

# CHAPTER 1

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## WAVES—A GENERAL DESCRIPTION

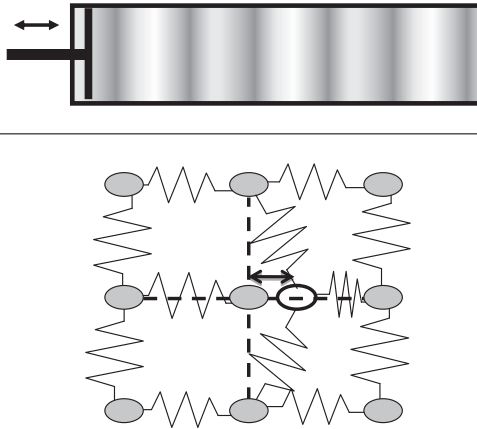
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*Synopsis:* This chapter deals with the basic definitions of waves in general and acoustic waves in particular. The objective of this chapter is to “touch base” with wave properties and their mathematical description. We shall start with a qualitative description and define the various types of waves. We shall examine what properties can be used to describe waves, introduce the wave equation in a homogeneous medium, learn about “group” and “standing” waves, and describe in detail spherical and cylindrical waves. Finally, we shall study the wave equation in a nonhomogeneous medium and the Born and Rytov approximations associated with its solution.

(*Note:* It is likely that many of the items presented in this chapter are trivial or familiar to most readers. Thus, a reader who is well acquainted with the material may hop to the next chapter.)

### 1.1 GENERAL DEFINITIONS OF WAVES— A QUALITATIVE DESCRIPTION

When a mechanical wave propagates through matter, energy is transferred from one location to another [1, 2]. However, this energy is not associated with mass transfer. For example, one may consider the propagating waves formed when moving an end point of a jumping rope or when the first piece from a set of dominoes aligned in a row falls down. Clearly, energy is transferred by motion of matter, but mass is not really transferred from one location to another. The nature of the propagating wave is determined by the type of perturbation causing it to appear but also and mainly by the properties of the



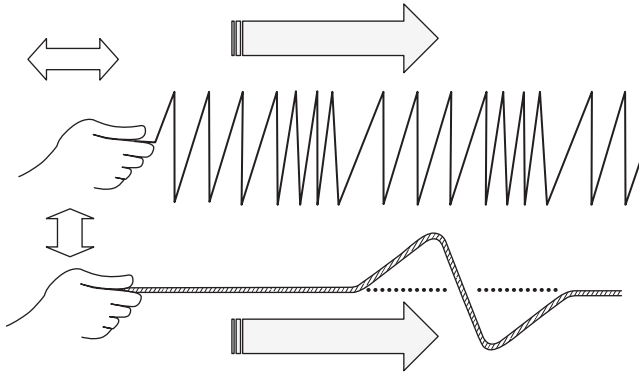
**Figure 1.1.** Schematic depiction of two possible configurations of mechanical wave propagation. **(Top)** The wave propagation direction is limited to only one direction. **(Bottom)** Although the small masses' motion is horizontal, waves can propagate along the vertical direction as well.

media through which it travels. For clarity, let us observe the two examples shown in Fig. 1.1:

**Example 1.** A cylinder filled with gas or fluid has a piston on its left side. At a certain time point the piston is pushed back and forth. Consequently, a pressure wave will propagate through the gas or fluid along the cylinder's axis. In this case the wave may be considered as one-dimensional.

**Example 2.** A matrix consisting of springs and spherical masses is at rest. At a certain time point, one mass is pushed back and forth. As a result, the masses along the line of perturbation will vibrate and a mechanical wave will propagate through the matrix. However, in this case the perturbation will also be associated with deformation of springs from both sides of the moving mass. Hence, one may expect mechanical waves to propagate along directions that are perpendicular to the induced motion as well.

From Example 2, one can conclude that several types of waves, each characterized by different properties, can be generated at the same time and even from the same perturbation. (This issue will be discussed in Chapter 4, where wave propagation in solids is analyzed.) Also, it is important to note that the motion direction of the particles constituting the medium does not have to align with the wave propagation direction. Using this fact, we can apply a preliminary division of wave types based on the relation between the wave propagation direction and the motion direction of the particles constituting the medium (see also Fig. 1.2):



**Figure 1.2.** Division of waves into two types based on the relation between the wave propagation direction and the motion direction of the particles constituting the medium. **(Top)** In a longitudinal wave the particles motion is parallel to the wave propagation direction. **(Bottom)** In a transverse wave the particles motion is perpendicular to the wave propagation direction.

*A Longitudinal Wave*—A wave for which the direction of displacement for the medium's particles is *parallel* to the direction of wave propagation

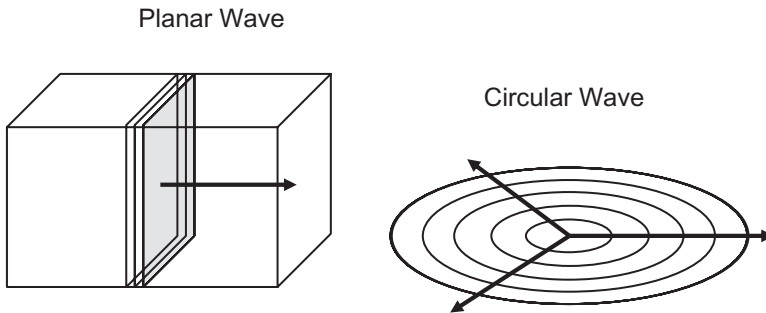
*A Transverse Wave*—A wave for which the direction of displacement for the medium's particles is *perpendicular* to the direction of wave propagation.

This basic division is frequently used in ultrasound, but it is important to note that there are also wave types for which the direction of motion for the medium's particles is not fixed relative to the wave propagation direction. For example, in surface waves (as in sea waves) the angle between the two directions changes continuously.

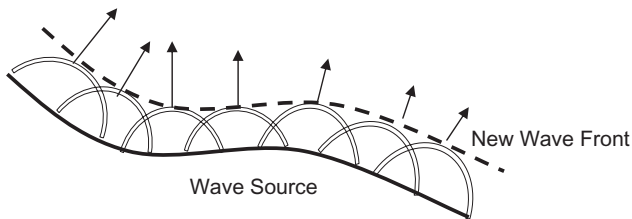
A second and important division is based on the wave front geometry. Using this approach we can again divide the waves into two basic types (Fig. 1.3):

A. *A Planar Wave*—A wave for which the wave front is located on a plane that propagates in space.

B. *A Circular Wave*—A wave that propagates symmetrically around a reference point (as a sphere or a ring), or around a reference line (as a cylinder).



**Figure 1.3.** Division of waves into two types based on the geometry of the wave front. **(Left)** A planar wave. **(Right)** a circular wave.



**Figure 1.4.** Schematic demonstration of the Huygens' Principle implementation to an arbitrarily shaped wave source (or wave front). Each point on the wave front is assumed to emanate a spherical wave (only a few are depicted for clarity). The superposition of all these small spherical waves forms the new wave front.

