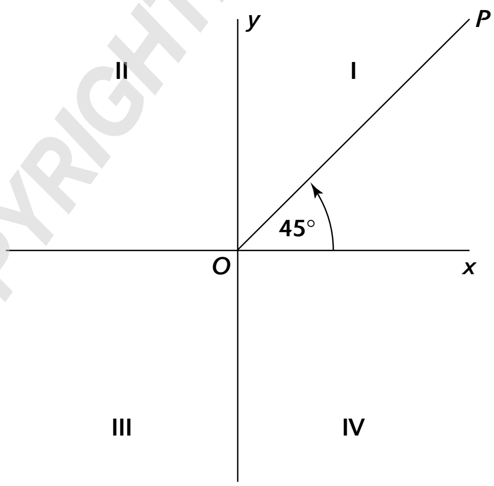
Chapter 1

Trigonometric Ideas

The word *trigonometry* comes from two Greek words, *trigonon*, meaning triangle, and *metria*, meaning measurement. This is the branch of mathematics that deals with the ratios between the sides of right triangles with reference to either of its acute angles and enables you to use this information to find unknown sides or angles of any triangle. Trigonometry is not just an intellectual exercise, but has uses in the fields of engineering, surveying, navigation, architecture, and yes, even rocket science.

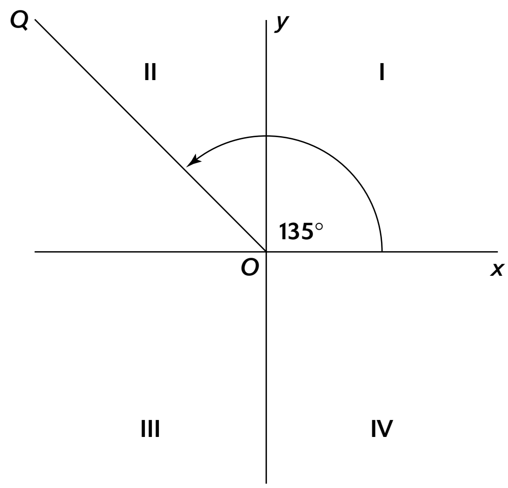
Angles and Quadrants

An angle is a measure of rotation and is expressed in degrees or radians. For now, we'll stick with degrees, and we'll examine working with radians in the next chapter. Consider any angle in standard position to have its vertex at the origin (the place where the *x*- and *y*-axes cross), labeled *O* in the diagrams. Angle measure is the amount of rotation between the two rays forming the angle.



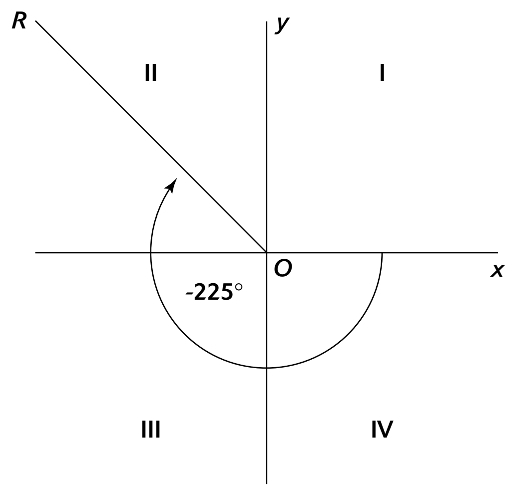
A first quadrant angle in standard position.

Figure the **initial side** of the angle above beginning on the *x*-axis to the right of the origin. Consider the **terminal side** of the angle to be hinged at *O*. The terminal side of the angle, *OP,* was rotated counterclockwise from the *x*-axis through an angle of less than 90° to form the first quadrant angle shown above. Notice the Roman numerals. They mark the quadrants I, or first; II, or second; III, or third; and IV, or fourth. Notice that the quadrant numbers rotate *counterclockwise* around the origin. Because the angle in the above figure has its initial side on the *x*-axis, it is said to be in **standard position.** Had the terminal side made a full turn and come back to the *x*-axis, it would have rotated 360°.



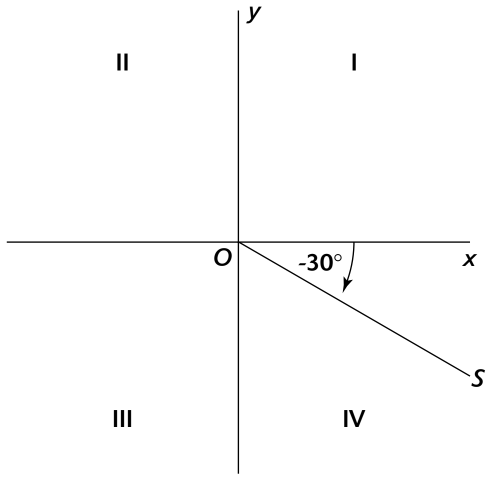
A second quadrant angle in standard position.

The above figure is called a second quadrant angle because its terminal side is in the second quadrant. When the magnitude of an angle is measured in a counterclockwise direction, the angle's measure is positive. The above figure shows an angle of 135° measure.



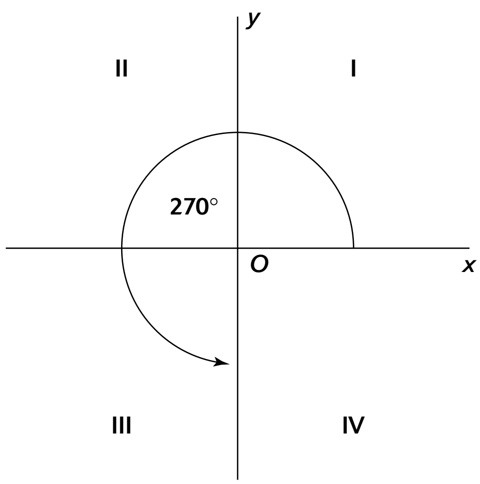
A second quadrant angle measured clockwise.

The angle in the above figure is identical to the figure that precedes it in every way except how the angle was measured. Since it was measured clockwise rather than counterclockwise, it has a measure of −225°. Notice that the absolute value of that angle is obtained by subtracting 135° from 360°. The negative sign marks the direction in which it was measured.



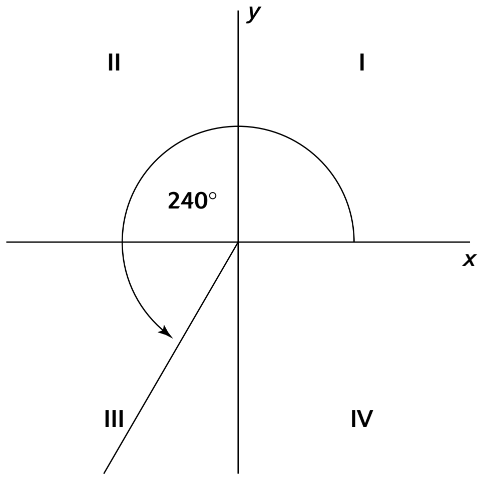
A fourth quadrant negative angle.

Notice that the fourth quadrant angle in the above figure, if measured counterclockwise, would have measured 330°. Can you see why? Moving counterclockwise, it would have been 30° shy of a full 360° rotation.



A quadrantal angle.

When an angle is in standard position, and its terminal side coincides with one of the axes, it is referred to as a **quadrantal angle.** Angles of 90°, 180°, and 270° are three examples of quadrantal angles. They are by no means all the quadrantal angles that are possible, but we'll get to that in the next lesson.



A third quadrant angle.

The one angle remaining to be shown is a Q-III angle (see above). A third quadrant (or Q-III) angle is any angle with its terminal side being in the third quadrant. Because this angle was formed by a counterclockwise rotation, it is positive.

Example Problems

These problems show the answers and solutions.

1. In which quadrant is the terminal side of a 95° angle in standard position?

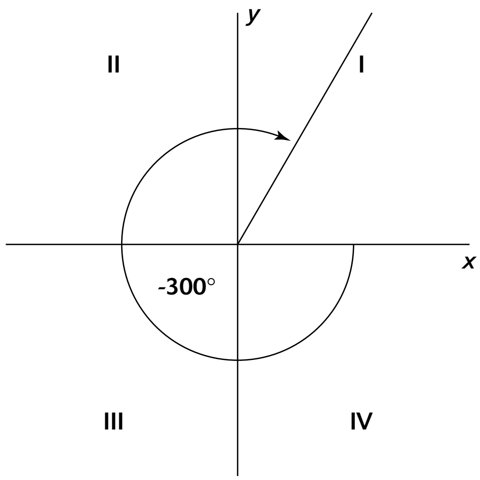
   ***answer:* II** This also breaks with the style from *CliffsStudySolver Geometry* and *CliffsStudySolver Algebra.*

   Since the angle is in standard position, its initial side is on the x-axis to the right of the origin. The y-axis forms a right (90°) angle with the initial side, so a 95° angle's terminal side must sweep past the vertical y-axis and into quadrant II.

2. In which quadrant is the terminal side of a −320° angle?

   ***answer:* I**

   Since the angle is in standard position, its initial side is on the x-axis to the right of the origin. Since its sign is negative, its terminal side rotates clockwise past the y-axis at −90°, on past the x-axis at −180°, past the y-axis again at −270°, and continues on another 50° to terminate in the first quadrant. See the figure that follows.



A −320° angle.

