

# Contents

<b>1</b>	<b>Infinite Sequences and Series</b>	<b>1</b>
1.1	Real and Complex Numbers	3
1.1.1	Arithmetic	3
1.1.2	Algebraic Equations	4
1.1.3	Infinite Sequences; Irrational Numbers	5
1.1.4	Sets of Real and Complex Numbers	7
1.2	Convergence of Infinite Series and Products	8
1.2.1	Convergence and Divergence; Absolute Convergence	8
1.2.2	Tests for Convergence of an Infinite Series of Positive Terms	10
1.2.3	Alternating Series and Rearrangements	11
1.2.4	Infinite Products	13
1.3	Sequences and Series of Functions	14
1.3.1	Pointwise Convergence and Uniform Convergence of Sequences of Functions	14
1.3.2	Weak Convergence; Generalized Functions	15
1.3.3	Infinite Series of Functions; Power Series	16
1.4	Asymptotic Series	19
1.4.1	The Exponential Integral	19
1.4.2	Asymptotic Expansions; Asymptotic Series	20
1.4.3	Laplace Integral; Watson's Lemma	22
A	Iterated Maps, Period Doubling, and Chaos	26
	Bibliography and Notes	30
	Problems	31
<b>2</b>	<b>Finite-Dimensional Vector Spaces</b>	<b>37</b>
2.1	Linear Vector Spaces	41
2.1.1	Linear Vector Space Axioms	41
2.1.2	Vector Norm; Scalar Product	43
2.1.3	Sum and Product Spaces	47
2.1.4	Sequences of Vectors	49
2.1.5	Linear Functionals and Dual Spaces	49
2.2	Linear Operators	51
2.2.1	Linear Operators; Domain and Image; Bounded Operators	51
2.2.2	Matrix Representation; Multiplication of Linear Operators	54

2.2.3	The Adjoint Operator . . . . .	56
2.2.4	Change of Basis; Rotations; Unitary Operators . . . . .	57
2.2.5	Invariant Manifolds . . . . .	61
2.2.6	Projection Operators . . . . .	63
2.3	Eigenvectors and Eigenvalues . . . . .	64
2.3.1	Eigenvalue Equation . . . . .	64
2.3.2	Diagonalization of a Linear Operator . . . . .	65
2.3.3	Spectral Representation of Normal Operators . . . . .	67
2.3.4	Minimax Properties of Eigenvalues of Self-Adjoint Operators . . . . .	71
2.4	Functions of Operators . . . . .	75
2.5	Linear Dynamical Systems . . . . .	77
A	Small Oscillations . . . . .	80
	Bibliography and Notes . . . . .	83
	Problems . . . . .	84
<b>3</b>	<b>Geometry in Physics</b> . . . . .	<b>93</b>
3.1	Manifolds and Coordinates . . . . .	97
3.1.1	Coordinates on Manifolds . . . . .	97
3.1.2	Some Elementary Manifolds . . . . .	98
3.1.3	Elementary Properties of Manifolds . . . . .	101
3.2	Vectors, Differential Forms, and Tensors . . . . .	104
3.2.1	Smooth Curves and Tangent Vectors . . . . .	104
3.2.2	Tangent Spaces and the Tangent Bundle $\mathcal{T}(\mathcal{M})$ . . . . .	105
3.2.3	Differential Forms . . . . .	106
3.2.4	Tensors . . . . .	109
3.2.5	Vector and Tensor Fields . . . . .	110
3.2.6	The Lie Derivative . . . . .	114
3.3	Calculus on Manifolds . . . . .	116
3.3.1	Wedge Product: $p$ -Forms and $p$ -Vectors . . . . .	116
3.3.2	Exterior Derivative . . . . .	120
3.3.3	Stokes' Theorem and its Generalizations . . . . .	123
3.3.4	Closed and Exact Forms . . . . .	128
3.4	Metric Tensor and Distance . . . . .	130
3.4.1	Metric Tensor of a Linear Vector Space . . . . .	130
3.4.2	Raising and Lowering Indices . . . . .	131
3.4.3	Metric Tensor of a Manifold . . . . .	132
3.4.4	Metric Tensor and Volume . . . . .	133
3.4.5	The Laplacian Operator . . . . .	134
3.4.6	Geodesic Curves on a Manifold . . . . .	135
3.5	Dynamical Systems and Vector Fields . . . . .	139
3.5.1	What is a Dynamical System? . . . . .	139
3.5.2	A Model from Ecology . . . . .	140
3.5.3	Lagrangian and Hamiltonian Systems . . . . .	142
3.6	Fluid Mechanics . . . . .	148