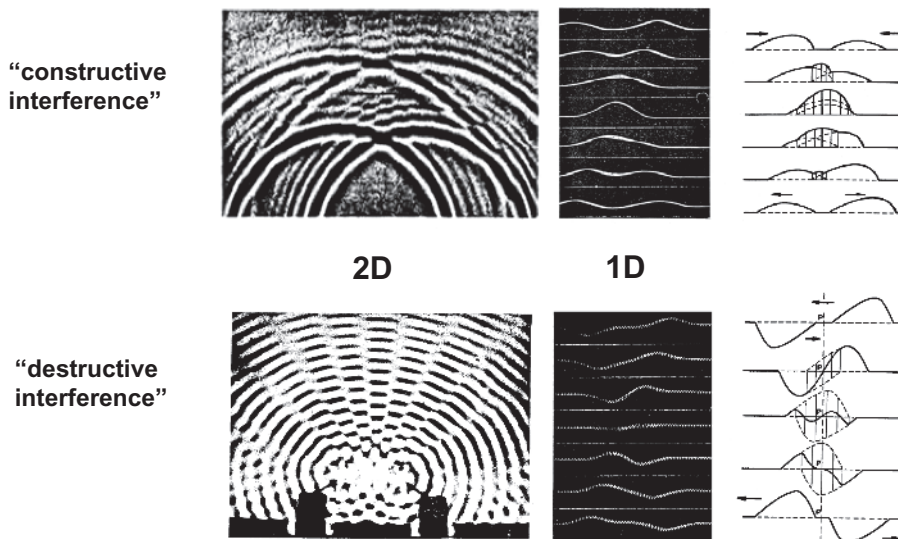
Of course in nature one can encounter also intermediate type of waves. For example, the acoustic beam transmitted from a disc-shaped transducer (see Chapter 8) has planar wave characteristics (particularly around its center). However, as the ratio between the transducer diameter and the wavelength is decreased, the wave will have more and more spherical wave characteristics.

In the general case, one can use Huygens' Principle to investigate the propagating wave front of an arbitrary shape. Huygens' Principle states that any wave source (or wave front for that matter) can be considered as an infinite collection of spherical wave sources [3]. This is schematically demonstrated in Fig. 1.4.

## 1.2 GENERAL PROPERTIES OF WAVES— A QUALITATIVE DESCRIPTION

### 1.2.1 Interference and the Superposition Principle

One interesting property of wave interaction is the interference phenomenon. When two waves collide, they can form “constructive” interference and their



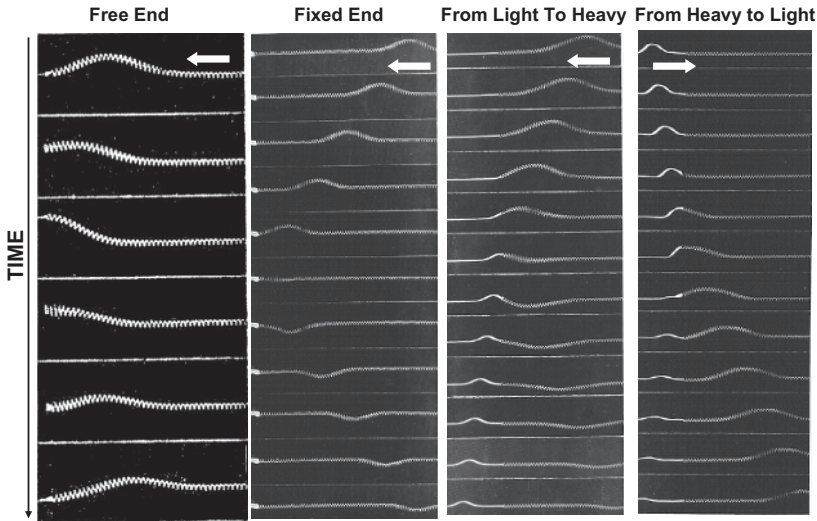
**Figure 1.5. (Top)** Demonstration of constructive interference in a water bath (2D case—*left*) and for strings (1D case—*middle*). For clarity, the 1D case is also schematically depicted on the right side. **(Bottom)** Demonstration of destructive interference. Reprinted with permission from *PSSC Physics*, 7th edition, by Haber-Schaim, Dodge, Gardner, and Shore, Kendall/Hunt Publishing Company, 1991.

amplitudes seem to enhance each other or “destructive” interference and their amplitudes seem to attenuate each other (see Fig. 1.5) [4]. However, after passing through each other, each wave proceeds as though “nothing has happened” (if nonlinear effects are negligible). The superposition principle states that within an interference zone the net amplitude is the sum of all the interacting wave amplitudes.

### 1.2.2 Reflection and Transmission of Waves

When a wave passes from one material to another or when it encounters a discontinuity in the medium in which it travels, part of its energy is reflected and part of it is through-transmitted with or without a change in direction. For an acoustic wave the reflected part is usually referred to as an “echo.” It is worth noting that in ultrasound we distinguish between the waves reflected along a straight line to the source which are referred to as “backscatter” and which are scattered to other directions. The energy partition and the change in wave properties of the transmitted and scattered waves contain a lot of information that can be utilized for imaging or diagnosis (as indicated in the introduction chapter).

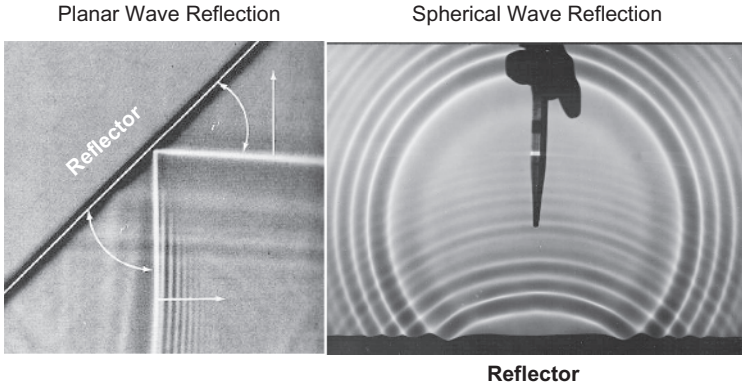
In the one-dimensional (1D) case, when a wave passes from a spring that has certain mechanical properties to another spring that has different



**Figure 1.6.** A demonstration of wave transmission and reflection phenomenon in springs. The arrow indicates the propagation direction of the impinging wave. Each picture indicates a new time frame. **(Left column)** A wave reaching a free end. **(Second column)** A wave reaching a fixed end. **(Third column)** A wave traveling in a light (low density) spring encounters a heavy (high-density) spring. **(Right column)** A wave traveling in a heavy (high-density) spring encounters a light (low-density) spring. Reprinted with permission from *PSSC Physics*, 7th edition, by Haber-Schaim, Dodge, Gardner, and Shore, Kendall/Hunt Publishing Company, 1991.

properties, or when the wave reaches an end point, reflection occurs and, if possible, transmission occurs as well. In Fig. 1.6, four different scenarios of a wave propagating in a spring and encountering a discontinuity point are demonstrated (from left to right): (a) a wave reaching a free end, (b) a wave reaching a fixed end, (c) a wave traveling in a low-density spring reaching a point connecting the spring with a high-density spring, and (d) a wave traveling in a high-density spring reaching a point connecting the spring with a low-density spring. It is important to note that when the wave encounters a “tough” medium, such as a heavy spring or a fixed end point, the phase of its echo changes (i.e., its amplitude changes signs). Also, it is important to note that when the wave travels from a heavy spring to a light one, its amplitude is *increased* (energy, however, is conserved as will be explained in the next chapter).

