Chapter 1

The Basics

Algebra is a language. You need to know the rules and definitions to understand this language and its many manipulations. In this chapter is a review of some of the important basics of algebra: rules for exponents and operations involving polynomials. These should be reviewed before going on to some of the advanced topics in Algebra II.

Rules for Exponents

A power or exponent tells how many times a number multiplies itself. Many opportunities exist in algebra for combining and simplifying expressions with two or more of these exponential terms in them. The rules used here to combine numbers and variables work for any expression with exponents. They are found in formulas and applications in science, business, and technology, as well as math. The term a^4 has an exponent of 4 and a *base* of *a*. The base is what gets multiplied repeatedly. The exponent tells how many times that *repeatedly* is.

Laws for Using Exponents

$a^n \cdot a^m = a^{n+m}$	When multiplying two numbers that have the same base, add their exponents.
$\frac{a^n}{a^m} = a^{n-m}$	When dividing two numbers that have the same base, subtract their exponents.
$(a^n)^m = a^{n \cdot m}$	When raising a value that has an exponent to another power, multiply the two exponents.
$(a \cdot b)^n = a^n \cdot b^n$	The product of two numbers raised to a power is equal to raising each number to that power and then multiplying them together.
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	The quotient of two numbers raised to a power is equal to raising each of the numbers to that power and then dividing them.
$a^{-n} = \frac{1}{a^n}$	A value raised to a negative power can be written as a fraction with the positive power of that number in the denominator.
$a^0 = 1$	Any number (except 0) raised to the 0 power is equal to 1.

Example Problems

These problems show the answers and solutions.

1. Simplify: $x^4\left(\frac{x^3}{x^{-2}}\right)^3$

answer: x^{19}

In this case, the course of action is to simplify the expressions inside the parentheses first, raise that result to the third power, and finally multiply by the first factor.

$$x^4\left(\frac{x^3}{x^{-2}}\right)^3 = x^4\left(x^3\cdot x^2\right)^3 = x^4\left(x^5\right)^3 = x^4\left(x^{15}\right) = x^{19}$$

2. Simplify: $y^{-3}\left(y^2\right)^4\cdot\frac{y^4}{yy^3}$

answer: y^5

The denominator reads yy^3, which implies that the first factor has an exponent of 1, reading y^1y^3.

$$y^{-3}\left(y^2\right)^4\cdot\frac{y^4}{yy^3} = y^{-3}\cdot y^8\cdot\frac{y^4}{y^4} = y^5\cdot 1 = y^5$$

3. Simplify: $\left(\frac{a^4b^2}{a^{-1}b^4}\right)^5\cdot\left(\frac{a^3}{ab^4}\right)^{-1}$

answer: $\frac{a^{23}}{b^6}$

A nice property of fractions is that when they're raised to a negative power, you can rewrite the expression and change the power to a positive if you "flip" the fraction. So $\left(\frac{a^3}{ab^4}\right)^{-1} = \left(\frac{ab^4}{a^3}\right)^1$. First, rewrite the second fraction without the negative exponent. Then simplify the fractions inside the parentheses. The next step is to raise the factors in the parentheses to the powers. Lastly, multiply the terms in the two numerators and denominators.

$$\left(\frac{a^4b^2}{a^{-1}b^4}\right)^5\left(\frac{a^3}{ab^4}\right)^{-1} = \left(\frac{a^4b^2}{a^{-1}b^4}\right)^5\left(\frac{ab^4}{a^3}\right)^1 = \left(\frac{a^4a^1}{b^2}\right)^5\left(\frac{b^4}{a^2}\right)^1 = \left(\frac{a^5}{b^2}\right)^5\left(\frac{b^4}{a^2}\right)^1 = \frac{a^{25}\cdot b^4}{b^{10}\cdot a^2} = \frac{a^{\not{25}\,23}}{b^{\not{10}\,6}}\cdot\frac{\not{b^4}}{\not{a^2}} = \frac{a^{23}}{b^6}$$

Work Problems

Use these problems to give yourself additional practice.

1. Simplify: $\left(\frac{x^3}{x^{-3}}\right)^2$

2. Simplify: $\left(\frac{y^4}{3x^2}\right)^{-3}$

3. Simplify: $\frac{a^{-2}x^2}{a^4x}\cdot\left(\frac{a^3}{x^3}\right)^4$

4. Simplify: $\frac{\left(abc^2\right)^4}{a^2bc^{-1}}$

5. Simplify: $\left(\frac{x^2y}{2w^4z}\right)^4\left(\frac{w^4z^6}{xy}\right)^4$

Worked Solutions

1. $\boldsymbol{x^{12}}$ First simplify inside the parentheses. Then raise the result to the second power.

$$\left(\frac{x^3}{x^{-3}}\right)^2=\left(x^6\right)^2=x^{12}$$

2. $\boldsymbol{\frac{27x^6}{y^{12}}}$ First "flip" the fraction and change the power to positive.