A	Calculus of Variations . . . . .	152
B	Thermodynamics . . . . .	153
	Bibliography and Notes . . . . .	158
	Problems . . . . .	159
<b>4</b>	<b>Functions of a Complex Variable</b>	<b>167</b>
4.1	Elementary Properties of Analytic Functions . . . . .	169
4.1.1	Cauchy–Riemann Conditions . . . . .	169
4.1.2	Conformal Mappings . . . . .	171
4.2	Integration in the Complex Plane . . . . .	176
4.2.1	Integration Along a Contour . . . . .	176
4.2.2	Cauchy’s Theorem . . . . .	177
4.2.3	Cauchy’s Integral Formula . . . . .	178
4.3	Analytic Functions . . . . .	179
4.3.1	Analytic Continuation . . . . .	179
4.3.2	Singularities of an Analytic Function . . . . .	182
4.3.3	Global Properties of Analytic Functions . . . . .	184
4.3.4	Laurent Series . . . . .	186
4.3.5	Infinite Product Representations . . . . .	188
4.4	Calculus of Residues: Applications . . . . .	190
4.4.1	Cauchy Residue Theorem . . . . .	190
4.4.2	Evaluation of Real Integrals . . . . .	191
4.5	Periodic Functions; Fourier Series . . . . .	195
4.5.1	Periodic Functions . . . . .	195
4.5.2	Doubly Periodic Functions . . . . .	197
A	Gamma Function; Beta Function . . . . .	199
A.1	Gamma Function . . . . .	199
A.2	Beta Function . . . . .	203
	Bibliography and Notes . . . . .	204
	Problems . . . . .	205
<b>5</b>	<b>Differential Equations: Analytical Methods</b>	<b>211</b>
5.1	Systems of Differential Equations . . . . .	213
5.1.1	General Systems of First-Order Equations . . . . .	213
5.1.2	Special Systems of Equations . . . . .	215
5.2	First-Order Differential Equations . . . . .	216
5.2.1	Linear First-Order Equations . . . . .	216
5.2.2	Ricatti Equation . . . . .	218
5.2.3	Exact Differentials . . . . .	220
5.3	Linear Differential Equations . . . . .	221
5.3.1	$n$ th Order Linear Equations . . . . .	221
5.3.2	Power Series Solutions . . . . .	222
5.3.3	Linear Independence; General Solution . . . . .	223
5.3.4	Linear Equation with Constant Coefficients . . . . .	225