In the two-dimensional (2D) case and naturally in the three dimensions (3D), one has to distinguish between circular and planar waves. In the case of a planar wave, the rule is simple: The angle of reflection from a reflector is equal to the angle of incidence. However, for a circular wave (e.g., spherical or cylindrical), things are a little bit more complicated. The reflected wave seems to emanate from a virtual source which is located on the other side of the reflector (see Fig. 1.7).



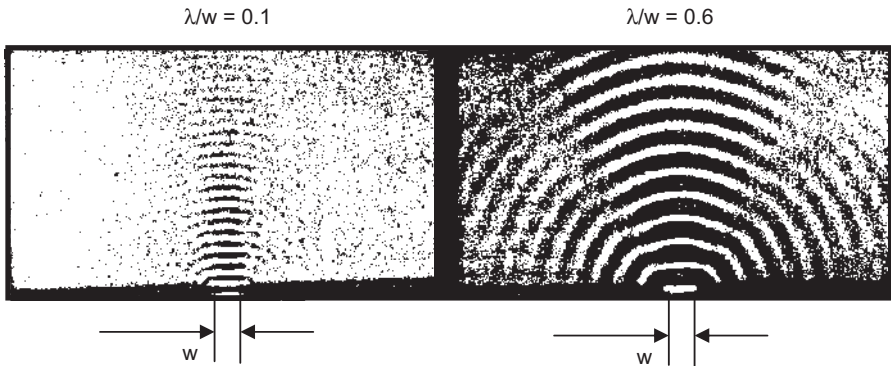
**Figure 1.7.** A demonstration of the reflection phenomenon for a planar wave (**left**) and for a circular wave (**right**). Reprinted with permission from *PSSC Physics*, 7th edition, by Haber-Schaim, Dodge, Gardner, and Shore, Kendall/Hunt Publishing Company, 1991.

### 1.2.3 Diffraction

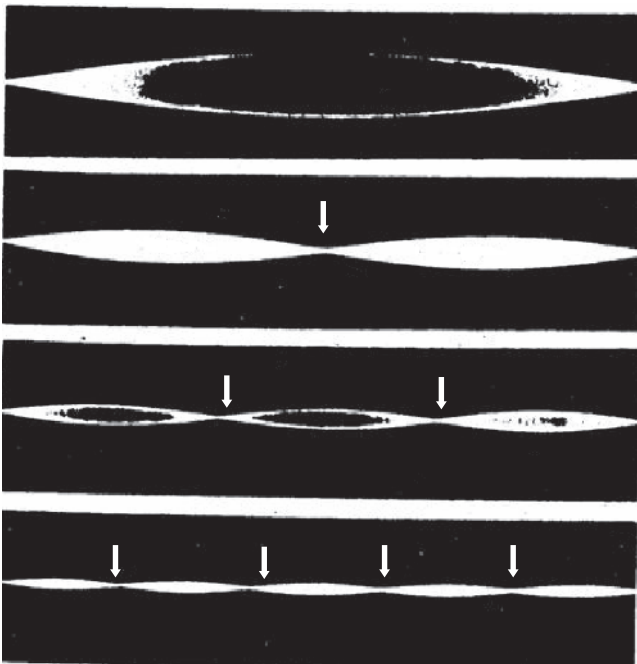
Another phenomenon that is associated with waves is “diffraction.” Waves that impinge upon a corner or pass through a slot in a screen bend their trajectory and propagate into zones that should have been “shadowed.” This phenomenon is enhanced for wavelengths that are relatively long relative to the geometry of the obstacle. Diffraction may be significant in medical ultrasound since the typical wavelengths are in the scale of a millimeter (for example, for a frequency of 1 MHz, the wavelength is about 1.5 mm in soft tissues), and so is the needed resolution. The diffraction phenomenon is demonstrated in Fig. 1.8 for clarity.

### 1.2.4 Standing Waves

Another phenomenon that needs to be noted is the appearance of “standing” waves. When waves rush back and forth within a confined medium (e.g., when strong reflectors are located at each end point of the medium), the waves interfere with each other and with their reflections. Consequently, a motion pattern is formed which is spatially fixed but temporally changes its amplitude (see, for example, Fig. 1.9). This pattern is referred to as a “standing” wave, since an illusion is formed where the wave seems to stand still while its amplitude changes. This type of wave is most probably observed in string-based musical instruments. They may be noticed in vibrating strings and in resonance boxes. In medical applications, this type of wave may be significant when considering the design of instruments and particularly the design of instruments that use constant wave (CW) transmission.



**Figure 1.8.** Demonstration of the diffraction phenomenon for a planar wave propagating from the bottom toward the top and passing through a slot for which width is  $W$ . The ratio between the wavelength  $\lambda$  and the slot width  $W$  is 0.1 for the case shown on the left and 0.6 for the case shown on the right. As can be noted, the phenomenon is much more enhanced on the right side. Reprinted with permission from *PSSC Physics*, 7th edition, by Haber-Schaim, Dodge, Gardner, and Shore, Kendall/Hunt Publishing Company, 1991.

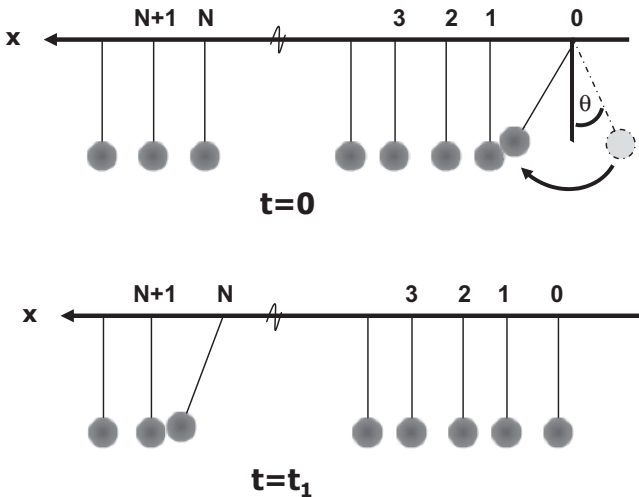


**Figure 1.9.** A demonstration of the standing wave phenomenon. In these four pictures taken with a long exposure time, four vibration modes for a string tied at both end points are depicted. As can be noted, certain nodal points (marked by the arrows) do not move. Reprinted with permission from *PSSC Physics*, 7th edition, by Haber-Schaim, Dodge, Gardner, and Shore, Kendall/Hunt Publishing Company, 1991.

