

# Contents

|   |             |
|---|-------------|
| <b>Preface</b>  | <b>xiii</b> |
| <b>Acknowledgments</b>  | <b>xv</b>   |
| <b>About the Authors</b>  | <b>xvii</b> |
| <b>CHAPTER 1</b>  |             |
| <b>Concepts of Probability</b>  | <b>1</b>    |
| 1.1 Introduction  | 1           |
| 1.2 Basic Concepts  | 2           |
| 1.3 Discrete Probability Distributions  | 2           |
| 1.3.1 Bernoulli Distribution  | 3           |
| 1.3.2 Binomial Distribution   | 3           |
| 1.3.3 Poisson Distribution  | 4           |
| 1.4 Continuous Probability Distributions  | 5           |
| 1.4.1 Probability Distribution Function, Probability<br>Density Function, and Cumulative Distribution<br>Function | 5           |
| 1.4.2 The Normal Distribution   | 8           |
| 1.4.3 Exponential Distribution  | 10          |
| 1.4.4 Student's $t$ -distribution   | 11          |
| 1.4.5 Extreme Value Distribution  | 12          |
| 1.4.6 Generalized Extreme Value Distribution  | 12          |
| 1.5 Statistical Moments and Quantiles   | 13          |
| 1.5.1 Location  | 13          |
| 1.5.2 Dispersion  | 13          |
| 1.5.3 Asymmetry   | 13          |
| 1.5.4 Concentration in Tails  | 14          |
| 1.5.5 Statistical Moments   | 14          |
| 1.5.6 Quantiles   | 16          |
| 1.5.7 Sample Moments  | 16          |
| 1.6 Joint Probability Distributions   | 17          |
| 1.6.1 Conditional Probability   | 18          |
| 1.6.2 Definition of Joint Probability Distributions   | 19          |
|   | <b>vii</b>  |

|       |                                  |    |
|-------|----------------------------------|----|
| 1.6.3 | Marginal Distributions           | 19 |
| 1.6.4 | Dependence of Random Variables   | 20 |
| 1.6.5 | Covariance and Correlation       | 20 |
| 1.6.6 | Multivariate Normal Distribution | 21 |
| 1.6.7 | Elliptical Distributions         | 23 |
| 1.6.8 | Copula Functions                 | 25 |
| 1.7   | Probabilistic Inequalities       | 30 |
| 1.7.1 | Chebyshev's Inequality           | 30 |
| 1.7.2 | Fréchet-Hoeffding Inequality     | 31 |
| 1.8   | Summary                          | 32 |

**CHAPTER 2****Optimization****35**

|       |   |    |
|-------|---|----|
| 2.1   | Introduction                                      | 35 |
| 2.2   | Unconstrained Optimization                        | 36 |
| 2.2.1 | Minima and Maxima of a Differentiable<br>Function | 37 |
| 2.2.2 | Convex Functions                                  | 40 |
| 2.2.3 | Quasiconvex Functions                             | 46 |
| 2.3   | Constrained Optimization                          | 48 |
| 2.3.1 | Lagrange Multipliers                              | 49 |
| 2.3.2 | Convex Programming                                | 52 |
| 2.3.3 | Linear Programming                                | 55 |
| 2.3.4 | Quadratic Programming                             | 57 |
| 2.4   | Summary   | 58 |

**CHAPTER 3****Probability Metrics****61**

|       |  |    |
|-------|--|----|
| 3.1   | Introduction                           | 61 |
| 3.2   | Measuring Distances: The Discrete Case | 62 |
| 3.2.1 | Sets of Characteristics                | 63 |
| 3.2.2 | Distribution Functions                 | 64 |
| 3.2.3 | Joint Distribution                     | 68 |
| 3.3   | Primary, Simple, and Compound Metrics  | 72 |
| 3.3.1 | Axiomatic Construction                 | 73 |
| 3.3.2 | Primary Metrics                        | 74 |
| 3.3.3 | Simple Metrics                         | 75 |
| 3.3.4 | Compound Metrics                       | 84 |
| 3.3.5 | Minimal and Maximal Metrics            | 86 |
| 3.4   | Summary                                | 90 |
| 3.5   | Technical Appendix                     | 90 |

|       |  |    |
|-------|--|----|
| 3.5.1 | Remarks on the Axiomatic Construction of Probability Metrics | 91 |
| 3.5.2 | Examples of Probability Distances                            | 94 |
| 3.5.3 | Minimal and Maximal Distances                                | 99 |