$$\left(\frac{y^4}{3x^2}\right)^{-3}=\left(\frac{3x^2}{y^4}\right)^3=\frac{27x^6}{y^{12}}$$

3. $\boldsymbol{\frac{a^6}{x^{11}}}$ First raise the factors in the parentheses to the fourth power. Then simplify the first fraction before multiplying the two fractions together.

$$\frac{a^{-2}x^2}{a^4x}\cdot\left(\frac{a^3}{x^3}\right)^4=\frac{a^{-2}x^2}{a^4x}\cdot\frac{a^{12}}{x^{12}}=\frac{\cancel{x}}{\cancel{a^6}}\cdot\frac{a^{\cancel{12}\,6}}{x^{\cancel{12}\,11}}=\frac{a^6}{x^{11}}$$

4. $\boldsymbol{a^2b^3c^9}$ First raise the numerator to the fourth power. Then simplify the fraction.

$$\frac{\left(abc^2\right)^4}{a^2bc^{-1}}=\frac{a^4b^4c^8}{a^2bc^{-1}}=\frac{a^{\cancel{4}\,2}b^{\cancel{4}\,3}c^{\cancel{8}\,9}}{\cancel{a^2}\cancel{b}\cancel{c^{-1}}}=a^2b^3c^9$$

5. $\boldsymbol{\frac{x^4z^{20}}{16}}$ Since both fractions are raised to the fourth power, it is easier to combine them in the same parentheses and then later raise the result to the fourth power.

$$\left(\frac{x^2y}{2w^4z}\right)^4\left(\frac{w^4z^6}{xy}\right)^4=\left(\frac{x^2y}{2w^4z}\cdot\frac{w^4z^6}{xy}\right)^4=\left(\frac{x^{\cancel{2}\,1}\cancel{y}}{2\cancel{w^4}\cancel{z}}\cdot\frac{\cancel{w^4}z^{\cancel{6}\,5}}{\cancel{x}\cancel{y}}\right)^4=\left(\frac{xz^5}{2}\right)^4=\frac{x^4z^{20}}{16}$$

Adding and Subtracting Polynomials

One major objective of working with algebraic expressions is to write them as simply as possible and in a logical, generally accepted arrangement. When more than one term exists (a term consists of one or more factors multiplied together and separated from other terms by + or −), then you check to see whether they can be combined with other terms that are *like* them. Numbers by themselves without letters or variables are *like* terms. You can combine 14 and 8 because you know what they are and know the rules. For instance, 14 + 8 = 22, 14 − 8 = 6, 14(8) = 112, and so on. Numbers can be written so they can combine with one another. They can be added, subtracted, multiplied, and divided, as long as you don't divide by zero. Fractions can be added if they have a common denominator. Algebraic expressions involving variables or letters have to be dealt with carefully. Since the numbers that the letters represent aren't usually known, you can't add or subtract terms with different letters. The expression $2a + 3b$ has to stay that way. That's as simple as you can write it, but the expression $4c + 3c$ can be simplified. You don't know what c represents, but you can combine the terms to tell how many of them you have (even though you don't know what they are!): $4c + 3c = 7c$. Here are some other examples:

$$5ab + 9ab = 14ab$$
$$5x^2y - x^2y + 6xy^2 + 2xy^2 = 4x^2y + 8xy^2$$

Notice that there are two different kinds of terms, one with the x squared and the other with the y squared. Only those that have the letters exactly alike with the exact same powers can be combined. The only thing affected by adding and subtracting these terms is the coefficient.

Example Problems

These problems show the answers and solutions.

1. Simplify $5xy^2 + 8x - 9y^2 - 2x + 3y^2 - 8xy^2$.

   answer: $-3xy^2 + 6x - 6y^2$

   There are three different kinds of terms. First rearrange the terms so that the like terms are together. Then combine the like terms.

$$5xy^2 + 8x - 9y^2 - 2x + 3y^2 - 8xy^2 = 5xy^2 - 8xy^2 + 8x - 2x - 9y^2 + 3y^2 = -3xy^2 + 6x - 6y^2$$

2. Simplify $3x^4 - 2x^3 + x^2 - x + 5 - 2x^2 + 3x^3 + 11$.

   answer: $3x^4 + x^3 - x^2 - x + 16$

   Rearrange the terms so that the like ones are together. By convention, you write terms that have different powers of the same variable in either decreasing or increasing order of their powers.

$$\begin{aligned} 3x^4 - 2x^3 + x^2 - x + 5 - 2x^2 + 3x^3 + 11 &= 3x^4 - 2x^3 + 3x^3 + x^2 - 2x^2 - x + 5 + 11 \\ &= 3x^4 + x^3 - x^2 - x + 16 \end{aligned}$$

Work Problems

Use these problems to give yourself additional practice.