### 1.3 MECHANICAL ONE-DIMENSIONAL WAVES

Consider the familiar entertaining device that consists of a set of pendulums made from small metal balls tied in a compact row as shown in Fig. 1.10. At time point  $t = -\Delta t_0$  the first pendulum (marked as #0) is tilted by an angle  $\theta_0$  and is allowed to swing back. The initial potential energy of its metal ball will be converted into both kinetic and potential energy as the ball swings toward its neighbor. At time  $t = 0$  the ball will hit the ball of pendulum #1 and the impact will transfer some of its energy. The ball of pendulum #0 will then swing slightly (if there is a gap between adjacent balls) and resume its original equilibrium position. The ball of pendulum #1 will hit the ball of pendulum #2 and in turn will transfer to it some of its energy and so forth.

If we would take a camera and photograph the set of pendulums at time  $t = 0$  and at time  $t = t_1$ , we would be able to notice that there are “calm” regions (i.e., where the balls are in their equilibrium position) and the are “stormy” regions where the balls swing back and forth. The “stormy” region that was created by the initial perturbation propagates along the medium (toward its left side in the case shown in Fig. 1.10). Intuitively, we can relate this experiment to waves. Hence, we shall refer to the propagating perturbation of pendulums from their equilibrium state as a “wave.” Comparing our observation to the definition given in the previous sections, we shall indeed note that it suits our criteria. We have in this example a propagating mechanical energy that propagates through matter and that is not associated with mass transfer



**Figure 1.10.** A demonstration of a propagating mechanical wave in a “one-dimensional” medium. The perturbation from equilibrium which is induced at time  $t = 0$  by swinging the first pendulum will propagate and eventually reach the  $N$ th pendulum at time  $t = t_1$ .

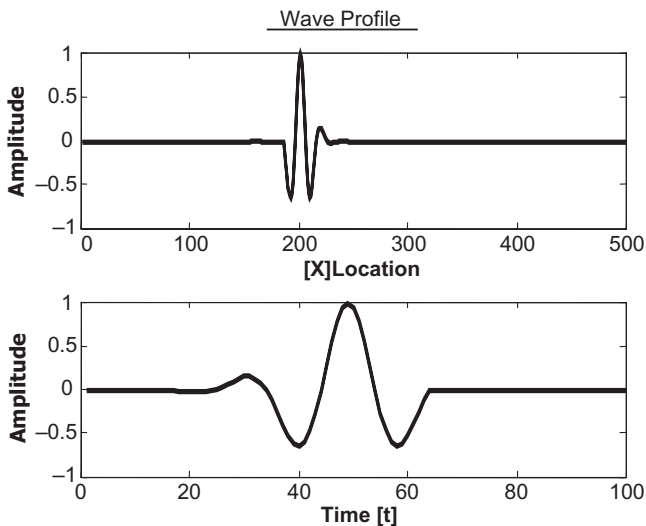
and that stems from a perturbation induced to the medium. This perturbation, which we refer to as a “wave,” varies with time and space and therefore can be designated in the most general case by the term

$$U = U(x, y, z, t) \quad (1.1)$$

where the function  $U$  can designate any physical property that characterizes the propagation of the propagating perturbation. In the example given above,  $U$  could express the pendulum angle relative to its equilibrium state, the metal ball velocity, or its energy, and so forth. In the one-dimensional case the wave can be represented by

$$U = U(x, t) \quad (1.2)$$

If we set the time to be constant and we plot  $U$  as a function of location, we shall obtain a profile of the property described by  $U = U(x, t = \text{const})$ . On the other hand, we can set the location to be a constant and plot the profile described by  $U = U(x = \text{const}, t)$ . Comparing the two obtained plots, we shall realize that the two plots characteristics are identical except for the reverse in directions and the natural change in scaling. This plot is called the “wave profile” (see Fig. 1.11).



**Figure 1.11.** A demonstration of a wave profile when plotted as a function of location for a given time point (**top**) or as a function of time for a given spatial location (**bottom**). As can be noted, the two profile characteristics are identical except for scaling and directions.

### 1.4 THE WAVE FUNCTION

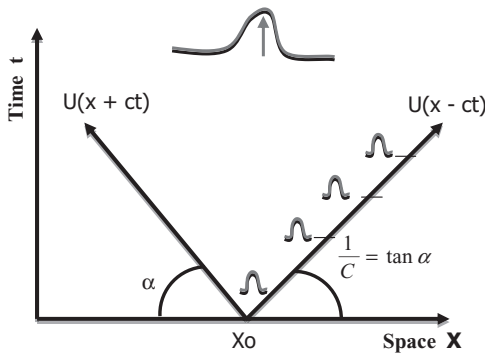
Generally speaking, a wave propagating along the positive direction of the  $X$  axis can be described by the general function [5]

$$\begin{array}{c} \longrightarrow \\ U(x, t) = U(x - ct) \end{array} \tag{1.3}$$

whereas a wave propagating along the negative direction of the  $X$  axis can be described by

$$\begin{array}{c} \longleftarrow \\ U(x, t) = U(x + ct) \end{array} \tag{1.4}$$

In order to validate this statement, let us examine a function  $U(x, t) = U(x - ct)$  representing a wave that has a profile of maximal amplitude  $U_0$  and a single peak, such as shown for example in the following icon:  $U_0 \left[ \text{peak} \right]$ . At time  $t = 0$  the peak of the wave profile is found at some point  $x_0$ , that is,  $U(x_0, t = 0) = U_0$ . Clearly, if for some other point  $x$  in time  $t = \Delta t$  we can find a location for which  $x - c \cdot \Delta t = x_0$ , its value must be  $U(x - c \cdot \Delta t) = U(x_0, 0) = U_0$ . However, since by definition we have  $C > 0$ , and time is always positive ( $t > 0$ ), it follows that the value of  $x$  which fulfills the condition  $x - c \cdot \Delta t = x_0$  must continuously increase. Or in other words, the peak of the wave profile must move along the positive direction of  $x$ . This can also be demonstrated graphically by plotting the location of the wave profile peak (or any other landmark for that matter) in the plane  $(x, t)$ . There it will move along a straight line given by  $x - x_0 = c \cdot \Delta t$  and whose slope is  $1/c = \Delta t / (x - x_0)$  as shown in Fig. 1.12. Similarly, it can be shown that  $U(x + ct)$  describes a wave that propagates along the negative direction of  $x$ .



**Figure 1.12.** Propagation of waves defined by the general function  $U(x \mp ct)$  in the space–time  $(x, t)$  plane.

