CHAPTER 1

The Volatility Problem

Suppose we use the standard deviation of possible future returns on a stock as a measure of its volatility. Is it reasonable to take that volatility as a constant over time? I think not.

— Fischer Black

INTRODUCTION

It is widely accepted today that an assumption of a constant volatility fails to explain the existence of the volatility smile as well as the leptokurtic character (fat tails) of the stock distribution. The above Fischer Black quote, made shortly after the famous constant-volatility Black-Scholes model was developed, proves the point.

In this chapter, we will start by describing the concept of Brownian motion for the stock price return as well as the concept of historic volatility. We will then discuss the derivatives market and the ideas of hedging and risk neutrality. We will briefly describe the Black-Scholes partial derivatives equation (PDE) in this section. Next, we will talk about jumps and level dependent volatility models. We will first mention the jump diffusion process and introduce the concept of leverage. We will then refer to two popular level dependent approaches: the constant elasticity variance (CEV) model and the Bensoussan-Crouhy-Galai (BCG) model. At this point, we will mention local volatility models developed in the recent past by Dupire and Derman-Kani, and we will discuss their stability.

Following this, we will tackle the subject of stochastic volatility, where we will mention a few popular models, such as the square-root model and the general autoregressive conditional heteroskedasticity (GARCH) model. We will then talk about the pricing PDE under stochastic volatility and the risk-neutral version of it. For this we will need to introduce the concept of market price of risk.

The generalized Fourier transform is the subject of the following section. This technique was used by Alan Lewis extensively for solving stochastic volatility problems. Next, we will discuss the mixing solution, both in correlated and uncorrelated cases. We will mention its link to the fundamental transform and its usefulness for Monte Carlo–based methods. We will then describe the long-term asymptotic case, where we get closed-form approximations for many popular methods, such as the square-root model. Lastly, we will talk about pure-jump models, such as variance gamma and variance gamma with stochastic arrival.

THE STOCK MARKET

The Stock Price Process

The relationship between the stock market and the mathematical concept of Brownian motion goes back to Bachelier [18]. A Brownian motion corresponds to a process, the increments of which are independent stationary normal random variables. Given that a Brownian motion can take negative values, it cannot be used for the stock price. Instead, Samuelson [211] suggested using this process to represent the *return* of the stock price, which will make the stock price a geometric (or exponential) Brownian motion.

In other words, the stock price S follows a log-normal process[1]

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{1.1}$$

where dB_t is a Brownian motion process, μ the instantaneous expected total return of the stock (possibly adjusted by a dividend yield), and σ the instantaneous standard deviation of stock price returns, called the *volatility* in financial markets.

Using Ito's lemma,[2] we also have

$$d\ln(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t \tag{1.2}$$

The stock return μ could easily become time dependent without changing any of our arguments. For simplicity, we will often refer to it as μ even if we mean μ_t. This remark holds for other quantities, such as r_t, the interest-rate, or q_t, the dividend yield.

Equation (1.1) represents a continuous process. We can either take this as an approximation of the real discrete tick-by-tick stock movements or

[1]For an introduction to stochastic processes, see Karatzas [167] or Oksendal [197].
[2]See, for example, Hull [146].

consider it the real unobservable dynamics of the stock price, in which case the discrete prices constitute a *sample* from this continuous ideal process. Either way, the use of a continuous equation makes the pricing of financial instruments more analytically tractable.

The discrete equivalent of (1.2) is

$$\ln S_{t+\Delta t} = \ln S_t + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}B_t \tag{1.3}$$

where (B_t) is a sequence of independent normal random variables with zero mean and variance of 1.

Historic Volatility

This suggests a first simple way to estimate the volatility, σ, namely the *historic volatility*. Considering $S_1, \ldots, S_N$ as a sequence of known historic daily stock close prices, calling $R_n = \ln(S_{n+1}/S_n)$ the stock price return between two days and $\bar{R} = \frac{1}{N}\sum_{n=0}^{N-1} R_n$ the mean return, the historic volatility would be the annualized standard deviation of the returns, namely

$$\sigma_{hist} = \sqrt{\frac{252}{N-1}\sum_{n=0}^{N-1}(R_n - \bar{R})^2} \tag{1.4}$$

Because we work with annualized quantities, and we are using daily stock closing prices, we needed the factor 252, supposing that there are approximately 252 business days in a year.[3]

Note that N, the number of observations, can be more or less than one year; therefore when talking about a historic volatility, it is important to know what time horizon we are considering. We can indeed have three-month historic volatility or three-year historic volatility. Needless to say, taking too few prices would give an inaccurate estimation. Similarly, the begin and end date of the observations matter. It is preferable to take the end date as close as possible to today so that we include recent observations.

An alternative was suggested by Parkinson [200] in which instead of daily closing prices we use the high and the low prices of the stock on that day, and $R_n = \ln(S_n^{high}/S_n^{low})$. The volatility would then be

$$\sigma_{parkinson} = \sqrt{\frac{252}{N-1}\frac{1}{4\ln(2)}\sum_{n=0}^{N-1}(R_n - \bar{R})^2}$$

This second moment estimation derived by Parkinson is based upon the fact that the range R_n of the asset follows a *Feller* distribution.

[3]Clearly the observation frequency does not have to be daily.

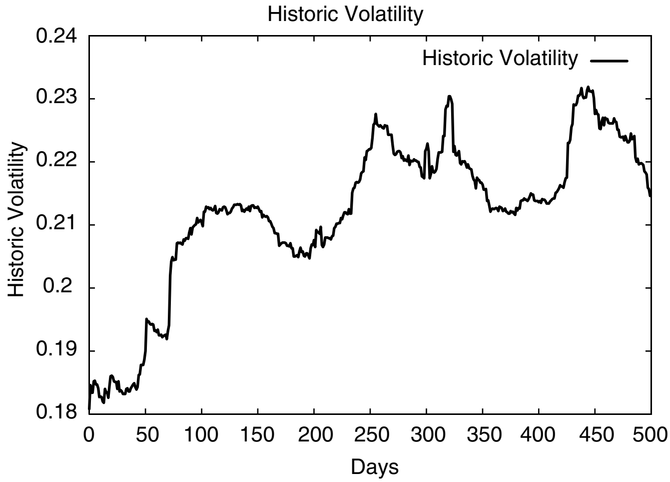

FIGURE 1.1 The SPX Historic Rolling Volatility from 01/03/2000 to 12/31/2001. As we can see, the volatility is clearly nonconstant.

Plotting, for instance, the one-year rolling[4] historic volatility (1.4) of the S&P 500 Stock Index, it is easily seen that this quantity is *not* constant over time (Figure 1.1). This observation was made as early as the 1960s by many financial mathematicians and followers of the *chaos theory*. We therefore need time-varying volatility models.

One natural extension of the constant volatility approach is to make σ_t a deterministic function of time. This is equivalent to giving the volatility a term structure, by analogy with interest rates.

THE DERIVATIVES MARKET

Until now, we have mentioned the stock price movements independently from the derivatives market, but we now are going to include the financial derivatives (especially options) prices as well. These instruments became very popular and as liquid as the stocks themselves after Black and Scholes introduced their risk-neutral pricing formula in [38].

[4]By *rolling* we mean that the one-year interval slides within the total observation period.

The Black-Scholes Approach

The Black-Scholes approach makes a number of reasonable assumptions about markets being frictionless and uses the log-normal model for the stock price movements. It also supposes a constant or deterministically time-dependent stock drift and volatility. Under these conditions, they prove that it is possible to hedge a position in a contingent claim dynamically by taking an offsetting position in the underlying stock and hence become *immune* to the stock movements. This risk neutrality is possible because, as they show, we can replicate the financial derivative (for instance, an option) by taking positions in cash and the underlying security. This condition of the possibility of replication is called *market completeness*.

In this situation, everything happens as if we were replacing the stock drift μ_t with the risk-free rate of interest r_t in (1.1) or $r_t - q_t$ if there is a dividend-yield q_t. The contingent claim $f(S,t)$ having a payoff $G(S_T)$ will satisfy the famous Black-Scholes equation

$$rf = \frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \tag{1.5}$$

Indeed the hedged portfolio $\Pi = f - \frac{\partial f}{\partial S}S$ is immune to the stock random movements and, according to Ito's lemma, verifies

$$d\Pi = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt$$

which must also be equal to $r\Pi dt$ or else there would be possibility of Riskless arbitrage.[5]

Note that this equation is closely related to the Feynman-Kac equation satisfied by $F(S,t) = \mathbf{E}_t(h(S_T))$ for any function h under the risk-neutral measure; $F(S,t)$ must be a Martingale[6] under this measure and therefore must be driftless, which implies $dF = \sigma S \frac{\partial F}{\partial S} dB_t$ and

$$0 = \frac{\partial F}{\partial t} + (r-q)S\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}$$

This would indeed be a different way to reach the same Black-Scholes equation, by using $f(S,t) = \exp(-rt)F(S,t)$, as was done, for instance, in Shreve [218].

Let us insist again on the fact that the real drift of the stock price does not appear in the preceding equation, which makes the volatility σ_t the only

[5]For a detailed discussion, see Hull [146].
[6]For an explanation, see Shreve [218] or Karatzas [167].

unobservable quantity. As we said, the volatility could be a deterministic function of time without changing the foregoing argument, in which case all we need to do is to replace σ^2 with $\frac{1}{t}\int_0^t \sigma_s^2 ds$, and keep everything else the same.

For calls and puts, where the payoffs $G(S_T)$ are respectively $MAX(0, S_T - K)$ and $MAX(0, K - S_T)$ and where K is the strike price and T the maturity of the option, the Black-Scholes partial derivatives equation is solvable and gives the celebrated Black-Scholes formulae

$$call_t = S_t e^{-q(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{1.6}$$

and

$$put_t = -S_t e^{-q(T-t)}\Phi(-d_1) + Ke^{-r(T-t)}\Phi(-d_2) \tag{1.7}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

is the cumulative standard normal function and

$$d_1 = d_2 + \sigma\sqrt{T-t} \qquad \text{and} \qquad d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Note that using the well-known symmetry property for normal distributions $\Phi(-x) = 1 - \Phi(x)$ in the above formulae, we could reach the *put-call parity* relationship

$$call_t - put_t = S_t e^{-q(T-t)} - Ke^{-r(T-t)} \tag{1.8}$$

which we can also rearrange as

$$S_t e^{-q(T-t)} - call_t = Ke^{-r(T-t)} - put_t$$

The left-hand side of this last equation is called a *covered call* and is equivalent to a short position in a put combined with a bond.

The Cox-Ross-Rubinstein Approach

Later, Cox, Ross, and Rubinstein [66] developed a simplified approach using the binomial law to reach the same pricing formulae. The approach commonly referred to as the *binomial tree* uses a tree of recombining spot prices, in which at a given time step n we have $n+1$ possible $S[n][j]$ spot prices, with $0 \le j \le n$. Calling p the upward transition probability and $1-p$ the downward transition probability, S the stock price today, and $S_u = uS$

and $S_d = dS$ upper and lower possible future spot prices, we can write the expectation equation[7]

$$\mathbf{E}[S] = puS + (1-p)dS = e^{r\Delta t}S$$

which immediately gives us

$$p = \frac{a-d}{u-d}$$

with $a = \exp(r\Delta t)$.

We can also write the variance equation

$$Var[S] = pu^2S^2 + (1-p)d^2S^2 - e^{2r\Delta t}S^2 \approx \sigma^2 S^2 \Delta t$$

which after choosing a centering condition, such as $ud = 1$, will provide us with $u = \exp\big(\sigma\sqrt{\Delta t}\big)$ and $d = \exp\big(-\sigma\sqrt{\Delta t}\big)$. Using the values for u, d, and p we can build the tree, and using the final payoff we can calculate the option price by backward induction.[8] We can also build this tree by applying an explicit finite difference scheme to the PDE (1.5), as was done in Wilmott [238]. An important advantage of the tree method is that it can be applied to American options (with early exercise) as well.

It is possible to deduce the *implied* volatility of call and put options by solving a reverse Black-Scholes equation, that is, find the volatility that would equate the Black-Scholes price to the market price of the option. This is a good way to see how derivatives markets *perceive* the underlying volatility. It is easy to see that if we change the maturity and strike prices of options (and keep everything else fixed) the implied volatility will *not* be constant. It will have a linear skew and a convex form as the strike price changes. This famous "smile" cannot be explained by simple time dependence, hence the necessity of introducing new models (Figure 1.2).[9]

JUMP DIFFUSION AND LEVEL-DEPENDENT VOLATILITY

In addition to the volatility smile observable from the implied volatilities of the options, there is evidence that the assumption of a pure normal distribution (also called pure *diffusion*) for the stock return is not accurate. Indeed "fat tails" have been observed away from the mean of the stock return. This

[7]The expectation equation is written under the risk-neutral probability.
[8]For an in-depth discussion on binomial trees, see Cox [67].
[9]It is interesting to note that this smile phenomenon was practically nonexistent prior to the 1987 stock-market crash. Many researchers therefore believe that the markets have *learnt to factor-in* a crash possibility, which creates the volatility smile.

