CAT

Part 1

Whole Numbers

Terms You Should Know

After each word in the Glossary, the lesson where it first appears is cited. Occasionally a second or third appearance is also given if there are additional major references, but not all appearances of each word are necessarily cited.

append (Lesson 12). Stick on to the end; Example: to multiply 43 by 100, append two zeroes to the 43. [$43 \times 100 = 43\mathbf{00}$].

associative property (Lessons 2, 4). Applies to addition and multiplication only and says however you group the numbers for addition or multiplication, the answer is not affected.

cardinal numbers (Lesson 1). The numbers we count with; a.k.a. counting numbers or natural numbers.

commutative property for addition (Lesson 2). No matter which way you add two numbers, the answer is always the same. That may be abbreviated as $a + b = b + a = c$.

commutative property for multiplication (Lesson 4). No matter which way you multiply two numbers, the answer is always the same. That may be abbreviated as $a \times b = b \times a = c$.

composite number (Lesson 14). The name applied to a number with more than 2 factors.

counting numbers (Lesson 1). See cardinal numbers.

decade (Lesson 13). A group of ten things; another name for the tens place (in addition, of course, to being ten years).

difference (Lesson 3). The answer in a subtraction; see remainder.

dividend (Lesson 10). In a division, the number being divided into.

divisible (Lesson 14). Can be perfectly divided by another.

divisor (Lesson 5). The number being divided by in a division.

exchange (Lessons 2, 3). See "rename"; also used to refer to money or barter.

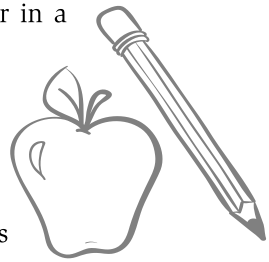
factor (Lessons 9, 12, 14). **1.** *n.* A number multiplied to form another. **2.** *v.t.* To divide a quantity out of another.

ladder division (Lesson 10). A form of division in which groups of the divisor are repeatedly subtracted and written on steps below the division bracket.

magnitude (Lesson 1). A fancy word for size.

minuend (Lesson 8). The top number in a place value subtraction, or the number being subtracted from.

multiplicand (Lesson 9). The number being multiplied; the top number in a place value multiplication.



multiplier (Lesson 9). The number being multiplied by; the bottom number in a place value multiplication.

natural numbers (Lesson 1). See cardinal numbers.

ordinal numbers (Lesson 1). The numbers that are used to show position, that is 1st, 2nd, 3rd, . . . 25th, and so on.

parameter (Lesson 13). Is a word with several different meanings, but it is used in this book to mean a boundary or limit.

prime number (Lesson 14). A number which has exactly two factors, itself and one.

product (Lesson 4). The answer in a multiplication.

quotient (Lesson 5). The answer in a division.

regroup (Lesson 6). See "rename."

remainder (Lesson 3). **1.** The answer in a subtraction; see difference. **2.** The leftover amount in a division; the part not divided.

rename (Lesson 6). **1.** In addition, regroup ten of one quantity for one of the group under the column heading to its immediate left. **2.** In subtraction, regroup one of a quantity for ten in the column to its immediate right; also called "exchange."

running total (Lesson 12). An arithmetic operation where subtotals are found using two numbers at a time, and then another number is added to, subtracted from, used to multiply, or divided into the prior total; (often done with a calculator).

Sieve of Eratosthenes (Lesson 14). A hundreds square 10 across and 10 high naming the first 100 natural numbers and used to find prime numbers.

subtrahend (Lesson 8). The amount being subtracted, or the bottom number in a place value subtraction.

sum (Lesson 2). The answer in an addition.

tolerance (Lesson 11). The allowable amount of variation from a preset standard.

trial dividend (Lesson 10). Created by rounding the first two digits of a division's dividend or partial dividend to the nearest 10 for use with the trial divisor.

trial divisor (Lesson 10). Created by rounding the divisor to the nearer 10, 100, or whatever and then using only the leftmost digit to estimate a partial quotient in conjunction with the partial dividend.

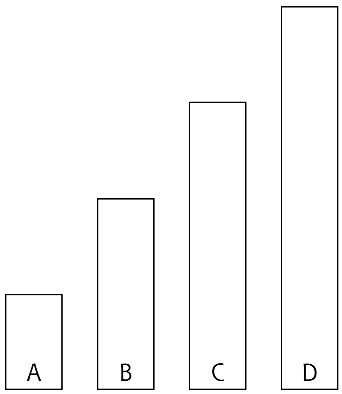
within tolerance (Lesson 11). Within acceptable limits; see "tolerance."