## 1.5 THE WAVE EQUATION

The wave function  $U(x \pm ct)$  described above is related to a differential equation for which it can serve as a solution. This equation is defined as the “wave equation” and for the one-dimensional (1D) case in a homogeneous medium is given by [5]

$$\frac{\partial^2 U}{\partial X^2} = \frac{1}{C^2} \cdot \frac{\partial^2 U}{\partial t^2} \quad (\text{one dimension}) \quad (1.5)$$

We can prove this statement for the function  $U(x - ct)$ . Let us define a new parameter  $x - ct \equiv g$  and its corresponding derivative as  $\frac{\partial u}{\partial g} = U'$ . Using the chain rule, we can write the relation  $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial g} \cdot \frac{\partial g}{\partial x}$  and also the relation  $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial g} \cdot \frac{\partial g}{\partial t}$ . And since  $\frac{\partial u}{\partial t} = -C \cdot \frac{\partial u}{\partial g}$ , it is easy to see that the relation

$$U'' = \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

is valid.

For the three-dimensional (3D) case we can use the term  $U = U(\hat{n} \cdot \bar{r} - ct)$  to describe the wave, where  $\hat{n}$  is a unit vector along the wave propagation direction and  $\bar{r}$  is the spatial location vector. The corresponding wave equation in this case is given by

$$\nabla^2 U = \frac{1}{C^2} \cdot \frac{\partial^2 U}{\partial t^2} \quad (\text{three dimensions}) \quad (1.6)$$

## 1.6 HARMONIC WAVES

Each wave can be represented using the Fourier transform by a series of harmonic waves. Harmonic waves are waves with a defined periodic profile (such as sine or cosine functions). These types of waves are mathematically convenient to work with and can be described (for the 1D case) by

$$u = A e^{j[\omega t - kx]} = A [\cos(\omega t - kx) + j \sin(\omega t - kx)] \quad (1.7)$$

or more generally (for 3D) by

$$u = Ae^{j[\omega t - \bar{k} \cdot \bar{R}]} \quad (1.8)$$

where the wave temporal (also called angular) frequency is given by

$$\omega = 2\pi f = \frac{2\pi}{T} \quad (1.9)$$

and the wave spatial frequency is given by

$$\bar{k} \triangleq k \cdot \hat{n} \quad (1.10)$$

where  $\hat{n}$  is a unit vector along the wave propagation direction and  $k$  is called the “wave number.” The parameter  $k$  is required for two reasons:

- (a) To cancel the physical dimension of the exponential term.
- (b) To ensure the periodicity of the function with a wavelength of  $\lambda$ .

The value of  $k$  can be determined by studying a simple harmonic wave, for example:

$$u = A[\cos(\omega t - kx + \varphi)] \quad (1.11)$$

where  $\varphi$  represents the phase of the wave. Naturally, a peak of the wave profile will appear at time  $t = 0$  wherever the following relation applies:

$$\cos(\varphi - kx) = 1 \quad (1.12)$$

Thus, for two consecutive peak points  $x_n$  and  $x_{n+1}$  on the wave profile at time  $t = 0$ , the following relations must apply:

$$\varphi - kx_n = n \cdot 2\pi \quad (1.13)$$

$$\varphi - kx_{n+1} = (n+1) \cdot 2\pi \quad (1.14)$$

By subtracting the two equations, we shall obtain

$$k(x_{n+1} - x_n) = 2\pi \quad (1.15)$$

And since the distance between two adjacent peak points for a harmonic wave equals the wavelength  $\lambda = c/f$ , it follows that

$$\boxed{k = \frac{2\pi}{\lambda}} \quad (1.16)$$

It is important to note that the physical units of  $k$  are radians/length. Hence, it is actually a parameter defining spatial frequency. The full spatial frequency spectrum of a given wave can be obtained by applying the Fourier transform to the wave profile at a given time point.

### 1.6.1 Equivalent Presentations

It is also worth noting that harmonic waves can be presented mathematically using alternative arguments. However, the physical interpretation is the same. Three exemplary presentations are given below:

$$U = A \cos \left[ 2\pi \left( \frac{x}{\lambda} - \frac{t}{T} \right) + \varphi \right], \quad (1.17)$$

$$U = A \cos \left[ \frac{2\pi}{\lambda} (x - ct) + \varphi \right], \quad (1.18)$$

$$U = A \cos \left[ 2\pi \left( \frac{x}{\lambda} - ft \right) + \varphi \right] \quad (1.19)$$

## 1.7 GROUP WAVES

When a group of waves travel together in a medium, their amplitudes and phases interfere with each other and a pattern is formed which seems to propagate in a speed that differs from the speed of its components. In order to demonstrate this phenomenon, let us consider the simplest case where two waves having the same amplitude, but with different temporal (angular) and spatial frequencies, travel together. For example, if the first wave is given by

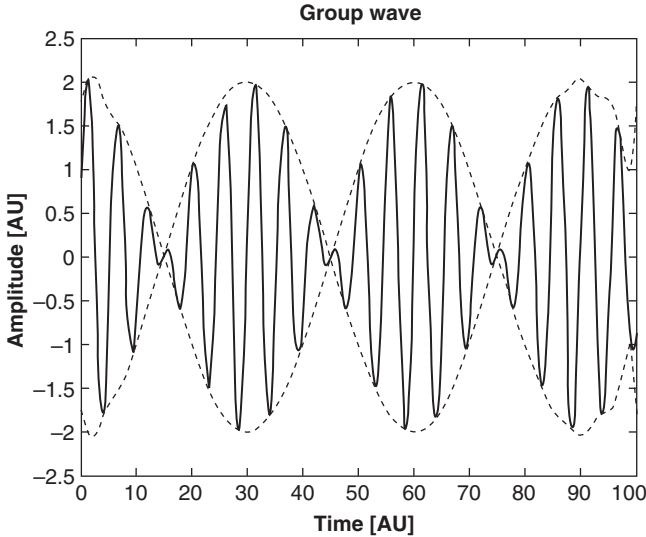
$$U_1 = A \cos(k_1 x - \omega_1 t) \quad (1.20)$$

and the second wave that accompanies it is given by

$$U_2 = A \cos(k_2 x - \omega_2 t) \quad (1.21)$$

then the pattern given in Fig. 1.13 is obtained. Marking the envelope of this combination, it would appear as though the envelope travels as a new wave in the medium. This pattern is called a “group” wave and is given mathematically for this specific example by

$$U = 2A \left\{ \underbrace{\cos \left[ \left( \frac{k_1 + k_2}{2} \right) x - \left( \frac{\omega_1 + \omega_2}{2} \right) t \right]}_{\text{High-frequency components}} \cdot \underbrace{\cos \left[ \left( \frac{k_1 - k_2}{2} \right) x - \left( \frac{\omega_1 - \omega_2}{2} \right) t \right]}_{\text{Low-frequency components}} \right\} \quad (1.22)$$



**Figure 1.13.** A group wave obtained by combining two sinusoidal waves of different frequencies. (The envelope is depicted by the dashed line for clarity.)

As can be noted, the combined pattern is represented by a multiplication between a high-frequency-components (both spatial and temporal) wave (the carrier wave) and a low-frequency-components (modulating) wave.

