &= b \\
\mathfrak{R}_j(q) &= c \\
\mathfrak{R}_k(q) &= d
\end{align*} \tag{1.4}$$

so that a quaternion $q \in \mathbb{H}$ can be written as:

$$q = \mathfrak{R}(q) + i\mathfrak{R}_i(q) + j\mathfrak{R}_j(q) + k\mathfrak{R}_k(q) \tag{1.5}$$

The Cartesian notation for a quaternion $q \in \mathbb{H}$ is, in fact, its expression in a specific 4D basis of the algebra $\mathbb{H}$, namely in the basis $\{1, i, j, k\}$. Recall that, as an algebra, $\mathbb{H}$ possesses a vector space structure that allows the expression of any of its elements in terms of its components in a basis of $\mathbb{H}$. The basis $\{1, i, j, k\}$ is the most common and popular basis to express a quaternion. However, we may encounter some other bases later on in this book, leading to alternate notations for a quaternion $q \in \mathbb{H}$. Before introducing these notations, we first review some remarkable properties of quaternions.

### 1.2. Properties

Here, we list some of the properties of quaternions that will be used throughout the book.

From the algebra structure of $\mathbb{H}$, the sum of two quaternions is trivial. Given two quaternions $q$ and $p$, we have:

$$q + p = S(q) + S(p) + V(p) + V(q) \tag{1.6}$$

Expressing the two quaternions in their Cartesian forms, $q = a + ib + jc + kd$ and $p = e + if + jg + kh$, their sum is:

$$q + p = (a + e) + i(b + f) + j(c + g) + k(d + h) \tag{1.7}$$
and their product takes the form:

\[
qp = (a + ib + jc + kd) (e + if + jg + kh) \\
= ae - (bf + cg + dh) + \\
a (if + jg + kh) + e (ib + jc + kd) + \\
i (ch - dg) + j (df - bh) + k (bg - cf)
\]

[1.8]

Using the scalar/vector notation, this product takes the following form:

\[
qp = S(q) S(p) - \langle V(q), V(p) \rangle + S(q) V(p) + S(p) V(q) + V(q) \times V(p)[1.9]
\]

where \( \langle \cdot, \cdot \rangle \) is the scalar product and \( \times \) is the vector cross product. These are understood in the classical sense of the three-dimensional (3D) vector cross and inner products, which means that:

\[
\langle V(q), V(p) \rangle = bf + cg + dh \tag{1.10}
\]

which is scalar valued, i.e. \( \langle V(q), V(p) \rangle \in \mathbb{R} \), and that:

\[
V(q) \times V(p) = i(ch - dg) + j(df - bh) + k(bg - cf) \tag{1.11}
\]

where the result is a pure quaternion, i.e. \( (V(q) \times V(p)) \in V(\mathbb{H}) \).

A very noticeable property is that the product of two quaternions is not commutative so that in general:

\[
qp \neq pq \tag{1.12}
\]

This can be inferred from the presence of the non-commutative cross product in [1.9]. Note, however, that the product of quaternions is associative so that for any three quaternions \( q, p, r \in \mathbb{H} \), the following is true:

\[
(qp) r = q (pr) \tag{1.13}
\]

The norm of a quaternion \( q \) is defined as:

\[
\|q\| = a^2 + b^2 + c^2 + d^2 \tag{1.14}
\]

A quaternion \( q \in \mathbb{H} \) with \( \|q\| = 1 \) is said to be a unit quaternion. As previously mentioned, \( \mathbb{H} \) is one of the four existing normed division algebras. As a result, given any two quaternions \( p, q \in H \), then:

\[
\|qp\| = \|q\| \|p\| \tag{1.15}
\]
It can also be easily checked that \( \|qp\| = \|pq\| \). A related quantity that will be used in the following is the modulus of a quaternion. It is defined as the length of the quaternion in Euclidean 4D space. The modulus of \( q \in \mathbb{H} \) is denoted by \( |q| \) and is expressed as:

\[
|q| = \sqrt{a^2 + b^2 + c^2 + d^2} = \|q\|^{\frac{1}{2}} \tag{1.16}
\]

Obviously, \( |q| \in \mathbb{R}^+ \) and \( |q| = 0 \) if and only if \( q = 0 \). Like the norm, the modulus of a product of two quaternions \( p \) and \( q \) has the following property:

\[
|pq| = |p| \ |q| = |qp| \tag{1.17}
\]

Just as with the complex numbers, the conjugate of a quaternion \( q \) is obtained by negating its imaginary part. However, in \( \mathbb{H} \) the imaginary part is 3D and consists of the entire vector part \( V(q) \). Denoted by \( \overline{q} \), the conjugate of \( q \) is thus defined as:

\[
\overline{q} = a - ib - jc - kd = S(q) - V(q) \tag{1.18}
\]

It follows that the scalar and vector parts of any quaternion \( q \in \mathbb{H} \) can be obtained by:

\[
S(q) = \frac{1}{2} (q + \overline{q}) \quad \text{and} \quad V(q) = \frac{1}{2} (q - \overline{q}) \tag{1.19}
\]

Conjugation in \( \mathbb{H} \) has the following property:

\[
\overline{\overline{q}} = q \tag{1.20}
\]

