So I pass from a task, which has filled the greater part of many years of my life, which has broadened in my view as they passed, and which has suffered interruptions that threatened to end it before its completion. Many of its defects are known to me; after it has gone from me, others will become apparent. Nevertheless, my hope is that my work will ease the labour of those who, coming after me, may desire to possess a systematic account of this branch of pure mathematics.

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October, 1906.

Introduction

This work is about geometrical optics though it shall extend into some fundamental areas of physical optics as well. It makes heavy use of several branches of mathematics which, perhaps, the reader will find disturbingly unfamiliar. These I will describe with some care but with only lip service to mathematical rigor and vigor.

Keep in mind that geometrical optics is a peculiar science. Its fundamental artifacts are rays, which do not exist, and wavefronts, which indeed do exist but are not directly observable. A third item is the caustic, a surface in image space which is certainly observable, defined variously as the envelope of an array of rays associated with some point object, the locus of the principal centers of wavefront curvatures, or as the locus of points where the differential element of area of a wavefront vanishes. Of course, these wavefronts must be in a wavefront train generated by a lens and associated with some fixed object point.

The peculiarities of geometrical optics go even further. Rays, which do not exist, are trajectories of corpuscles, which also do not exist. These trajectories, according to the principle of Fermat, are those paths over which the time of transit of a corpuscle, passing from one point to another, is either a maximum or a minimum.

Yet it works. Geometrical optics, anachronistic as it is, remains the basis for modern optical design, the highly successful engineering application built on the sandiest foundation imaginable. There is hardly one area of modern science in which instruments are used whose design depends ultimately on Fermat's postulate on the intrinsic laziness of mother nature.

In what follows I shall use a method best described as axiomatic, the axiom being Fermat's principle. This we must modify, however. Since point-to-point transit times can be maxima as well as minima we must use, in the language of the Calculus of Variations, *extrema* (singular: *extremum*) as our criterion in applying Fermat's principle.

Indeed the interpretation of the principle of Fermat in terms of the language of the variational calculus will lead us to ray paths in *inhomogeneous* media; media in which the refractive index

The Mathematics of Geometrical and Physical Optics: The k-function and its Ramifications. O.N. Stavroudis

ISBN: 3-527-40448-1

is a continuous function of position. These ray paths will be expressed in the form of a system of ordinary differential equations that can be applied to any specified media.

These ray paths are then subject to analysis using the techniques of the differential geometry of space curves. Using these differential equations for a ray path we can deduce its shape and its relationship to the refracting medium itself. From these results we can determine, quickly and easily, the nature of rays in, say, Maxwell's fish eye.

From here we pass on to the Hilbert integral, developed originally for dealing with the problem of the variable end point in the Calculus of Variations. This very rich theory leads us to a number of very important deductions in geometrical optics; conditions for the existence of wavefronts, Snell's law, The Hamilton-Jacobi equations (though both Hamilton and Jacobi preceded Hilbert by as much as a half century), the eikonal equation, among others. In this context the theorem of Malus becomes trivial. From this context Herzberger recognized the importance of the *normal congruence* or the *orthotomic system* of rays.

With the concept of the wavefront in hand we proceed to the differential geometry of surfaces and to partial differential equations of the first order. One such is the eikonal equation, mentioned above, obtained from the Hamilton-Jacobi equation, for which we find a general solution descriptive of any wavefront train in a *homogeneous optical medium*; one with a constant refractive index.

In terms of the differential geometry of surfaces we can find, for the general wavefront train, wavefront principal directions and curvatures. This leads to the important concept of the caustic, that surface that is the locus of the principal centers of wavefront curvature. In the caustic resides all of the monochromatic aberrations associated with a wavefront train and, ultimately, with the lens and object point that give rise to it. The structure of this caustic describes completely the *image errors*: *spherical aberration, coma* and *astigmatism*. Its location in space indicates the *field errors*; *distortion* and *field curvature*.

Along the way we look at *generalized ray tracing*, more properly, a generalization of the *Coddington equations*, that determines the principal directions and principal curvatures at any point on a wavefront through which a traced ray passes.

This we apply to the prolate spheroid, a rotationally symmetric ellipsoid generated by rotating an ellipse about its major axis. This leads to a reflecting optical system, consisting of two confocal spheroids, that I have called the *modern schiefspiegler*.

We also look at Herzberger's fundamental optical invariant and his diapoint theory and apply it to the representation of wavefronts obtained form the solution of the eikonal equation. This leads to a hierarchial system of aberrations.

The canon that I have described here, based on Fermat's principle, omits many important items. Outstanding among these is *paraxial theory* and *paraxial ray tracing*. Although is it is of tremendous practical importance, it is based on an approximation that, in my opinion, does not belong here.

A far more fundamental omission is Gaussian optics, in particularly, its model as developed by Maxwell. He began with certain assumptions about perfect lenses from which he represented perfect image formation by a fractional-linear transformation. Upon assuming that his perfect lens is rotationally symmetric, he was able to derive its cardinal points; the foci, the nodal points and the principal points.

Other omissions are the *Seidel aberrations* and their higher order extensions. These are from a solution of the eikonal equation in the form of a power series that has never been shown to converge.

Huygens' principle is omitted. It is clearly independent of any corpuscular concepts and is based on wavefront propagation as the envelope of spherical wavelets, which also do not exist, centered on a previous position of the wavefront. It also leads to Snell's law. It was for many centuries the main competitor to Fermat's corpuscules.

But nowadays the photon incorporates the best of both the corpuscule and the wavelet, a compromise that has resulted in a far more useful theory with applications far beyond the dreams of Fermat and Huygens.