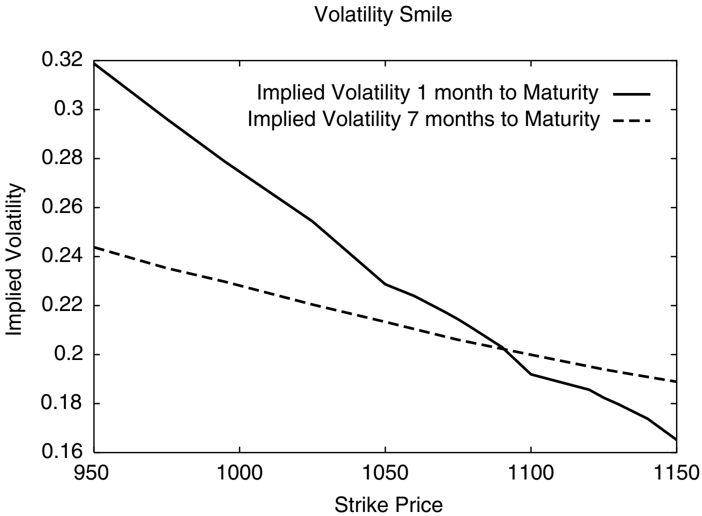

FIGURE 1.2 The SPX Volatility Smile on February 12, 2002 with Index = \$1107.50, 1 Month and 7 Months to Maturity. The negative skewness is clearly visible. Note how the smile becomes *flatter* as time to maturity increases.

phenomenon is called *leptokurticity* and could be explained in many different ways.

Jump Diffusion

Some try to explain the smile and the leptokurticity by changing the underlying stock distribution from a diffusion process to a jump-diffusion process. A jump diffusion is *not* a level-dependent volatility process; however, we are mentioning it in this section to demonstrate the importance of the *leverage effect*. Merton [190] was first to actually introduce jumps in the stock distribution. Kou [172] recently used the same idea to explain both the existence of fat tails and the volatility smile.

The stock price will follow a modified stochastic process under this assumption. If we add to the Brownian motion, dB_t; a Poisson (jump) process[10] dq with an intensity[11] λ, and then calling $k = \mathbf{E}(Y - 1)$ with $Y - 1$

[10] See, for instance, Karatzas [167].

[11] The intensity could be interpreted as the mean number of jumps per time unit.

the random variable percentage change in the stock price, we will have

$$dS_t = (\mu - \lambda k)S_t dt + \sigma S_t dB_t + S_t dq \tag{1.9}$$

or equivalently,

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2} - \lambda k\right)t + \sigma B_t\right] Y_n$$

where $Y_0 = 1$ and $Y_n = \prod_{j=1}^{n} Y_j$, with Y_j's independently identically distributed random variables and n a Poisson random variable with a parameter λt.

It is worth noting that for the special case where the jump corresponds to total ruin or *default*, we have $k = -1$, which will give us

$$dS_t = (\mu + \lambda)S_t dt + \sigma S_t dB_t + S_t dq \tag{1.10}$$

and

$$S_t = S_0 \exp\left[\left(\mu + \lambda - \frac{\sigma^2}{2}\right)t + \sigma B_t\right] Y_n$$

Given that in this case $\mathbf{E}(Y_n) = \mathbf{E}(Y_n^2) = e^{-\lambda t}$, it is fairly easy to see that in the risk-neutral world

$$\mathbf{E}(S_t) = S_0 e^{rt}$$

exactly as in the pure diffusion case, but

$$Var(S_t) = S_0^2 e^{2rt}\left(e^{(\sigma^2+\lambda)t} - 1\right) \approx S_0^2\left(\sigma^2 + \lambda\right)t \tag{1.11}$$

unlike the pure diffusion case, where $Var(S_t) \approx S_0^2 \sigma^2 t$.

Proof: Indeed

$$\begin{aligned}
\mathbf{E}(S_t) &= S_0 \exp((r + \lambda)t) \exp\left(-\frac{\sigma^2}{2}t\right)\mathbf{E}[\exp(\sigma B_t)]\mathbf{E}(Y_n) \\
&= S_0 \exp((r + \lambda)t) \exp\left(-\frac{\sigma^2}{2}t\right) \exp\left(\frac{\sigma^2}{2}t\right) \exp(-\lambda t) = S_0 \exp(rt)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}(S_t^2) &= S_0^2 \exp(2(r + \lambda)t) \exp\left(-\sigma^2 t\right)\mathbf{E}[\exp(2\sigma B_t)]\mathbf{E}(Y_n^2) \\
&= S_0^2 \exp(2(r + \lambda)t) \exp\left(-\sigma^2 t\right) \exp\left(\frac{(2\sigma)^2}{2}\right) \exp(-\lambda t) \\
&= S_0^2 \exp((2r + \lambda)t) \exp\left(\sigma^2 t\right)
\end{aligned}$$

and as usual

$$Var(S_t) = \mathbf{E}\left(S_t^2\right) - \mathbf{E}^2(S_t)$$

(QED)

Link to Credit Spread Note that for a zero-coupon risky bond Z with no recovery, a credit spread C and a face value X paid at time t we have

$$Z = e^{-(r+C)t}X = e^{-\lambda t}(e^{-rt}X) + (1 - e^{-\lambda t})(0)$$

consequently $\lambda = C$ and using (1.11) we can write

$$\tilde{\sigma}^2(C) = \sigma^2 + C$$

where σ is the fixed (pure diffusion) volatility and $\tilde{\sigma}$ is the modified jump diffusion volatility. The preceding equation relates the volatility and *leverage*, a concept we will see later in level-dependent models as well.

Also, we could see that everything happens as if we were using the Black-Scholes pricing equation but with a modified "interest rate," which is $r + C$. Indeed the hedged portfolio $\Pi = f - \frac{\partial f}{\partial S}S$ now satisfies

$$d\Pi = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt$$

under the no-default case, which occurs with a probability of $e^{-\lambda dt} \approx 1 - \lambda dt$ and

$$d\Pi = -\Pi$$

under the default case, which occurs with a probability of $1 - e^{-\lambda dt} \approx \lambda dt$.

We therefore have

$$\mathbf{E}(d\Pi) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - \lambda\Pi\right) dt$$

and using a diversification argument we can always say that $\mathbf{E}(d\Pi) = r\Pi dt$ which provides us with

$$(r + \lambda)f = \frac{\partial f}{\partial t} + (r + \lambda)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \qquad (1.12)$$

which again is the Black-Scholes PDE with a "risky rate."

A generalization of the jump diffusion process would be the use of the *Levy process*. A Levy process is a stochastic process with independent and stationary increments. Both the Brownian motion and the Poisson process are included in this category. For a description, see Matacz [186].

Level-Dependent Volatility

Many assume that the smile and the fat tails are due to the level dependence of the volatility. The idea would be to make σ_t level dependent or a function of the spot itself; we would therefore have

$$dS_t = \mu_t S_t dt + \sigma(S, t)S_t dB_t \qquad (1.13)$$

Note that to be exact, a level-dependent volatility is a function of the spot price alone. When the volatility is a function of the spot price *and* time, it is referred to as *local volatility*, which we shall discuss further.

The Constant Elasticity Variance Approach One of the very first attempts to use this approach was the constant elasticity variance (CEV) method realized by Cox [64] and [65] (Figure 1.3). In this method we would suppose an equation of the type

$$\sigma(S, t) = CS_t^{\gamma} \tag{1.14}$$

where C and γ are parameters to be calibrated either from the stock price returns themselves or from the option prices and their implied volatilities. The CEV method was recently analyzed by Jones [165] in a paper in which he uses two γ exponents.

This level-depending volatility represents an important feature that is observed in options markets as well as in the underlying prices: the negative correlation between the stock price and the volatility, also called the *leverage effect*.

The Bensoussan-Crouhy-Galai Approach Bensoussan, Crouhy, and Galai (BCG) [33] try to find the level dependence of the volatility in a manner that differs from that of Cox and Ross (Figure 1.4). Indeed in the CEV model, Cox and

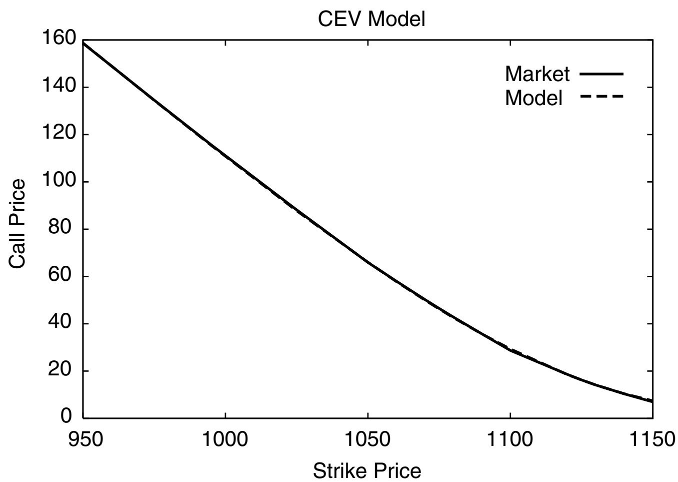

FIGURE 1.3 The CEV Model for SPX on February 12, 2002 with Index = $1107.50, 1 Month to Maturity. The smile is fitted well, but the model assumes a perfect (negative) correlation between the stock and the volatility.

Ross *first* suppose that $\sigma(S,t)$ has a certain exponential form and only then try to calibrate the model parameters to the market. Alternatively, BCG try to deduce the functional form of $\sigma(S,t)$ by using a firm structure model.

The idea of firm structure is not new and goes back to Merton [189], when he considers that the firm assets follow a log-normal process

$$dV = \mu_V V\,dt + \sigma_V V\,dB_t \tag{1.15}$$

where μ_V and σ_V are the asset's return and volatility. One important point is that σ_V is considered *constant*. Merton then argues that the equity S of the firm could be considered a call option on the assets of the firm with a strike price K equal to the face value of the firm liabilities and an expiration T equal to the average liability maturity.

Using Ito's lemma, it is fairly easy to see that

$$\begin{aligned} dS &= \mu S dt + \sigma(S,t) S dB_t \\ &= \left(\frac{\partial S}{\partial t} + \mu_V V\frac{\partial S}{\partial V} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 S}{\partial V^2}\right)dt + \sigma_V V\frac{\partial S}{\partial V}dB_t \end{aligned} \tag{1.16}$$

which immediately provides us with

$$\sigma(S,t) = \sigma_V \frac{V}{S}\frac{\partial S}{\partial V} \tag{1.17}$$

which is an implicit functional form for $\sigma(S,t)$.

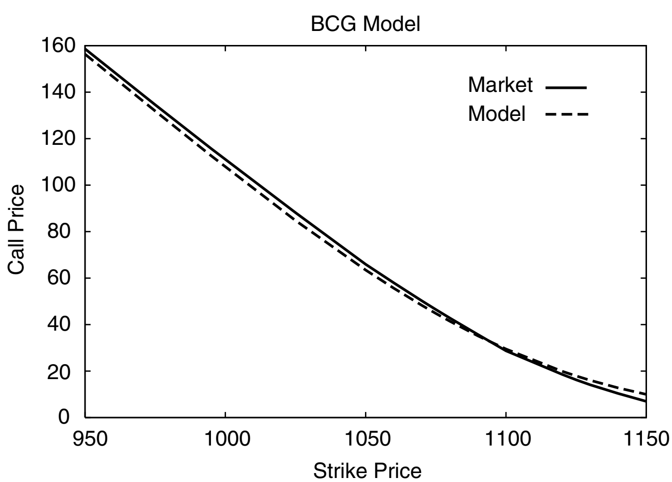

FIGURE 1.4 The BCG Model for SPX on February 12, 2002 with Index = \$1107.50, 1 Month to Maturity. The smile is fitted well.

Next, BCG eliminate the asset term in the preceding functional form and end up with a nonlinear PDE

$$\frac{\partial \sigma}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \sigma}{\partial S^2} + \left(r + \sigma^2\right) S \frac{\partial \sigma}{\partial S} = 0 \qquad (1.18)$$

This PDE gives the dependence of σ on S and t.