5.4	Linear Second-Order Equations . . . . .	226
5.4.1	Classification of Singular Points . . . . .	226
5.4.2	Exponents at a Regular Singular Point . . . . .	226
5.4.3	One Regular Singular Point . . . . .	229
5.4.4	Two Regular Singular Points . . . . .	229
5.5	Legendre's Equation . . . . .	231
5.5.1	Legendre Polynomials . . . . .	231
5.5.2	Legendre Functions of the Second Kind . . . . .	235
5.6	Bessel's Equation . . . . .	237
5.6.1	Bessel Functions . . . . .	237
5.6.2	Hankel Functions . . . . .	239
5.6.3	Spherical Bessel Functions . . . . .	240
A	Hypergeometric Equation . . . . .	241
A.1	Reduction to Standard Form . . . . .	241
A.2	Power Series Solutions . . . . .	242
A.3	Integral Representations . . . . .	244
B	Confluent Hypergeometric Equation . . . . .	246
B.1	Reduction to Standard Form . . . . .	246
B.2	Integral Representations . . . . .	247
C	Elliptic Integrals and Elliptic Functions . . . . .	249
	Bibliography and Notes . . . . .	254
	Problems . . . . .	255
<b>6</b>	<b>Hilbert Spaces</b> . . . . .	<b>261</b>
6.1	Infinite-Dimensional Vector Spaces . . . . .	264
6.1.1	Hilbert Space Axioms . . . . .	264
6.1.2	Convergence in Hilbert space . . . . .	267
6.2	Function Spaces; Measure Theory . . . . .	268
6.2.1	Polynomial Approximation; Weierstrass Approximation Theorem . . . . .	268
6.2.2	Convergence in the Mean . . . . .	270
6.2.3	Measure Theory . . . . .	271
6.3	Fourier Series . . . . .	273
6.3.1	Periodic Functions and Trigonometric Polynomials . . . . .	273
6.3.2	Classical Fourier Series . . . . .	274
6.3.3	Convergence of Fourier Series . . . . .	275
6.3.4	Fourier Cosine Series; Fourier Sine Series . . . . .	279
6.4	Fourier Integral; Integral Transforms . . . . .	281
6.4.1	Fourier Transform . . . . .	281
6.4.2	Convolution Theorem; Correlation Functions . . . . .	284
6.4.3	Laplace Transform . . . . .	286
6.4.4	Multidimensional Fourier Transform . . . . .	287
6.4.5	Fourier Transform in Quantum Mechanics . . . . .	288
6.5	Orthogonal Polynomials . . . . .	289
6.5.1	Weight Functions and Orthogonal Polynomials . . . . .	289
6.5.2	Legendre Polynomials and Associated Legendre Functions . . . . .	290
6.5.3	Spherical Harmonics . . . . .	292

6.6	Haar Functions; Wavelets . . . . .	294
A	Standard Families of Orthogonal Polynomials . . . . .	305
	Bibliography and Notes . . . . .	310
	Problems . . . . .	311
<b>7</b>	<b>Linear Operators on Hilbert Space</b>	<b>319</b>
7.1	Some Hilbert Space Subtleties . . . . .	321
7.2	General Properties of Linear Operators on Hilbert Space . . . . .	324
7.2.1	Bounded, Continuous, and Closed Operators . . . . .	324
7.2.2	Inverse Operator . . . . .	325
7.2.3	Compact Operators; Hilbert–Schmidt Operators . . . . .	326
7.2.4	Adjoint Operator . . . . .	327
7.2.5	Unitary Operators; Isometric Operators . . . . .	329
7.2.6	Convergence of Sequences of Operators in $\mathcal{H}$ . . . . .	329
7.3	Spectrum of Linear Operators on Hilbert Space . . . . .	330
7.3.1	Spectrum of a Compact Self-Adjoint Operator . . . . .	330
7.3.2	Spectrum of Noncompact Normal Operators . . . . .	331
7.3.3	Resolution of the Identity . . . . .	332
7.3.4	Functions of a Self-Adjoint Operator . . . . .	335
7.4	Linear Differential Operators . . . . .	336
7.4.1	Differential Operators and Boundary Conditions . . . . .	336
7.4.2	Second-Order Linear Differential Operators . . . . .	338
7.5	Linear Integral Operators; Green Functions . . . . .	339
7.5.1	Compact Integral Operators . . . . .	339
7.5.2	Differential Operators and Green Functions . . . . .	341
	Bibliography and Notes . . . . .	344
	Problems . . . . .	345
<b>8</b>	<b>Partial Differential Equations</b>	<b>353</b>
8.1	Linear First-Order Equations . . . . .	356
8.2	The Laplacian and Linear Second-Order Equations . . . . .	359
8.2.1	Laplacian and Boundary Conditions . . . . .	359
8.2.2	Green Functions for Laplace’s Equation . . . . .	360
8.2.3	Spectrum of the Laplacian . . . . .	363
8.3	Time-Dependent Partial Differential Equations . . . . .	366
8.3.1	The Diffusion Equation . . . . .	367
8.3.2	Inhomogeneous Wave Equation: Advanced and Retarded Green Functions . . . . .	369
8.3.3	The Schrödinger Equation . . . . .	373
8.4	Nonlinear Partial Differential Equations . . . . .	376
8.4.1	Quasilinear First-Order Equations . . . . .	376
8.4.2	KdV Equation . . . . .	378
8.4.3	Scalar Field in 1 + 1 Dimensions . . . . .	380
8.4.4	Sine-Gordon Equation . . . . .	383
A	Lagrangian Field Theory . . . . .	384