LESSON 1

Order and Magnitude

Most people think of numbers as a way of counting things, such as the number of students in a class or the number of lemon seeds in a glass of iced tea. Counting certainly is one way to use numbers. The numbers we count with are known as **cardinal numbers.** They have other names too, such as **counting numbers** and **natural numbers.** Totally distinct from the cardinal numbers, but obviously related to them, are the **ordinal numbers.** These are the numbers that are used to show position, that is 1st, 2nd, 3rd, . . . 25th, and so on. Some elementary math teachers will dwell upon these distinctions, but others will not. It is, however, a distinction that you should make sure that your child knows.

A much more important distinction exists in the meaning of numbers that some children are not taught, but are left to stumble upon, if they ever do. Rather than spelling it out immediately, I'd like you to consider the following diagram.



Some Rectangles

Suppose that we let the rectangle labeled "A" have a value of 1. Which rectangle would have a value of 2? You didn't have to think about that for very long, did you? If "A" has the value of 1, then the next one, "B" would be two. It's simple enough. Two comes after one; one comes before two.

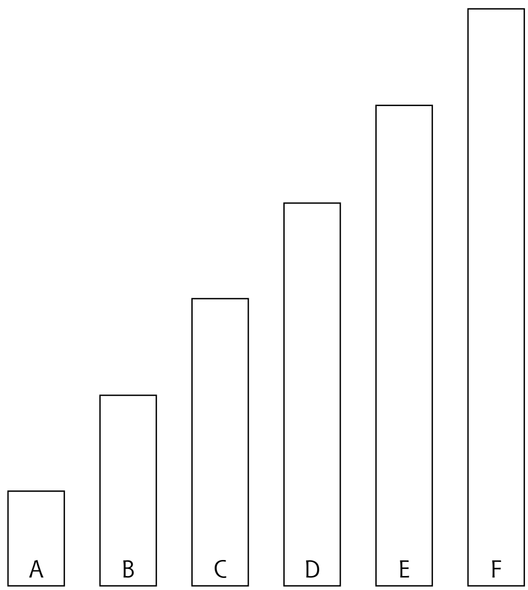
Now, with that in mind, let's assign the value 1 to rectangle "B." Which rectangle would have the value 2? Hmm, this one may require a bit more thought. I'm certain that you're not considering rectangle "A" for a multitude of reasons, which I shall not yet share with you. How about rectangle "C?" C is next in line. Just as "B" followed "A", "C" follows "B." Well, that might or might not look good to you, but I'm afraid that "C" won't do. That's because there's more to the concept of numbers than just order or sequence. Two doesn't just come after one. Its **magnitude** is twice that of one. Which rectangle follows "B" and is twice as big as "B?" That would be rectangle "D." Therefore, if "B" is 1, "D" is 2.

Order and magnitude are the two essential properties of numbers. The reasons I was so sure you would not select “A” for 2 in the last mental exercise was that you would never move to the left to get a larger number, and you would never pick a smaller bar to represent a larger quantity. I know from many years of teaching experience that more than 50% of students will disregard relative magnitude and just go for the next larger in line, “C.”

By the way, if “B” is 1 and “D” is 2, what are “A” and “C?” Take a moment and think about it. Since “A” is half the size of “B,” it would stand for the number $\frac{1}{2}$. “C” is as big as “B” added to “A”, and so would represent the number $1\frac{1}{2}$, but that’s a topic for later concern.

Suppose that we assigned the value 1 to the rectangle “C.” Suppose also that we added rectangles that grew at the same rate as the four pictured previously and were sequentially lettered. Which invisible lettered rectangle would have the value 2?

Did you think about it? It’s going to be twice the size of “C”. Just in case you need help, here’s another figure. If you got the correct answer without need for the figure, good job.