Proof: A quick sketch of the proof is as follows: With S being a contingent claim on V, we have the risk-neutral Black-Scholes PDE

$$\frac{\partial S}{\partial t} + rV\frac{\partial S}{\partial V} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 S}{\partial V^2} = rS$$

and using $\frac{\partial S}{\partial V} = 1/\frac{\partial V}{\partial S}$ as well as $\frac{\partial S}{\partial t} = -\frac{\partial S}{\partial V}\frac{\partial V}{\partial t}$ and $\frac{\partial^2 S}{\partial V^2} = -\frac{\partial^2 V}{\partial S^2}/\left(\frac{\partial V}{\partial S}\right)^3$ we have the reciprocal Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

Now posing $\Psi(S,t) = \ln V(S,t)$, we have $\frac{\partial V}{\partial t} = V\frac{\partial \Psi}{\partial t}$ as well as $\frac{\partial V}{\partial S} = V\frac{\partial \Psi}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2} = V\left(\frac{\partial^2 \Psi}{\partial S^2} + (\frac{\partial \Psi}{\partial S})^2\right)$, and we will have the new PDE

$$r = \frac{\partial \Psi}{\partial t} + rS\frac{\partial \Psi}{\partial S} + \frac{1}{2}\sigma^2 S^2\left(\frac{\partial^2 \Psi}{\partial S^2} + \left(\frac{\partial \Psi}{\partial S}\right)^2\right)$$

and the equation

$$\sigma = \sigma_V / \left(S\frac{\partial \Psi}{\partial S}\right)$$

This last identity implies that $\frac{\partial \Psi}{\partial S} = \frac{\sigma_V}{S\sigma}$ as well as $\frac{\partial^2 \Psi}{\partial S^2} = \frac{-\sigma_V(\sigma + S\frac{\partial \sigma}{\partial S})}{S^2\sigma^2}$, and therefore the PDE becomes

$$r = \frac{\partial \Psi}{\partial t} + r\sigma_V/\sigma + \frac{1}{2}\left(\sigma_V^2 - \sigma_V\left(\sigma + S\frac{\partial \sigma}{\partial S}\right)\right)$$

taking the derivative with respect to S and using $\frac{\partial^2 \Psi}{\partial S \partial t} = -\frac{\sigma_V}{S\sigma^2}\frac{\partial \sigma}{\partial t}$ we get the final PDE

$$\frac{\partial \sigma}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \sigma}{\partial S^2} + \left(r + \sigma^2\right) S \frac{\partial \sigma}{\partial S} = 0$$

as previously stated. *(QED)*

We therefore have an implicit functional form for $\sigma(S,t)$, and, just as for the CEV case, we need to calibrate the parameters to the market data.

LOCAL VOLATILITY

In the early 1990s, Dupire [89], as well as Derman and Kani [74], developed a concept called *local volatility*, in which the volatility smile was retrieved from the option prices.

The Dupire Approach

The Breeden & Litzenberger Identity This approach uses the options prices to get the implied distribution for the underlying stock. To do this we can write

$$V(S_0, K, T) = call(S_0, K, T) = e^{-rT} \int_0^{+\infty} (S - K)^+ p(S_0, S, T) dS \quad (1.19)$$

where S_0 is the stock price at time $t = 0$ and K the strike price of the call, and $p(S_0, S, T)$ is the *unknown* transition density for the stock price. As usual, $x^+ = MAX(x, 0)$

Using Equation (1.19) and differentiating with respect to K twice, we get the Breeden and Litzenberger [44] implied distribution

$$p(S_0, K, T) = e^{rT} \frac{\partial^2 V}{\partial K^2} \quad (1.20)$$

Proof: The proof is straightforward if we write

$$e^{rT} V(S_0, K, T) = \int_K^{+\infty} Sp(S_0, S, T) dS - K \int_K^{+\infty} p(S_0, S, T) dS$$

and take the first derivative

$$e^{rT} \frac{\partial V}{\partial K} = -Kp(S_0, K, T) + Kp(S_0, K, T) - \int_K^{+\infty} p(S_0, S, T) dS$$

and the second derivative in the same manner. *(QED)*

The Dupire Identity Now, according to the Fokker-Planck (or forward Kolmogorov) equation[12] for this density, we have

$$\frac{\partial p}{\partial T} = \frac{1}{2} \frac{\partial^2 (\sigma^2(S, t) S^2 p)}{\partial S^2} - r \frac{\partial (Sp)}{\partial S}$$

[12]See, for example, Wilmott [237] for an explanation on Fokker-Planck equation.

and therefore after a little rearrangement have

$$\frac{\partial V}{\partial T} = \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 V}{\partial K^2} - rK\frac{\partial V}{\partial K}$$

which provides us with the local volatility formula

$$\sigma^2(K,T) = \frac{\frac{\partial V}{\partial T} + rK\frac{\partial V}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 V}{\partial K^2}} \tag{1.21}$$

Proof: For a quick proof of the above let us use the zero interest rates case (the general case could be done similarly). We would then have

$$p(S_0, K, T) = \frac{\partial^2 V}{\partial K^2}$$

as well as Fokker-Planck

$$\frac{\partial p}{\partial T} = \frac{1}{2}\frac{\partial^2(\sigma^2(S,t)S^2 p)}{\partial S^2}$$

Now

$$\begin{aligned}\frac{\partial V}{\partial T} &= \int_0^{+\infty}(S_T - K)^+\frac{\partial p}{\partial T}dS_T\\ &= \int_0^{+\infty}(S_T - K)^+\frac{1}{2}\frac{\partial^2\left(\sigma^2(S,T)S^2 p\right)}{\partial S^2}dS_T\end{aligned}$$

and integrating by parts twice and using the fact that

$$\frac{\partial^2(S_T - K)^+}{\partial K^2} = \delta(S_T - K)$$

with $\delta(.)$, the Dirac function, we will have

$$\frac{\partial V}{\partial T} = \frac{1}{2}\sigma^2(K,T)K^2 p(S_0,K,T) = \frac{1}{2}K^2\sigma^2(K,T)\frac{\partial^2 V}{\partial K^2}$$

as stated. *(QED)*

It is also possible to use the implied volatility, σ_{BS}, from the Black-Scholes formula (1.6) and express the foregoing local volatility in terms of σ_{BS} instead of V. For a detailed discussion, we could refer to Wilmott [237].

Local Volatility vs. Instantaneous Volatility Clearly, the local volatility is related to the instantaneous variance v_t, as Gatheral [113] shows; the relationship could be written as

$$\sigma^2(K,T) = \mathbf{E}[v_T|S_T = K] \tag{1.22}$$

that is, local variance is the risk-neutral expectation of the instantaneous variance conditional on the final stock price being equal to the strike price.[13]

Proof: Let us show the above identity for the case of zero interest rates.[14] As mentioned, we have

$$\sigma^2(K,T) = \frac{\frac{\partial V}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 V}{\partial K^2}}$$

On the other hand, using the call payoff $V(S_0, K, t = T) = \mathbf{E}[(S_T - K)^+]$ we have

$$\frac{\partial V}{\partial K} = \mathbf{E}[H(S_T - K)]$$

with $H(.)$, the heaviside function and

$$\frac{\partial^2 V}{\partial K^2} = \mathbf{E}[\delta(S_T - K)]$$

with $\delta(.)$, the Dirac function.

Therefore the Ito lemma at $t = T$ would provide

$$d(S_T - K)^+ = H(S_T - K)dS_T + \frac{1}{2}v_T S_T^2 \delta(S_T - K)dT$$

Using the fact that the forward price (here with zero interest rates, the stock price) is a Martingale under the risk-neutral measure

$$dV = d\mathbf{E}[(S_T - K)^+] = \frac{1}{2}\mathbf{E}\left[v_T S_T^2 \delta(S_T - K)\right] dT$$

Now we have

$$\begin{aligned}\mathbf{E}[v_T S_T^2 \delta(S_T - K)] &= \mathbf{E}[v_T|S_T = K]K^2\mathbf{E}[\delta(S_T - K)] \\ &= \mathbf{E}[v_T|S_T = K]K^2\frac{\partial^2 V}{\partial K^2}\end{aligned}$$

---

[13]Note that this is independent from the process for v_t, meaning that *any* stochastic volatility model satisfies this property, which is an attractive feature of local volatility models.

[14]For the case of nonzero rates, we need to work with the forward price instead of the stock price.

Putting all this together

$$\frac{\partial V}{\partial T} = \frac{1}{2}K^2\frac{\partial^2 V}{\partial K^2}\mathbf{E}[v_T|S_T = K]$$

and by the preceding expression of $\sigma^2(K, T)$, we will have

$$\sigma^2(K, T) = \mathbf{E}[v_T|S_T = K]$$

as claimed. *(QED)*

The Derman-Kani Approach

The Derman-Kani technique is very similar to the above approach, except that it uses the binomial (or trinomial) tree framework instead of the continuous one. Using the binomial tree notations, their upward transition probability p_i from the spot s_i at time t_n to the upper node S_{i+1} at the following time-step t_{n+1}, is obtained from the usual

$$p_i = \frac{F_i - S_i}{S_{i+1} - S_i} \tag{1.23}$$

where F_i is the stock forward price known from the market and S_i the lower spot at the step t_{n+1}.

In addition, we have for a call expiring at time step t_{n+1}

$$C(K, t_{n+1}) = e^{-r\Delta t}\sum_{j=1}^{n}\left[\lambda_j p_j + \lambda_{j+1}\left(1 - p_{j+1}\right)\right]MAX(S_{j+1} - K, 0)$$

where λ_j's are the known Arrow-Debreu prices corresponding to the discounted probability of getting to the point s_j at time t_n from S_0, the initial stock price. These probabilities could easily be derived iteratively.

This allows us after some calculation to obtain S_{i+1} as a function of s_i and S_i, namely

$$S_{i+1} = \frac{S_i[e^{r\Delta t}C(s_i, K, t_{n+1}) - \Sigma] - \lambda_i s_i(F_i - S_i)}{[e^{r\Delta t}C(s_i, K, t_{n+1}) - \Sigma] - \lambda_i(F_i - S_i)}$$

where the term Σ represents the sum $\sum_{j=i+1}^{n}\lambda_j(F_j - s_i)$. This means that after choosing the usual centering condition for the binomial tree

$$s_i^2 = S_i S_{i+1}$$

we have all the elements to build the tree and deduce the implied distribution from the Arrow-Debreu prices.

Stability Issues

The local volatility models are very elegant and theoretically sound; however, they present in practice many stability issues. They are *ill-posed inversion* problems and are extremely sensitive to the input data.[15] This might introduce arbitrage opportunities and in some cases negative probabilities or variances. Derman and Kani suggest overwriting techniques to avoid such problems.

Andersen [13] tries to improve this issue by using an implicit finite difference method; however, he recognizes that the negative variance problem could still happen.

One way to make the results smoother is to use a constrained optimization. In other words, when trying to fit theoretical results C_{theo} to the market prices C_{mrkt}, instead of minimizing

$$\sum_{j=1}^{N}\left(C_{theo}(K_j) - C_{mrkt}(K_j)\right)^2$$

we could minimize

$$\lambda\frac{\partial\sigma}{\partial t} + \sum_{j=1}^{N}\left(C_{theo}(K_j) - C_{mrkt}(K_j)\right)^2$$

where λ is a constraint parameter, which could also be interpreted as a Lagrange multiplier. However, this is an artificial way to smoothen the results and the real issue remains that, once again, we have an inversion problem that is inherently unstable. Furthermore, local volatility models imply that future implied volatility smiles will be flat relative to today's, which is another limitation.[16] As we will see in the following section, stochastic volatility models offer more time-homogeneous volatility smiles.

An alternative approach suggested in [16] would be to choose a prior risk-neutral distribution for the asset (based on a subjective view) and then minimize the relative entropy distance between the desired surface and this prior distribution. This approach uses the Kullback-Leibler distance (which we will discuss in the context of maximum likelihood estimation [MLE]) and performs the minimization via dynamic programming [35] on a tree.

[15] See Tavella [226] or Avellaneda [16].
[16] See Gatheral [114].

Calibration Frequency

One of the most attractive features of local-vol models is their ability to match plain-vanilla puts and calls *exactly*. This will avoid arbitrage situations, or worse, market manipulations by traders to create "phantom" profits. As explained in Hull [147], these arbitrage-free models were developed by researchers with a single calibration (SC) methodology assumption. However, in practice, traders use them with a continual recalibration (CR) strategy. Indeed if they used the SC version of the model, significant errors would be introduced from one week to the following as shown by Dumas et al. [88]. However, once this CR version is used, there is no guarantee that the no-arbitrage property of the original SC model is preserved. Indeed the Dupire equation determines the marginal stock distribution at different points in time, but not the joint distribution of these stock prices. Therefore a path-dependent option could very well be mispriced, and the more path-dependent this option, the greater the mispricing.