1. Simplify by combining like terms: $m^2 + 3mn + m + 8 + 9mn - m^2 + 14m$.
2. Simplify by combining like terms: $3a + 4b - 6 + 2a - 11$.
3. Simplify by combining like terms: $2\pi r^2 + 8\pi r^2 - 6\pi r + 7$.
4. Simplify by combining like terms: $a + ab + ac + ad + ae + 1$.
5. Simplify by combining like terms: $5x^2y - 2x^2y + x^2y + xy^2 - 8y^2$.

Worked Solutions

1. $\mathbf{12mn + 15m + 8}$ Rearrange the terms so that the terms that can be combined are together.

$$\begin{aligned} m^2 + 3mn + m + 8 + 9mn - m^2 + 14m &= m^2 - m^2 + 3mn + 9mn + m + 14m + 8 \\ &= 12mn + 15m + 8 \end{aligned}$$

2. $\mathbf{5a + 4b - 17}$ $3a + 4b - 6 + 2a - 11 = 3a + 2a + 4b - 6 - 11 = 5a + 4b - 17$

3. $\mathbf{10\pi r^2 - 6\pi r + 7}$ Only the terms with the πr^2 will combine.

4. $\boldsymbol{a + ab + ac + ad + ae + 1}$ This is already simplified. None of the terms have exactly the same variables. There's nothing more to do.

5. $\boldsymbol{4x^2y + xy^2 - 8y^2}$ There are three terms with the same variables raised to the same powers. Be very careful with problems like this.

Multiplying Polynomials

Multiplying polynomials requires that each term in one polynomial multiplies each term in the other polynomial. When a monomial (one term) multiplies another polynomial, the distributive property is used, and the result quickly follows. Multiplying polynomials with more than one term can be very complicated or tedious, but some procedures or methods can be used to provide better organization and accuracy. For instance, to multiply a binomial times another binomial, such as $(x + 2)(a + 3)$, or to multiply a binomial times a trinomial, such as $(x + 2)(x + y + 3)$, you can use the distributive property. Distribute the $(x + 2)$ over the other terms.

$$\begin{aligned}(x + 2)(a + 3) &= (x + 2)(a) + (x + 2)(3)\\ &= x(a) + 2(a) + x(3) + 2(3)\\ &= ax + 2a + 3x + 6\end{aligned}$$

$$\begin{aligned}(x + 2)(x + y + 3) &= (x + 2)(x) + (x + 2)(y) + (x + 2)(3)\\ &= x(x) + 2(x) + x(y) + 2(y) + x(3) + 2(3)\\ &= x^2 + 2x + xy + 2y + 3x + 6\\ &= x^2 + xy + 5x + 2y + 6\end{aligned}$$

None of the terms are alike, so this can't be simplified further.

The FOIL Method

Multiplying two binomials together is a very common operation in algebra. The FOIL method is preferred when multiplying most types of binomials.

The letters in FOIL stand for First, Outer, Inner, and Last. These words describe the positions of the terms in the two binomials. Each term actually will have two different names, because each term is used twice in the process. In the multiplication problem, $(x + 2)(a + 3)$:

The x and the a are the First terms of each binomial.
The x and the 3 are the Outer terms in the two binomials.
The 2 and the a are the Inner terms in the two binomials.
The 2 and the 3 are the Last terms of each binomial.

These pairings tell you what to multiply.

F: multiply $x \cdot a$
O: multiply $x \cdot 3$
I: multiply $2 \cdot a$
L: multiply $2 \cdot 3$

Add these together, $x \cdot a + x \cdot 3 + 2 \cdot a + 2 \cdot 3 = ax + 3x + 2a + 6$. It's the same result, in a slightly different order, as the one obtained previously with distribution.

When using this FOIL method, you'll notice that, when the two binomials are alike—that is they have the same types of terms—the Outer and Inner terms combine, and the result is a trinomial. If they aren't alike, as shown by this last example, then none of the terms in the solution will combine, and you'll have a four-term polynomial.

Example Problems

These problems show the answers and solutions.

1. $(x + 3)(x - 8)$

   answer: $x^2 - 5x - 24$

   Using FOIL,

   F: multiply $x \cdot x$
   O: multiply $x(-8)$
   I: multiply $3 \cdot x$
   L: multiply $3(-8)$

   Add the products together: $x^2 - 8x + 3x - 24 = x^2 - 5x - 24$

2. $(3x - 1)(x - y)$

   answer: $3x^2 - 3xy - x + y$

   Using FOIL,

   F: multiply $3x \cdot x$
   O: multiply $3x(-y)$
   I: multiply $-1 \cdot x$
   L: multiply $-1(-y)$

   Add the products together: $3x^2 - 3xy - x + y$. Notice that none of the terms combine—they're not quite alike.