In contrast to the complex case, conjugation is not an involution\(^2\) but an anti-involution, such that for \( q, p \in \mathbb{H} \):

\[
\overline{qp} = \overline{p} \overline{q} \tag{1.21}
\]

that is, the order of the factors in a quaternion product is reversed by the conjugation operator. Note that the modulus (and also the norm) of a quaternion \( q \) can be expressed using the conjugate of \( q \) as:

\[
|q| = \sqrt{q \overline{q}} \quad \text{and} \quad \|q\| = q \overline{q} \tag{1.22}
\]

---

\(^2\) An involution \( f : \mathbb{H} \to \mathbb{H} \) is such that for \( q, p \in \mathbb{H} \):

\[
\begin{align*}
 f(f(q)) &= q \\
 f(p + q) &= f(p) + f(q) \\
 f(pq) &= f(p)f(q)
\end{align*}
\]
It is also possible to define involutions over \( \mathbb{H} \). Involutions are defined with respect to a pure unit quaternion \( \mu (\mu \in \mathbb{V}(\mathbb{H}) \) and \( |\mu| = 1 \). The most general case (together with many properties) is presented in [ELL 07d]. As special cases, one can choose \( \mu \) as one of the standard basis elements of \( \mathbb{H} \), i.e. \( i, j \) or \( k \). Involutions with respect to these three unit pure quaternions can be called canonical. Given a quaternion \( q \), its three canonical involutions are:

\[
\begin{align*}
q^i &= a + ib - jc - kd = -iqi \\
q^j &= a - ib + jc - kd = -jqj \\
q^k &= a - ib - jc + kd = -kqk
\end{align*}
\]

[1.23]

Clearly, \( q \) and its three canonical involutions allow us to recover the four components \( a, b, c, d \) of \( q \) by linear combination. Now, the most general definition for involution is:

\[
\overline{q}^\mu = -\mu q \mu
\]

[1.24]

where \( \mu \in \mathbb{V}(\mathbb{H}) \) and \( |\mu| = 1 \).

Involutions in \( \mathbb{H} \) possess many properties (see [ELL 07d] for details) among which, for \( q, p \in \mathbb{H} \) and \( \mu \in \mathbb{V}(\mathbb{H}) \) and \( \mu^2 = -1 \) (\( i, j \) and \( k \) are possible choices for \( \mu \)):

\[
\begin{align*}
\overline{qp}^\mu &= \overline{q}^\mu \overline{p}^\mu \\
\overline{q}^{\mu p} &= q \\
\overline{q}^{\mu j} &= \overline{q}^k
\end{align*}
\]

[1.25]

As \( \mathbb{H} \) is a division algebra, any non-null quaternion possesses an inverse. The inverse of a given quaternion \( q \in \mathbb{H} \) is given by:

\[
q^{-1} = \frac{\overline{q}}{||q||}
\]

[1.26]

where it can be easily checked that \( qq^{-1} = 1 \) because of [1.22]. Note that for a pure unit quaternion \( \mu \), \( (||\mu|| = 1 \) and \( S(\mu) = 0 \)), the following holds: \( \mu^{-1} = -\mu \).

Now that we have introduced the inverse of a quaternion, we are ready to look at the ratio of two quaternions \( p \) and \( q \). Ratios must be handled with care in \( \mathbb{H} \) (indeed, in any non-commutative algebra), and it is preferable to avoid the \( p/q \) notation when possible, as it is ambiguous, since \( p/q \) can be interpreted as the product of \( p \) by \( q^{-1} \);
the notation \( p/q \) does not specify the order of the product so that it leaves the possibility for \( pq^{-1} \) and \( q^{-1}p \). The ambiguity arises from the fact that in general:

\[
pq^{-1} \neq q^{-1}p
\]  

[1.27]

The above non-equality arises from the fact that:

\[
\frac{pq}{\|q\|} = \frac{p\bar{q}}{\|q\|} \text{ while } \frac{q^{-1}p}{\|q\|} = \frac{\bar{q}p}{\|q\|}
\]  

[1.28]

Thus, it is important to consider ratios as products with the inverse and to take care of the order of the product.

Now, the modulus of a ratio is an interesting quantity to consider, as it does not suffer from the order in the multiplication, due to the property of the modulus given in [1.17]. It thus follows that:

\[
|q^{-1}p| = |pq^{-1}| = \frac{|p|}{|q|}
\]  

[1.29]

It can be useful to write a given quaternion \( q \in \mathbb{H} \) as a product of a scalar positive number (its modulus) and a unit quaternion. This can be done in the following way:

\[
q = |q| \tilde{q} = |q| \left( \frac{a}{|q|} + i \frac{b}{|q|} + j \frac{c}{|q|} + k \frac{d}{|q|} \right)
\]  

[1.30]

where we used the notation \( \tilde{q} \) for the unit modulus version of \( q \), i.e. \( \tilde{q} = q/|q| \) so that \( |	ilde{q}| = 1 \). Note that the decomposition of \( q \) into the product of its modulus and its normalized version is unique. The normalized version of \( q \), denoted by \( \tilde{q} \), is also sometimes called a versor. Finally, it must be emphasized that if \( q \) is a pure quaternion, i.e. \( q \in \mathbf{V}(\mathbb{H}) \), then it is uniquely written as \( q = |q| \mu \) where we have denoted \( \tilde{q} = \mu \) to highlight the fact that it is a pure unit quaternion.