Hull [147] takes the example of a bet option, a compound option, and a barrier option. The bet option depends on the distribution of the stock at one point in time and therefore is correctly priced with a continually recalibrated local vol model. The compound option has some path dependency, and hence a certain amount of mispricing compared with a stochastic volatility (SV) model. Finally, the barrier option has a strong degree of path dependency and will introduce large errors. Note that this is due to the discrete nature of the data. Indeed, the maturities we have are limited. If we had all possible maturities in a continuous way, the joint distribution would be determined completely. Also, when interpolating in time, it is customary to interpolate upon the true variance $t\sigma_t^2$ rather than the volatility σ_t given the equation

$$T_2\sigma^2(T_2) = T_1\sigma^2(T_1) + (T_2 - T_1)\sigma^2(T_1, T_2)$$

Interpolating upon the true variance will provide smoother results as shown by Jackel [152].

Proof: Indeed, calling for $0 \leq T_1 \leq T_2$, the spot return variances

$$Var(0, T_2) = T_2\sigma^2(T_2)$$

$$Var(0, T_1) = T_1\sigma^2(T_1)$$

for a Brownian motion, we have independent increments and therefore a forward variance $Var(T_1, T_2)$ such that

$$Var(0, T_1) + Var(T_1, T_2) = Var(0, T_2)$$

which demonstrates the point. *(QED)*

STOCHASTIC VOLATILITY

Unlike nonparametric local volatility models, parametric stochastic volatility (SV) models define a specific stochastic differential equation for the unobservable instantaneous variance. As we shall see, the previously defined CEV model could be considered a special case of these models.

Stochastic Volatility Processes

The idea would be to use a different stochastic process for σ altogether. Making the volatility a deterministic function of the spot is a special "degenerate" two-factor, a natural generalization of which would precisely be to have two stochastic processes with an imperfect correlation.[17]

Several different stochastic processes have been suggested for the volatility. A popular one is the *Ornstein-Uhlenbeck* (OU) process:

$$d\sigma_t = -\alpha\sigma_t dt + \beta dZ_t \tag{1.24}$$

where α and β are two parameters, remembering the stock equation

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t$$

there is a (usually negative) correlation ρ between dZ_t and dB_t, which can in turn be time or level dependent. Heston [134] and Stein [223] were among those who suggested the use of this process. Using Ito's lemma, we can see that the stock-return variance $v_t = \sigma_t^2$ satisfies a *square-root* or Cox-Ingersoll-Ross (CIR) process

$$dv_t = (\omega - \theta v_t)dt + \xi\sqrt{v_t}dZ_t \tag{1.25}$$

with $\omega = \beta^2$, $\theta = 2\alpha$, and $\xi = 2\beta$.

Note that the OU process has a closed-form solution

$$\sigma_t = \sigma_0 e^{-\alpha t} + \beta\int_0^t e^{-\alpha(t-s)}dZ_s$$

[17]Note that here the *instantaneous* volatility is stochastic. Recent work by researchers such as Schonbucher supposes a stochastic implied-volatility process, which is a completely different approach. See, for instance, [213]. On the other hand, Avellaneda et al. [17] use the concept of *uncertain volatility* for pricing and hedging derivative securities. They make the volatility switch between two extreme values based on the convexity of the derivative contract and obtain a nonlinear *Black-Scholes-Barenblatt* equation, which they solve on a grid.

which means that σ_t follows in law $\Phi\big(\sigma_0 e^{-\alpha t}, \frac{\beta^2}{2\alpha}\big(1-e^{-2\alpha t}\big)\big)$, with Φ again the normal distribution. This was discussed in Fouque [104] and Shreve [218].

Heston and Nandi [137] show that this process corresponds to a special case of the general auto regressive conditional heteroskedasticity (GARCH) model, which we will discuss next. Another popular process is the GARCH (1,1) process, where we would have

$$dv_t = (\omega - \theta v_t)dt + \xi v_t dZ_t \tag{1.26}$$

GARCH and Diffusion Limits

The most elementary GARCH process, called GARCH(1,1), was developed originally in the field of econometrics by Engle [94] and Bollerslev [40] in a *discrete* framework. The stock discrete equation (1.3) could be rewritten by taking $\Delta t = 1$ and $v_n = \sigma_n^2$ as

$$\ln S_{n+1} = \ln S_n + \left(\mu - \frac{1}{2}v_{n+1}\right) + \sqrt{v_{n+1}}B_{n+1} \tag{1.27}$$

calling the mean adjusted return

$$u_n = \ln\left(\frac{S_n}{S_{n-1}}\right) - \left(\mu - \frac{1}{2}v_n\right) = \sqrt{v_n}B_n \tag{1.28}$$

the variance process in GARCH(1,1) is supposed to be

$$v_{n+1} = \omega_0 + \beta v_n + \alpha u_n^2 = \omega_0 + \beta v_n + \alpha v_n B_n^2 \tag{1.29}$$

where α and β are weight parameters and ω_0 is a parameter related to the long-term variance.[18]

Nelson [194] shows that as the time interval length decreases and becomes infinitesimal, Equation (1.29) becomes precisely the previously cited Equation (1.26). To be more accurate, there is a *weak convergence* of the discrete GARCH process to the continuous diffusion limit.[19] For a GARCH(1,1) continuous diffusion, the correlation between dZ_t and dB_t is zero.

---

[18]It is worth mentioning that as explained in [100], a GARCH(1,1) model could be rewritten as an autoregressive moving average model of first order, ARMA(1,1), and therefore an auto regressive model of infinite order, AR($+\infty$). GARCH is therefore a parsimonious model that can fit the data with only a few parameters. Fitting the same data with an ARCH or AR model would require a much larger number of parameters. This feature makes the GARCH model very attractive.

[19]For an explanation on weak convergence, see, for example, Varadhan [230].

It might appear surprising that even if the GARCH(1,1) process has only *one* source of randomness, namely B_n, the continuous version has two independent Brownian motions. This is understandable if we consider B_n a standard normal random variable and $A_n = B_n^2 - 1$ another random variable. It is fairly easy to see that A_n and B_n are uncorrelated even if A_n is a function of B_n. As we go toward the continuous version, we can use Donsker's theorem,[20] by letting the time interval $\Delta t \to 0$, to prove that we end up with two uncorrelated and therefore independent Brownian motions. This is a limitation of the GARCH(1,1) model–hence the introduction of the nonlinear asymmetric GARCH (NGARCH) model.

Duan [83] attempts to explain the volatility smile by using the NGARCH process expressed by

$$v_{n+1} = \omega_0 + \beta v_n + \alpha\left(u_n - c\sqrt{v_n}\right)^2 \tag{1.30}$$

where c is a parameter to be determined.

The NGARCH process was first introduced by Engle [97]. The continuous counterpart of NGARCH is the same equation (1.26), except unlike the equation resulting from GARCH(1,1) there *is* a nonzero correlation between the stock process and the volatility process. This correlation is created precisely because of the parameter c that was introduced, and is once again called the *leverage* effect. The parameter c is sometimes referred to as the *leverage parameter*.

We can find the following relationships between the discrete process and the continuous diffusion limit parameters by letting the time interval become infinitesimal

$$\omega = \frac{\omega_0}{dt^2}$$

$$\theta = \frac{1 - \alpha\left(1 + c^2\right) - \beta}{dt}$$

$$\xi = \alpha\sqrt{\frac{\kappa - 1 + 4c^2}{dt}}$$

and the correlation between dB_t and dZ_t

$$\rho = \frac{-2c}{\sqrt{\kappa - 1 + 4c^2}}$$

where κ represents the Pearson kurtosis[21] of the mean adjusted returns (u_n). As we can see, the sign of the correlation ρ is determined by the parameter c.

[20]For a discussion on Donsker's theorem, similar to the central limit theorem, see, for instance, Whitt [235].

[21]The kurtosis corresponds to the fourth moment. The Pearson kurtosis for a normal distribution is equal to 3.

Proof: A quick proof of the convergence to diffusion limit could be outlined as follows. Let us assume that $c = 0$ for simplicity; we therefore are dealing with the GARCH(1,1) case. As we saw

$$v_{n+1} = \omega_0 + \beta v_n + \alpha v_n B_n^2$$

therefore

$$v_{n+1} - v_n = \omega_0 + \beta v_n - v_n + \alpha v_n - \alpha v_n + \alpha v_n B_n^2$$

or

$$v_{n+1} - v_n = \omega_0 - (1 - \alpha - \beta)v_n + \alpha v_n(B_n^2 - 1)$$

Now, allowing the time-step Δt to become variable and posing $Z_n = (B_n^2 - 1)/\sqrt{\kappa - 1}$

$$v_{n+\Delta t} - v_n = \omega\Delta t^2 - \theta\Delta t v_n + \xi v_n\sqrt{\Delta t}Z_n$$

and annualizing v_n by posing $v_t = v_n/\Delta t$, we shall have

$$v_{t+\Delta t} - v_t = \omega\Delta t - \theta\Delta t v_t + \xi v_t\sqrt{\Delta t}Z_n$$

and as $\Delta t \to 0$, we get

$$dv_t = (\omega - \theta v_t)dt + \xi v_t dZ_t$$

as claimed. *(QED)*

Note that the discrete GARCH version of the square-root process (1.25) is

$$v_{n+1} = \omega_0 + \beta v_n + \alpha(B_n - c\sqrt{v_n})^2 \tag{1.31}$$

as Heston and Nandi show[22] in [137] (Figure 1.5).

Also, note that having a diffusion process $dv_t = b(v_t)dt + a(v_t)dZ_t$ we can apply an Euler approximation[23] to discretize and obtain a Monte Carlo process, such as $v_{n+1} - v_n = b(v_n)\Delta t + a(v_n)\sqrt{\Delta t}Z_n$. It is important to note that if we use a GARCH process and go to the continuous diffusion limit, and then apply an Euler approximation, we will *not necessarily* find the original GARCH process again. Indeed, there are many different ways to discretize the continuous diffusion limit and the GARCH process corresponds to one special way. In particular, if we use (1.31) and allow $\Delta t \to 0$ to get to the continuous diffusion limit, we shall obtain (1.25). As we will see later in

[22]For a detailed discussion on the convergence of different GARCH models toward their diffusion limits, also see Duan [85].

[23]See, for instance, Jones [165].

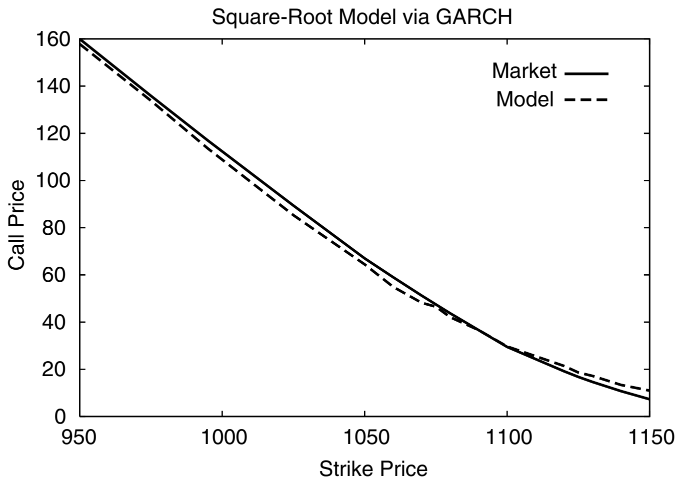

FIGURE 1.5 The GARCH Monte Carlo Simulation with the Square-Root Model for SPX on February 12, 2002 with Index = \$1107.50, 1 Month to Maturity. The Powell optimization method was used for least-square calibration.

the section on *mixing solutions*, we can then apply a discretization to this process and obtain a Monte Carlo simulation

$$v_{n+1} = v_n + (\omega - \theta v_n)\Delta t + \xi\sqrt{v_n}\sqrt{\Delta t}Z_n$$

which is again different from (1.31) but obviously has to be consistent in terms of pricing. However, we should know which method we are working with from the very beginning to perform our calibration on the corresponding specific process.

Corradi [61] explains this in the following manner: The discrete GARCH model could converge either toward a two-factor continuous limit if one chooses the Nelson parameterization, or could very well converge to a one-factor diffusion limit if one chooses another parameterization. Furthermore, an appropriate Euler discretization of the one-factor continuous model will provide a GARCH discrete process, while as previously mentioned the discretization of the two-factor diffusion model provides a two-factor discrete process. This distinction is fundamental and could explain why GARCH and SV behave differently in terms of parameter estimation.