## 1.8 WAVE VELOCITY

Wave velocity is not so trivial to define. The more intuitive concept is to track the speed at which the energy carried by the wave propagates. This can be estimated by tracking a specific point of reference on the envelope of a propagating wavepacket (for a group wave). Such point of reference could be, for example, the front edge of the profile envelope or its point of maximal value. In the example given in Fig. 1.10, we can refer to the ball located on the left side of the “stormy” region as the front end of the wave envelope. By measuring the propagation  $\Delta x$  of this landmark at time interval  $\Delta t$ , the wave velocity can be estimated from

$$c = \frac{\Delta x}{\Delta t} \quad (1.23)$$

This type of wave propagation speed estimation by envelope amplitude tracking is called “group velocity.” For a group comprised of a set of harmonic waves, the group velocity  $c_g$  is given by

$$c_g = \frac{\partial \omega(k)}{\partial k} \quad (1.24)$$

In case of a single-frequency wave, one can track the phase propagation speed. This can be done, for example, by selecting any point of reference on the wave profile. In the case of a wavepacket (a group wave), one can track the phase of one of its high frequency components. In such a case the obtained value is defined as the “phase velocity”  $c_p$  and is given by

$$c_p = \frac{\omega(k)}{k} \quad (1.25)$$

If dispersion is negligible or not existent (such as the case for electromagnetic waves in vacuum), the angular (temporal) frequency and the wave number are linearly related, that is,

$$\omega = c \cdot k \quad (1.26)$$

Hence for this case the group velocity and the phase velocity are the same, that is,  $c_g = c_p$ . If the relation between the angular (temporal) frequency and the wave number is not linear, the speed of sound varies with the wave frequency. This case is discussed in Chapter 5 (Section 5.5).

## 1.9 STANDING WAVES (A MATHEMATICAL DESCRIPTION)

As was presented in Section 1.2.4, the standing wave phenomenon occurs when a string (or some other type of bounded medium such as a beam or a tube) fixed on both sides vibrates. While most of the particles on the string (or medium) move periodically, there are specific points, called “nodal points,” that do not move (as shown in Fig. 1.9). This phenomenon is explained by the existence of two waves of the same amplitude and frequency propagating in the same medium toward opposite directions. Such waves can be generated by the reflections caused by the boundaries (see Fig. 1.6). As a result, the two waves continuously interfere with each other. For a specific length of the medium and for specific frequencies (see Chapter 2), destructive interference occurs constantly at the nodal points and prevent them from moving.

Let us define the wave propagating along the positive  $x$  direction as  $U_1$ ,

$$U_1 = A \cos(kx - \omega t) \quad (1.27)$$

and define the wave propagating along the negative  $x$  direction as  $U_2$ ,

$$U_2 = A \cos(kx + \omega t) \quad (1.28)$$

The resulting displacements within the medium will be given by  $U = U_1 + U_2$ . And recalling the trigonometric relation,

$$\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cdot \cos\left(\frac{\alpha - \beta}{2}\right) \quad (1.29)$$

we obtain

$$U = U_1 + U_2 = \underbrace{2A \cos(kx)}_{X(x)} \cdot \underbrace{\cos(\omega t)}_{\Theta(t)} \quad (1.30)$$

or, in a more general form,

$$U = X(x) \cdot \Theta(t) \quad (1.31)$$

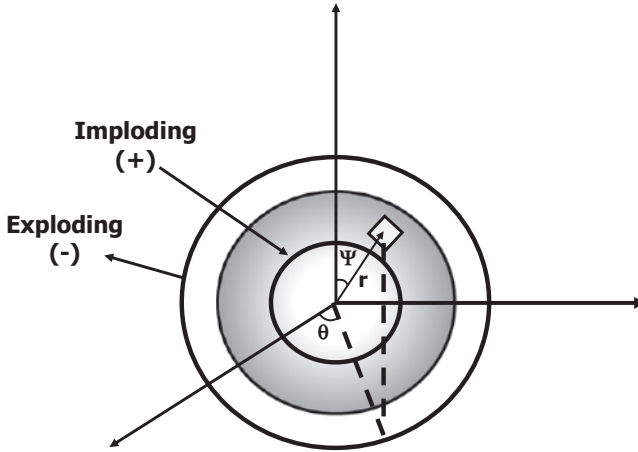
As can be noted, the combined displacement pattern is comprised of a multiplication between two functions: (i) the function  $\Theta(t)$ , which depends solely on time and which modulates only the wave amplitude, and (ii) the function  $X(x)$ , which depends solely on location and exhibit a *constant profile* in space. And since at the nodal points we have  $X(x) = 0$ , no displacement will occur at these points, and hence we have the resulting impression of a “standing wave.” (This phenomenon will be discussed again in Chapter 2.)

## 1.10 SPHERICAL WAVES

In daily life we occasionally encounter waves with circular symmetry (see Fig. 1.14). Most commonly, we associate such waves with the ripples pattern formed when a pebble is thrown into a water pond. In such cases the symmetry is two-dimensional (2D).

However, waves with three-dimensional (3D) symmetry which propagate as a sphere in space (and are therefore referred to as “spherical” waves) are also very common. In fact, such waves are created from almost any noise- or stress-generating event. For example, the noise generated when two objects collide, generated by a ringing bell, or generated by an explosion can all produce spherical waves. If the source for the waves may be considered small enough relative to the wavelength, we can assume that the wave stems from a single-point source and hence may be considered as spherical. Moreover, Huygens’ Principle (see above) states that any wave source can be considered as an infinite collection of spherical wave sources.

A spherical wave is characterized by 3D spatial symmetry around its source (in an isotropic medium). Therefore it is natural to describe this type of wave in a spherical coordinate system. Defining the function describing the wave as  $\phi$ , the three-dimensional wave equation is given by



**Figure 1.14.** Schematic depiction of a spherical wave spreading symmetrically from (or toward) a central point.

$$\nabla^2 \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} \tag{1.32}$$

Rewriting the left-hand side of Eq. (1.32) in a spherical coordinate system  $r, \theta, \Psi$ , we obtain [6]

$$\nabla^2 \varphi \triangleq \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \cdot \frac{\partial^2 \varphi}{\partial \Psi^2} \tag{1.33}$$

And since function  $\varphi$  should be symmetric relative to  $\theta$  and  $\Psi$ , it follows that

$$\frac{\partial^2 \varphi}{\partial \Psi^2} = 0 \tag{1.34}$$

$$\frac{\partial \varphi}{\partial \theta} = 0 \tag{1.35}$$

And therefore we shall obtain

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{2}{r} \frac{\partial \varphi}{\partial r} = \frac{1}{r} \left( \frac{\partial^2 (r\varphi)}{\partial r^2} \right) = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} \tag{1.36}$$

It can be easily shown that a solution to this wave equation is given by the generic function:

$$\varphi = \frac{A}{r} f\left(t \pm \frac{r}{c}\right) \quad (1.37)$$

where the “+” sign designates an imploding wave (i.e., propagating toward the central point) and the “-” sign designates an exploding wave.  $A$  is the wave amplitude, whose units are the physical units of  $\varphi$  (pressure, for example) multiplied by length.