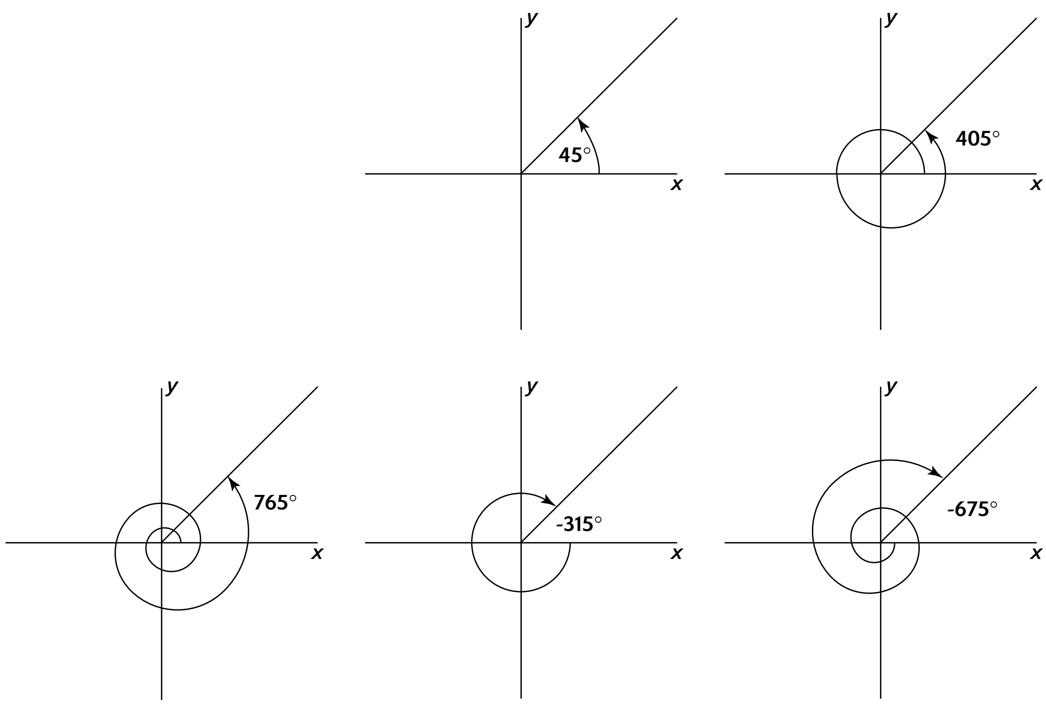
3. What is the special name given to a right angle in standard position?

answer: **quadrantal angle**

A right angle in standard position will have its terminal side on the *y*-axis. That makes it a quadrantal angle.

Coterminal Angles

Two angles that are in standard position and share a common terminal side are said to be coterminal angles. All of the angles in the following figure are coterminal with an angle of degree measure of 45°. The arrow shows the direction and the number of rotations through which the terminal side goes.



Angles coterminal with 45°.

All angles that are coterminal with an angle measuring $d°$ may be represented by the following equation:

$$d° + n \cdot 360°$$

Example Problems

These problems show the answers and solutions.

1. Name four angles that are coterminal with 80°.

   ***answer:* –640°, –280°, 440°, 800°, . . .**

   Any angle coterminal with 80° must have a multiple of 360° added to it; according to the formula $d° + n \cdot 360°$, the following are some angles coterminal with 80°:

   $d° + n \cdot 360° = 80° + (1)(360°) = 80° + 360° = 440°$

   $d° + n \cdot 360° = 80° + (2)(360°) = 80° + 720° = 800°$

   $d° + n \cdot 360° = 80° + (3)(360°) = 80° + 1080° = 1160°$

   $d° + n \cdot 360° = 80° + (4)(360°) = 80° + 1440° = 1520°$

   Of course, any number could have been substituted for n, and that means negative as well as positive values, for example:

   $d° + n \cdot 360° = 80° + (-1)(360°) = 80° - 360° = -280°$

   $d° + n \cdot 360° = 80° + (-2)(360°) = 80° - 720° = -640°$

   $d° + n \cdot 360° = 80° + (-3)(360°) = 80° - 1080° = -1000°$

   $d° + n \cdot 360° = 80° + (-4)(360°) = 80° - 1440° = -1360°$

   So, the answer is really the following:

   Four angles coterminal with 80° are . . ., −640°, −280°, 80°, 440°, 800°, . . .

   The angle 80° itself was included in the series so as not to break the pattern. Notice that as you move from left to right, each angle measure is 360° greater than the one to its left.

2. Is an angle measuring 220° coterminal with an angle measuring 960°?

   ***answer:* No**

   If angles measuring 220° and 960° were coterminal, then

$$960° = 220° + n \cdot 360°$$
$$740° = n \cdot 360°$$

But 740° is *not* a multiple of 360°, so the angles *cannot* be coterminal.

Work Problems

Use these problems to give yourself additional practice.

1. In which quadrant does the terminal side of an angle of degree measure $1200°$ fall?
2. In order for an angle to be a quadrantal angle, by what number must it be capable of being divided?
3. What is the lowest possible positive degree measure for an angle that is coterminal with an angle of $-1770°$?
4. In which quadrant does the terminal side of an angle of $990°$ degree measure fall?
5. Name two positive and two negative angles that are coterminal with an angle of degree measure $135°$.

Worked Solutions

1. **II** First see how many times $360°$ can be subtracted or divided out of the total.

   $3 \cdot 360° = 1080°$

   $1200° - 1080° = 120°$

   A $120°$ angle is larger than $90°$ and less than $180°$, and so its terminal side falls in the second quadrant.

2. **90** Quadrantal angles' terminal sides fall on the axes. Therefore, no matter what the size of the coterminal angle, it must be capable of being in the positions of $90°$, $180°$, $270°$, or $0°$. All are divisible by 90.

3. **30°** Just keep adding $360°$ until the sum goes from a negative to a positive value:

   $5 \cdot 360° = 1800°$

   $-1770° + 1800° = 30°$

4. **None** It is quadrantal. $990°$ is two full revolutions from the starting position, and then an additional $270°$:

   $2 \cdot 360° = 720°$

   $990° - 720° = 270°$

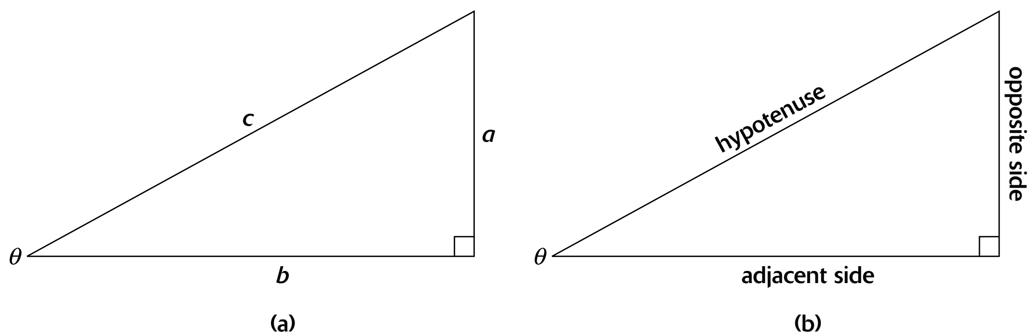
   That places the terminal side on the y-axis, south of the origin (negative).

5. **–585°, –225°, 495°, 855°** Needless to say, there are an infinite number of other solutions, each of which is determined by substituting into the expression: $d° + n \cdot 360°$.

Trigonometric Functions of Acute Angles

The building blocks of trigonometry are based on the characteristics of similar triangles that were first formulated by Euclid. He discovered that if two triangles have two angles of equal measure, then the triangles are similar. In similar triangles, the ratios of the corresponding sides of one to the other are all equal. Since all right triangles contain a 90° angle, proving two of them similar only requires having one acute angle of one triangle equal in measure to one acute angle of the second. Having established that, we easily find that in two similar right triangles, the ratio of each side to another in one triangle is equal to the ratio between the two corresponding sides of the other triangle. It is no long stretch from there to realize that this must be true of all similar triangles. Those relationships led to the **trigonometric ratios.** It is customary to use lowercase Greek letters to designate the angle measure of specific angles. It doesn't matter which Greek letter is used, but the most common are α (alpha), β (beta), ϕ (phi), and θ (theta).

The trigonometric ratios that follow are based upon the following reference triangle, which is drawn in two different ways.



Figures (a) and (b).

Both figures show the same triangle with sides a, b, and c, and with angle θ at the left end of the base. The difference is that in figure (b), the two legs are labeled with respect to $\angle\theta$. That is to say, side a is marked as opposite to $\angle\theta$, and side b is adjacent to $\angle\theta$. You might correctly argue that side c is also adjacent to $\angle\theta$, but that side already has a name (you learned about this in plane geometry). Being the side opposite the right angle, it's the hypotenuse, hence it is the nonhypotenuse adjacent side to $\angle\theta$ that is assigned the name "adjacent."

That leads us to the first three trigonometric functions:

The sine of θ is: $\sin\theta = \frac{a}{c} = \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}}$

The cosine of θ is: $\cos\theta = \frac{b}{c} = \frac{\text{length of side adjacent } \theta}{\text{length of hypotenuse}}$

The tangent of θ is: $\tan\theta = \frac{a}{b} = \frac{\text{length of side opposite } \theta}{\text{length of side adjacent } \theta}$

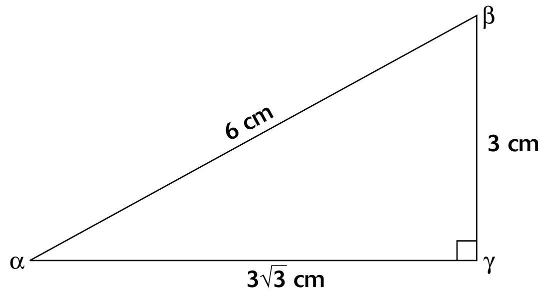
In the early days of American history, way before the days of political correctness, some ambitious trigonometry student in search of a mnemonic device by which to remember his or her trigonometric ratios dreamed up the SOHCAHTOA Indian tribe, which today would be the SOHCAHTOA tribe of Native Americans. SOHCAHTOA is an acronym for the basic trig ratios and their components; that is:

Sin-**O**pposite/**H**ypotenuse-**C**os-**A**djacent/**H**ypotenuse-**T**an-**O**pposite/**A**djacent

Keep in mind that as long as the angles remain the same, the ratios of their pairs of sides will remain the same, regardless of how big or small they are in length. The trigonometric ratios in right triangles depend exclusively on the angle measurements of the triangles and have no dependence on the lengths of their sides.

Example Problems

These problems show the answers and solutions. All refer to the following figure.



1. Find the sine of α.

   ***answer:* 0.5**

   The sin ratio is $\frac{\text{opposite}}{\text{hypotenuse}}$. 3 cm is the length of the side opposite $\angle\alpha$, and the hypotenuse is 6 cm.

   So, $\sin\alpha = \frac{3}{6} = \frac{1}{2} = 0.5$.

2. Find $\cos\beta$.

   ***answer:* 0.5**

   The cos ratio is $\frac{\text{adjacent}}{\text{hypotenuse}}$. 3 cm is the length of the side adjacent $\angle\beta$, and the hypotenuse is 6 cm.

   So, $\cos\beta = \frac{3}{6} = \frac{1}{2} = 0.5$.