3. $(3x^2 - 1)(x^2 - 2)$

   answer: $3x^4 - 7x^2 + 2$

   Using FOIL,

   F: multiply $3x^2 \cdot x^2$
   O: multiply $3x^2(-2)$
   I: multiply $-1 \cdot x^2$
   L: multiply $-1(-2)$

   Add the products together: $3x^4 - 6x^2 - x^2 + 2 = 3x^4 - 7x^2 + 2$

Work Problems

Use these problems to give yourself additional practice. Find the products and simplify the answers.

1. $(x-3)(x+7)$
2. $(2y+3)(3y-2)$
3. $(6x-3)(6x-5)$
4. $(ax-1)(bx-2)$
5. $(m^3-3)(m^3+11)$

Worked Solutions

1. $\mathbf{x^2+4x-21}$ Using FOIL, $(x-3)(x+7) = (x-3)(x+7) = x^2+7x-3x-21$. The middle two terms combine.
2. $\mathbf{6y^2+5y-6}$ Using FOIL, $(2y+3)(3y-2) = 6y^2-4y+9y-6$. Again, the middle two terms combine.
3. $\mathbf{36x^2-48x+15}$ Using FOIL, $(6x-3)(6x-5) = 36x^2-30x-18x+15$.
4. $\mathbf{abx^2-(2a+b)x+2}$ At first, it appears that the middle two terms can't combine. If the a and b had been numbers, you could have added them together. They still can be added and the result multiplies the x: $(ax-1)(bx-2) = ax\cdot bx-2ax-bx+2 = abx^2-(2a+b)x+2$.
5. $\mathbf{m^6+8m^3-33}$ Using FOIL, $(m^3-3)(m^3+11) = m^6+11m^3-3m^3-33$.

Special Products

It's nice to have the FOIL method to multiply two binomials together. Unfortunately, no other really handy tricks exist for multiplying other polynomials. Basically, you just distribute the smaller of the two polynomials over the other polynomial. Any polynomials can be multiplied together. The different types of multiplications are classified by the number of terms in the multiplier. Some products, however, are easier to perform because of patterns that exist in them. These patterns largely are due to the special types of polynomials that are being multiplied together. Whenever you can recognize a special situation and can take advantage of a pattern, you'll save time and be less likely to make an error. Here are the special products:

1. $(a+b)(a-b) = a^2-b^2$ — Multiplying the sum and difference of the same two numbers
2. $(a+b)^2 = a^2+2ab+b^2$ or $(a-b)^2 = a^2-2ab+b^2$ — Squaring a binomial
3. $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ or $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$ — Cubing a binomial sum or difference
4. $(a+b)(a^2-ab+b^2) = a^3+b^3$ or $(a-b)(a^2+ab+b^2) = a^3-b^3$ — Resulting in the sum or difference of two cubes

Special Product #1

$$(a + b)(a - b) = a^2 - b^2$$

This equation represents the product of two binomials that have the same two terms, but the terms are added in one binomial and subtracted in the other. It's better known as, "The sum and the difference of the same two numbers." Notice that if FOIL is used here, the Outer product, $-ab$, and the Inner product, ab, are the opposites of one another. This means that they add up to 0, leaving just the first term squared minus the last term squared.

Example Problems

These problems show the answers and solutions.

1. $(x + 8)(x - 8)$

answer: $x^2 - 64$

Just square the first and last terms and take the difference.

2. $(3x + 2z)(3x - 2z)$

answer: $9x^2 - 4z^2$

Again, we have a difference of two squares.

3. $(xyz + 2p^3)(xyz - 2p^3)$

answer: $x^2y^2z^2 - 4p^6$

$(xyz)^2 = x^2y^2z^2$ and $\left(2p^3\right)^2 = 4p^6$

Special Product #2

$$(a + b)^2 = a^2 + 2ab + b^2$$

This product is of a binomial times itself. It's known as, "The perfect square trinomial." The pattern is that the first and last terms in the trinomial are the squares of the two terms in the binomial. The middle term of the trinomial is twice the product of the two original terms. Because the first and last terms are squares, they'll always be positive. The sign of the middle term will depend upon whatever the operation is in the binomial.

Using FOIL on the square of a binomial, $(a + b)^2 = (a + b)(a + b) = a \cdot a + a \cdot b + b \cdot a + b \cdot b = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$.

Using the special product on $(a + b)^2$, first write down the squares of the two terms:

$$a^2 \qquad b^2$$

Then double the product of the two terms, $2ab$. That's the middle term.

$$a^2 + 2ab + b^2$$

Example Problems

These problems show the answers and solutions.

1. $(y + 6)^2$

   answer: $y^2 + 12y + 36$

   The first term is the square of y. The middle term is twice the product of the two terms y and 6. The last term is the square of 6.

2. $(9z - 4)^2$

   answer: $81z^2 - 72z + 16$

   Notice the middle term has a minus sign.

3. $\left(m + \frac{1}{2}\right)^2$

   answer: $m^2 + m + \frac{1}{4}$

   The middle term is twice the product of m and $\frac{1}{2}$: $2 \cdot m \cdot \frac{1}{2} = m$.