Bibliography and Notes . . . . .	386
Problems . . . . .	387
<b>9 Finite Groups</b>	<b>391</b>
9.1 General Properties of Groups . . . . .	393
9.1.1 Group Axioms . . . . .	393
9.1.2 Cosets and Classes . . . . .	395
9.1.3 Algebras; Group Algebra . . . . .	397
9.2 Some Finite Groups . . . . .	399
9.2.1 Cyclic Groups . . . . .	399
9.2.2 Dihedral Groups . . . . .	399
9.2.3 Tetrahedral Group . . . . .	400
9.3 The Symmetric Group $S_N$ . . . . .	401
9.3.1 Permutations and the Symmetric Group $S_N$ . . . . .	401
9.3.2 Permutations and Partitions . . . . .	404
9.4 Group Representations . . . . .	406
9.4.1 Group Representations by Linear Operators . . . . .	406
9.4.2 Schur's Lemmas and Orthogonality Relations . . . . .	410
9.4.3 Kronecker Product of Representations . . . . .	417
9.4.4 Permutation Representations . . . . .	418
9.4.5 Representations of Groups and Subgroups . . . . .	422
9.5 Representations of the Symmetric Group $S_N$ . . . . .	424
9.5.1 Irreducible Representations of $S_N$ . . . . .	424
9.5.2 Outer Products of Representations of $\mathcal{S}_m \otimes \mathcal{S}_n$ . . . . .	426
9.5.3 Kronecker Products of Irreducible Representations of $S_N$ . . . . .	428
9.6 Discrete Infinite Groups . . . . .	431
A Frobenius Reciprocity Theorem . . . . .	435
B $S$ -Functions and Irreducible Representations of $S_N$ . . . . .	437
B.1 Frobenius Generating Function for the Simple Characters of $S_N$ . . . . .	437
B.2 Graphical Calculation of the Characters $\chi_{(m)}^{(\lambda)}$ . . . . .	442
B.3 Outer Products of Representations of $\mathcal{S}_m \otimes \mathcal{S}_n$ . . . . .	446
Bibliography and Notes . . . . .	451
Problems . . . . .	451
<b>10 Lie Groups and Lie Algebras</b>	<b>457</b>
10.1 Lie Groups . . . . .	460
10.2 Lie Algebras . . . . .	461
10.2.1 The Generators of a Lie Group . . . . .	461
10.2.2 The Lie Algebra of a Lie Group . . . . .	462
10.2.3 Classification of Lie Algebras . . . . .	465
10.3 Representations of Lie Algebras . . . . .	469
10.3.1 Irreducible Representations of $SU(2)$ . . . . .	469
10.3.2 Addition of Angular Momenta . . . . .	471
10.3.3 $S_N$ and the Irreducible Representations of $SU(2)$ . . . . .	474
10.3.4 Irreducible Representations of $SU(3)$ . . . . .	476

A	Tensor Representations of the Classical Lie Groups . . . . .	482
A.1	The Classical Lie Groups . . . . .	482
A.2	Tensor Representations of $U(n)$ and $SU(n)$ . . . . .	483
A.3	Irreducible Representations of $SO(n)$ . . . . .	487
B	Lorentz Group; Poincaré Group . . . . .	489
B.1	Lorentz Transformations . . . . .	489
B.2	$SL(2, C)$ and the Homogeneous Lorentz Group . . . . .	493
B.3	Inhomogeneous Lorentz Transformations; Poincaré Group . . . . .	496
	Bibliography and Notes . . . . .	498
	Problems . . . . .	499
<b>Index</b>		<b>507</b>