THE PRICING PDE UNDER STOCHASTIC VOLATILITY

A very important issue to underline here is that, because of the unhedgeable second source of randomness, the concept of market completeness is lost.

We can no longer have a straightforward risk-neutral pricing. This is where the *market price of risk* will come into consideration.

The Market Price of Volatility Risk

Indeed, taking a more general form for the variance process

$$dv_t = b(v_t)dt + a(v_t)dZ_t \tag{1.32}$$

as we previously said, using the Black-Scholes risk-neutrality argument, Equation (1.1) could be replaced with

$$dS_t = (r_t - q_t)S_t dt + \sigma_t S_t dB_t \tag{1.33}$$

This is equivalent to changing the probability measure from the real one to the *risk-neutral* one.[24] We therefore need to use (1.33) together with the risk-adjusted volatility process

$$dv_t = \tilde{b}(v_t)dt + a(v_t)dZ_t \tag{1.34}$$

where

$$\tilde{b}(v_t) = b(v_t) - \lambda a(v_t)$$

with λ the market price of volatility risk. This quantity is closely related to the market price of risk for the stock $\lambda_e = (\mu - r)/\sigma$. Indeed, as Hobson [140] and Lewis [177] both show, we have

$$\lambda = \rho\lambda_e + \sqrt{1-\rho^2}\lambda^* \tag{1.35}$$

where λ^* is the market price of risk associated with $dB_t - \rho dZ_t$, which can also be regarded as the market price of risk for the hedged portfolio.

The passage from Equation (1.32) to Equation (1.34) and the introduction of the market price of volatility risk could also be explained by the Girsanov theorem, as was done for instance in Fouque [104].

It is important to underline the difference between the real and the risk-neutral measures here. If we use historic stock prices together with the real stock-return drift μ to estimate the process parameters, we will obtain the real volatility drift $b(v)$. An alternative method would be to estimate $\tilde{b}(v)$ by using current option prices and performing a least-square estimation. These calibration methods will be discussed in detail in the following chapters.

[24] See Hull [146] or Shreve [218] for more detail.

The risk-neutral version for a discrete NGARCH model would also involve the market price of risk and instead of the usual

$$\begin{aligned} \ln S_{n+1} &= \ln S_n + \left(\mu - \frac{1}{2}v_{n+1}\right) + \sqrt{v_{n+1}}B_{n+1} \\ v_{n+1} &= \omega_0 + \beta v_n + \alpha v_n(B_n - c)^2 \end{aligned}$$

we would have

$$\begin{aligned} \ln S_{n+1} &= \ln S_n + \left(r - \frac{1}{2}v_{n+1}\right) + \sqrt{v_{n+1}}\tilde{B}_{n+1} \\ v_{n+1} &= \omega_0 + \beta v_n + \alpha v_n\left(\tilde{B}_n - c - \lambda_e\right)^2 \end{aligned} \tag{1.36}$$

where $\tilde{B}_n = B_n + \lambda_e$, which could be regarded as the discrete version of the Girsanov theorem. Note that the market price of risk for the stock λ_e is *not* separable from the leverage parameter c in the above formulation. Duan shows in [84] and [86] that risk-neutral GARCH system (1.36) will indeed converge toward the continuous risk-neutral GARCH

$$\begin{aligned} dS_t &= S_t r dt + S_t\sqrt{v_t}dB_t \\ dv_t &= (\omega - \tilde{\theta}v_t)dt + \xi v_t dZ_t \end{aligned}$$

as we expected.

The Two-Factor PDE

From here, writing a two-factor PDE for a derivative security f becomes a simple application of the two-dimensional Ito's lemma. The PDE will be[25]

$$\begin{aligned} rf &= \frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2 f}{\partial S^2} + \tilde{b}(v)\frac{\partial f}{\partial v} \\ &\quad + \frac{1}{2}a^2(v)\frac{\partial^2 f}{\partial v^2} + \rho a(v)\sqrt{v}S\frac{\partial^2 f}{\partial S \partial v} \end{aligned} \tag{1.37}$$

Therefore, it is possible, after calibration, to apply a finite difference method[26] to the above PDE to price the derivative $f(S,t,v)$. An alternative would be to use directly the stochastic processes for dS_t and dv_t and apply a two-factor Monte Carlo simulation. Later in the chapter we will also mention other possible methods, such as the mixing solution or asymptotic approximations.

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[25] For a proof of the derivation see Wilmott [237] or Lewis [177].

[26] See, for instance, Tavella [227] or Wilmott [237] for a discussion on finite difference methods.

Other possible approaches for incomplete markets and stochastic volatility assumption include *super-replication* and *local risk minimization*.[27] The super-replication strategy is the cheapest self-financing strategy with a terminal value no less than the payoff of the derivative contract. This technique was primarily developed by El-Karoui and Quenez in [91]. Local risk minimization involves a partial hedging of the risk. The risk is reduced to an "intrinsic component" by taking an offsetting position in the underlying security as usual. This method was developed by Follmer and Sondermann in [102].

THE GENERALIZED FOURIER TRANSFORM

The Transform Technique

One useful technique to apply to the PDE (1.37) is the *generalized Fourier transform*.[28] First, we can use the variable $x = \ln S$ in which case, using Ito's lemma, Equation (1.37) could be rewritten as

$$rf = \frac{\partial f}{\partial t} + \left(r - q - \frac{1}{2}v\right)\frac{\partial f}{\partial x} + \frac{1}{2}v\frac{\partial^2 f}{\partial x^2} + \tilde{b}(v)\frac{\partial f}{\partial v} + \frac{1}{2}a^2(v)\frac{\partial^2 f}{\partial v^2} + \rho a(v)\sqrt{v}\frac{\partial^2 f}{\partial x \partial v} \tag{1.38}$$

Calling

$$\hat{f}(k, v, t) = \int_{-\infty}^{+\infty} e^{ikx} f(x, v, t)dx \tag{1.39}$$

where k is a *complex* number,[29] $\hat{f}$ will be defined in a complex *strip* where the imaginary part of k is between two real numbers α and β. Once $\hat{f}$ is suitably defined, meaning that $k_i = \mathcal{I}(k)$ (the imaginary part of k) is within the appropriate strip, we can write the inverse Fourier transform

$$f(x, v, t) = \frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikx}\hat{f}(k, v, t)dk \tag{1.40}$$

where we are integrating for a *fixed* k_i parallel to the real axis.

Each derivative satisfying (1.37) or equivalently (1.38) has a known payoff $G(S_T)$ at maturity. For instance, as we said before, a call option has a payoff $MAX(0, S_T - K)$ where K is the call strike price. It is easy to see

[27] For a discussion on both these techniques, see Frey [107].

[28] See Lewis [177] for a detailed discussion on this technique.

[29] As usual we note $i = \sqrt{-1}$.

that for $k_i > 1$ the Fourier transform of a call option exists and the payoff transform is

$$-\frac{K^{ik+1}}{k^2 - ik} \tag{1.41}$$

Proof: Indeed, we can write

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{ikx}(e^x - K)^+ dx &= \int_{\ln K}^{+\infty} e^{ikx}(e^x - K)dx \\ &= 0 - \left(\frac{K^{ik+1}}{ik+1} - K\frac{K^{ik}}{ik}\right) \\ &= -K^{ik+1}\left(\frac{1}{ik+1} - \frac{1}{ik}\right) = -K^{ik+1}\frac{1}{k^2 - ik}\end{aligned}$$

as stated. *(QED)*

The same could be applied to a put option or other derivative securities. In particular, a covered call (stock minus call) having a payoff $MIN(S_T, K)$ will have a transform for $0 < k_i < 1$ equal to

$$\frac{K^{ik+1}}{k^2 - ik} \tag{1.42}$$

Applying the transform to the PDE (1.38) and introducing $\tau = T - t$ and

$$\hat{h}(k, v, \tau) = e^{(r+ik(r-q))\tau}\hat{f}\ (k, v, \tau) \tag{1.43}$$

and posing[30] $c(k) = \frac{1}{2}(k^2 - ik)$, we get the new PDE equation

$$\frac{\partial \hat{h}}{\partial \tau} = \frac{1}{2}a^2(v)\frac{\partial^2 \hat{h}}{\partial v^2} + (\tilde{b}(v) - ik\rho(v)a(v)\sqrt{v})\frac{\partial \hat{h}}{\partial v} - c(k)v\hat{h} \tag{1.44}$$

Lewis calls the *fundamental transform* a function $\hat{H}(k, v, \tau)$ satisfying the PDE (1.44) and satisfying the initial condition $\hat{H}(k, v, \tau = 0) = 1$. If we know this fundamental transform, we can then multiply it by the derivative security's payoff transform and then divide it by $e^{(r+ik(r-q))\tau}$ and apply the inverse Fourier technique by keeping k_i in an appropriate strip and finally get the derivative as a function of $x = \ln S$.

Special Cases

There are cases where the fundamental transform is known. The case of a constant (or deterministic) volatility is the most elementary one. Indeed,

[30]We are following Lewis [177] notations.

using (1.44) together with $dv_t = 0$, we can easily find

$$\hat{H}(k, v, \tau) = e^{-c(k)v\tau}$$

which is analytic in k over the entire complex plane. Using the call payoff transform (1.41), we can rederive the Black-Scholes equation. The same can be done if we have a deterministic volatility $dv_t = b(v_t)dt$ by using the function $Y(v, t)$ where $dY = b(Y)dt$.

The square-root model (1.25) is another important case where $\hat{H}(k, v, \tau)$ is known and analytic. We have for this process

$$dv_t = (\omega - \theta v_t)dt + \xi\sqrt{v_t}dZ_t$$

or under the risk-neutral measure

$$dv_t = (\omega - \tilde{\theta} v_t)dt + \xi\sqrt{v_t}dZ_t$$

with $\tilde{\theta} = (1-\gamma)\rho\xi + \sqrt{\theta^2 - \gamma(1-\gamma)\xi^2}$, where $\gamma \leq 1$ represents the risk-aversion factor.

For the fundamental transform, we get

$$\hat{H}(k, v, \tau) = \exp\left[f_1(t) + f_2(t)v\right] \tag{1.45}$$

with

$$t = \frac{1}{2}\xi^2\tau \quad \tilde{\omega} = \frac{2}{\xi^2}\omega \quad \tilde{c} = \frac{2}{\xi^2}c(k) \qquad \text{and}$$

$$f_1(t) = \left[tg - \ln\left(\frac{1-he^{td}}{1-h}\right)\right]\tilde{\omega}$$

$$f_2(t) = \left[\frac{1-e^{td}}{1-he^{td}}\right]g$$

where

$$d = \sqrt{\bar{\theta}^2 + 4\tilde{c}} \quad g = \frac{1}{2}(\bar{\theta} + d) \quad h = \frac{\bar{\theta} + d}{\bar{\theta} - d} \qquad \text{and}$$

$$\bar{\theta} = \frac{2}{\xi^2}\left[(1 - \gamma + ik)\rho\xi + \sqrt{\theta^2 - \gamma(1-\gamma)\xi^2}\right]$$

The above transform has a cumbersome expression, but it can be seen that it is analytic in k and therefore always exists. For a proof of the foregoing refer to Lewis [177].

TABLE 1.1 SPX Implied Surface as of 03/09/2004. T is the maturity and $M = K/S$ the inverse of the moneyness

T / M	0.70	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20	1.30
1.000	24.61	21.29	19.73	18.21	16.81	15.51	14.43	13.61	13.12	12.94	13.23
2.000	21.94	18.73	18.68	17.65	16.69	15.79	14.98	14.26	13.67	13.22	12.75
3.000	20.16	18.69	17.96	17.28	16.61	15.97	15.39	14.86	14.38	13.96	13.30
4.000	19.64	18.48	17.87	17.33	16.78	16.26	15.78	15.33	14.92	14.53	13.93
5.000	18.89	18.12	17.70	17.29	16.88	16.50	16.13	15.77	15.42	15.11	14.54
6.000	18.46	17.90	17.56	17.23	16.90	16.57	16.25	15.94	15.64	15.35	14.83
7.000	18.32	17.86	17.59	17.30	17.00	16.71	16.43	16.15	15.88	15.62	15.15
8.000	17.73	17.54	17.37	17.17	16.95	16.72	16.50	16.27	16.04	15.82	15.40

The inversion of the Fourier transform for the square-root (Heston) model is a very popular and powerful approach. It is appealing because of its robustness and speed. The following example is based on SPX options as of 03/09/2004 expiring in 1 to 8 years from the calibration date (Tables 1.1 and 1.2).