In [1.5], we introduced the Cartesian form of a quaternion \( q \in \mathbb{H} \), in which it is expressed using the sum of a real part \( \Re(q) \), an \( i \)-imaginary part \( \Im_i(q) \), a \( j \)-imaginary part \( \Im_j(q) \) and a \( k \)-imaginary part \( \Im_k(q) \). This expression is a special case of the expansion of a quaternion over a 4D basis. The specific basis used in [1.5] is \( \{1, i, j, k\} \). This is the classical basis used by most authors. Now, it is possible to use a different basis and it turns out that there is an infinite amount of choices for a basis in \( \mathbb{H} \). Given two pure unit quaternions \( \mu \) and \( \eta \), i.e. \( \mu^2 = \eta^2 = -1 \), which are orthogonal to each other: \( \mu \perp \eta \), the set \( \{1, \mu, \eta, \mu \eta\} \) is a basis of \( \mathbb{H} \). The infinite number of possibilities arises from the infinite number of possible choices for the
pair of orthogonal pure unit quaternions $\mu$ and $\eta$. Note that the orthogonality between pure unit quaternions should be understood as:

$$S(\mu \eta) = \langle \mu, \eta \rangle = 0 \quad [1.31]$$

which is the scalar part of the quaternion product of two pure quaternions (see the expression for the quaternion product in [1.9]). Also note that the orthogonality condition does not mean that the quaternion product between $\mu$ and $\eta$ is equal to zero. On the contrary, as can be seen from [1.9], we have:

$$\mu \eta = V(\mu \eta) = \mu \times \eta \quad [1.32]$$

which is indeed a pure unit quaternion orthogonal to both $\mu$ and $\eta$. This makes $\mu \eta$ the third pure unit quaternion that, together with 1, $\mu$ and $\eta$, forms the quaternion basis. Quaternion bases play an important role in the quaternion notation as well as in the computation of quaternion Fourier transforms (QFT) with arbitrary axis, as will be presented in Chapter 3.

The list of quaternion properties presented in this section is not intended to be exhaustive. More properties can be found in various references [CON 03, KAN 89, WAR 97, KUI 02, HAN 06].

1.3. Exponential and logarithm of a quaternion

Among the functions with quaternion-valued arguments that will be considered in the sequel, two will be of major importance: the exponential and logarithm functions. The former will be central to the study and use of QFTs in subsequent chapters.

1.3.1. Exponential of a pure quaternion

The exponential function $\exp : V(\mathbb{H}) \to \mathbb{H}$ can be defined through its power series expansion; thus, for a given (non-null) pure quaternion $\xi \in V(\mathbb{H})$ written as $\xi = |\xi| \xi$ with $\xi$ a pure unit quaternion$^3$ (i.e. $\xi \in V(\mathbb{H})$ and $\xi^2 = -1$) and $|\xi| \in \mathbb{R}^+$, we can write:

$$e^\xi = \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} = \sum_{n=0}^{+\infty} \frac{|\xi|^n \xi^n}{n!} \quad [1.33]$$

3 Just like any quaternion, a pure quaternion $\xi \in V(\mathbb{H})$ can be uniquely written as $\xi = |\xi| \xi$. $\xi$ is unit quaternion (in this case, pure) and $|\xi|$ is its modulus.
where we make use of the fact that $|\xi|$ and $\xi$ commute as any quaternion commutes with a real number. Now, just like the familiar complex imaginary number $I$ does, a pure unit quaternion behaves, when it is raised to an integer power of $n$, as:

$$\xi^n = \begin{cases} (-1)^k & \text{if } n = 2k \\ (-1)^k \xi & \text{if } n = 2k + 1 \end{cases}$$  \[1.34\]

With this in mind, the exponential $e^\xi$ simply is:

$$e^\xi = \sum_{n=0}^{+\infty} (-1)^k \frac{|\xi|^{2k}}{(2k)!} + \xi \sum_{n=0}^{+\infty} (-1)^k \frac{|\xi|^{2k+1}}{(2k+1)!}$$  \[1.35\]

In this equation, the right-hand side terms are the series expansions of the classical $\cos$ and $\sin$ functions. We have established that for a pure quaternion $\xi = |\xi| \xi$, the exponential of $\xi$ is:

$$e^{\xi} = \cos |\xi| + \xi \sin |\xi|$$  \[1.36\]

This shows that the exponential of a pure quaternion can be expressed easily in terms of cosine and sine functions just as in the complex case. The difference lies in the axis $\xi$ which is a unit pure quaternion, while the argument is the modulus of $\xi$. Thus, the exponential of a pure quaternion is a full quaternion, with real/scalar part $\cos |\xi|$ and vector part $\xi \sin |\xi|$.

It must also be noticed that the exponential of a pure quaternion is always of unit modulus:

$$|e^{\xi}| = 1, \quad \forall \xi \in \mathbf{V}(\mathbb{H})$$  \[1.37\]

which is easily verified using [1.36] as well as property [1.34] with $n = 2$.

Thus, we conclude that the exponential of a pure quaternion is a full unit quaternion. Now, the reciprocal property is interesting: any full unit quaternion can be expressed as the exponential of a pure quaternion\(^4\). This is a remarkable property that will be tackled in section 1.3.3 for the expression of the logarithm of a quaternion, and especially for the polar form and Euler formula over $\mathbb{H}$ (sections 1.4.1 and 1.4.1.1).