**CHAPTER 4**

**Ideal Probability Metrics**

**103**

|       |   |     |
|-------|---|-----|
| 4.1   | Introduction  | 103 |
| 4.2   | The Classical Central Limit Theorem                   | 105 |
| 4.2.1 | The Binomial Approximation to the Normal Distribution | 105 |
| 4.2.2 | The General Case                                      | 112 |
| 4.2.3 | Estimating the Distance from the Limit Distribution   | 118 |
| 4.3   | The Generalized Central Limit Theorem                 | 120 |
| 4.3.1 | Stable Distributions                                  | 120 |
| 4.3.2 | Modeling Financial Assets with Stable Distributions   | 122 |
| 4.4   | Construction of Ideal Probability Metrics             | 124 |
| 4.4.1 | Definition  | 125 |
| 4.4.2 | Examples  | 126 |
| 4.5   | Summary   | 131 |
| 4.6   | Technical Appendix                                    | 131 |
| 4.6.1 | The CLT Conditions                                    | 131 |
| 4.6.2 | Remarks on Ideal Metrics                              | 133 |

**CHAPTER 5**

**Choice under Uncertainty**

**139**

|       |   |     |
|-------|---|-----|
| 5.1   | Introduction  | 139 |
| 5.2   | Expected Utility Theory                             | 141 |
| 5.2.1 | St. Petersburg Paradox                              | 141 |
| 5.2.2 | The von Neumann–Morgenstern Expected Utility Theory | 143 |
| 5.2.3 | Types of Utility Functions                          | 145 |
| 5.3   | Stochastic Dominance                                | 147 |
| 5.3.1 | First-Order Stochastic Dominance                    | 148 |
| 5.3.2 | Second-Order Stochastic Dominance                   | 149 |
| 5.3.3 | Rothschild–Stiglitz Stochastic Dominance            | 150 |
| 5.3.4 | Third-Order Stochastic Dominance                    | 152 |
| 5.3.5 | Efficient Sets and the Portfolio Choice Problem     | 154 |
| 5.3.6 | Return versus Payoff                                | 154 |

|       |   |     |
|-------|---|-----|
| 5.4   | Probability Metrics and Stochastic Dominance  | 157 |
| 5.5   | Summary                                       | 161 |
| 5.6   | Technical Appendix                            | 161 |
| 5.6.1 | The Axioms of Choice                          | 161 |
| 5.6.2 | Stochastic Dominance Relations of Order $n$   | 163 |
| 5.6.3 | Return versus Payoff and Stochastic Dominance | 164 |
| 5.6.4 | Other Stochastic Dominance Relations          | 166 |

**CHAPTER 6****Risk and Uncertainty****171**

|       |   |     |
|-------|---|-----|
| 6.1   | Introduction                                | 171 |
| 6.2   | Measures of Dispersion                      | 174 |
| 6.2.1 | Standard Deviation                          | 174 |
| 6.2.2 | Mean Absolute Deviation                     | 176 |
| 6.2.3 | Semistandard Deviation                      | 177 |
| 6.2.4 | Axiomatic Description                       | 178 |
| 6.2.5 | Deviation Measures                          | 179 |
| 6.3   | Probability Metrics and Dispersion Measures | 180 |
| 6.4   | Measures of Risk                            | 181 |
| 6.4.1 | Value-at-Risk                               | 182 |
| 6.4.2 | Computing Portfolio VaR in Practice         | 186 |
| 6.4.3 | Backtesting of VaR                          | 192 |
| 6.4.4 | Coherent Risk Measures                      | 194 |
| 6.5   | Risk Measures and Dispersion Measures       | 198 |
| 6.6   | Risk Measures and Stochastic Orders         | 199 |
| 6.7   | Summary                                     | 200 |
| 6.8   | Technical Appendix                          | 201 |
| 6.8.1 | Convex Risk Measures                        | 201 |
| 6.8.2 | Probability Metrics and Deviation Measures  | 202 |

**CHAPTER 7****Average Value-at-Risk****207**

|       |                                      |     |
|-------|--------------------------------------|-----|
| 7.1   | Introduction                         | 207 |
| 7.2   | Average Value-at-Risk                | 208 |
| 7.3   | AVaR Estimation from a Sample        | 214 |
| 7.4   | Computing Portfolio AVaR in Practice | 216 |
| 7.4.1 | The Multivariate Normal Assumption   | 216 |
| 7.4.2 | The Historical Method                | 217 |
| 7.4.3 | The Hybrid Method                    | 217 |
| 7.4.4 | The Monte Carlo Method               | 218 |
| 7.5   | Backtesting of AVaR                  | 220 |