As we shall see further, the optimal Heston parameter set to fit this surface could be found via a least-square estimation approach and for the index at $S = \$1156.86$ we find the optimal parameters $\hat{v}_0 = 0.1940$ and

$$\hat{\Psi} = (\hat{\omega}, \hat{\theta}, \hat{\xi}, \hat{\rho}) = (0.052042332, 1.8408, 0.4710, -0.4677)$$

THE MIXING SOLUTION

The Romano-Touzi Approach

The idea of *mixing solutions* was probably presented for the first time by Hull and White [149] for a zero correlation case. Later, Romano and Touzi

TABLE 1.2 Heston Prices Fitted to the 03/09/2004 Surface

T / M	0.70	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20	1.30
1.000	30.67	21.44	17.09	13.01	9.33	6.18	3.72	2.03	1.03	0.50	0.13
2.000	31.60	22.98	18.98	15.25	11.87	8.89	6.37	4.35	2.83	1.78	0.66
3.000	32.31	24.18	20.44	16.98	13.82	11.00	8.55	6.47	4.77	3.43	1.66
4.000	33.21	25.48	21.93	18.66	15.63	12.91	10.50	8.39	6.61	5.10	2.93
5.000	33.87	26.54	23.20	20.09	17.22	14.63	12.30	10.21	8.39	6.82	4.36
6.000	34.56	27.55	24.34	21.36	18.60	16.08	13.79	11.73	9.89	8.26	5.64
7.000	35.35	28.61	25.52	22.64	19.96	17.49	15.24	13.19	11.35	9.70	6.97
8.000	35.77	29.34	26.39	23.64	21.07	18.69	16.51	14.51	12.68	11.04	8.24

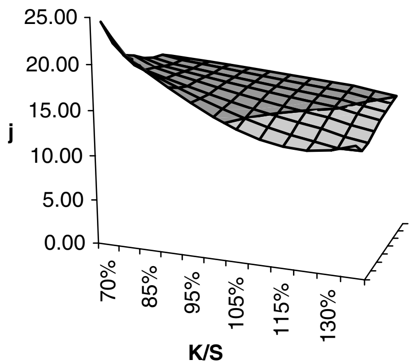

FIGURE 1.6 The SPX implied surface as of 03/09/2004. We can observe the negative skewness as well as the flattening of the slope with maturity.

[209] generalized this approach for a correlated case. The basic idea is to *separate* the random processes of the stock and the volatility, integrate the stock process conditionally upon a given volatility, and finally end up with a one-factor problem. Let us be reminded of the two processes we had:

$$dS_t = (r_t - q_t)S_t dt + \sigma_t S_t dB_t$$

and

$$dv_t = \tilde{b}(v_t)dt + a(v_t)dZ_t$$

under a risk-neutral measure.

Given a correlation ρ_t between dB_t and dZ_t, we can introduce the Brownian motion dW_t independent of dZ_t and write the usual Cholesky[31] factorization:

$$dB_t = \rho_t dZ_t + \sqrt{1-\rho_t^2}\, dW_t$$

We can then introduce the same $X_t = \ln S_t$ and write the new system of equations:

$$dX_t = (r-q)dt + dY_t - \frac{1}{2}\left(1-\rho_t^2\right)\sigma_t^2 dt + \sqrt{1-\rho_t^2}\,\sigma_t dW_t \qquad (1.46)$$
$$dY_t = -\frac{1}{2}\rho_t^2\sigma_t^2 dt + \rho_t\sigma_t dZ_t$$
$$dv_t = \tilde{b}_t dt + a_t dZ_t$$

where, once again, the two Brownian motions are independent.

[31] See, for example, Press [204].

It is now possible to integrate the stock process for a given volatility and end up with an expectation on the volatility process only. We can think of (1.46) as the limit of a discrete process, while the time step $\Delta t \to 0$.

For a derivative security $f(S_0, v_0, T)$ with a payoff[32] $G(S_T)$, using the bivariate normal density for two uncorrelated variables, we can write

$$
\begin{aligned}
f(S_0, v_0, T) &= e^{-rT}\mathbf{E}_0[G(S_T)] \qquad (1.47)\\
&= e^{-rT}\lim_{\Delta t \to 0}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} G(S_T)\prod_{t=0}^{T-\Delta t}\exp\left[-\frac{1}{2}\left(Z_t^2 + W_t^2\right)\right]\frac{dZ_t dW_t}{2\pi}
\end{aligned}
$$

If we know how to integrate the above over dW_t for a given volatility and we know the result $f^*(S, v, T)$ (for instance, for a European call option, we know the Black-Scholes formula (1.6), there are many other cases where we have closed-form solutions), then we can introduce the auxiliary variables[33]

$$S^{eff} = S_0 e^{Y_T} = S_0 \exp\left(-\frac{1}{2}\int_0^T \rho_t^2\sigma_t^2 dt + \int_0^T \rho_t\sigma_t dZ_t\right) \qquad (1.48)$$

and

$$v^{eff} = \frac{1}{T}\int_0^T \left(1 - \rho_t^2\right)\sigma_t^2 dt \qquad (1.49)$$

and as Romano and Touzi prove in [209], we will have

$$f(S_0, v_0, T) = \mathbf{E}_0[f^*(S^{eff},\ v^{eff},\ T)] \qquad (1.50)$$

where this last expectation is being taken on dZ_t only. Note that in the zero correlation case discussed by Hull and White [149] we have $S^{eff} = S_0$ and $v^{eff} = v_T = \frac{1}{T}\int_0^T \sigma_t^2 dt$, which makes the expression (1.50) a natural weighted average.

A One-Factor Monte Carlo Technique

As Lewis suggests, this will enable us to run a single-factor Monte Carlo simulation on the dZ_t and apply the known closed form for each simulated path. The method does suppose, however, that the payoff $G(S_T)$ does *not* depend on the volatility. Indeed, going back to (1.46) we can do a simulation on Y_t and v_t using the random sequence of (Z_t); then, after one path is generated, we can calculate $S^{eff} = S_0\exp(Y_T)$ and $v^{eff} = \frac{1}{T}\sum_{t=0}^{T-\Delta t}(1 - \rho_t^2)v_t\Delta t$

---

[32] The payoff should *not* depend on the volatility process.

[33] Again, all notations are taken from Lewis [177].

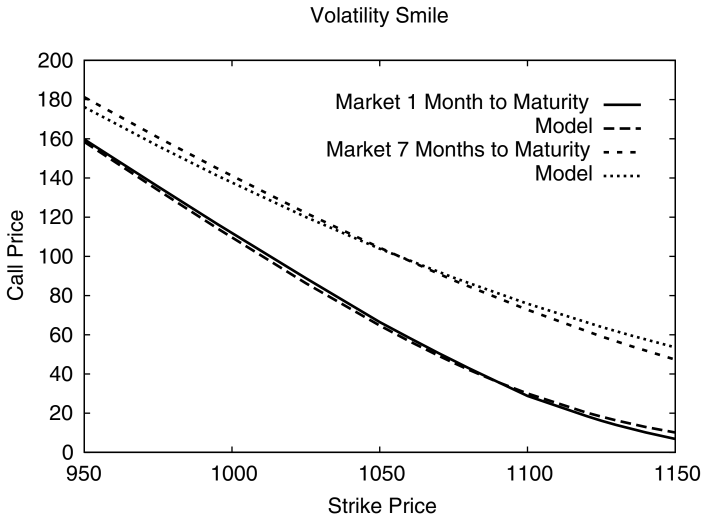

FIGURE 1.7 Mixing Monte Carlo Simulation with the Square-Root Model for SPX on February 12, 2002 with Index = $1107.50, 1 month and 7 months to Maturity. The Powell optimization method was used for least-square calibration. As we can see, both maturities are fitted fairly well.

and then apply the known closed form (e.g. Black-Scholes for a call or put) with S^{eff} and v^{eff}. Repeating this procedure for a large number of times and averaging over the paths, as we usually do in Monte-Carlo methods, we will have $f(S_0, v_0, T)$. This will give us a way to calibrate the model parameters to the market data. For instance, using the square-root model

$$dv_t = (\omega - \theta v_t)dt + \xi\sqrt{v_t}dZ_t$$

we can estimate ω, θ, ξ, and ρ from the market prices via a least-square estimation applied to theoretical prices obtained from the preceding Monte Carlo method (Figure 1.7). We can either use a single calibration and suppose we have time-independent parameters or perform one calibration per maturity. The single calibration method is known to provide a bad fit, hence the idea of adding jumps to the stochastic volatility process as described by Matytsin [187]. However, this method will introduce new parameters for calibration.[34]

[34]Eraker et al. [98] claim that a model containing jumps in the return *and* the volatility process will fit the options and the underlying data well, and will have no misspecification left.

THE LONG-TERM ASYMPTOTIC CASE

In this section we will discuss the case in which the contract time to maturity is very large, $t \to \infty$. We will focus on the special case of a square-root process because this is the model we will use in many cases.

The Deterministic Case

We shall start with the case of deterministic volatility and use that for the more general case of the stochastic volatility.

We know that under the square-root model the variance follows

$$dv_t = (\omega - \theta v_t)dt + \xi\sqrt{v_t}dZ_t$$

As an *approximation*, we can drop the stochastic term and obtain

$$\frac{dv_t}{dt} = \omega - \theta v_t$$

which is an ordinary differential equation providing us immediately with

$$v_t = \frac{\omega}{\theta} + \left(v - \frac{\omega}{\theta}\right)e^{-\theta t} \tag{1.51}$$

where v is the initial variance for $t = 0$.

Using the results from the fundamental transform for a covered call option and put-call parity, we have for $0 < k_i < 1$

$$call(S, v, \tau) = Se^{-q\tau} - Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\frac{\hat{H}(k, v, \tau)}{k^2 - ik}dk \tag{1.52}$$

where $\tau = T - t$ and $X = \ln\left(\frac{Se^{-q\tau}}{Ke^{-r\tau}}\right)$ represent the adjusted moneyness of the option. For the special "at-the-money"[35] case, where $X = 0$, we have

$$call(S, v, \tau) = Ke^{-r\tau}\left[1 - \frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{\hat{H}(k, v, \tau)}{k^2 - ik}dk\right] \tag{1.53}$$

As we previously said for a deterministic volatility case, we know the fundamental transform

$$\hat{H}(k, v, \tau) = \exp[-c(k)U(v, \tau)]$$

[35] This is different from the usual definition of at-the-money calls, where $S = K$. This vocabulary is borrowed from Alan Lewis.

With $U(v,\tau) = \int_0^\tau v(t)dt$ and as before $c(k) = \frac{1}{2}(k^2 - ik)$, which in the special case of the square-root model (1.51), will provide us with

$$U(v,\tau) = \frac{\omega}{\theta}\tau + \left(v - \frac{\omega}{\theta}\right)\left(\frac{1 - e^{-\theta\tau}}{\theta}\right)$$

This shows once again that $\hat{H}(k)$ is analytic in k over the entire complex plane.

Now if we let $\tau \to \infty$, we can write the approximation

$$\frac{call(S,v,\tau)}{Ke^{-r\tau}} \approx 1 - \frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} \exp\left[-c(k)\frac{\omega}{\theta}\tau - c(k)\frac{1}{\theta}\left(v - \frac{\omega}{\theta}\right)\right]\frac{dk}{k^2 - ik} \tag{1.54}$$

We can either calculate the above integral exactly using the Black-Scholes theory, or take the minimum where $c'(k_0) = 0$, meaning $k_0 = \frac{i}{2}$, and perform a Taylor approximation parallel to the real axis around the point $k = k_r + \frac{i}{2}$, which will give us

$$\frac{call(S,v,\tau)}{Ke^{-r\tau}} \approx 1 - \frac{2}{\pi}\exp\left(-\frac{\omega}{8\theta}\tau\right)\exp\left[-\frac{1}{8\theta}\left(v - \frac{\omega}{\theta}\right)\right]\int_{-\infty}^{\infty}\exp\left(-k_r^2\frac{\omega}{2\theta}\tau\right)dk_r$$

the integral being a Gaussian we will get the result

$$\frac{call(S,v,\tau)}{Ke^{-r\tau}} \approx 1 - \sqrt{\frac{8\theta}{\pi\omega\tau}}\exp\left[-\frac{1}{8\theta}\left(v - \frac{\omega}{\theta}\right)\right]\exp\left(-\frac{\omega}{8\theta}\tau\right) \tag{1.55}$$

which finishes our deterministic approximation case.