\(^4\) A full unit quaternion $q$ is uniquely expressed as the exponential of a pure quaternion, i.e. $q = \exp(\xi)$ with $\xi = |\xi| \xi$, iff $|\xi| \in [0, 2\pi[$.
Another important property, which differs drastically from the complex exponential case, is that the product of two exponentials of pure quaternions is not an exponential with argument equal to the sum of the arguments of the original exponentials. This means that for \( \alpha, \beta \in \mathbb{R}^+ \) and any two distinct pure unit quaternions \( \mu, \nu \), we have:

\[
e^{\nu \alpha} e^{\mu \beta} \neq e^{\nu \alpha + \mu \beta}
\]  

[1.38]

One should note that in the special case where \( \mu = \nu \), the equality is fulfilled, meaning that for \( \alpha, \beta \in \mathbb{R}^+ \) and \( \mu \) a pure unit quaternion, one has \( \exp (\mu \alpha) \exp (\mu \beta) = \exp (\mu (\alpha + \beta)) \). In fact, this is a well-known property of exponential functions over non-commutative algebras (for example, the exponentials of matrices) and a consequence of the Baker–Campbell–Hausdorff formula (see, for example, [GIL 08] for illustrations of this formula).

The exponential function of a pure quaternion will be of use when considering polar representations of quaternions as well as when defining QFTs.

### 1.3.2. Exponential of a full quaternion

We have already introduced the exponential of a pure quaternion in section 1.3.1 for use later in the definition of Fourier transforms. Here, we consider the exponential of full quaternions, i.e. the function \( \exp : \mathbb{H} \to \mathbb{H} \). Just as in section 1.3.1, the exponential function is directly defined through its series expansion that is absolutely convergent. Thus, for a quaternion \( q \in \mathbb{H} \), its exponential is given by:

\[
e^q = \sum_{n=0}^{+\infty} \frac{q^n}{n!}
\]  

[1.39]

Now, recalling that any quaternion can be written as \( q = S(q) + V(q) \), it follows directly that:

\[
e^q = e^{S(q)} e^{V(q)}
\]  

[1.40]

leading to the special case of the exponential of a pure quaternion if \( S(q) = 0 \). Now, it is also possible to expand the \( e^{V(q)} \) part into \( \cos \) and \( \sin \), as \( V(q) \in V(\mathbb{H}) \), leading to the following expression for the exponential of \( q \):

\[
e^q = e^{S(q)} \left( \cos |V(q)| + \overline{V(q)} \sin |V(q)| \right)
\]  

[1.41]

---

5 This generalizes to the exponential of full quaternions.
with $V(q) = |V(q)|\tilde{V}(q)$, where $\tilde{V}(q)$ is the normalized vector part of $q$, i.e. the versor associated with $V(q)$. Note that due to the fact that $e^{S(q)}$ is real-valued, it commutes with the exponential of the vector part $V(q)$. Finally, note that given a quaternion $q \in \mathbb{H}$ and a complex number $z \in \mathbb{C}$, their exponentials $e^{q}$ and $e^{z}$ are isomorphic, provided that $|q| = |z|$ and $S(q) / |V(q)| = \mathbb{R}(z) / \mathbb{I}(z)$.

1.3.3. Logarithm of a quaternion

The logarithm of a quaternion $q$ is simply defined as the inverse of the exponential function, so that for $q, p \in \mathbb{H}$, we have:

$$p = \ln q \quad [1.42]$$

which means that $e^{p} = q$. However, it is possible to obtain an expression for the logarithm of $q$ in terms of its elements. First, we recall that $q \in \mathbb{H}$ can be expressed as $q = |q|\tilde{q}$, where $\tilde{q}$ is the normalized version of $q$. Now, it follows directly that the logarithm of $q$ is:

$$\ln q = \ln |q| + \ln \tilde{q} \quad [1.43]$$

Now, the second term on the right-hand side of the equation can be found to have an interesting literal expression. There is no doubt that $\ln |q| \in \mathbb{R}$; now, we are going to show that $\ln \tilde{q}$ is a pure quaternion. First, remember from section 1.3.1 that, as $\tilde{q}$ is a full unit quaternion, it can be expressed as the exponential of a pure (non-unit) quaternion $\xi$:

$$\tilde{q} = e^{\xi} = e^{|\xi|\xi} = e^{\phi_{q}\mu_{q}} \quad [1.44]$$

where we used the following notation:

$$\begin{cases} \phi_{q} = |\xi| & \in \mathbb{R}^{+} \\ \mu_{q} = \xi & \in V(\mathbb{H}) \end{cases} \quad [1.45]$$

This notation was chosen as $\phi_{q}$ and $\mu_{q}$ will be identified later in section 1.4.

Now, with this notation, and by direct substitution of [1.44] into [1.43], we have:

$$\ln q = \ln |q| + \mu_{q}\phi_{q} \quad [1.46]$$

where we use the fact that $\mu_{q}\phi_{q} = \phi_{q}\mu_{q}$ as $\phi_{q}$ is real-valued. This expression of the logarithm of a quaternion can be seen as a generalization of the well-known logarithm
of a complex number. In particular, the fact that we fixed the range of values to be taken by the argument $\phi_q$ between 0 and $2\pi$ ensures that the logarithm function with a quaternion argument is not multivalued.