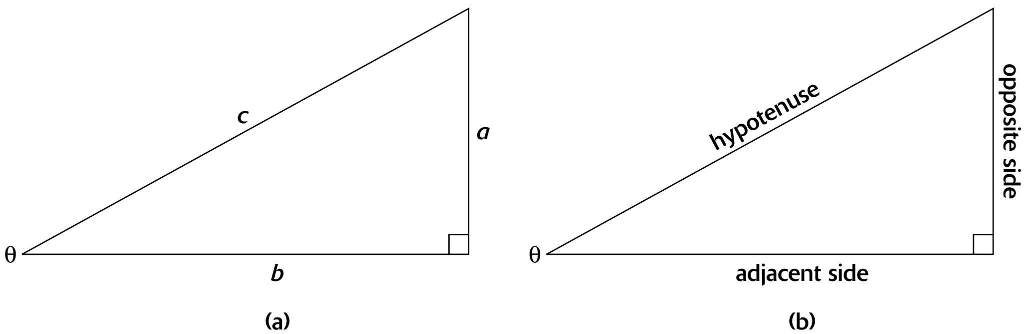
3. Find $\tan\beta$.

   ***answer:* $\sqrt{3}$**

   The tan ratio is $\frac{\text{opposite}}{\text{adjacent}}$. $3\sqrt{3}$ cm is the length of the side opposite $\angle\beta$, and its adjacent side is 3 cm. So, $\tan\beta = \frac{3\sqrt{3}}{3} = \sqrt{3}$.

Reciprocal Trigonometric Functions

The three remaining trigonometric ratios are the reciprocals of the first three. You may think of them as the first three turned upside down, or what you must multiply the first three by in order to get a product of 1. The reciprocal of the sine is cosecant, abbreviated csc. Secant is the reciprocal of cosine and is abbreviated sec. Finally, cotangent, abbreviated cot, is the reciprocal of tangent.



Models for reciprocal trigonometric ratios.

The cosecant of θ is: $\csc\theta = \frac{c}{a} = \frac{\text{length of hypotenuse}}{\text{length of side opposite }\theta}$

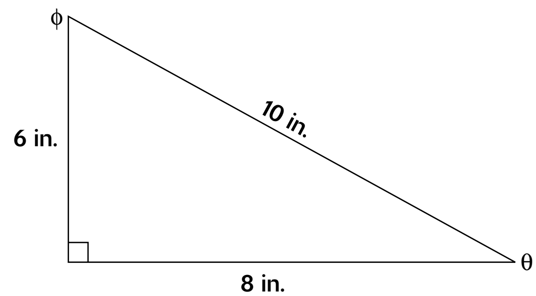
The secant of θ is: $\sec\theta = \frac{c}{b} = \frac{\text{length of hypotenuse}}{\text{length of side adjacent }\theta}$

The cotangent of θ is: $\cot\theta = \frac{b}{a} = \frac{\text{length of side adjacent }\theta}{\text{length of side opposite }\theta}$

There is no SOHCAHTOA tribe here to help out, but you shouldn't need one. Just remember the pairings, find the right combination for its reciprocal (that is for secant; remember it pairs with cosine), and flip it over.

Example Problems

These problems show the answers and solutions. All problems refer to the following figure.



Model for example problems.

1. Find $\csc\phi$.

***answer:* 1.25**

$$\csc\phi = \frac{\text{length of hypotenuse}}{\text{length of side opposite }\phi} = \frac{10}{8} = 1.25$$

2. Find $\sec\theta$.

***answer:* 1.25**

$$\sec\theta = \frac{\text{length of hypotenuse}}{\text{length of side adjacent }\theta} = \frac{10}{8} = 1.25$$

3. Find $\cot\theta$.

***answer:* 1.33**

$$\cot\theta = \frac{\text{length of side adjacent } \theta}{\text{length of side opposite } \theta} = \frac{8}{6} = 1.33$$

Work Problems

Use these problems to give yourself additional practice.

1. An angle has a sine of 0.3.
   a. What other function's value can you determine?
   b. What is that value?
2. An angle has a cosine of 0.6.
   a. What other function's value can you determine?
   b. What is that value?
3. An angle has a tangent of 2.5.
   a. What other function's value can you determine?
   b. What is that value?
4. An angle has a secant of 1.8.
   a. What other function's value can you determine?
   b. What is that value?
5. An angle has a cosecant of 5.3.
   a. What other function's value can you determine?
   b. What is that value?

Worked Solutions

1. **cosecant, 3.33** Sine's reciprocal function is cosecant. It is the reciprocal of sine; that is, $\csc\theta = \frac{1}{\sin\theta}$, but you know that the value of sin is 0.3, therefore $\csc\theta = \frac{1}{0.3}$.

   Divide 1 by 0.3, and you get 3.33 (rounding to the hundredths).

2. **secant, 1.67** Cosine's reciprocal function is secant. Since it's the reciprocal of cosine, $\sec\theta = \frac{1}{\cos\theta}$, but you were given the cos as 0.6, therefore $\sec\theta = \frac{1}{0.6}$.

   Divide 1 by 0.6 to find a rounded value of 1.67.

3. **cotangent, 0.4** Tangent's reciprocal function is cotangent. It is the reciprocal of tangent, so $\cot\theta = \frac{1}{\tan\theta}$, but you were given tan = 2.5, so $\cot\theta = \frac{1}{2.5}$.

   Divide 1 by 2.5 and get 0.4.

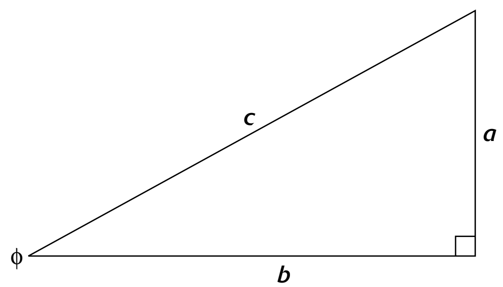
4. **cosine, 0.56** Since secant's reciprocal function is cosine, $\cos\theta = \frac{1}{\sec\theta}$, but you know that the value of sec is 1.8; therefore $\cos\theta = \frac{1}{1.8}$. Divide 1 by 1.8, and you get 0.56 (rounding to the hundredths).

5. **sine, 0.19** Since cosecant's reciprocal function is sine, $\sin\theta = \frac{1}{\csc\theta}$, but you know that the value of csc is 5.3; therefore $\sin\theta = \frac{1}{5.3}$.

   Divide 1 by 5.3, and you'll get 0.188, which you'll round to 0.19.

Introducing Trigonometric Identities

When trigonometric functions of an angle ϕ are related in an equation, and that equation is true for all values of ϕ, then the equation is known as a **trigonometric identity.** The following trigonometric identities can be constructed from the trigonometric ratios that you just reviewed.



Triangle referenced by identities below.

Referring to the above figure, since $\sin\phi = \frac{a}{c}$, $\cos\phi = \frac{b}{c}$, and $\tan\phi = \frac{a}{b}$, it follows that

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\frac{a}{c}}{\frac{b}{c}}.$$

The second part of which you can simplify like this: $\frac{a}{c} \cdot \frac{c}{b} = \frac{a}{\phi} \cdot \frac{\phi}{b} = \frac{a}{b}$

Which serves to prove the identity: $\tan\phi = \frac{\sin\phi}{\cos\phi}$

You could also prove the identity: $\cot\phi = \frac{\cos\phi}{\sin\phi}$

These two identities are extremely useful and should be memorized. There's a third very handy identity, but first, you must become familiar with some conventional notation. The symbol $(\sin\theta)^2$ and $\sin^2\theta$ mean the same thing and may be used interchangeably. With that in mind, the third identity referred to is $\sin^2\phi + \cos^2\phi = 1$. If you would like to see that proven, refer to the previous figure and the Pythagorean theorem, stated as $a^2 + b^2 = c^2$.

$$\sin^2\phi + \cos^2\phi = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

You can add the two fractions to get $\frac{a^2+b^2}{c^2}$.

But you already know that $a^2 + b^2 = c^2$, so you substitute in the numerator and simplify:

$$\frac{c^2}{c^2} = 1$$

Therefore, $\sin^2\phi + \cos^2\phi = 1$.

The importance of these three identities cannot be over-stressed. You will deal much more extensively with identities in the fourth chapter, but for now, try to learn these three.

Example Problems

These problems show the answers and solutions. Where applicable, round each answer to the nearest thousandth.

1. Find the sin and tangent of $\angle\lambda$ if λ is an acute angle ($0° < \lambda < 90°$) and $\cos\lambda = \cos\lambda = \frac{1}{5}$.

   ***answer:* 0.980, 4.9**

   Since $\sin^2\phi + \cos^2\phi = 1$, then $\sin^2\lambda + \cos^2\lambda = 1$.

   Substitute: $\sin^2\lambda + \left(\frac{1}{5}\right)^2 = 1$

   Square and subtract from both sides: $\sin^2\lambda = 1 - \frac{1}{25}$

   Subtract: $\sin^2\lambda = \frac{24}{25}$

   Take the square root of both sides: $\sin\lambda = \sqrt{\frac{24}{25}} = 0.980$

   Next, find the tangent using what you just found: $\cos\lambda = \tan\lambda = \frac{\sin\lambda}{\cos\lambda}$ and

   $\cos\lambda = \frac{1}{5} = 0.2$.

   Substituting, you find that $\tan\lambda = \frac{0.980}{0.2} = 4.9$.

2. Find the cos and tangent of $\angle\phi$ if ϕ is an acute angle ($0 < \phi < 90°$) and $\sin\phi = 0.867$.

   ***answer:* 0.498, 1.741**

   First, use the identity $\sin^2\phi + \cos^2\phi = 1$ to find $\cos\phi$.

   Substitute: $(0.867)^2 + \cos^2\phi = 1$

   Subtract $(0.867)^2$ from both sides: $\cos^2\phi = 1 - (0.867)^2$

   Square the quantity in parentheses: $\cos^2\phi = 1 - 0.752$

   Subtract 0.752 from 1: $\cos^2\phi = 0.248$

   Take the square root of both sides: $\cos\phi = 0.498$

   Now, solve for the tangent.

First write the relevant identity: $\tan\phi = \dfrac{\sin\phi}{\cos\phi}$

Substitute for sin and cos: $\tan\phi = \dfrac{0.867}{0.498}$

And divide: $\tan\phi = 0.741$

3. Find the sin of $\angle\theta$ if θ is an acute angle ($0 < \theta < 90°$), $\tan\theta = 1.192$, and $\cos\phi = 0.643$.