Special Product #3

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

or

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Methods exist for finding all powers of a binomial, such as $(a + b)^4$, $(a + b)^7$, $(a - b)^{30}$, and so on. Cubing a binomial is a common task, and the pattern in this special product helps. When the *sum* of two terms (a binomial) is cubed, there's a 1-3-3-1 pattern coupled with decreasing powers of the first term and increasing powers of the second term, and all the terms in the result are added. If there's a *difference* that's being cubed, the only change in the basic pattern is that the terms have alternating signs.

Example Problems

These problems show the answers and solutions.

1. $(1 + z)^3$

   answer: $(1 + z)^3 = 1 + 3z + 3z^2 + z^3$

   Using the 1-3-3-1 pattern as the base, the powers of 1 ($1^3, 1^2, 1^1, 1^0$) are placed with their decreasing powers, right after the numbers. Note that the 1^0 isn't written that way, but just as a 1. Then the increasing powers of z (z^0, z^1, z^2, z^3) are placed with the numbers and 1s. The terms are then simplified.

$$1 \cdot 1^3 + 3 \cdot 1^2 \cdot z^1 + 3 \cdot 1^1 \cdot z^2 + 1 \cdot z^3 = 1 + 3z + 3z^2 + z^3$$

2. $(y - 3)^3$

answer: $y^3 - 9y^2 + 27y - 27$

This time the two terms have their zero-exponents for emphasis—especially on the number −3. This shows how the powers increase with each step. Now, simplify the expression.

$$\begin{aligned}(y-3)^3 &= 1\cdot y^3(-3)^0 + 3\cdot y^2(-3)^1 + 3\cdot y^1(-3)^2 + 1\cdot y^0(-3)^3\\ &= 1\cdot y^3\cdot 1 + 3\cdot y^2(-3) + 3\cdot y^1(9) + 1\cdot 1(-27)\\ &= y^3 - 9y + 27y - 27\end{aligned}$$

As you see, the 1-3-3-1 pattern has disappeared, but using it to build the product was simpler than multiplying this out the "long way" with distribution.

Special Product #4

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3$$

or

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3$$

These products are paired because of their results. These are very specific types of products. The terms in the binomial and trinomial have to be just so. You wouldn't necessarily expect to see such strange combinations of factors to be multiplied except that these wonderful results do occur. Also, these combinations of multipliers will show up again when factoring binomials.

The first multiplier is a binomial. The second multiplier is a trinomial with the squares of the two terms in the binomial in the first and third positions. The middle term of the trinomial multiplier is the opposite of the product of the two terms in the binomial. The result is always the sum or difference of two cubes, and the operation between the two terms in the answer is the same as the operation in the original binomial.

Example Problems

These problems show the answers and solutions.

1. $(x + 2)(x^2 - 2x + 4)$

answer: $x^3 + 8$

Recognizing the pattern—the sum of the cube root of x^3 and the cube root of 8 multiplying the trinomial containing the squares of each of these roots and the opposite of their product—saves time. Here's how it looks if you have to multiply it all out:

$$(x + 2)(x^2 - 2x + 4) = (x + 2)(x^2) + (x + 2)(-2x) + (x + 2)(4)$$
$$= x\cdot x^2 + 2\cdot x^2 + x(-2x) + 2(-2x) + x\cdot 4 + 2\cdot 4 = x^3 + 2x^2 - 2x^2 - 4x + 4x + 8 = x^3 + 8$$

Notice how the four middle terms are opposites of one another and disappear. That's what always happens with this particular product.

2. $(3y - 1)(9y^2 + 3y + 1)$

answer: $27y^3 - 1$

The first term in the trinomial is the square of the first term in the binomial. The last term in the trinomial is the square of the second term in the binomial. The middle term is the opposite of the product of the two terms in the binomial. Since this fits the pattern, the product is the difference of the cubes of the two terms in the binomial.

Work Problems

Use these problems to give yourself additional practice. Find the products and simplify the answers.

1. $(p + 70)(p - 70)$
2. $(2z - 9)^2$
3. $(m + 4)^3$
4. $(5x - 1)^3$
5. $(8 - y)(64 + 8y + y^2)$

Worked Solutions

1. $\mathbf{p^2 - 4{,}900}$ This is the product of the sum and difference of the same two values. The result is always the difference of their squares.

2. $\mathbf{4z^2 - 36z + 81}$ This is a perfect square trinomial. It's the result of multiplying a binomial by itself. The first and last terms are the squares of the two terms in the binomial. The middle term is twice their product.