1.4. Representations

Here we describe several existing representations of quaternions that will be used throughout the book.

1.4.1. Polar forms

1.4.1.1. Euler formula

In addition to the Cartesian form of $q$ given in [1.1], there are several other representations for quaternions that have been introduced since their discovery. One of the most important notations, which was introduced by Hamilton himself and is called the polar form of a quaternion $q$, is the quaternion equivalent of what Richard Feynman called the “jewel” of complex numbers, namely the Euler formula. It encapsulates the link between geometry and algebra. For a quaternion $q \in \mathbb{H}$, it reads as:

$$q = |q| e^{\mu_q \phi_q} = |q| (\cos \phi_q + \mu_q \sin \phi_q) \quad [1.47]$$

where $|q|$ is the modulus, $\mu_q$ is called the axis and $\phi_q$ is the phase/argument of $q$. The elements of the polar form are given in terms of the Cartesian components as:

$$\begin{align*}
|q| &= \sqrt{a^2 + b^2 + c^2 + d^2} \\
\mu_q &= \frac{bi + cj + dk}{\sqrt{b^2 + c^2 + d^2}} \\
\phi_q &= \arctan \left( \frac{\sqrt{b^2 + c^2 + d^2}}{a} \right) \quad [1.48]
\end{align*}$$

where $\mu_q \in \mathbb{V}(\mathbb{H})$ and $|\mu_q| = 1$. From this notation, it follows that unit quaternions have a modulus equal to 1, and pure quaternions (i.e. with $a = 0$) have a phase equal to $\pi/2$. The polar form allows us to link geometrical concepts such as rotations in 3D and 4D to the algebra of quaternions. This will be elaborated upon in Chapter 2. It is interesting to look at the special case of quaternions with unit modulus. In such a case, the Euler form simply reads as:

$$e^{\mu \phi} = \cos \phi + \mu \sin \phi \quad [1.49]$$
still with \( \mu \in V(H) \) and \( |\mu| = 1 \). Note that \( \cos \phi \in \mathbb{R} \) and \( \mu \sin \phi \) is the vector part of this quaternion. As a result, taking its conjugate consists of simply negating the vector part, giving the following identity:

\[
\cos \phi - \mu \sin \phi = e^{-\mu \phi} \quad [1.50]
\]

where one can see that conjugating a unit quaternion consists of either reversing its axis or negating its phase. Note that this could have been guessed from the fact that for unit quaternions \( q \in \mathbb{H} \), i.e. \( |q| = 1 \), then \( \overline{q} = q^{-1} \). Now, due to expressions [1.49] and [1.50], we can express the \( \sin \) and \( \cos \) functions in the following way:

\[
\begin{aligned}
\cos \phi &= \frac{1}{2} (e^{\mu \phi} + e^{-\mu \phi}) \\
\sin \phi &= -\frac{1}{2} \mu (e^{\mu \phi} - e^{-\mu \phi})
\end{aligned}
\quad [1.51]
\]

which will prove to be useful later in the study of QFTs in Chapter 3.

1.4.1.2. The Euler angle parameterization polar form

Another polar form that was used in [BÜL 01] has a direct connection to Euler angles [ALT 86]. As there are 12 possible conventions when using Euler angles, there are equivalently 12 possible polar forms. Here, we illustrate the polar form with the \( XYZ \) convention as used in [BÜL 01]. With this convention, a quaternion \( q \in \mathbb{H} \) can be expressed as:

\[
q = |q| e^{\eta i} e^{\kappa j} e^{\phi k} \quad [1.52]
\]

The three angles of \( q \), i.e. \( \eta \in [0, 2\pi) \), \( \kappa \in [0, \pi) \), \( \phi \in [0, 2\pi) \), can be identified as three phases, which are related to Euler angles when \( |q| = 1 \).

1.4.1.3. The Cayley–Dickson form

It can also be useful to consider a quaternion as a pair of complex numbers with specific multiplication rules. This is the idea behind the Cayley–Dickson (CD) form of a quaternion \( q \), which can be obtained using the doubling procedure [KAN 89]. Thus, a quaternion \( q \in \mathbb{H} \) has a CD form that reads:

\[
q = z_1 + z_2 j \quad [1.53]
\]

where \( z_1 = a + ib \in \mathbb{C}_i \) and \( z_2 = c + id \in \mathbb{C}_i \). Obviously, there is an infinite number of CD forms, depending on the unit pure quaternion that is used to split the quaternion into two different planes. This will be detailed in section 1.4.1.4. All the quaternion properties could be rephrased by this notation. However, we will not do so as it is not of use in the following. As an example, we give the expression of the quaternion conjugation in the CD notation, which is:

\[
\overline{q} = z_1^* - z_2 j \quad [1.54]
\]

where * denotes the classical complex conjugation.
1.4.1.4. The ortho-split or symplectic form