***answer:* 0.766**

This time, you only need: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Substitute what you know: $1.192 = \dfrac{\sin\theta}{0.643}$

Now multiply both sides by 0.643: $(0.643)(1.192) = \left(\dfrac{\sin\theta}{0.643}\right)(0.643)$

Reversing the results, you get: $\sin\theta = 0.766.$

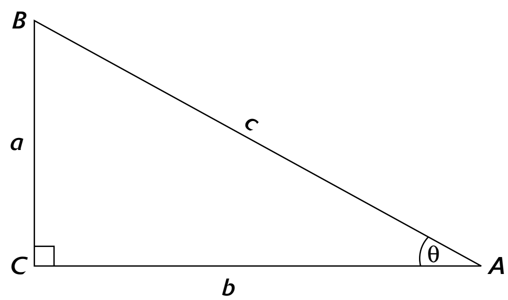
Trigonometric Cofunctions

Trigonometric functions are often considered in pairs, known as cofunctions. Sine and cosine are cofunctions. So are secant and cosecant. The final pair of cofunctions are tangent and cotangent. From the right triangle *ABC*, the following identities can be seen:

$\sin A = \frac{a}{c} = \cos B$ $\sin B = \frac{b}{c} = \cos A$

$\sec A = \frac{c}{b} = \csc B$ $\sec B = \frac{c}{a} = \csc A$

$\tan A = \frac{a}{b} = \cot B$ $\tan B = \frac{b}{a} = \cot A$



Reference triangle for cofunctions.

To refresh your memory, all three angles of a triangle are supplementary (sum to 180°), and angles that sum to 90° are known as complementary. Since one of the three angles in a right triangle measures 90°, the sum of the remaining acute angles must be complementary. Refer to the above reference triangle to confirm the following relationships:

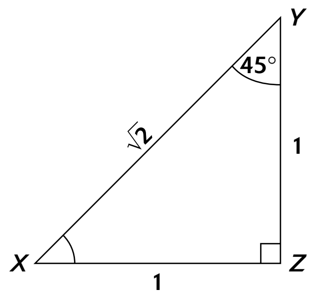
$\sin\theta = \cos(90° - \theta)$ $\cos\theta = \sin(90° - \theta)$

$\sec\theta = \csc(90° - \theta)$ $\csc\theta = \sec(90° - \theta)$

$\tan\theta = \cot(90° - \theta)$ $\cot\theta = \tan(90° - \theta)$

Two Special Triangles

The figure below shows an isosceles right triangle with each leg having a length of 1. Can you figure out the measure of angle *X*?



An isosceles right triangle.

That last question was intended as a joke, since in any isosceles triangle, the angles opposite the equal legs are always equal in measure. What is different, however, is that in an isosceles right triangle, the acute angles always measure 45°. That's because they must be both complementary and equal. What is also different in an isosceles right triangle is that the hypotenuse always has the same relationship to the legs. That relationship, of course, may be found using the Pythagorean theorem.

In this case:	$z^2 = x^2 + y^2$
Substituting:	$z^2 = 1^2 + 1^2$
Squaring and adding:	$z^2 = 2$
Therefore:	$z = \sqrt{2}$

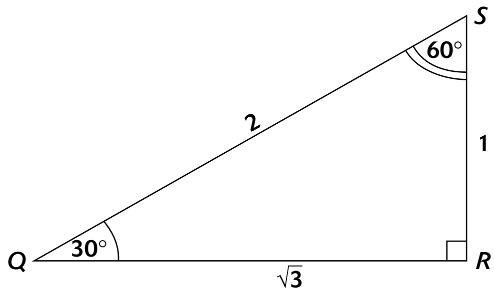
All right, you already knew that, because it shows that in the above figure, but what if instead of 1, *x* and *y* had been 2?

Substituting:	$z^2 = 2^2 + 2^2$
Squaring and adding:	$z^2 = 8$
Therefore:	$z = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$

Do you see the pattern yet? Try one more, just to make sure. This time, let *x* and *y* be 5.

Substituting:	$z^2 = 5^2 + 5^2$
Squaring and adding:	$z^2 = 50$
Therefore:	$z = \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$

To sum it all up, when dealing with an isosceles right triangle, the hypotenuse is always the length of the leg times the square root of 2.



The 30-60-90 right triangle.

Students and teachers of trigonometry are quite fond of a second special right triangle. That's the one with acute angles of 30° and 60°, or as it's often referred to, the 30-60-90 right triangle. The relationship among the sides are spelled out in the above figure. Notice that the side opposite the 30° angle is half the length of the hypotenuse, and the length of the side opposite the 60° angle is half the hypotenuse times the square root of three.

From the two special triangles, you can compile a table of frequently used trigonometric functions.

Table of Trigonometric Ratios for 30°, 45°, and 60° Angles

θ	$\sin\theta$	$\cos\theta$	$\sec\theta$	$\csc\theta$	$\tan\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\sqrt{2}$	1	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$

Example Problems

These problems show the answers and solutions.

1. One leg of an isosceles right triangle is 4 cm long.
   a. How long is the other leg?
   b. How long is the hypotenuse?

   answer: **4 cm, $4\sqrt{2}$**

   For part a, both legs of an isosceles right triangle are the same length. As for part b, you have seen that the hypotenuse of an isosceles right triangle is equal in length to a side times the square root of two.

2. The shortest leg of a 30-60-90 triangle is 5 inches long.
   a. How long is the other leg?
   b. How long is the hypotenuse?

   answer: **$5\sqrt{3}$, 10 in.**

It might be simpler to find the answer to part b first. In a 30-60-90 triangle, the shortest side is the one opposite the smallest angle, that is, the one opposite 30°. That side is half the length of the hypotenuse, so the hypotenuse must be twice the length of that side, or 10 in. Finally, the side opposite the 60° angle (the other leg) is half the hypotenuse times the square root of three.

3. In triangle *ABC*, with right angle at *B*, the cosine of $\angle A$ is $\frac{11}{15}$. What is the sine $\angle C$?

answer: $\frac{11}{15}$

The cofunction identities tell us that in a given right triangle, the sine of one acute angle is the cosine of the other.

Work Problems

Use these problems to give yourself additional practice.

1. In triangle *ABC*, with right angle at *B*, the cosine of $\angle A$ is $\frac{3}{5}$. What are the sine of $\angle A$ and the tangent of $\angle A$?
2. One leg of an isosceles right triangle is 8 cm long. What is the length of the hypotenuse?
3. The hypotenuse of a 30-60-90 triangle is 10 inches long. The shorter leg of the triangle is *a*, and the longer *b*. Find the lengths of *a* and *b*.
4. In triangle *PQR*, with right angle at *P*, the sine of $\angle Q$ is 0.235. What is the tangent of $\angle R$?

Worked Solutions

1. $\mathbf{\frac{4}{5}, \frac{4}{3}}$

Remember: $\sin^2 A + \cos^2 A = 1$

Therefore: $\sin^2 A + \left(\frac{3}{5}\right)^2 = 1$

Square and subtract from both sides: $\sin^2 A = 1 - \frac{9}{25}$

Make 1 an equivalent fraction: $\sin^2 A = \frac{25}{25} - \frac{9}{25}$

Subtract: $\sin^2 A = \frac{16}{25}$

Now get the square root of both sides: $\sin A = \frac{4}{5}$

As for tangent, use the ratio identity: $\tan A = \frac{\sin A}{\cos A}$

Substitute: $\tan A = \frac{\frac{4}{5}}{\frac{3}{5}}$

Rewrite as a reciprocal multiplication: $\tan A = \frac{4}{5} \times \frac{5}{3}$

The 5s cancel, so you get: $\tan A = \frac{4}{\cancel{5}} \times \frac{\cancel{5}}{3} = \frac{4}{3}$

2. **$8\sqrt{2}$ cm** You learned in the "Two Special Triangles" section that the hypotenuse of an isosceles right triangle is equal to the length of a side times the square root of two. If you did not recall that, then use the Pythagorean theorem, which in the case of an isosceles right triangle may be written

$c^2 = a^2 + a^2$ (Remember, both legs are equal.)

Substitute: $c^2 = 8^2 + 8^2$

Square and add: $c^2 = 64 + 64 = 128$

Solve for c: $c = 8\sqrt{2}$

3. **5 inches, $5\sqrt{3}$ inches** In a 30-60-90 triangle, the shorter leg is opposite the 30° angle and is half the length of the hypotenuse. Half of 10 inches is 5 inches. The longer leg is opposite the 60° angle and is equal to half the hypotenuse times the square root of 3. That's $5\sqrt{3}$ inches.

4. **17.67** The sine of $\angle Q$ is 0.235, but you want the tangent of $\angle R$, so you'll use sine's cofunction, $\cos\angle R = 0.235$. Next, you need to find $\sin\angle R$ so that you may relate sin and cos with the tangent identity.

First write the equation: $\sin^2 R + \cos^2 R = 1$

Next, substitute: $\sin^2 R + (0.235)^2 = 1$

Clear the parentheses: $\sin^2 R + 0.055 = 1$

Collect the constants: $\sin^2 R = 1 - 0.055$

Subtract: $\sin^2 R = 0.945$

Solve for $\sin R$: $\sin R = 0.972$

Now for the tangent identity: $\tan R = \dfrac{\sin R}{\cos R}$

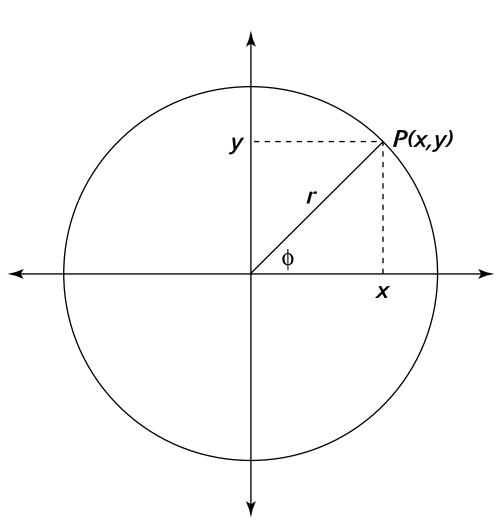
Substitute: $\tan R = \dfrac{0.972}{0.055}$

And divide: $\tan R = 17.67$

Functions of General Angles

When an acute angle is written in standard position, it is always in the first quadrant, and there, all trigonometric functions exist and are positive. This is not true, however, of angles in general. Some of the six trigonometric functions are undefined for quadrantal angles, and some have negative values in certain quadrants. In standard position, an angle is considered to have its starting position in quadrant I on the x-axis and its terminal side in or between one of the four quadrants.

Consider the angle, ϕ in the following figure. Point P is on the terminal side of the angle, r, and has coordinates (x, y). The radius of the circle is 1.