3. $\mathbf{m^3 + 12m^2 + 48m + 64}$ This is the cube of a binomial. The 1-3-3-1 pattern was used with decreasing powers of the first term and increasing powers of the second term.

$$(m+4)^3 = 1 \cdot m^3 \cdot 4^0 + 3 \cdot m^2 \cdot 4^1 + 3 \cdot m^1 \cdot 4^2 + 1 \cdot m^0 \cdot 4^3 = m^3 + 12m^2 + 48m + 64$$

4. $\mathbf{125x^3 - 75x^2 + 15x - 1}$ This is also the cube of a binomial. This time the binomial is a difference, so there are alternating signs in the answer.

$$\begin{aligned}(5x-1)^3 &= 1 \cdot (5x)^3 \cdot (-1)^0 + 3 \cdot (5x)^2 \cdot (-1)^1 + 3 \cdot (5x)^1 \cdot (-1)^2 + 1 \cdot (5x)^0 \cdot (-1)^3 \\ &= 125x^3 - 75x^2 + 15x - 1\end{aligned}$$

5. $\mathbf{512 - y^3}$ This is the special product that results in the difference between two perfect cubes. The first term in the binomial is 8, and its square, 64, is the first term in the trinomial. The second term in the binomial is $-y$, and its square is y^2. The middle term in the trinomial is the opposite of the product of the two terms in the binomial. So, rather than distributing the binomial over the trinomial, just write this special product as $8^3 - y^3 = 512 - y^3$.

Dividing Polynomials

The method used in the division of polynomials depends on the divisor—the term or terms that divide into the other expression. If the divisor has only one term, then the division can be done by splitting up the dividend (the expression being divided into) into separate terms and making a fraction of each term in the dividend with the divisor in the denominator. If the divisor has more than one term, however, the previous method does not work. You can use a shortcut with some special types of long division, but long division will always work.

Divisors with One Term

Where just one term is in the divisor, split the problem up into fractions and simplify each term.

Example Problems

These problems show the answers and solutions.

1. Divide: $(5y^5 - 10y^4 + 20y^2 - y + 15) \div (5y^2)$.

   answer: $y^3 - 2y^2 + 4 - \frac{1}{5y} + \frac{3}{y^2}$

   The divisor, $5y^2$, will be the denominator of each fraction.

$$\frac{5y^5 - 10y^4 + 20y^2 - y + 15}{5y^2} = \frac{5y^5}{5y^2} - \frac{10y^4}{5y^2} + \frac{20y^2}{5y^2} - \frac{y}{5y^2} + \frac{15}{5y^2} = y^3 - 2y^2 + 4 - \frac{1}{5y} + \frac{3}{y^2}$$

2. Divide: $(6xy - 2x^2y^2 + 8xy^3) \div (2xy^2)$.

   answer: $\frac{3}{y} - x + 4y$

   The divisor will be the denominator of each fraction.

$$\frac{6xy - 2x^2y^2 + 8xy^3}{2xy^2} = \frac{6xy}{2xy^2} - \frac{2x^2y^2}{2xy^2} + \frac{8xy^3}{2xy^2} = \frac{3}{y} - x + 4y$$

Long Division

When the divisor has two or more terms, you can do the operation with long division. This looks very much like the division done on whole numbers. It uses the same setup and operations.

Example Problems

These problems show the answers and solutions.

1. Divide using long division: $(4y^3 - 3y^2 + 2y - 7) \div (y - 2)$.

   answer: $\left(4y^3 - 3y^2 + 2y - 7\right) \div \left(y - 2\right) = 4y^2 + 5y + 12 + \frac{17}{y-2}$

Write the dividend in decreasing powers, leaving spaces for any missing terms (skipped powers).

$$y-2\overline{)4y^3-3y^2+2y-7}$$

Focus on the first term in the divisor, the y. Then determine what must multiply that y so you get the first term in the dividend, the $4y^3$. Multiplying y by $4y^2$ will give you $4y^3$. Write the $4y^2$ above the $4y^3$.

$$\begin{array}{r l} & 4y^2 \\ y-2 & \overline{)4y^3-3y^2+2y-7} \end{array}$$

Now multiply both terms in the divisor by $4y^2$ and put the results under the terms in the dividend that are alike.

$$\begin{array}{r l} & 4y^2 \\ y-2 & \overline{)4y^3-3y^2+2y-7} \\ & \;\;4y^3-8y^2 \end{array}$$

Next, subtract. The easiest way is to change each term in the expression that you're subtracting to its opposite and add. The subtraction of signed numbers is done by changing the sign of the number being subtracted.

$$\begin{array}{r l} & 4y^2 \\ y-2 & \overline{)4y^3-3y^2+2y-7} \\ & \underline{-4y^3+8y^2} \\ & \qquad +5y^2+2y-7 \end{array}$$

The remaining terms in the dividend are brought down and then the process is repeated. Determine what you have to multiply y by to get the new first term. Multiplying by $5y$ will do it.

$$\begin{array}{r l} & 4y^2+5y \\ y-2 & \overline{)4y^3-3y^2+2y-7} \\ & \underline{-4y^3+8y^2} \\ & \qquad +5y^2+2y-7 \\ & \qquad\;\; 5y^2-10y \end{array}$$

Again, change the signs and add.