A more general representation exists, in the spirit of the CD form, that allows the interpretation of a quaternion in terms of two complex numbers found in two non-intersecting two-dimensional (2D) planes in 4D space\(^6\). Consider a basis in \( \mathbb{H} \):
\[
\{1, \mu, \nu, \mu \nu\},
\]
where \( \mu^2 = \nu^2 = -1 \) and \( \text{S}(\mu \nu) = 0 \), i.e. \( \mu \perp \nu \). \( \mu \) and \( \nu \) are pure unit quaternions (square roots of \(-1\)). As a result, \( \mu \nu \perp \mu \) and \( \mu \nu \perp \nu \). Then, any quaternion \( q \in \mathbb{H} \) can be written as:
\[
q = (a' + b' \mu) + (c' + d' \mu) \nu
\]  
[1.55]

where the two above-mentioned planes are spanned by \( \{1, \mu\} \) and \( \{\nu, \mu \nu\} \). This representation of a quaternion is sometimes referred to as its ortho-split representation or symplectic form\(^7\) as in [ELL 07c]. It is also known as the decomposition of \( q \) into its simplex part and perplex part. Using the quantities defined in [1.55] the simplex part of \( q \) is \( (a' + b' \mu) \) and the perplex part of \( q \) is \( (c' + d' \mu) \).

Later in this book, we will sometimes make use of the notation:
\[
q = q_s + q_p \nu
\]  
[1.56]

with notations \( q_s \in \mathbb{C}_{\mu} \) for the simplex part of \( q \in \mathbb{H} \) and \( q_p \in \mathbb{C}_{\mu} \) for its perplex part, both understood with respect to the axis \( \nu \) satisfying \( \mu \perp \nu \). Using this symplectic notation, a quaternion \( q \) can be written as:
\[
q = q_+ + q_-
\]  
[1.57]

where:
\[
\begin{align*}
q_+ &= \frac{1}{2} (q + \mu q \nu) \\
q_- &= \frac{1}{2} (q - \mu q \nu)
\end{align*}
\]  
[1.58]

which can be seen as a generalization of the expressions in [1.19] using a quaternion \( q \) and its conjugate.

We also mention here the swap-rule for the symplectic notation, which will be of use in the even-odd split study of the QFT in section 3.1.3. If, in a symplectic

---

\(^6\) The two planes are called non-intersecting even though it is not strictly true as they intersect at the origin. However, as this is the only point in 4D space where the planes intersect, we will keep the terminology “non-intersecting”.

\(^7\) Note that the terminology symplectic used here is different from the classically used term “symplectic” used in differential geometry, topology or group theory where it designates a special non-degenerate 2-form.
decomposition, one needs to have the imaginary axis $\nu$ to be placed on the left, then one can simply write $q$ as:

$$q = (a' + b' \mu) + (c' - d' \mu) \nu = q_s + \nu q_p^*$$  \[1.59\]

where the complex conjugation $^*$ is used on $q_p$ despite its quaternionic nature as it is isomorphic to a complex number, i.e. $q_p \in \mathbb{C}_\mu$.

Note finally that choosing a basis in $\mathbb{H}$ to express a quaternion $q$ induces a choice of symplectic representation. The symplectic form is thus a generalization of the CD form presented in section 1.4.1.3.

1.4.1.5. The polar Cayley–Dickson form

Recently, a new representation was introduced for elements of $\mathbb{H}$, called the **polar Cayley–Dickson** representation [SAN 10]. Given a quaternion $q \in \mathbb{H}$, its polar CD form is:

$$q = \rho_q e^{\Phi_q \hat{j}}$$  \[1.60\]

with $\rho_q \in \mathbb{C}$ the **complex modulus** of $q$ and $\Phi_q \in \mathbb{C}$ its **complex phase**. This representation of a quaternion is unique. Its construction is given in detail in [SAN 10]. Here, we present the main expressions of the polar CD form, without details of the sign ambiguity that is known to exist during the construction of the polar CD form of a quaternion from its Cartesian form. For a complete discussion of this sign issue, one should refer to [SAN 10]. Now, given a quaternion $q \in \mathbb{H}$, with CD form $q = z_1 + z_2 \hat{j}$, its complex modulus and phase $\rho_q$ and $\Phi_q$ are obtained by:

$$\begin{cases} 
\rho_q & = z_1 \frac{|q|}{|z_1|} \\
\Phi_q & = -\ln \left( \frac{z_1^* q}{\|z_1\| \hat{j}} \right) 
\end{cases}$$  \[1.61\]

Note that in the expression for $\Phi_q$, the argument of the logarithm is a product between a complex number and a quaternion, meaning that the order is important and that $q$ should be multiplied from the left by $z_1^*$. Note also that the product $z_1^* q$ is rather special in that if we replace $q$ by its CD form, we get $z_1^* q = |z_1|^2 + z_1^* z_2 \hat{j}$, which is a **degenerate** quaternion having a real part, an $\Im_j$ part and an $\Im_k$ part, but its $\Im_i$ part is null. As a result, taking the logarithm of such a degenerate quaternion leads to a quaternion with only a $\Im_j$ and a $\Im_k$ part. The negation and right multiplication by $\hat{j}$ finally lead to the fact that $\Phi_q$ is a complex number from $\mathbb{C}_\hat{i}$. It is also interesting to note that any ortho-split form as introduced in section 1.4.1.3 will lead to another possible polar CD form based on the axis used for the split.