|                                    |   |            |
|------------------------------------|---|------------|
| 7.6                                | Spectral Risk Measures                            | 222        |
| 7.7                                | Risk Measures and Probability Metrics             | 224        |
| 7.8                                | Summary   | 227        |
| 7.9                                | Technical Appendix                                | 227        |
| 7.9.1                              | Characteristics of Conditional Loss Distributions | 228        |
| 7.9.2                              | Higher-Order AVaR                                 | 230        |
| 7.9.3                              | The Minimization Formula for AVaR                 | 232        |
| 7.9.4                              | AVaR for Stable Distributions                     | 235        |
| 7.9.5                              | ETL versus AVaR                                   | 236        |
| 7.9.6                              | Remarks on Spectral Risk Measures                 | 241        |
| <b>CHAPTER 8</b>                   |   |            |
| <b>Optimal Portfolios</b>          |   | <b>245</b> |
| 8.1                                | Introduction                                      | 245        |
| 8.2                                | Mean-Variance Analysis                            | 247        |
| 8.2.1                              | Mean-Variance Optimization Problems               | 247        |
| 8.2.2                              | The Mean-Variance Efficient Frontier              | 251        |
| 8.2.3                              | Mean-Variance Analysis and SSD                    | 254        |
| 8.2.4                              | Adding a Risk-Free Asset                          | 256        |
| 8.3                                | Mean-Risk Analysis                                | 258        |
| 8.3.1                              | Mean-Risk Optimization Problems                   | 259        |
| 8.3.2                              | The Mean-Risk Efficient Frontier                  | 262        |
| 8.3.3                              | Mean-Risk Analysis and SSD                        | 266        |
| 8.3.4                              | Risk versus Dispersion Measures                   | 267        |
| 8.4                                | Summary   | 274        |
| 8.5                                | Technical Appendix                                | 274        |
| 8.5.1                              | Types of Constraints                              | 274        |
| 8.5.2                              | Quadratic Approximations to Utility Functions     | 276        |
| 8.5.3                              | Solving Mean-Variance Problems in Practice        | 278        |
| 8.5.4                              | Solving Mean-Risk Problems in Practice            | 279        |
| 8.5.5                              | Reward-Risk Analysis                              | 281        |
| <b>CHAPTER 9</b>                   |   |            |
| <b>Benchmark Tracking Problems</b> |   | <b>287</b> |
| 9.1                                | Introduction                                      | 287        |
| 9.2                                | The Tracking Error Problem                        | 288        |
| 9.3                                | Relation to Probability Metrics                   | 292        |
| 9.4                                | Examples of r.d. Metrics                          | 296        |
| 9.5                                | Numerical Example                                 | 300        |
| 9.6                                | Summary   | 304        |

|       |   |     |
|-------|---|-----|
| 9.7   | Technical Appendix  | 304 |
| 9.7.1 | Deviation Measures and r.d. Metrics                           | 305 |
| 9.7.2 | Remarks on the Axioms   | 305 |
| 9.7.3 | Minimal r.d. Metrics  | 307 |
| 9.7.4 | Limit Cases of $\mathcal{L}_p^*(X, Y)$ and $\Theta_p^*(X, Y)$ | 310 |
| 9.7.5 | Computing r.d. Metrics in Practice                            | 311 |

**CHAPTER 10****Performance Measures****317**

|        |  |     |
|--------|--|-----|
| 10.1   | Introduction   | 317 |
| 10.2   | Reward-to-Risk Ratios                                      | 318 |
| 10.2.1 | RR Ratios and the Efficient Portfolios                     | 320 |
| 10.2.2 | Limitations in the Application of<br>Reward-to-Risk Ratios | 324 |
| 10.2.3 | The STARR  | 325 |
| 10.2.4 | The Sortino Ratio  | 329 |
| 10.2.5 | The Sortino-Satchell Ratio                                 | 330 |
| 10.2.6 | A One-Sided Variability Ratio                              | 331 |
| 10.2.7 | The Rachev Ratio   | 332 |
| 10.3   | Reward-to-Variability Ratios                               | 333 |
| 10.3.1 | RV Ratios and the Efficient Portfolios                     | 335 |
| 10.3.2 | The Sharpe Ratio   | 337 |
| 10.3.3 | The Capital Market Line and the Sharpe Ratio               | 340 |
| 10.4   | Summary  | 343 |
| 10.5   | Technical Appendix   | 343 |
| 10.5.1 | Extensions of STARR  | 343 |
| 10.5.2 | Quasiconcave Performance Measures                          | 345 |
| 10.5.3 | The Capital Market Line and Quasiconcave<br>Ratios         | 353 |
| 10.5.4 | Nonquasiconcave Performance Measures                       | 356 |
| 10.5.5 | Probability Metrics and Performance Measures               | 357 |

**Index****361**