More Rectangles

The rectangle that’s twice the height of “C” is “F,” so when “C” has a value of 1, “F” has a value of 2. Remember, sequence *and* magnitude. Make sure that your child learns that. If you would like to know the values of “A,” “B,” “D,” and “E,” after you’ve thought about it, look at the footnote at the bottom of this page.*

*Answers: “A” = $\frac{1}{3}$ “B” = $\frac{2}{3}$ “D” = $1\frac{1}{3}$ and “E” = $1\frac{2}{3}$

LESSON 2

Addition Facts

Addition (or adding) is the first and simplest combining operation. It is based upon counting and then naming the highest number counted. For example, if there were two sheep to your left and one sheep to your right, you might start on the right and count one and then turn left and continue two, three, hence 1 + 2 = 3. I'm sure you know that " + " is the plus sign indicating addition and " = " means "is equal to," but I don't want to take anything for granted.

It is essential that **addition facts** be committed to memory. By addition facts, I mean all **sums** (results of additions) totaling 10 or less. It is best to begin using counters of some type. Actually more than a single type is best. Pennies are readily available, as are pencils, matchsticks, nickels, paperclips, and so forth. Start your child out with 10 of any one of these, and let her show and tell you all combinations adding up to 10 or less, two at a time. Then do the same with a different counter. After four or five such exercises, move on to paper and pencil. There are 100 addition facts, and they follow this paragraph.

1. 1 + 0 = 1	**19.** 2 + 8 = 10	**37.** 5 + 2 = 7
2. 1 + 1 = 2	**20.** 3 + 0 = 3	**38.** 5 + 3 = 8
3. 1 + 2 = 3	**21.** 3 + 1 = 4	**39.** 5 + 4 = 9
4. 1 + 3 = 4	**22.** 3 + 2 = 5	**40.** 5 + 5 = 10
5. 1 + 4 = 5	**23.** 3 + 3 = 6	**41.** 6 + 0 = 6
6. 1 + 5 = 6	**24.** 3 + 4 = 7	**42.** 6 + 1 = 7
7. 1 + 6 = 7	**25.** 3 + 5 = 8	**43.** 6 + 2 = 8
8. 1 + 7 = 8	**26.** 3 + 6 = 9	**44.** 6 + 3 = 9
9. 1 + 8 = 9	**27.** 3 + 7 = 10	**45.** 6 + 4 = 10
10. 1 + 9 = 10	**28.** 4 + 0 = 4	**46.** 7 + 0 = 7
11. 2 + 0 = 2	**29.** 4 + 1 = 5	**47.** 7 + 1 = 8
12. 2 + 1 = 3	**30.** 4 + 2 = 6	**48.** 7 + 2 = 9
13. 2 + 2 = 4	**31.** 4 + 3 = 7	**49.** 7 + 3 = 10
14. 2 + 3 = 5	**32.** 4 + 4 = 8	**50.** 8 + 0 = 8
15. 2 + 4 = 6	**33.** 4 + 5 = 9	**51.** 8 + 1 = 9
16. 2 + 5 = 7	**34.** 4 + 6 = 10	**52.** 8 + 2 = 10
17. 2 + 6 = 8	**35.** 5 + 0 = 5	**53.** 9 + 0 = 9
18. 2 + 7 = 9	**36.** 5 + 1 = 6	**54.** 9 + 1 = 10

"What?" you're no doubt gasping. "I thought the book said 100."

Well, I kind of lied to you—twice. First of all, I said "I call addition facts sums to 10 and less." I do, but your child's teacher is probably going to call them sums to 20 or less. But don't get mad yet. I told you that you needed 100 addition facts, although, in fact, the 54 above make the ones we haven't yet gotten to much easier. If you learn the sums we've already done, you'll recognize which combinations make 10. That will make the rest much easier, for reasons you'll see later.

Before going on, I want to call your attention to the addition of "0". For reasons that should be apparent, "0" is known as the identity element for addition. Adding "0" to a number results in a sum that is identical to what you began with. All right. Let's finish the job we started.

55. $2 + 9 = ?$

Well, we know from having learned the preceding facts that $2 + 8 = 10$. 9 is 1 more than 8, so $2 + 9 = (2 + 8) + 1$. The parentheses are used for grouping purposes, tying together what we are going to add next. $(2 + 8) + 1 = 10 + 1 = 11$. Did you follow that? We group to 10 and then add on whatever's left over. Here's the next one.

56. $3 + 8 = ?$ Well, $2 + 8 = 10$, so $3 + 8 = (2 + 8) + 1 = 11$.