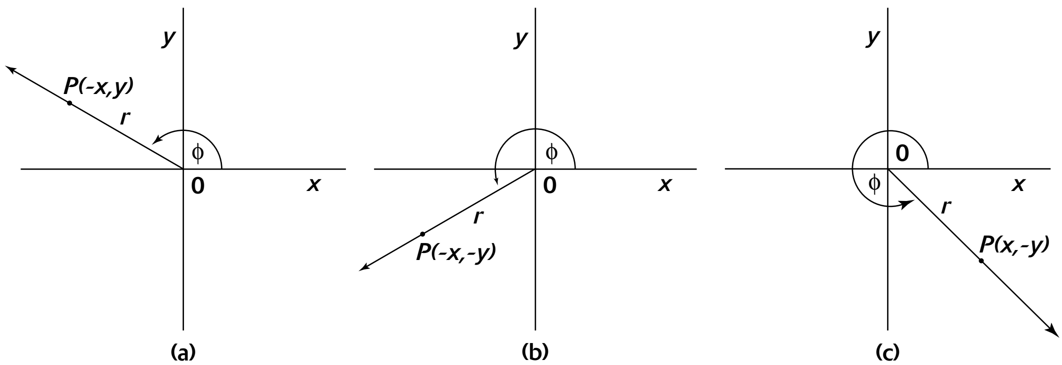


A unit circle ($r = 1$).

Look it over carefully, and you'll see that $\sin\phi = y$, $\cos\phi = x$, and $\tan\phi = \frac{y}{x}$.

This is a tangible proof of the $\tan\phi = \frac{\sin\phi}{\cos\phi}$ ratio identity.

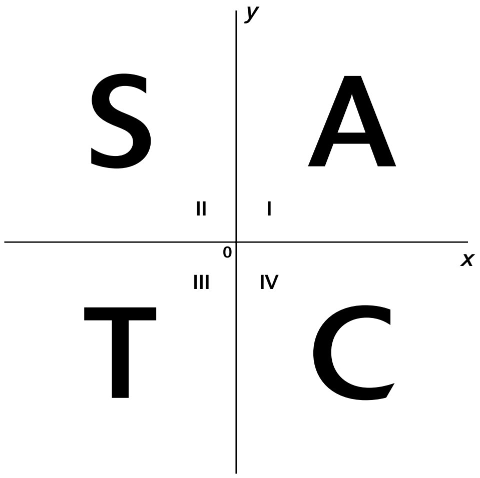
Alas, not all standard angles terminate in the first quadrant. Look at these three angles:



Angles terminating in quadrants other than I: (a) Angle terminating in II; (b) Angle terminating in III; (c) Angle terminating in IV

In all four quadrants, the value of r is positive. In quadrant II, only sine is positive $\left(\frac{y}{r}\right)$, while cosine $\left(\frac{-x}{r}\right)$ and tangent $\left(\frac{y}{-x}\right)$ are negative. Cosecant, secant, and tangent will always have the same signs as their inverses. We leave it to you to figure out what functions are positive or negative in quadrants III and IV, and why.

To help in the future, there's a little mnemonic scheme you might want to remember, and it's represented in the following figure.



Clockwise ACTS, or counter CAST.

You may remember this as being the word “CAST” reading counterclockwise and beginning in quadrant IV, or “ACTS” reading clockwise from quadrant I. The letters tell you which ratio (and its inverse) is positive in that quadrant: “C” for cosine in IV, “T” for tangent in III, “S” for sine in II, and “A” for all in I.

Should $\angle\phi$ be a quadrantal angle, then either x or y may be equal to 0. If that 0 is in the numerator, then the trigonometric ratio will have a value of 0, but if it’s in the denominator, the ratio is undefined. Yet other times, the ratio equals 1. That’s shown in the following table.

Trigonometric Ratios for Quadrantal Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\csc\theta$	$\cot\theta$
0°	0	1	0	1	—	—
90°	1	0	—	—	1	0
180°	0	–1	0	–1	—	—
270°	–1	0	—	—	–1	0

Example Problems

These problems show the answers and solutions.

1. What is the sign of tan 230°?

   ***answer:* positive**

   Refer to the previous figure. 230° falls in quadrant III, where tangent is positive.

2. What is the sign of sin 300°?

   ***answer:* negative**

   Refer to the previous figure. 300° falls in quadrant IV, where sine is negative.

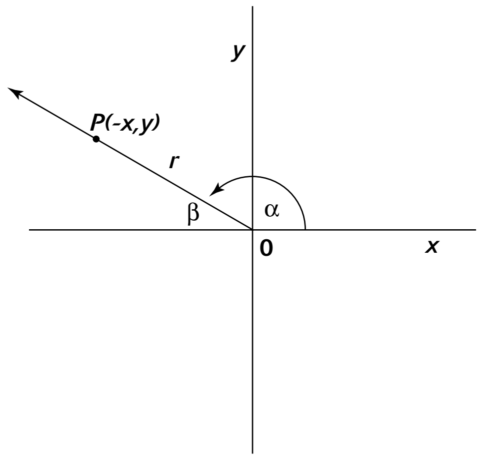
3. What is the sign of sec 320°?

answer: **positive**

Secant is the reciprocal of cosine. 320° falls in quadrant IV, where cosine is positive, so secant must also be positive in that quadrant.

Reference Angles

The trigonometric functions of nonacute angles may be converted so that they correspond to the functions of acute angles. See the following figure.

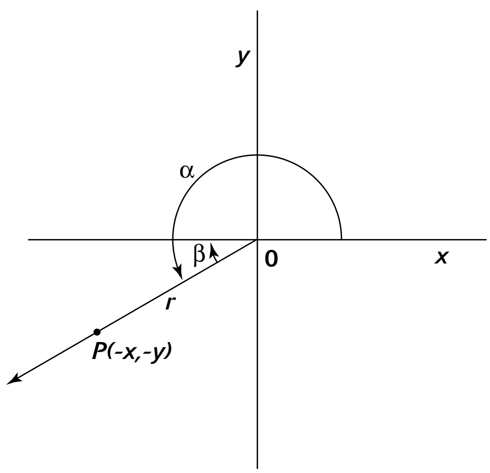


$180° - \alpha = \text{reference}\angle\beta.$

α is a second quadrant angle in standard position. By subtracting $\angle\alpha$ from 180°, you get the acute reference angle, $\angle\beta$. You can now find the trigonometric ratios using the reference angle, but bear in mind that since the original angle terminated in the second quadrant, all of the ratios will be negative except for sine and cosecant.

$$\sin\beta = \frac{y}{r} \qquad \cos\beta = \frac{-x}{r} \qquad \tan\beta = \frac{y}{-x}$$

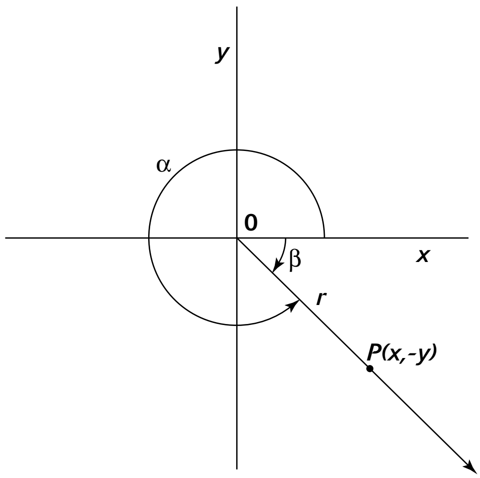
To find cosecant, secant, and cotangent, just flip over the three preceding ratios.



$\alpha - 180° = \text{reference}\angle\beta.$

In the third quadrant, reference $\angle\beta$ is found by subtracting 180° from α, as shown in the previous figure.

$$\sin\beta = \frac{-y}{r} \qquad \cos\beta = \frac{-x}{r} \qquad \tan\beta = \frac{-y}{-x}$$



$$360° - \alpha = \text{reference}\angle\beta.$$

In quadrant IV, $\angle\beta$ is found by subtracting α from 360°.

$$\sin\beta = \frac{-y}{r}\ \cos\beta = \frac{x}{r}\ \tan\beta = \frac{-y}{x}$$

Notice that the reference angle always sits above or below the *x*-axis.

Example Problems

These problems show the answers and solutions.

1. What is the cosine of 150°?

***answer:* –0.867 or** $-\frac{\sqrt{3}}{2}$

To find the reference angle, subtract 150° from 180°. That's a 30° angle. In a 30-60-90 triangle of hypotenuse 2, the side opposite the 30° angle is 1, and the side opposite the 60° angle is $\sqrt{3}$. That makes $\cos 30° = \frac{\sqrt{3}}{2}$, but cosine is negative in the second quadrant. $\sqrt{3} \approx 1.732$, hence the two possible answers.

2. What is the sine of 225°?

***answer:* –0.707 or** $-\frac{\sqrt{2}}{2}$

To find a quadrant III reference angle, subtract 180°. 225° – 180° = 45°. When you studied an isosceles right triangle, you found both sin and $\cos 45° = \frac{1}{\sqrt{2}}$, which you rationalize by multiplying it by $\frac{\sqrt{2}}{\sqrt{2}}$, thus getting $\frac{\sqrt{2}}{2}$. It's considered poor form to leave a radical in the denominator. $\sqrt{2} \approx 1.414$, so half of it is 0.707, but since sine is negative in the third quadrant, the solution is negative.

3. Find the tangent of 300°.

***answer:* –1.732 or $-\sqrt{3}$**

The angle is in quadrant IV, so subtract it from 360°. $360^\circ - 300^\circ = 60^\circ$. Refer back to solution 1, to find that the $\tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$. In quadrant IV, tan is negative, hence the answers shown.

Squiggly versus Straight

We presume that you noticed the squiggly equal signs used in the explanations of example problems 1 and 2. Mathematicians use that symbol to indicate that one quantity is a close approximation of another.

For example, $\sqrt{3}$ is easily shown on a calculator to equal 1.732050808, and then some, so mathematicians would much prefer to write $\sqrt{3} \approx 1.732$ than $\sqrt{3} = 1.732$.

On the other hand, a mathematician would never write $5 + 7 \approx 12$. In fact, $5 + 7 = 12$, with no approximating or rounding involved. The squiggly equal sign is also likely to be used when the numbers are readings from scientific instruments, which only approximate true conditions.

Work Problems

Use these problems to give yourself additional practice.

1. A 135° angle is angle in standard position. What is its cosecant?
2. Find the cotangent of a 240° angle in standard position.
3. What is the sine of a 675° angle in standard position?
4. Find the secant of a 330° angle in standard position.
5. Angle χ is in standard position, and its terminal side passes through the point with coordinates (−12,5). Find all six of its trigonometric functions.
6. If $\sin\theta = \frac{4}{5}$, and $\cos\theta$ is negative, what are the values of the five remaining trigonometric functions?

Worked Solutions

1. $\sqrt{2}$ **or 1.414** The reference angle for 135° is 45°. If necessary, refer back to the figure on p. 35 ("An isosceles right triangle"), where you'll find that csc 45° is $\frac{\sqrt{2}}{1}$, which, of course, is simply $\sqrt{2}$.