The Stochastic Case

For the stochastic volatility case, Lewis uses the same Taylor expansion. He notices that for the deterministic case we had

$$\hat{H}(k,v,\tau) = \exp\left[-c(k)U(v,\tau)\right] \approx \exp[-\lambda(k)\tau]u(k,v)$$

for $\tau \to \infty$, where

$$\lambda(k) = c(k)\frac{\omega}{\theta}$$

and

$$u(k,v) = \exp\left[-c(k)\frac{1}{\theta}\left(v - \frac{\omega}{\theta}\right)\right]$$

If we *suppose* that this identity holds for the stochastic volatility case as well, we can use the PDE (1.44) and interpret the result as an *eigenvalue-eigenfunction* identity with the eigenvalue $\lambda(k)$ and the eigenfunction $u(k,v)$.

This assumption is reasonable because the first Taylor approximation term for the stochastic process *is* deterministic. Indeed, introducing the operator

$$\Lambda(u) = -\frac{1}{2}a^2(v)\frac{d^2u}{dv^2} - \left[\tilde{b}(v) - ik\rho(v)a(v)\sqrt{v}\right]\frac{du}{dv} + c(k)vu$$

we have

$$\Lambda(u) = \lambda(k)u \tag{1.56}$$

Now the idea would be to perform a Taylor expansion around the minimum k_0 where $\lambda'(k_0) = 0$. Lewis shows that such k_0 is always situated on the imaginary axis. This property is referred to as the "ridge" property.

The Taylor expansion along the real axis could be written as

$$\lambda(k) = \lambda(k_0 + k_r) \approx \lambda(k_0) + \frac{1}{2}k_r^2\lambda''(k_0)$$

Note that we are dealing with a *minimum*, and therefore $\lambda''(k_0) > 0$. Using the above second-order approximation for $\lambda(k)$, we get

$$\frac{call(S,v,\tau)}{Ke^{-r\tau}} \approx 1 - \frac{u(k_0,v)}{k_0^2 - ik_0}\frac{1}{\sqrt{2\pi\lambda''(k_0)\tau}}\exp[-\lambda(k_0)\tau]$$

We can then move from the special "at-the-money" case to the general case by reintroducing $X = \ln\left(\frac{Se^{-q\tau}}{Ke^{-r\tau}}\right)$, and we will finally obtain

$$\frac{call(S,v,\tau)}{Ke^{-r\tau}} \approx e^X - \frac{u(k_0,v)}{k_0^2 - ik_0}\frac{1}{\sqrt{2\pi\lambda''(k_0)\tau}}\exp[-\lambda(k_0)\tau - ik_0X] \tag{1.57}$$

which completes our determination of the asymptotic closed form in the general case.

For the special case of the square-root model, taking the risk-neutral case $\gamma = 1$, we have[36]

$$\lambda(k) = -\omega g^*(k) = \frac{\omega}{\xi^2}\left[\sqrt{(\theta + ik\rho\xi)^2 + (k^2 - ik)\xi^2} - (\theta + ik\rho\xi)\right]$$

which also allows us to calculate $\lambda''(k)$. Also

$$u(k,v) = \exp[g^*(k)v]$$

[36]We can go back to the general case $\gamma \leq 1$ by replacing θ with $\sqrt{\theta^2 - \gamma(1-\gamma)\xi^2} + (1-\gamma)\rho\xi$ because this transformation is independent from k altogether.

where we use the notations from (1.45) and we pose

$$g^* = g - d$$

The k_0 such that $\lambda'(k_0) = 0$ is

$$k_0 = \frac{i}{1-\rho^2}\left(\frac{1}{2} - \frac{\rho}{\xi}\left[\theta - \frac{1}{2}\sqrt{4\theta^2 + \xi^2 - 4\rho\theta\xi}\right]\right)$$

which together with (1.57) provides us with the result for $call(S, v, \tau)$ in the asymptotic case under the square-root stochastic volatility model.

Note that for $\xi \to 0$ and $\rho \to 0$, we find again the deterministic result $k_0 \to \frac{i}{2}$.

A Series Expansion on Volatility-of-Volatility

Another asymptotic approach for the stochastic volatility model suggested by Lewis [177] is a Taylor expansion on the volatility-of-volatility. There are two possibilities for this: We can perform the expansion *either* for the option price *or* for the implied volatility directly. In what follows, we consider the former approach. Once again, we use the fundamental transform $H(k, V, \tau)$ with $H(k, V, 0) = 1$ and

$$\frac{\partial H}{\partial \tau} = \frac{1}{2}a^2(v)\frac{\partial^2 H}{\partial v^2} + \left(\tilde{b}(v) - ik\rho(v)a(v)\sqrt{v}\right)\frac{\partial H}{\partial v} - c(k)vH$$

and $c(k) = \frac{1}{2}(k^2 - ik)$. We then pose $a(v) = \xi\eta(v)$ and expand $H(k, V, \tau)$ on powers of ξ and finally apply the inverse Fourier transform to obtain an expansion on the call price.
With our usual notations $\tau = T - t$, $X = \ln(\frac{S}{K}) + (r - q)\tau$ and $Z(V) = V\tau$, the series will be

$$\begin{aligned} C(S, V, \tau) &= c_{BS}(S, v, \tau) + \xi\tau^{-1}J_1\tilde{R}_{11}\frac{\partial c_{BS}(S, v, \tau)}{\partial V} \\ &\quad + \xi^2\left[\tau^{-2}J_3\tilde{R}_{20} + \tau^{-1}J_4\tilde{R}_{12} + \frac{1}{2}\tau^{-2}J_1^2\tilde{R}_{22}\right]\frac{\partial c_{BS}(S, v, \tau)}{\partial V} + O(\xi^3) \end{aligned}$$

where $v(V, \tau)$ is the deterministic variance

$$v(V, \tau) = \frac{\omega}{\theta} + \left(V - \frac{\omega}{\theta}\right)\left(\frac{1 - e^{-\theta\tau}}{\theta\tau}\right)$$

and $\tilde{R}_{pq} = R_{pq}(X, v(V, \tau), \tau)$ with R_{pq} given polynomials of (X, Z) of degree four at most, and J_n's known functions of (V, τ).

The explicit expressions for all these functions are given in the third chapter of the Lewis book [177].

The obvious advantages of this approach are its speed and stability. The issue of lack of time homogeneity of the parameters $\Psi = (\omega, \theta, \xi, \rho)$ could be addressed by performing one calibration per time interval. In this case, for each time interval $[t_n, t_{n+1}]$ we will have one set of parameters $\Psi_n = (\omega_n, \theta_n, \xi_n, \rho_n)$ and depending on what maturity T we are dealing with, we will use one or the other parameter set.

We compare the values obtained from this series-based approach with those from a mixing Monte Carlo method in Figure 1.8. We are taking the example that Heston studied in [134]. The graph shows the difference $C(S, V, \tau) - c_{BS}(S, V, \tau)$ for a fixed $K = \$100$ and $\tau = 0.50$ year. The other inputs are $\omega = 0.02$, $\theta = 2.00$, $\xi = 0.10$, $\rho = -0.50$, $V = 0.01$, and $r = q = 0$. As we can see, the true value of the call is *lower* than the Black-Scholes value for the out-of-the-money (OTM) region. The higher ξ and $|\rho|$ are, the larger this difference will be.

In Figures 1.9 and 1.10, we reset the correlation ρ to zero to have a symmetric distribution, but we use a volatility-of-volatility of $\xi = 0.10$ and $\xi = 0.20$ respectively. As discussed, the parameter ξ is the one creating the leptokur-

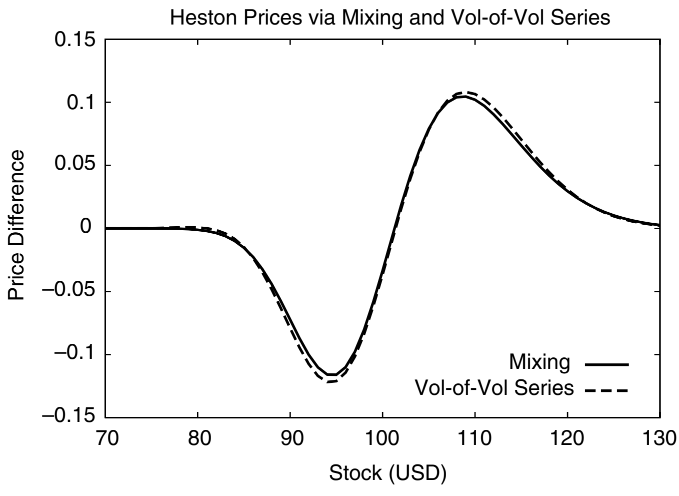

FIGURE 1.8 Comparing the Volatility-of-Volatility Series Expansion with the Monte Carlo Mixing Model. The graph shows the price difference $C(S, V, \tau) - c_{BS}(S, V, \tau)$. We are taking $\xi = 0.10$ and $\rho = -0.50$. This example was used in the original Heston paper.

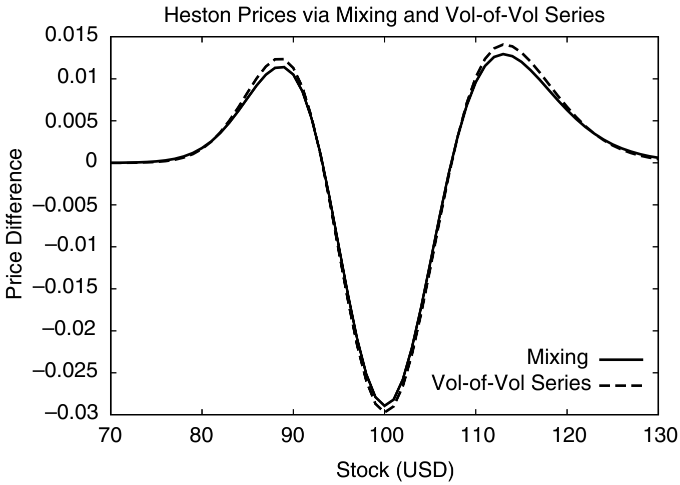

FIGURE 1.9 Comparing the Volatility-of-Volatility Series Expansion with the Monte Carlo Mixing Model. The graph shows the price difference $C(S, V, \tau) - c_{BS}(S, V, \tau)$. We are taking $\xi = 0.10$ and $\rho = 0$. This example was used in the original Heston paper.

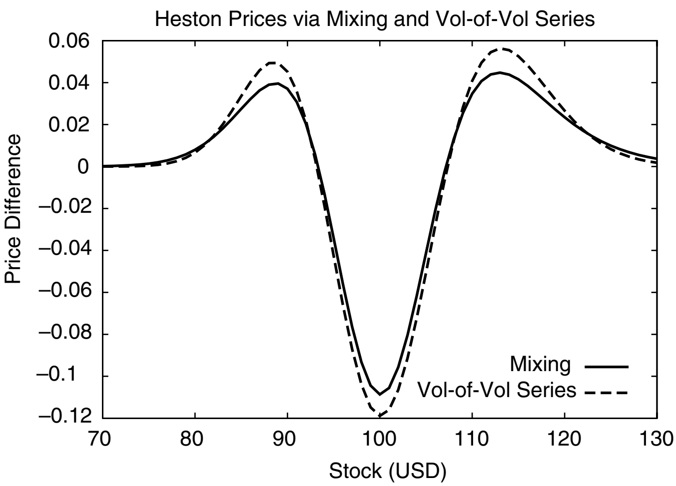

FIGURE 1.10 Comparing the Volatility-of-Volatility Series Expansion with the Monte Carlo Mixing Model. The graph shows the price difference $C(S, V, \tau) - c_{BS}(S, V, \tau)$. We are taking $\xi = 0.20$ and $\rho = 0$. This example was used in the original Heston paper.

ticity phenomenon. A higher volatility-of-volatility causes higher valuation for far-from-the-money options.[37]

Unfortunately, the foregoing series approximation becomes poor as soon as the volatility-of-volatility becomes larger than 0.40 and the maturity becomes of the order of 1 year. This case is not unusual at all and therefore makes the use of this method limited. This is why the method of choice remains the inversion of the Fourier transform, as previously described.

PURE-JUMP MODELS

Variance Gamma

An alternative point of view is to drop the diffusion assumption altogether and replace it with a pure-jump process. Note that this is different from the jump-diffusion process previously discussed. Madan et al. suggested the following framework, called *variance-gamma* (VG) in [182]. We would have the log-normal-like stock process

$$d\ln S_t = (\mu_S + \omega)dt + X(dt; \sigma, \nu, \theta)$$

where as before μ_S is the real-world statistical drift of the stock log return and $\omega = \frac{1}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2)$.