An illustration of the usefulness of the polar CD form is made in section 4.3.4.
1.4.2. The $\mathbb{C}_j$-pair notation

In the style of the *ortho-split* notation or CD form, it is possible to think of a quaternion as an *ordered pair of complex numbers* and to manipulate them as such. Here, we mention this notation, called the $\mathbb{C}_j$-pair associated with a quaternion, as it highlights a very special product, namely the *bicomplex product* of two quaternions. This product will be of use when studying convolution for the one-dimensional (1D) QFT in Chapter 4.

Consider the quaternion $q = a + ib + jc + kd \in \mathbb{H}$ and let us write it as $q = q_1 + iq_2$. In this way, $q$ can be considered as the pair $q = (q_1, q_2)$ with $q_1 = a + jc$ and $q_2 = b + jd$ being elements of $\mathbb{C}_j$. Obviously, it is a special case of an *ortho-split* decomposition, with axis $i$, where the quaternion $q$ is simply understood as a pair of complex numbers.

Now, with this notation, we could rewrite anything related to quaternion algebra. For example, the conjugate of $q$ is $\overline{q} = (q_1^*, -q_2)$. More interestingly, if we consider the two quaternions $q = (q_1, q_2)$ and $p = (p_1, p_2)$, then their (quaternion) product is:

$$qp = (q_1p_1 - q_2^*p_2, q_2p_1 + q_1^*p_2)$$  \[1.62\]

and the inverse of $q$ is simply:

$$q^{-1} = \left( \frac{q_1^*}{|q|^2}, \frac{-q_2}{|q|^2} \right)$$  \[1.63\]

Now, the involutions of $q$ with respect to the canonical axis of $\mathbb{H}$ can be expressed as:

$$
\begin{align*}
\overline{q}^i &= (q_1, -q_2) \\
\overline{q}^j &= (q_1^*, q_2^*) \\
\overline{q}^k &= (q_1^*, -q_2^*)
\end{align*}
$$  \[1.64\]

Indeed, all the quaternion operations can be handled with this notation, but we will make special use in the following of one particular operation that comes in a very useful form when using the complex pair notation: the *bicomplex product* denoted by $\odot$. Given two quaternions in their complex pair forms $q = (q_1, q_2), p = (p_1, p_2) \in \mathbb{H}$, their *bicomplex product* is:

$$q \odot p = (q_1p_1 - q_2^*p_2, q_2p_1 + q_1^*p_2)$$  \[1.65\]
where we can see the difference with the standard quaternion product in [1.62]. A very remarkable property of the bicomplex product of two quaternions is that it is **commutative**. This can be easily seen from the expression of \( p \odot q \):

\[
p \odot q = (p_1 q_1 - p_2 q_2, p_2 q_1 + p_1 q_2)
\]

and by remembering that \( p_{1,2} \) and \( q_{1,2} \) are complex numbers and thus \( p_i q_j = q_j p_i \) for any pair \( i, j = 1, 2 \). This leads to the following interesting property:

\[
q \odot p = p \odot q \forall p, q \in \mathbb{H} \tag{1.67}
\]

Note that the commutativity of this product is associated with the \( \mathbb{C}_j \)-pairing notation. If we use a different way to divide a quaternion into two complex numbers (a \( \mathbb{C}_\mu \) pairing with \( \mu \) a square root of \(-1\)), then we need to carefully look at how to define an associated commutative product.

Rather than providing a lengthy study of the commutative product, we give here some properties of this special product over \( \mathbb{H} \), for any quaternion \( q = (q_1, q_2) = (a + j c, b + j d) \), any \( \mathbb{C}_i \) number \( z = \Re(z) + j \Im_i(z) = (\Re(z), \Im_i(z)) \), any \( \mathbb{C}_j \) number \( w = \Re(w) + j \Im_j(w) = (w, 0) \) and \( \mathbb{C}_k \) number \( s = \Re(s) + k \Im_k(s) = (\Re(s), j \Im_k(s)) \).

Then the following properties hold:

\[
\begin{align*}
q \odot w &= (q_1, q_2) \odot (w, 0) = (q_1 w, q_2 w) \\
q \odot z &= (q_1, q_2) \odot (\Re(z), \Im_i(z)) \\
&= (q_1 \Re(z) - q_2 \Im_i(z), q_2 \Re(z) + q_1 \Im_i(z)) \\
q \odot s &= (q_1, q_2) \odot (\Re(s), j \Im_k(s)) \\
&= (q_1 \Re(s) - j q_2 \Im_k(s), q_2 \Re(s) + j q_1 \Im_k(s)) \\
q \odot i &= (q_1, q_2) \odot (0, 1) = (-q_2, q_1) \\
q \odot j &= (q_1, q_2) \odot (j, 0) = (q_1 j, q_2 j) \\
q \odot k &= (q_1, q_2) \odot (0, j) = (-q_2 j, q_1 j) \\
i \odot j &= j \odot i = k \\
k \odot j &= j \odot k = -i \\
i \odot k &= k \odot i = -j \\
i \odot i &= -1 \\
j \odot j &= -1 \\
k \odot k &= 1
\end{align*}
\]

where all the given results can be checked by direct calculation using the bicomplex product expression given in [1.65].
As previously mentioned, the *bicomplex product* will not be encountered many times in this book, but it will be of use in section 4.3 when studying the convolution theorem for the QFT of complex-valued signals. The interested reader is referred to [PRI 91, ROC 04] for more materials on bicomplex numbers.