   Since sine is positive in the second quadrant, so is cosecant.

2. $\frac{\sqrt{3}}{3}$ **or 0.577** To find the reference angle, subtract 180°. That makes 60°. Next, if necessary, look back at the figure on p. 36 ("The 30-60-90 right triangle").

   $\cot 60° = \frac{1}{\sqrt{3}}$, which you rationalize: $\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

   Finally, consider that the angle originally terminated in Q-III (third quadrant), where tangent, and therefore, cotangent, is positive.

3. $-\frac{\sqrt{2}}{2}$ **or –.707** 675° goes one full 360° turn, and then 45° less than another one. That makes it a 45° angle in Q-IV, just below the *x*-axis. sin 45°, as you may recall from the figure on p. 35 ("An isosceles right triangle") or the table on p. 36 (Table of Trigonometric Ratios for 30°, 45°, and 60° Angles), $= \frac{\sqrt{2}}{2}$. Sine is not positive in Q-IV, so the answer will be negative.

4. $\frac{2\sqrt{3}}{3}$ **or 1.155** Subtract from 360°, and you find that this is a 30° angle in Q-IV. Secant 30° is $\frac{2\sqrt{3}}{3}$. Since cosine is positive in Q-IV, so is secant.

5. Since the angle's terminal side passes through (−12, 5), build a right triangle about that point in Q-II, with the angle to be dealt with at the origin. The side opposite that angle is the *y*-coordinate, 5. The adjacent side of the triangle is the absolute value of the *x*-coordinate, $|-12| = 12$. To find the hypotenuse of the triangle, you can either use the Pythagorean theorem, or remember the Pythagorean triple, 5-12-13. Either way, the hypotenuse has a length of 13. So, take care to remember that the terminal side of the angle is in Q-II, where only sine and its reciprocal, cosecant, are positive. With that in mind, you find:

   $\sin\chi = \frac{5}{13}$, $\cos\chi = -\frac{12}{13}$, $\tan\chi = -\frac{5}{12}$, $\csc\chi = \frac{13}{5}$, $\sec\chi = -\frac{13}{12}$, and $\cot\chi = -\frac{12}{5}$

6. **Since** $\sin\theta = \frac{4}{5}$ **and** $\cos\theta$ **is negative, the angle must be in the second quadrant; since in Q-I, cosine would have been positive, and sine is positive in Q-III and Q-IV.** The hypotenuse of the triangle must be 5, and the opposite side 4 (from the sine), which makes the adjacent side 3, either by the Pythagorean theorem, or by remembering the Pythagorean triple, 3-4-5. So, $\cos\theta = -\frac{3}{5}$ and $\tan\theta = -\frac{4}{3}$, $\csc\theta = \frac{5}{4}$, $\sec\theta = -\frac{5}{3}$, and $\cot\theta = -\frac{3}{4}$.

Trig Tables versus Calculators

We've already taken note of the fact that the trigonometric ratios depend only on the value of the angle measure and are independent of the size of the triangle's sides. You would think that somebody would write down those values for each degree of measure. It should come as no surprise then, that somebody has. You'll find a table of trigonometric ratios in the back of this book, beginning on p. 297. In order to read the value of a trigonometric ratio for a certain angle, you move one finger down the column that says the name of the ratio you're looking for at the top, while moving your eyes down the angle values on the left until you find the one for which you're looking. Then move them across to the proper column. Where your eyes and finger meet should be the value you want. The tables show values for all six trigonometric functions in increments of 1°.

Now all of the angles we've spoken about so far have been in degrees, but the degree is not the smallest unit of angle measure. Just like that other round thing we're used to seeing on a wall, degrees can be broken down into minutes. There are, in fact, 60 minutes in one degree, and, in turn, 60 seconds in one minute. What that means is that it is possible to break the degree measurement of a circle into $360 \times 60 \times 60 = 1{,}296{,}000$ parts. All of those parts are not especially useful or accurate when dealing with the small circles in this book or on your graph paper, but consider how handy they can be when describing the location on a three-dimensional circular object, like the earth.

In addition to the traditional subdivisions mentioned in the last paragraph, degrees are also capable of being subdivided into decimal parts, like 33.5°, 40.75°, or even 27.625°. There is really no limit to the number of decimal places to which an angle's size may be carried out, although after the sixth, one might start to become a bit suspicious about how accurately that quantity was measured. It is also true, however, that scientific measuring techniques are continually being refined.

That brings us to scientific calculators. Scientific calculators have the capacity to calculate at least the three fundamental trigonometric functions: sine, cosine, and tangent. The method in which the data is entered varies from brand to brand, but for the most part, you press the button that names the ratio you want to find, enter the number of degrees and any decimal portion that may follow, and press <Enter> to display the ratio. Most calculators do not provide cosecant, secant, or cotangent keys, however. In order to find the cosecant of, say 30°, you would first press the <SIN> button, type 30, and press <Enter>. The screen now displays 0.5. Next, clear the screen or add the answer to memory and clear the screen. Next type 1÷, or 1/ depending upon how your calculator is labeled; they mean the same thing. Then either press your <RECALL> key or key in .5. Finally, press <Enter>. The screen will show 2, the cosecant of 30°. That's because, as you'll hopefully recall, cosecant and sine are inverse functions:

$$\left(\csc\phi = \frac{1}{\sin\phi}\right)$$

When using a table of trigonometric functions, it's possible to look at the function and see what the angle is. Go ahead. Look at the sines in the table on p. 297 (Trig Functions Table) and find the angle whose sine is 0.423. Go ahead, really do it. We'll wait. Well, if you actually went and looked under the sin column and found the value 0.423, you would have found that it belongs to the angle, 25°. You can't do the same thing with a calculator. Instead, you have to use the functions arcsin, arccos, and arctan, which are also written $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. That means, the expression $\sin\theta = 0.5$ may be written as $\sin^{-1}\theta = 0.5$ or $\arcsin\theta = 0.5$. Sometimes, it is referred to as "the inverse sine of 0.5." In any of those cases, the expression is used if the trigonometric ratio is known, and the angle that it belongs to is being sought. In the last case, when $\sin^{-1}\theta = 0.5$, then $\theta = 30°$.

Calculators handle arcsine in different ways. On some, you push an <arc> button and then press the appropriate trig ratio. On others, you must press the <2nd function> button and then press the appropriate trig ratio button.

Interpolation

Most trigonometric functions, whether listed in tables or found on calculators, are approximations. Nevertheless, every effort possible is made to get the numbers as close as possible to the actual value. Bearing that in mind, if you're using tables that give values to the nearest degree, and you need the cosine of a fractional angle, say 30.6°, you're going to need a way to find a value that is more accurate than 0.8660, the cosine for 30°. The method for approximating the closer value is known as **interpolation.** It works like this:

$$\cos 30° > \cos 30.6° > \cos 31° \text{ (which means } \cos 30.6° \text{ is between } \cos 30° \text{ and } \cos 31°)$$

$$\cos 30° = 0.8660,\ \cos 31° = 0.8572.$$

The difference between them is $0.8660 - 0.8572 = 0.0088$.

30.6° is 0.6, or $\frac{6}{10}$ of the way between the two angles, so cos 30.6° should be 0.6 of the difference between the two cosines less than cos 30°: $0.0088 \times 0.6 = 0.0053$.

Since cos 31° is lower than cos 30°, subtract from cos 30°:

$$0.8660 - 0.0053 = 0.8607$$

$\therefore \cos 30.6° = 0.8607$ (The symbol ∴ means "therefore.")

Let's try one more of those, so that you have a chance to see both possible scenarios. This time, find the sine of 42.3°.

$$42° < 42.3° < 43°$$

$$\sin 42° = 0.6691,\ \sin 43° = 0.6820.$$

The difference between them is $0.\ 6820 - 0.6691 = 0.0129$.

42.3° is 0.3, or $\frac{3}{10}$ of the way between the two angles, so sin 42.3° should be 0.3 of the difference between the two sines greater than sin 42°: $0.0129 \times 0.3 = 0.0039$.

Since sin 42° is lower than sin 43° add to sin 42°:

$$0.6691 + .0039 = 0.6730$$

$$\therefore \sin 42.3° = 0.6730$$

Interpolation may also be used to approximate the size of an angle to the nearest tenth of a degree. Suppose that you have $\sin\theta = 0.9690$, and you know that $\sin 75° = 0.9659$ and $\sin 76° = 0.9702$.

$$1.0\left\{\begin{array}{l} x\left\{\begin{array}{l}\sin 75^\circ = 0.9659\\ \sin\phi = 0.9690\end{array}\right\}0.0031 \\ \quad\sin 76^\circ = 0.9702\end{array}\right\}0.0043\,(x = \text{the difference})$$

Now, we'll use the variable x to set up a proportion:

$$\frac{x}{1.0} = \frac{0.0031}{0.0043}$$

$$0.0043x = 0.0031$$

$$x = \frac{0.0031}{0.0043}$$

$$x = 0.7$$

$$\therefore \phi = 75° + 0.7° = 75.7°$$

Example Problems

These problems show the answers and solutions. Use the tables that begin on p. 297. Do *not* use a calculator.

1. How many seconds are there in a 5° angle?

   ***answer:* 18,000**

   There are 60 minutes in one degree, and 60 seconds in each of those minutes.

   That's for one degree: $60 \times 60 = 3600$

   For 5 degrees, multiply it by 5: $5 \times 3600 = 18{,}000$

2. Find tan 46.8° to the nearest ten thousandth (four decimal places).

   ***answer:* 1.0650**

   $46° < 46.8° < 47°$

   $\tan 46° = 1.0355$, $\tan 47° = 1.0724$

   The difference between them is $1.0724 - 1.0355 = 0.0369$

   46.8° is 0.8, or $\frac{8}{10}$ of the way between the two angles, so tan 46.8° should be 0.8 of the difference between the two tans greater than tan 46°: $0.0369 \times 0.8 = 0.0295$.

   Since tan 46° is lower than tan 47° add to tan 46°:

   $$1.0355 + .0295 = 1.0650$$
   $$\therefore \tan 46.8° = 1.0650$$

3. For what acute angle does cosine have a value of 0.7826 if cos 38° = 0.7880 and cos 39° = 0.7771?

   ***answer:* 38.5°**

   $$1.0\left\{\begin{array}{l} x\left\{\begin{array}{l}\cos 75° = 0.7880\\ \cos\phi = 0.7826\end{array}\right\}0.0054 \\ \cos 76° = 0.7771\end{array}\right\}0.0109\,(x = \text{the difference})$$

   Now, we'll use the variable *x* to set up a proportion:

   $$\frac{x}{1.0} = \frac{0.0054}{0.0109}$$
   $$0.0109x = 0.0054$$
   $$x = \frac{0.0054}{0.0109}$$
   $$x = 0.5$$
   $$\therefore \phi = 38° + 0.5° = 38.5°$$

Work Problems

Use these problems to give yourself additional practice.