$$\begin{array}{r l} & 4y^2+5y \\ y-2 & \overline{)4y^3-3y^2+2y-7} \\ & \underline{-4y^3+8y^2} \\ & \qquad +5y^2+2y-7 \\ & \qquad \underline{-5y^2+10y} \\ & \qquad\qquad\quad 12y-7 \end{array}$$

You get the last part of the answer by multiplying the y in the divisor by 12 to get the new first term, the $12y$.

$$\begin{array}{r@{}l} & 4y^2 + 5y + 12 \\ y-2 & \overline{)4y^3 - 3y^2 + 2y - 7} \\ & \underline{-4y^3 + 8y^2} \\ & \quad + 5y^2 + 2y - 7 \\ & \quad \underline{-5y^2 + 10y} \\ & \qquad 12y - 7 \\ & \qquad 12y - 24 \end{array}$$

Do the last subtraction. Whatever is left over is the remainder. The remainder is usually written as a fraction with the divisor in the denominator.

$$\begin{array}{r@{}l} & 4y^2 + 5y + 12 \\ y-2 & \overline{)4y^3 - 3y^2 + 2y - 7} \\ & \underline{-4y^3 + 8y^2} \\ & \quad + 5y^2 + 2y - 7 \\ & \quad \underline{-5y^2 + 10y} \\ & \qquad 12y - 7 \\ & \qquad \underline{-12y + 24} \\ & \qquad\qquad 17 \end{array}$$

2. Use long division to divide $(8m^4 - 3m^2 + m - 12) \div (2m - 1)$.

answer: $4m^3 + 2m^2 - \frac{1}{2}m + \frac{1}{4} - \frac{11\frac{3}{4}}{2m-1}$

In this problem, terms are missing. The m^3 term is missing in the decreasing powers.

$$\begin{array}{r@{}l} & 4m^3 + 2m^2 - \frac{1}{2}m + \frac{1}{4} \\ 2m-1 & \overline{)8m^4 \qquad - 3m^2 + m - 12} \\ & \underline{8m^4 - 4m^3} \\ & \quad 4m^3 - 3m^2 + m - 12 \\ & \quad \underline{4m^3 - 2m^2} \\ & \qquad - m^2 + m - 12 \\ & \qquad \underline{- m^2 + \frac{1}{2}m} \\ & \qquad\qquad \frac{1}{2}m - 12 \\ & \qquad\qquad \underline{\frac{1}{2}m - \frac{1}{4}} \\ & \qquad\qquad\quad -11\frac{3}{4} \end{array}$$

$$(8m^4 - 3m^2 + m - 12) \div (2m - 1) = 4m^3 + 2m^2 - \frac{1}{2}m + \frac{1}{4} - \frac{11\frac{3}{4}}{2m-1}$$

Synthetic Division

Many division problems involving polynomials have special divisors of the form $x + a$ or $x - a$ where the coefficient of the variable is a 1. When this is the case, you can avoid long division and obtain the answer with a process called *synthetic division*. To do the problem $(3x^4 - 2x^3 + 91x + 11) \div (x + 3)$ using synthetic division, use the following steps:

1. Write just the coefficients of the terms in the dividend in decreasing order, inserting 0s for any missing terms (powers).

$$3 \quad -2 \quad 0 \quad 91 \quad 11$$

2. Put the opposite of the constant in the divisor in front of the row of coefficients.

$$\underline{-3}| \; 3 \quad -2 \quad 0 \quad 91 \quad 11$$

3. Draw a line two spaces below the row of coefficients and drop the first coefficient to below the line.

$$\begin{array}{r|rrrrr} -3 & 3 & -2 & 0 & 91 & 11 \\ & & & & & \\ \hline & 3 & & & & \end{array}$$

4. Multiply the constant in front times the dropped number and place the result below the second coefficient in the row. Add the two numbers together and put the result below the line.

$$\begin{array}{r|rrrrr} -3 & 3 & -2 & 0 & 91 & 11 \\ & & -9 & & & \\ \hline & 3 & -11 & & & \end{array}$$

5. Multiply that result times the constant in front, put the result below the third coefficient in the row. Add the two numbers and put the result below the line. Repeat this process until you run out of numbers in the row of coefficients.

$$\begin{array}{r|rrrrr} -3 & 3 & -2 & 0 & 91 & 11 \\ & & -9 & 33 & -99 & 24 \\ \hline & 3 & -11 & 33 & -8 & 35 \end{array}$$

6. Write the answer by using the new coefficients below the line and inserting powers of the variable, starting with a variable that has a power one less than the power of the dividend.

$$(3x^4 - 2x^3 + 91x + 11) \div (x + 3) = 3x^3 - 11x^2 + 33x - 8 + \frac{35}{x+3}$$

In general, synthetic division is much preferred over long division. It takes less time and less space, and you make fewer errors in signs and computations. The operations used are multiplication and addition of the signed numbers. The constant in the divisor is changed to its opposite right at the beginning, taking care of the subtraction process. This next example shows more on inserting the 0s for missed terms.