### 1.4.3. $\mathbb{R}$ and $\mathbb{C}$ matrix representations

In addition to scalar representations, there exist two matrix representations over $\mathbb{R}$ and $\mathbb{C}$ that are isomorphic to $\mathbb{H}$. Given a quaternion $q \in \mathbb{H}$, its real matrix representation, denoted as $\mathcal{M}_\mathbb{R}(q) \in \mathbb{R}^{4 \times 4}$, is given by:

$$
\mathcal{M}_\mathbb{R}(q) =
\begin{bmatrix}
  a & b & c & d \\
  -b & a & -d & c \\
  -c & d & a & -b \\
  -d & -c & b & a \\
\end{bmatrix}
$$

\[ \text{[1.69]} \]

and for any quaternion $q \in \mathbb{H}$, its complex matrix representation, denoted as $\mathcal{M}_\mathbb{C}(q) \in \mathbb{C}^{2 \times 2}$, is of the form:

$$
\mathcal{M}_\mathbb{C}(q) =
\begin{bmatrix}
  z_1 & -z_2^* \\
  z_2 & z_1^* \\
\end{bmatrix}
$$

\[ \text{[1.70]} \]

Comparing the real and complex matrix expressions for $q$ with the Cartesian form, one gets a direct identification of the imaginary units over $\mathbb{H}$, i.e. $i, j$ and $k$ with their real $4 \times 4$ and complex $2 \times 2$ matrix forms. As a real vector space, quaternions are spanned by these four real or complex matrices. For both of these matrix representations, the quaternion product is identified with the matrix product over $\mathbb{R}^{4 \times 4}$ and $\mathbb{C}^{2 \times 2}$. From a practical point of view, it is thus possible to manipulate

---

8 The transpose of the matrix representation given here is also valid.
quaternions through their matrix representations. However, as noted in section I.6.2, this is not optimal from a computational point of view as it requires much more computation and storage than necessary. Matrix representations of quaternions have been found to be of interest in the study of matrices with quaternion entries (for example [ZHA 97]), and in many more application fields.

1.5. Powers of a quaternion

In many situations, we need to multiply quaternions together. Due to the fact that \( \mathbb{H} \) is an algebra, the product of two quaternions is still a quaternion. Now, if we multiply a quaternion \( q \in \mathbb{H} \), say \( n \) times, by itself, it is interesting to look at how the components of the \( n \)th power of \( q \) can be expressed using the components of \( q \). First, note that a quaternion commutes with itself. This may seem trivial, but it is interesting to note that the right and left multiplications are equivalent in this special case. Now, assuming that \( q \) has the polar form given in [1.47], it follows directly that:

\[
q^n = |q|^n e^{n\mu_q \phi_q} = |q|^n \left( \cos(n\phi_q) + \mu_q \sin(n\phi_q) \right)
\]  

[1.71]

This is simply a generalization of the de Moivre formula for quaternions [KAN 89]. The demonstration can be carried out easily by using the series expansion of the exponential and the property of the \( n \)th power of the pure unit imaginary \( \mu_q \) (see [1.34]). Apart from its simplicity, this formula has useful consequences in the use of quaternions to represent rotations. Anticipating results from Chapter 2, multiplying\(^9\) \( n \) times by the same quaternion \( q \) will be shown to be equivalent to performing a rotation around the axis \( \mu_q \) by an angle \( n\phi_q/2 \).

1.6. Subfields

As previously mentioned, quaternions form a 4D algebra over \( \mathbb{R} \). A quaternion \( q \) with no vector part, \( V(q) = 0 \), is thus a real number, i.e. \( q \in \mathbb{R} \) if \( V(q) = 0 \). Similarly, any quaternion with two of its three imaginary components equal to zero is homomorphic to a complex number. This means that for \( q \in \mathbb{H} \) of the form:

\[
q = a + w\mu
\]

[1.72]

with \( \mu = i, j, k \), \( q \) is equivalent to a complex number. This is to say that numbers taking values in \( \mathbb{R} \oplus \mu \mathbb{R} \), with \( \mu^2 = -1 \), form an algebra homomorphic to complex numbers. \( q \) is said to be an element of \( \mathbb{C}_\mu \). Obviously, we can see that such a construction is the restriction of \( \mathbb{H} \) to a 2D plane spanned by \( \{1, \mu\} \), just as the

\(^9\) The exact way to perform rotations using quaternion multiplication is presented in detail in sections 2.1.2 and 2.1.6.
complex plane is classically known to be spanned by \( \{1, I\} \). Now, the above-mentioned \( \mu \) was required to be one of the three “classical” imaginary units from \( \mathbb{H} \). In fact, it may be any root of \(-1\) in \( \mathbb{H} \) so that the subfield (equivalent to the complex field) is now a different 2D plane inside the 4D space of \( \mathbb{H} \). Obviously, there are infinitely numerous such subspaces that can be defined within \( \mathbb{H} \). Arbitrarily choosing one such subfield yields a new basis for the expression of any quaternion \( q \in \mathbb{H} \) through the CD form given in section 1.4.1.4. This shows also that \( \mathbb{H} \) contains two copies of \( \mathbb{C} \) as subfields.