1. How many seconds are there in a 30° angle?
2. Find sin 35.7° to the nearest ten thousandth (four decimal places).
3. Find tan 61.3° to the nearest ten thousandth (four decimal places).
4. What is the secant of a 765° angle in standard position?
5. For what acute angle does sine have a value of 0.4602 if sin 27° = 0.4540 and sin 28° = 0.4695?

Worked Solutions

1. **108,000** 60 seconds per minute × 60 minutes per hour × 30 degrees gives

$$60 \times 60 \times 30 = 108{,}000$$

2. **0.5835** 35° < 35.7° < 36°

sin 35° = 0.5736, sin 36° = 0.5878.

The difference between them is 0.5878 − 0.5736 = 0.0142.

35.7° is 0.7, or $\frac{7}{10}$ of the way between the two angles, so sin 35.7° should be 0.7 of the difference between the two sines greater than sin 35°: 0.0142 × 0.7 = 0.0099.

Since sin 35° is lower than sin 36° add to sin 35°:

$$0.5736 + .0099 = 0.5835$$
$$\therefore \sin 35.7° = 0.5835$$

3. **1.8270** 61° < 61.3° < 62°

tan 61° = 1.8040, tan 62° = 1.8807.

The difference between them is 1.8807 − 1.8040 = 0.0767.

61.3° is 0.3, or $\frac{3}{10}$ of the way between the two angles, so sin 61.3° should be 0.3 of the difference between the two tans greater than tan 61°: 0.0767 × 0.3 = 0.0230.

Since tan 61° is lower than tan 62° add to tan 61°:

$$1.8040 + .0230 = 1.827$$
$$\therefore \tan 61.3° = 1.827$$

4. **1.4142** First of all, a 765° angle is a 45° angle after the two full rotations are removed. Secant is the inverse of cosine, so find cos 45° = .7071

Then make the fraction $\frac{1}{0.7071}$ and divide 1 ÷ 0.7071 = 1.4142.

Alternately, you could have recalled that $\cos 45^\circ = \frac{\sqrt{2}}{2}$, so $\sec 45^\circ = \frac{2}{\sqrt{2}}$.

You can rationalize the denominator by multiplying to get $\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$, or 1.4142

5. **27.4°** $1.0\left\{\begin{matrix} x\left\{\begin{matrix}\sin 27^\circ = 0.4540 \\ \sin\theta = 0.4602\end{matrix}\right\}0.0062 \\ \sin 28^\circ = 0.4695\end{matrix}\right\}0.0155\,(x = \text{the difference})$

Now, you'll use the variable *x* to set up a proportion:

$\frac{x}{1.0} = \frac{0.0062}{0.0155}$

$0.0155x = 0.0062$

$x = \frac{0.0062}{0.0155}$

$x = 0.4$

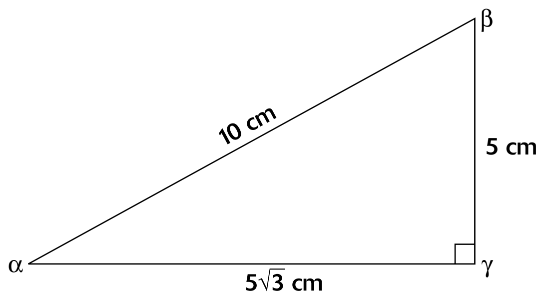
$\therefore \theta = 27^\circ + 0.4^\circ = 27.4^\circ$

Chapter Problems and Solutions

Problems

Solve these problems for more practice applying the skills from this chapter. Worked out solutions follow the problems.

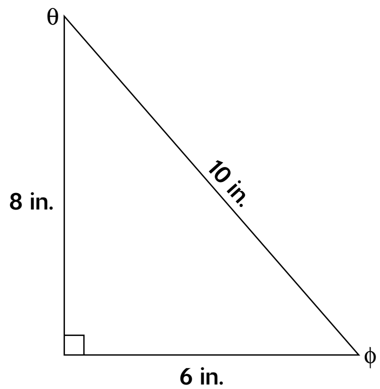
1. In which quadrant is the terminal side of a 105° angle in standard position?
2. What is the special name given to a straight angle in standard position?
3. Is an angle measuring 245° coterminal with an angle measuring 975°?
4. In order for an angle to be a quadrantal angle, by what number must it be capable of being divided?
5. In which quadrant does the terminal side of an angle of 1260° degree measure fall?



Problems 6–7 refer to the above figure.

6. Find the sin of α.

7. Find $\tan\beta$.



Problem 8 refers to the above figure.

8. Find $\sec\theta$.

9. An angle has a sine of 0.4. What is the value of its cosecant?

10. An angle has a tangent of 2.5. What is the value of its cotangent?

11. An angle has a cosecant of 3.8. What is the value of its sine?

12. Find the cos and tangent of $\angle\phi$ if ϕ is an acute angle ($0 < \phi < 90°$), and $\sin\phi = 0.7660$.

13. One leg of an isosceles right triangle is 8 cm long.

- **a.** How long is the other leg?
- **b.** How long is the hypotenuse?

14. In triangle *ABC*, with right angle at *B*, the cosine of $\angle A$ is $\frac{7}{18}$. What is the sin $\angle C$?

15. One leg of an isosceles right triangle is 12 in. long. What is the length of the hypotenuse?

16. In triangle *PQR*, with right angle at *P*, the sine of $\angle Q$ is 0.235. What is the tangent of $\angle R$?

17. What is the sign (positive or negative) of sin 290°?

18. What is the cosine of 240°?

19. Find the tangent of 330°.

20. Find the cotangent of a 240° angle in standard position.

21. Find the secant of a 300° angle in standard position.

22. If $\theta = \frac{4}{5}$, and $\cos\theta$ is negative, what are the values of the five remaining trigonometric functions?

Use the tables that begin on p. 297. Do *not* use a calculator.

23. Find tan 40.3° to the nearest ten thousandth (four decimal places).

24. Find sin 71.8° to the nearest ten thousandth (four decimal places).

25. For what acute angle does sine have a value of 0.5793 if sin 35° = 0.5736 and sin 36° = 0.5878?

Answers and Solutions

1. ***Answer:*** II Since the angle is in standard position, its initial side is on the *x*-axis to the right of the origin. the *y*-axis forms a right (90°) angle with the initial side, so a 105° angle's terminal side must sweep past the vertical and into quadrant II.

2. ***Answer:*** quadrantal A straight angle in standard position will have its terminal side on the *x*-axis. That makes it a quadrantal angle.

3. ***Answer:*** no If angles measuring 245° and 975° were coterminal, then

$$975° = 245° + n \cdot 360°$$

$$730° = n \cdot 360°$$

But 730° is *not* a multiple of 360°, so the angles *cannot* be coterminal.

4. ***Answer:*** 90 A quadrantal angles' terminal side falls on an axis. Therefore, no matter what the size of the coterminal angle, it must be capable of being in the positions of 90°, 180°, 270°, or 0°. All are divisible by 90.

5. ***Answer:*** None It is quadrantal. 1260° is three full revolutions from the starting position, and then an additional 180°:

$$3 \cdot 360° = 1080°$$

$$1260° - 1080° = 180°$$

That places the terminal side on the *x*-axis, left of the origin (negative).

6. ***Answer:*** 0.5 The sin ratio is $\frac{\text{opposite}}{\text{hypotenuse}}$. 5 cm is the length of the side opposite $\angle\alpha$, and the hypotenuse is 10 cm. So, $\sin\alpha = \frac{5}{10} = \frac{1}{2} = 0.5$.

7. ***Answer:*** $\sqrt{3}$ The tan ratio is $\frac{\text{opposite}}{\text{adjacent}}$.

   $5\sqrt{3}$ cm is the length of the side opposite $\angle\beta$, and its adjacent side is 5 cm. So, $\tan\beta = \frac{5\sqrt{3}}{5} = \sqrt{3}$.

8. ***Answer:*** 1.25 $\sec\theta = \frac{\text{length of hypotenuse}}{\text{length of side adjacent } \theta} = \frac{10}{8} = 1.25$

9. ***Answer:*** 2.5 Cosecant is the reciprocal of sine; that is, $\csc\theta = \frac{1}{\sin\theta}$, but you know that the value of sin is 0.4; therefore $\csc\theta = \frac{1}{0.4}$.

   Divide 1 by 0.4, and you get 2.5.

10. ***Answer:*** 0.4 Cotangent is the reciprocal function of tangent, so $\cot\theta = \frac{1}{\tan\theta}$, but you were given tan = 2.5, so $\cot\theta = \frac{1}{2.5}$.

    Divide 1 by 2.5 and get 0.4.

11. ***Answer:*** 0.2632 Sine is cosecant's reciprocal function, $\sin\theta = \frac{1}{\csc\theta}$, but you know that the value of csc is 3.8; therefore, $\sin\theta = \frac{1}{3.8}$.

    Divide 1 by 3.8, and you'll get 0.263157895, which you'll round to 0.2632.

12. ***Answer:*** 0.6307, 1.2304 First, use the identity $\sin^2\phi + \cos^2\phi = 1$ to find cos ϕ.

    Substitute: $(0.776)^2 + \cos^2\phi = 1$

    Subtract $(0.776)^2$ from both sides: $\cos^2\phi = 1 - (0.776)^2$

    Square the quantity in parentheses: $\cos^2\phi = 1 - 0.6022$

    Subtract 0.6022 from 1: $\cos^2\phi = 0.3978$

    Take the square root of both sides: $\cos^2\phi = 0.6307$

    Now, for the tangent.

    First, write the relevant identity: $\tan\phi = \frac{\sin\phi}{\cos\phi}$

    Substitute for sin and cos: $\tan\phi = \frac{0.7660}{0.6307}$

    And divide: $\tan\phi = 1.2304$

13. ***Answer:*** 8 cm, $8\sqrt{2}$ cm

    For part a, both legs of an isosceles right triangle are the same length. As for part b, you have seen that the hypotenuse of an isosceles right triangle is equal in length to a side times the square root of two, but if you forgot that there's always the Pythagorean theorem.

14. ***Answer:*** $\frac{7}{18}$

The cofunction identities tell us that in a given right triangle, the sine of one acute angle is the cosine of the other.