Example Problems

1. Divide $(5x^6 - 3x^5 + 2x - 7) \div (x + 2)$.

   answer: $5x^5 - 13x^4 + 26x^3 - 52x^2 + 104x - 206 + \frac{405}{x+2}$

   Set up the problem by writing just the coefficients of the terms in a row. Put in zeros for any terms that are missing in the list of decreasing powers of the variable.

   $$\begin{array}{r|rrrrrrr} -2 & 5 & -3 & 0 & 0 & 0 & 2 & -7 \\ & & -10 & 26 & -52 & 104 & -208 & 412 \\ \hline & 5 & -13 & 26 & -52 & 104 & -206 & 405 \end{array}$$

   $$\left(5x^6 - 3x^5 + 2x - 7\right) \div \left(x + 2\right) = 5x^5 - 13x^4 + 26x^3 - 52x^2 + 104x - 206 + \frac{405}{x+2}$$

2. Divide $(4y^3 - 3y^2 + 2y - 7) \div (y - 2)$.

   answer: $4y^2 + 5y + 12 + \frac{17}{y-2}$

   Look at the example for long division on page 35. This is the same problem, except that it is done with synthetic division.

   $$\begin{array}{r|rrrr} +2 & 4 & -3 & 2 & -7 \\ & & 8 & 10 & 24 \\ \hline & 4 & 5 & 12 & 17 \end{array}$$

   Which do you prefer?

Work Problems

Use these problems to give yourself additional practice.

1. Use long division to divide $(8x^4 - 2x^2 + 18x + 6) \div (2x - 1)$.
2. Use long division to divide $(3x^3 + 22x^2 - 7x - 6) \div (3x - 2)$.
3. Use synthetic division to divide $(4x^4 - 2x^3 + 3x^2 - x - 8) \div (x + 2)$.
4. Use synthetic division to divide $(x^4 + 3x + 1) \div (x - 1)$.
5. Use synthetic division to divide $(x^9 + 27x^6 + x + 3) \div (x + 3)$.

Worked Solutions

1. $\mathbf{4x^3 + 2x^2 + 9 + \frac{15}{2x-1}}$ Set up the long division problem. Focus on the first term of the divisor, the $2x$.

$$
\begin{array}{r|l}
 & 4x^3 + 2x^2 \qquad + 9 + \frac{15}{2x-1} \\
\hline
2x - 1 & 8x^4 \qquad - 2x^2 + 18x + 6 \\
 & \underline{8x^4 - 4x^3} \\
 & \quad + 4x^3 - 2x^2 + 18x + 6 \\
 & \quad \underline{4x^3 - 2x^2} \\
 & \qquad\qquad 18x + 6 \\
 & \qquad\qquad \underline{18x - 9} \\
 & \qquad\qquad\quad 15
\end{array}
$$

2. $\mathbf{x^2 + 8x + 3}$

$$
\begin{array}{r|l}
 & x^2 + \ \ 8x \ + 3 \\
\hline
3x - 2 & 3x^3 + 22x^2 - \ \ 7x - 6 \\
 & \underline{3x^3 - \ \ 2x^2} \\
 & \qquad 24x^2 - \ \ 7x - 6 \\
 & \qquad \underline{24x^2 - 16x} \\
 & \qquad\qquad\quad 9x - 6 \\
 & \qquad\qquad\quad \underline{9x - 6} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

In this case, there was no remainder. The binomial $3x - 2$ divided the expression evenly.

3. $\mathbf{4x^3 - 10x^2 + 23x - 47 + \frac{86}{x+2}}$

−2	**4**	**−2**	**3**	**−1**	**−8**
		−8	**20**	**−46**	**94**
	4	**−10**	**23**	**−47**	**86**

$$\left(4x^4 - 2x^3 + 3x^2 - x - 8\right) \div \left(x + 2\right) = 4x^3 - 10x^2 + 23x - 47 + \frac{86}{x+2}$$

4. $\mathbf{x^3 + x^2 + x + 4 + \frac{5}{x-1}}$

1	**1**	**0**	**0**	**3**	**1**
		1	**1**	**1**	**4**
	1	**1**	**1**	**4**	**5**

$$\left(x^4 - 3x + 1\right) \div \left(x - 1\right) = x^3 + x^2 + x + 4 + \frac{5}{x-1}$$

5. $\mathbf{x^8 - 3x^7 + 9x^6 + 1}$

−3	**1**	**0**	**0**	**27**	**0**	**0**	**0**	**0**	**1**	**3**
		−3	**9**	**−27**	**0**	**0**	**0**	**0**	**0**	**−3**
	1	**−3**	**9**	**0**	**0**	**0**	**0**	**0**	**1**	**0**

$$(x^9 + 27x^6 + x + 3) \div (x + 3) = x^8 - 3x^7 + 9x^6 + 1$$