As for $X(dt; \sigma, \nu, \theta)$, it has the following meaning:

$$X(dt; \sigma, \nu, \theta) = B(\gamma(dt, 1, \nu); \theta, \sigma)$$

where $B(dt; \theta, \sigma)$ would be a Brownian motion with drift θ and volatility σ. In other words

$$B(dt; \theta, \sigma) = \theta dt + \sigma\sqrt{dt}N(0, 1)$$

and $N(0, 1)$ is a standard Gaussian realization.

The time interval at which the Brownian motion is considered is not dt but $\gamma(dt, 1, \nu)$ which is a random realization following a gamma distribution with a mean 1 and variance rate ν. The corresponding probability density function is

$$f_\nu(dt, \tau) = \frac{\tau^{\frac{dt}{\nu}-1}e^{-\frac{\tau}{\nu}}}{\nu^{\frac{dt}{\nu}}\Gamma(\frac{dt}{\nu})}$$

where $\Gamma(x)$ is the usual gamma function.

Note that the stock log-return density could actually be *integrated* for the VG model, and the density of $\ln(S_t/S_0)$ is known and could be implemented

[37] Also note that the gap between the closed-form series and the Monte Carlo model increases with ξ. Indeed, the accuracy of the expansion decreases as ξ becomes larger.

via $K_\alpha(x)$, the modified Bessel function of the second kind. Indeed, calling $z = \ln(S_k/S_{k-1})$ and $h = t_k - t_{k-1}$ and posing $x_h = z - \mu_S h - \frac{h}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2)$ we have

$$p(z|h) = \frac{2\exp(\theta x_h/\sigma^2)}{\nu^{\frac{h}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{h}{\nu})}\left(\frac{x_h^2}{2\sigma^2/\nu + \theta^2}\right)^{\frac{h}{2\nu}-\frac{1}{4}} K_{\frac{h}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x_h^2(2\sigma^2/\nu + \theta^2)}\right)$$

Moreover, as Madan et al. show, the option valuation under VG is fairly straightforward and admits an analytically tractable closed form that can be implemented via the above modified Bessel function of second kind and a degenerate hypergeometric function. All details are available in [182].

Remark on the Gamma Distribution The gamma cumulative distribution function (CDF) could be defined as

$$P(a, x) = \frac{1}{\Gamma(a)}\int_0^x e^{-t}t^{a-1}dt$$

Note that with our notations

$$F_\nu(h, x) = F(h, x, \mu = 1, \nu)$$

with

$$F(h, x, \mu, \nu) = \frac{1}{\Gamma(\frac{\mu^2 h}{\nu})}\left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 h}{\nu}}\int_0^x e^{-\frac{\mu t}{\nu}}t^{\frac{\mu^2 h}{\nu}-1}dt$$

In other words

$$F(h, x, \mu, \nu) = P\left(\frac{\mu^2 h}{\nu}, \frac{\mu x}{\nu}\right)$$

The behavior of this CDF is displayed in Figure 1.11 for different values of the parameter $a > 0$ and for $0 < x < +\infty$.

Using the inverse of this CDF, we can have a simulated data set for the gamma law:

$$x^{(i)} = F_\nu^{-1}(h, \mathcal{U}^{(i)}[0, 1])$$

with $1 \leq i \leq N_{sims}$ and $\mathcal{U}^{(i)}[0, 1]$ a uniform random realization between zero and one.

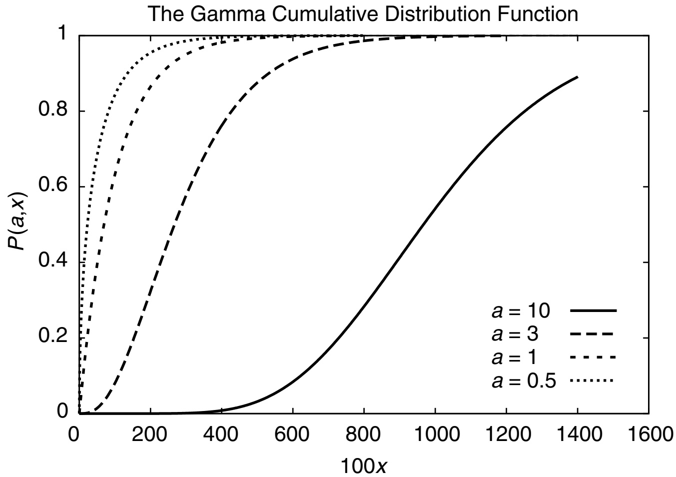

FIGURE 1.11 The Gamma Cumulative Distribution Function $P(a, x)$ for Various Values of the Parameter a. The implementation is based on code available in *Numerical Recipes in C* [204].

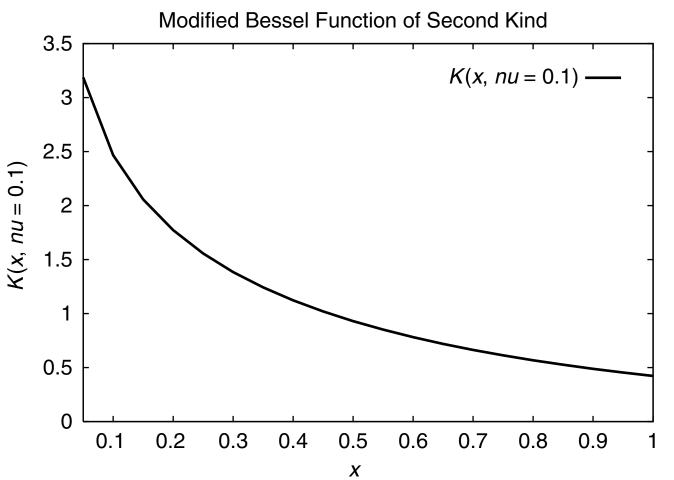

FIGURE 1.12 The Modified Bessel Function of Second Kind for a Given Parameter. The implementation is based on code available in *Numerical Recipes in C* [204].

Stochastic Volatility vs. Time-Changed processes As mentioned in [23], this alternative formulation leading to time-changed processes is closely related to the previously discussed stochastic volatility approach in the following way.

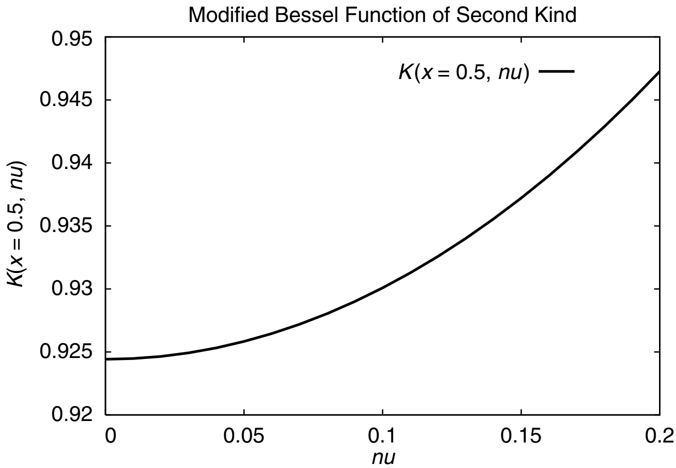

FIGURE 1.13 The Modified Bessel Function of Second Kind as a Function of the Parameter. The implementation is based on code available in *Numerical Recipes in C* [204].

Taking the foregoing VG stochastic differential equation

$$d \ln S_t = (\mu_S + \omega)dt + \theta\gamma(dt, 1, \nu) + \sigma\sqrt{\gamma(dt, 1, \nu)}N(0, 1)$$

one could consider $\sigma^2\gamma(t, 1, \nu)$ as the integrated variance and define $v_t(\nu)$, the instantaneous variance, as

$$\sigma^2\gamma(dt, 1, \nu) = v_t(\nu)dt$$

in which case, we would have

$$\begin{aligned} d \ln S_t &= (\mu_S + \omega)dt + (\theta/\sigma^2)v_t(\nu)dt + \sqrt{v_t(\nu)dt}N(0, 1) \\ &= (\mu_S + \omega + (\theta/\sigma^2)v_t(\nu))dt + \sqrt{v_t(\nu)}dZ_t \end{aligned}$$

where dZ_t is a Brownian motion. This last expression is a traditional stochastic volatility equation.

Variance Gamma with Stochastic Arrival

An extension of the VG model would be a variance gamma model with stochastic arrival (VGSA), which would include the volatility *clustering* effect. This phenomenon (also represented by GARCH) means that a high (low) volatility will be followed by a series of high (low) volatilities. In this approach, we replace the dt in the previously defined $f_\nu(dt, \tau)$ with $y_t dt$, where y_t follows a square-root (CIR) process

$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dW_t$$

where the Brownian motion dW_t is independent from other processes in the model. This is therefore a VG process in which the arrival time itself is stochastic. The mean reversion of the square-root process will cause the volatility persistence effect that is empirically observed. Note that (not counting μ_S) the new model parameter set is $\Psi = (\kappa, \eta, \lambda, \nu, \theta, \sigma)$.

Option Pricing under VGSA The option pricing could be carried out via a Monte Carlo simulation algorithm under the risk-neutral measure, where, as before, μ_S is replaced with $r - q$. We first would simulate the path of y_t by writing

$$y_k = y_{k-1} + \kappa(\eta - y_{k-1})\Delta t + \lambda\sqrt{y_{k-1}}\sqrt{\Delta t}Z_k$$

then calculate

$$Y_T = \sum_{k=0}^{N-1} y_k \Delta t$$

and finally apply *one-step* simulations

$$T^* = F_\nu^{-1}\left(Y_T, \mathcal{U}[0,1]\right)$$

and[38]

$$\ln S_T = \ln S_0 + (r - q + \omega)T + \theta T^* + \sigma\sqrt{T^*}B_k$$

Note that we have two normal random variables B_k, Z_k as well as a gamma-distributed random variable T^*, and that they are all uncorrelated. Once the stock price S_T is properly simulated, we can calculate the option price as usual.

The Characteristic Function As previously discussed, another way to tackle the option-pricing issue would be to use the characteristic functions. For VG, the characteristic function is

$$\Psi(u,t) = \mathbf{E}[e^{iuX(t)}] = \left(\frac{1}{1 - i\frac{\nu}{\mu}u}\right)^{\frac{\mu 2}{\nu}t}$$

Therefore the log-characteristic function could be written as

$$\psi(u,t) = \ln(\Psi(u,t)) = t\psi(u,1)$$

[38]This means that T in VG is replaced with Y_T. The rest remains identical.

In other words

$$\mathbf{E}[e^{iuX(t)}] = \Psi(u,t) = \exp(t\psi(u,1))$$

Using which, the VGSA characteristic function becomes

$$\mathbf{E}\left[e^{iuX(Y(t))}\right] = \mathbf{E}[\exp(Y(t)\psi(u,1))] = \phi(-i\psi(u,1))$$

with $\phi()$ the CIR characteristic function, namely

$$\phi(u_t) = \mathbf{E}[\exp(iuY_t)] = A(t,u)\exp(B(t,u)y_0)$$

where

$$A(t,u) = \frac{\exp(\kappa^2\eta t/\lambda^2)}{[\cosh(\gamma t/2) + \kappa/\gamma \sinh(\gamma t/2)]^{\frac{2\kappa\eta}{\lambda^2}}}$$
$$B(t,u) = \frac{2iu}{\kappa + \gamma\coth(\gamma t/2)}$$

and

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}$$

This allows us to determine the VGSA characteristic function, which we can use to calculate options prices via numeric Fourier inversion as described in [48] and [51].

Variance Gamma with Gamma Arrival Rate

For the variance gamma with gamma arrival rate (VGG), as before, the stock process under the risk-neutral framework is

$$d\ln S_t = (r - q + \omega)dt + X(h(dt);\sigma,\nu,\theta)$$

with $\omega = \frac{1}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2)$ and

$$X(h(dt);\sigma,\nu,\theta) = B(\gamma(h(dt),1,\nu);\theta,\sigma)$$

and the general gamma cumulative distribution function for $\gamma(h,\mu,\nu)$ is

$$F(\mu,\nu;h,x) = \frac{1}{\Gamma\left(\frac{\mu^2 h}{\nu}\right)}\left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 h}{\nu}}\int_0^x e^{-\frac{\mu t}{\nu}} t^{\frac{\mu^2 h}{\nu}-1}dt$$

and here $h(dt) = dY_t$ with Y_t is also gamma-distributed

$$dY_t = \gamma(dt,\mu_a,\nu_a)$$

The parameter set is therefore $\Psi = (\mu_a,\nu_a,\nu,\theta,\sigma)$.