15. ***Answer:*** $12\sqrt{2}$ in.

In the section "Two Special Triangles," the hypotenuse of an isosceles right triangle is equal to the length of a side times the square root of 2. If you did not recall that, then use the Pythagorean theorem, which—in the case of an isosceles right triangle—may be written $c^2 = a^2 + a^2$ (remember, both legs are equal).

Substitute: $c^2 = 12^2 + 12^2$

Square and add: $c^2 = 144 + 144 = 288$

Solve for c: $c = 12\sqrt{2}$

16. ***Answer:*** 2.4029 The sine of $\angle Q$ is 0.3842, but you want the tangent of $\angle R$, so you'll use sine's cofunction, $\cos\angle R = 0.3842$. Next, you need to find $\sin \angle R$ so that you may relate sin and cos with the tangent identity.

First write the equation: $\sin^2 R + \cos^2 R = 1$

Next, substitute: $\sin^2 R + (0.3842)^2 = 1$

Clear the parentheses: $\sin^2 R + 0.1476 = 1$

Collect the constants: $\sin^2 R = 1 - 0.1476$

Subtract: $\sin^2 R = 0.8524$

Solve for $\sin R$ $\sin R = 0.9232$

Now for the tangent identity: $\tan R = \frac{\sin R}{\cos R}$

Substitute: $\tan R = \frac{0.9232}{0.3842}$

And divide: $\tan R = 2.4029$

17. ***Answer:*** negative 290° falls in quadrant IV, where sine is negative. You may want to refer to the figure on p. 40 (Clockwise ACTS, or counter CAST).

18. ***Answer:*** $-\frac{1}{2}$

To find the reference angle, subtract 180° from 240°. That's a 60° angle. In a 30-60-90 triangle of hypotenuse 2, the side opposite the 30° angle is 1, and the side opposite the 60° angle is $\sqrt{3}$. That makes $\cos 60° = \frac{1}{2}$, but cosine is negative in the third quadrant, so the answer is $-\frac{1}{2}$.

19. ***Answer:*** $-\frac{\sqrt{3}}{3}$

The angle is in quadrant IV, so subtract it from 360°.

360° − 330° = 30°. Refer back to the 30-60-90 triangle to find the $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.
In Q-IV, tan is negative; hence the answer is $-\frac{\sqrt{3}}{3}$.

20. ***Answer:*** $\frac{\sqrt{3}}{3}$ or 0.577 To find the reference angle, subtract 180°. That makes 60°. Next, if necessary, look back at the figure on p. 36 ("The 30-60-90 right triangle").

$\cot 60° = \frac{1}{\sqrt{3}}$, which you can rationalize: $\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Finally, consider that the angle originally terminated in Q-III (third quadrant), where tangent, and therefore cotangent, is positive.

21. ***Answer:*** 2 Subtract 300° from 360°, and you find that this is a 60° angle in Q-IV.

Secant 60° is $\frac{2}{1} = 2$. Since cosine is positive in Q-IV, so is secant.

22. ***Answer:*** $\cos\theta = -\frac{3}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = \frac{5}{4}$, $\sec\theta = -\frac{5}{3}$, and $\cot\theta = -\frac{3}{4}$.

Since $\sin\theta = \frac{4}{5}$ and cos θ is negative, the angle must be in the second quadrant. Since in the first quadrant, the cosine would have been positive, and sine is positive in Q-III and Q-IV, the hypotenuse of the triangle must be 5, and the opposite side is 4 (from the sine), which makes the adjacent side 3, either by the Pythagorean theorem, or by remembering the Pythagorean triple, 3-4-5. So, $\cos\theta = -\frac{3}{5}$, $\tan\theta = -\frac{4}{3}$, $\csc\theta = \frac{5}{4}$, $\sec\theta = -\frac{5}{3}$, and $\cot\theta = -\frac{3}{4}$.

23. ***Answer:*** 0.8482 40° < 40.3° < 41°

tan 40° = 0.8391, tan 41° = 0.8693.

The difference between them is 1.0724 − 1.0355 = 0.0302.

40.3° is 0.3, or $\frac{3}{10}$ of the way between the two angles, so tan 40.3° should be 0.3 of the difference between the two tans: $0.0302 \times 0.3 = 0.0091$. Since tan 40° is lower than tan 41° add to tan 40°:

0.8391 + 0.0091 = 0.8482

$\therefore$ tan 40.3° = 0.8482

24. ***Answer:*** 0.9499 71° < 71.8° < 72°

sin 71° = 0.9455, sin 72° = 0.9510.

The difference between them is 0.9510 − 0.9455 = 0.0055.

71.8° is 0.8, or $\frac{8}{10}$ of the way between the two angles, so sin 71.8° should be 0.8 of the difference between the two sins: $0.0055 \times 0.8 = 0.0044$. Since sin 71° is lower than sin 72°, add to sin 71°:

$$0.9455 + 0.0044 = 0.9499$$

$$\therefore \sin 71.8° = 0.9499$$

25. ***Answer:*** 35.4° $\quad 1.0\left\{ x\left\{ \begin{array}{l} \sin 35° = 0.5736 \\ \sin\theta = 0.5793 \end{array} \right\} 0.0057 \atop \sin 36° = 0.5878 \right\} 0.0142\,(x = \text{the difference})$

Now, we'll use the variable x to set up a proportion:

$$\frac{x}{1.0} = \frac{0.0057}{0.0142}$$

$$0.0142x = 0.0057$$

$$x = \frac{0.0057}{0.0142}$$

$$x = 0.4$$

$$\therefore \theta = 35° + 0.4° - 35.4°$$

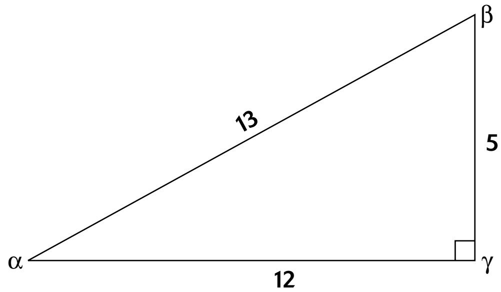
Supplemental Chapter Problems

Solve these problems for even more practice applying the skills from this chapter. The answer section will direct you to where you need to review.

Problems

1. In which quadrant is the terminal side of a −350° angle?
2. Name four angles that are coterminal with 70°.
3. In which quadrant does the terminal side of an angle of degree measure 1400° fall?
4. What is the lowest possible positive degree measure for an angle that is coterminal with one of −1950°?
5. Name two positive and two negative angles that are coterminal with an angle of degree measure 125°.

Problems 6–7 refer to the following figure.



6. Find $\cos\beta$.

7. Find $\csc\alpha$.

8. Find $\cot\beta$.

9. An angle has a cosine of 0.7. What is its secant?

10. An angle has a cosecant of 1.8. Name its inverse function and find its value.

11. Find the sin and tangent of $\angle\lambda$ if λ is an acute angle ($0 < \lambda < 90°$) and $\cos\lambda = \frac{3}{5}$.

12. Find the sin of $\angle\theta$ if θ is an acute angle ($0 < \theta < 90°$), $\tan\theta = 2.1445$ and $\cos\theta = 0.4226$.

13. The shortest leg of a 30-60-90 triangle is 10 inches long.
    a. How long is the other leg?
    b. How long is the hypotenuse?

14. The hypotenuse of a 30-60-90 triangle is 20 cm long. The shorter leg of the triangle is *a*, and the longer *b*. Find the lengths of *a* and *b*.

15. In triangle *ABC*, with right angle at *B*, the cosine of $\angle A$ is $\frac{12}{13}$. What are the sine of $\angle A$ and the tangent of $\angle A$?

16. What is the sign of tan 225°?

17. What is the sign of csc 310°?

18. What is the cosine of 225°?

19. A 315° angle is in standard position. What is its secant?

20. What is the sine of an 855° angle in standard position?

21. Angle χ is in standard position, and its terminal side passes through the point with coordinates (−18, 12). Find all six of its trigonometric functions.

22. How many seconds are there in an 18° angle?

23. For what acute angle does cosine have a value of 0.7615 if cos 40° = 0.7660 and cos 41° = 0.7547?

24. Find sin 28.7° to the nearest ten thousandth (four decimal places).

25. For what acute angle does sine have a value of 0.8261 if sin 55° = 0.8191 and sin 56° = 0.8290?

Answers

1. I (Angles and Quadrants, p. 21)
2. 430°, 790°, 1150°, 1510°, and so on. (Coterminal Angles, p. 25)
3. IV (Coterminal Angles, p. 25)
4. 150° (Coterminal Angles, p. 25)
5. −595°, −235°, 485°, 845°, and so on. (Coterminal Angles, p. 25)
6. $\frac{5}{13}$ (Trigonometric Functions of Acute Angles, p. 28)
7. $\frac{13}{5}$ (Trigonometric Functions of Acute Angles, p. 28)
8. $\frac{5}{12}$ (Reciprocal Trigonometric Functions, p. 29)
9. 1.4286 (Reciprocal Trigonometric Functions, p. 29)
10. sine, 0.5556 (Reciprocal Trigonometric Functions, p. 29)
11. $\frac{4}{5}, \frac{4}{3}$ (Trigonometric Cofunctions, p. 34)
12. 0.9063 (Trigonometric Cofunctions, p. 34)
13. $10\sqrt{3}$ in., 20 in. (Two Special Triangles, p. 35)
14. 10 cm, $10\sqrt{3}$ cm (Two Special Triangles, p. 35)
15. $\frac{5}{13}, \frac{5}{12}$ (Functions of General Angles, p. 38)
16. positive (Functions of General Angles, p. 38)
17. negative (Functions of General Angles, p. 38)
18. −0.707 or $\frac{\sqrt{2}}{2}$ (Reference Angles, p. 41)
19. 1.4142 or $\sqrt{2}$ (Reference Angles, p. 41)
20. 0.707 or $+\frac{\sqrt{2}}{2}$ (Reference Angles, p. 41)

21. $\sin = \frac{2\sqrt{13}}{13}$, $\cos = \frac{-3\sqrt{13}}{\sqrt{3}}$, $\tan = -\frac{2}{3}$, $\csc = \frac{\sqrt{13}}{2}$, $\sec = -\frac{\sqrt{13}}{3}$, and $\cot = -\frac{3}{2}$

(Reference Angles, p. 41)

22. 64,800 (Trig Tables versus Calculators, p. 44)

23. 40.4° (Interpolation, p. 45)

24. 0.4802 (Interpolation, p. 45)

25. 55.7° (Interpolation, p. 45)