This can be shown by substitution of Eq. (1.37) into the left-hand side of Eq. (1.36) (for simplicity we shall assume that the amplitude  $A = 1$ ),

$$\nabla^2 \varphi = \frac{1}{r} \left( \frac{\partial^2 (r\varphi)}{\partial r^2} \right) = \frac{1}{r} \left( \frac{\partial^2 \left[ f\left(t \pm \frac{r}{c}\right) \right]}{\partial r^2} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( \pm \frac{1}{c} f' \right) = \frac{1}{rc^2} f'' \quad (1.38)$$

while from Eq (1.32) we shall obtain the same term,

$$\nabla^2 \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{c^2} \cdot \frac{1}{r} \cdot \frac{\partial^2 f}{\partial t^2} = \frac{1}{c^2} \cdot \frac{1}{r} \cdot f'' \quad (1.39)$$

It should be pointed out that  $f\left(t \pm \frac{r}{c}\right)$  is equivalent to  $\tilde{f}(ct \pm r)$ . But it is very important to note that unlike a planar wave, the amplitude of a spherical wave *decays* with the distance from the source according to  $\frac{1}{r}$ , even when attenuation is negligible.

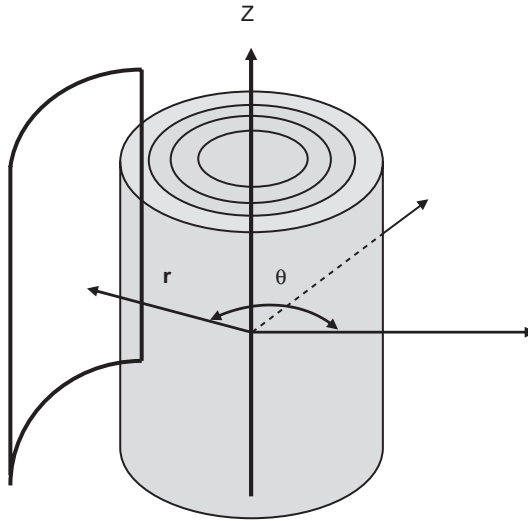
## 1.11 CYLINDRICAL WAVES

Cylindrical waves are those that propagate from a line source and that keep angular symmetry around the source axis. (The source could, for example, be an antenna pole.) A schematic depiction of a cylindrical wave front propagating from or toward a line of symmetry (in this case the  $Z$  axis) is shown in Fig. 1.15.

Rewriting the left-hand side of Eq. (1.32) in a cylindrical coordinate system  $r, z, \theta$ , we obtain [6]

$$\nabla^2 \varphi \triangleq \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} \quad (1.40)$$

The symmetry around  $Z$  and  $\theta$  yields



**Figure 1.15.** Schematic depiction of a cylindrical wave front propagating symmetrically from (or toward) a central line ( $Z$  axis in this case).

$$\frac{\partial^2 \varphi}{\partial z^2} = 0, \quad \frac{\partial^2 \varphi}{\partial \theta^2} = 0 \tag{1.41}$$

And therefore we obtain

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) \tag{1.42}$$

Substituting Eq. (1.42) into Eq. (1.32), we obtain the wave equation for this type of wave:

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} = \frac{1}{C^2} \frac{\partial^2 \varphi}{\partial t^2} \tag{1.43}$$

Multiplying both sides by  $r^2$  and moving all terms to the left-hand side yields

$$r^2 \frac{\partial^2 \varphi}{\partial r^2} + r \frac{\partial \varphi}{\partial r} - \frac{r^2}{C^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{1.44}$$

To find a solution, we shall assume that the space–time relationship could be presented by two functions multiplied by each other, that is,

$$\varphi = e^{i\omega t} \cdot \psi(r) \tag{1.45}$$

Thus, it follows that

$$\frac{\partial^2 \varphi}{\partial t^2} = -\omega^2 \cdot \varphi \quad (1.46)$$

And if we recall that

$$\frac{\omega^2}{C^2} = \frac{(2\pi f)^2}{(\lambda f)^2} = k^2 \quad (1.47)$$

we shall obtain

$$r^2 \frac{\partial^2 \varphi}{\partial r^2} + r \frac{\partial \varphi}{\partial r} + r^2 k^2 \varphi = 0 \quad (1.48)$$

As can be noted, this equation is actually a Bessel differential equation of order zero. The solution for this equation is given by the Hankel function of the first kind and of order zero  $H_0^1(kr)$ :

$$\varphi = H_0^1(kr) = A \cdot J_0(kr) + B \cdot Y_0(kr) \quad (1.49)$$

where  $J_0(kr)$  and  $Y_0(kr)$  are Bessel functions of the first and second types respectively, and  $A$  and  $B$  are two constants. It should be noted that this solution is a complex function.

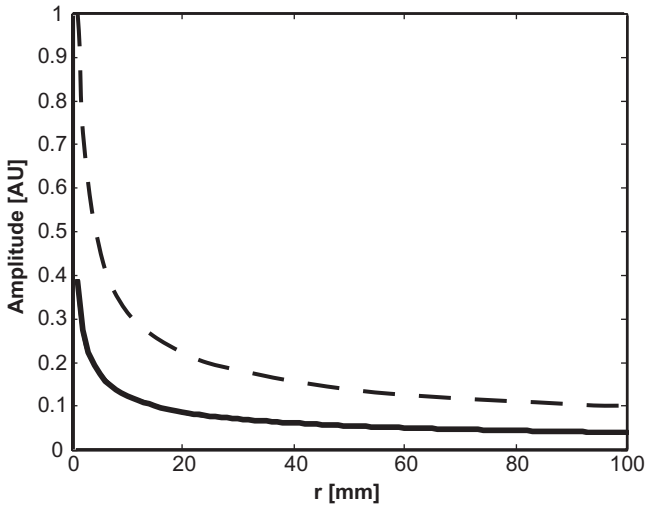
For simplicity the following approximation can be used:

$$\varphi \approx \frac{D}{\sqrt{r}} \psi(\omega t \pm kr) \quad (1.50)$$

The exact solution (absolute value) and its approximation are graphically compared in Fig. 1.16. The solutions were calculated for an expanding cylindrical wave propagating in water ( $C = 1500$  m/sec) with a frequency of 1 MHz. As can be observed, the approximated solution follows the pattern of the exact solution (particularly for large  $kr$  values); however, it has an overestimating offset (which could be partially reduced by subtracting a constant).

## 1.12 THE WAVE EQUATION IN A NONHOMOGENEOUS MEDIUM

Thus far we have assumed that the medium in which the waves propagate is uniform. However, in the majority of cases this assumption is not true, because the body consists of many complicated structures of tissues and blood vessels.



**Figure 1.16.** A graphic comparison between the absolute value of the exact solution (solid line) and the approximated solution (dashed line) for a cylindrical wave propagating with a frequency of 1 MHz in water. (Note: In order to avoid the singularity at the axes origin, the calculated range started at  $r = 1$  mm.)

Indeed the assumption of homogeneity is practically convenient; however, in certain applications (e.g., acoustic computed tomography), inhomogeneity has to be accounted for. Inhomogeneity can stem from spatial changes in density or elastic properties; however, since these properties affect the speed of sound, we may consider the speed of sound as the significant variable. Without affecting the generality of the following discussion, we shall consider only one harmonic wave with a given frequency in the equations (the reader should bear in mind that any wave can be expanded into a series of “monochromatic” harmonic waves using the Fourier transform). Consequently, the changes in the speed of sound will be manifested in changes of the wavelength. This spatial variation can thus be presented by the wave number  $k(\vec{r})$ , where the vector  $\vec{r}$  represents the spatial coordinates. Following the derivation presented in Kak and Slaney [7], the wave equation in a nonhomogeneous medium is given by

$$[\nabla^2 + k(\vec{r})^2]u(\vec{r}) = 0 \quad (1.51)$$

where  $u(\vec{r})$  is a function that represents the acoustic field. Using, for example, water (whose acoustic properties are close to that of soft tissues) as a reference medium, this equation can be rewritten as

$$[\nabla^2 + k_0^2]u(\vec{r}) = -k_0^2 [n(\vec{r})^2 - 1]u(\vec{r}) \quad (1.52)$$

where  $k_0$  is the wave number for that reference medium, and  $n(\bar{r})$  is the corresponding refractive index defined by

$$n(\bar{r}) = \frac{C_0}{C(\bar{r})} \quad (1.53)$$

where  $C_0$  is the speed of sound in the reference medium. Commonly, in the context of soft tissues, the changes in the speed of sound are minor. Hence, we may represent the object by a function  $F(\bar{r})$  that describes the changes as a perturbation with respect to the reference medium. Thus, the function describing the scanned object is given by

$$F(\bar{r}) = k_0^2 [n(\bar{r})^2 - 1] \quad (1.54)$$

The acoustic field  $u(\bar{r})$  within the medium may be presented by two components,

$$u(\bar{r}) = u_0(\bar{r}) + u_s(\bar{r}) \quad (1.55)$$

where  $u_0(\bar{r})$  is the (transmitted) incident field, which is the solution for the homogeneous wave equation in the reference medium,

$$[\nabla^2 + k_0^2]u_0(\bar{r}) = 0 \quad (1.56)$$

and the field  $u_s(\bar{r})$  describes the scattered field. Substituting into Eq. (1.52) yields

$$\boxed{[\nabla^2 + k_0^2]u_s(\bar{r}) = -F(\bar{r})u(\bar{r})} \quad (1.57)$$

This equation is called the Helmholtz equation. The object function cannot be derived explicitly from this equation; but since  $u_0(r^-)$  is assumed to be known and  $u_s(r^-)$  can usually be measured at defined locations around the object, we can write an implicit equation relating the measured field  $u_s(r^-)$  to the object function  $F(r^-)$ . To obtain this relation, we can use the Green function, defined as

$$g(\bar{r} - \bar{r}') = \frac{e^{jk_0|\bar{r} - \bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \quad (1.58)$$

Using this function, the following integral equation can be obtained:

$$\boxed{u_s(\bar{r}) = \int g(\bar{r} - \bar{r}') \cdot F(\bar{r}') \cdot u(\bar{r}') d\bar{r}'} \quad (1.59)$$

As noted above, this equation cannot be solved directly. However, we can try to approximate  $u_s(\bar{r})$  on the right-hand side of Eq. (1.59). There are two known approximations for the scattered field which are given in the following.

### 1.12.1 The Born Approximation

The idea behind this approximation is to utilize Eq. (1.55) and substitute it into Eq. (1.59). By doing so, we obtain

$$u_s(\bar{r}) = \int g(\bar{r} - \bar{r}') \cdot F(\bar{r}') \cdot u_0(\bar{r}') d\bar{r}' + \int g(\bar{r} - \bar{r}') \cdot F(\bar{r}') \cdot u_s(\bar{r}') d\bar{r}' \quad (1.60)$$

But since the scattered field is much weaker than the incident field (which is typically the case for soft tissues), we can omit the second term on the right-hand side of this equation and write the following approximation:

$$\boxed{u_s(\bar{r}) \approx \int g(\bar{r} - \bar{r}') \cdot F(\bar{r}') \cdot u_0(\bar{r}') d\bar{r}'} \quad (1.61)$$

Defining this approximation as the first Born approximation—that is,  $u_s(\bar{r}) \triangleq u_B(\bar{r})$ —we can go back and substitute its value again into Eq. (1.55) and obtain a better approximation for the scattered field in Eq. (1.60). This approximation is called the second Born approximation and is marked by  $u_B^{(2)}(\bar{r})$  and is given by

$$\boxed{u_B^{(2)}(\bar{r}) \approx \int g(\bar{r} - \bar{r}') \cdot F(\bar{r}') \cdot (u_0(\bar{r}') + u_B(\bar{r}')) d\bar{r}'} \quad (1.62)$$

Having this approximation at hand, it doesn't take much to realize that we can further improve it by re-substitution into Eq. (1.55) and Eq. (1.60) and then repeat the procedure over and over again. The general expression for the  $u_B^{(i+1)}(\bar{r})$  Born approximation is thus given by

$$\boxed{u_B^{(i+1)}(\bar{r}) \approx \int g(\bar{r} - \bar{r}') \cdot F(\bar{r}') \cdot (u_0(\bar{r}') + u_B^{(i)}(\bar{r}')) d\bar{r}'} \quad (1.63)$$

As noted above, this approximation is based on the assumption that the scattered field is much weaker than the incident field.

### 1.12.2 The Rytov Approximation

With this approximation the acoustic field is represented by a complex phase that is given by

$$u(\bar{r}) = e^{\phi(\bar{r})} \quad (1.64)$$

while the incident field is given by

$$u_0(\bar{r}) = e^{\phi_0(\bar{r})} \quad (1.65)$$

and the total field is given by

$$\phi(\bar{r}) = \phi_0(\bar{r}) + \phi_s(\bar{r}) \quad (1.66)$$

where  $\phi_s(\bar{r})$  designates the scattered field. Following some derivations and assumptions, it can be shown that the following relation can be obtained:

$$\phi_s(\bar{r}) \approx \frac{1}{u_0(\bar{r})} \int g(\bar{r} - \bar{r}') \cdot F(\bar{r}') \cdot u_0(\bar{r}') d\bar{r}' \quad (1.67)$$

The condition needed for the Rytov approximation to be valid is given by

$$F(\bar{r}) \gg [\nabla \phi_s(\bar{r})]^2 \quad (1.68)$$

As reported in the literature (see, for example, Keller [8]), the Rytov approximation is valid under less restrictive conditions than the Born approximation.

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