57. $3 + 9 = 2 + 10 = 12$
58. $4 + 7 = 1 + 10 = 11$
59. $4 + 8 = 2 + 10 = 12$
60. $4 + 9 = 3 + 10 = 13$
61. $5 + 6 = 1 + 10 = 11$
62. $5 + 7 = 2 + 10 = 12$
63. $5 + 8 = 3 + 10 = 13$
64. $5 + 9 = 4 + 10 = 14$
65. $6 + 5 = 1 + 10 = 11$
66. $6 + 6 = 2 + 10 = 12$
67. $6 + 7 = 3 + 10 = 13$
68. $6 + 8 = 4 + 10 = 14$
69. $6 + 9 = 5 + 10 = 15$
70. $7 + 4 = 1 + 10 = 11$
71. $7 + 5 = 2 + 10 = 12$
72. $7 + 6 = 3 + 10 = 13$
73. $7 + 7 = 4 + 10 = 14$
74. $7 + 8 = 5 + 10 = 15$
75. $7 + 9 = 6 + 10 = 16$
76. $8 + 3 = 1 + 10 = 11$
77. $8 + 4 = 2 + 10 = 12$
78. $8 + 5 = 3 + 10 = 13$
79. $8 + 6 = 4 + 10 = 14$
80. $8 + 7 = 5 + 10 = 15$
81. $8 + 8 = 6 + 10 = 16$
82. $8 + 9 = 7 + 10 = 17$
83. $9 + 2 = 1 + 10 = 11$
84. $9 + 3 = 2 + 10 = 12$
85. $9 + 4 = 3 + 10 = 13$
86. $9 + 5 = 4 + 10 = 14$
87. $9 + 6 = 5 + 10 = 15$
88. $9 + 7 = 6 + 10 = 16$
89. $9 + 8 = 7 + 10 = 17$
90. $9 + 9 = 8 + 10 = 18$

I think we can skip the $10 + 0$ through 9 that would round out the hundred, don't you?

You should know a couple of properties of addition. First is the **commutative property** that says no matter which way you add two numbers, the answer is always the same. That's usually abbreviated as $5 + 3 = 3 + 5 = 8$, or some such example.

Then there's the **associative property,** which is based upon addition (and all arithmetic operations) being **binary.** Binary is a fancy form of the word two. The point is, you can add only two numbers at a time. If you're going to add, say, $3 + 4 + 5$, you have to group them in pairs. You can group $(3 + 4) + 5$, meaning you'll add the $3 + 4$ first and then add 5 to the resulting 7, or you can group $3 + (4 + 5)$ and then add 3 to the resulting 9. Or, you could rewrite it as $(3 + 5) + 4$, and get $8 + 4$. The associative property says that however you group the numbers for addition, the answer is not affected. The preceding groups all total to 12. See? It all adds up!

LESSON 3

Subtraction Facts

Subtraction (or taking away) is the first uncombining (or inverse) operation, and the simpler of the two. It is based upon counting backward and then naming the highest number counted. For example, if you had five chickens and you gave away two, you might start with the five and count five, less one is four, less another is three, hence $5 - 2 = 3$. I'm sure you know that "–" is the minus sign indicating subtraction and " = " we've already encountered.

It is not at all essential that **subtraction facts** be committed to memory, since you can think of them as backward addition facts. Think about it: $8 - 3 = 5$. That's really another way of saying $5 + 3 = 8$. It's really more important to be able to relate subtraction facts to addition facts than to memorize them. Much as I hate to say it, it's possible to go through life using subtraction without ever learning the subtraction facts. You see it at the grocery store every day—if you go to a local non-Megamart. Buy a 35-cent pack of gum (if such a thing exists), and hand the cashier a dollar. He'll go "35 and 5, makes 40, and 10 makes 50, and 50 makes a dollar," and hand you 65 cents change. I call this "subtracting by adding up," or the "making change method."

I doubt that your kid's teacher is going to be placated by my musings here, so again, begin by using counters of some type. Start with a collection of 10 counters of one type and remove groups of varying numbers from that collection. Then do the same with a different counter. After many such exercises, move on to paper and pencil. Don't be shy about making flash cards on 3-×-5-inch index cards. The essential subtraction facts are listed here.

1. $1 - 0 = 0$
2. $2 - 0 = 2$
3. $3 - 0 = 3$
4. $4 - 0 = 4$
5. $5 - 0 = 5$
6. $6 - 0 = 6$
7. $7 - 0 = 7$
8. $8 - 0 = 8$
9. $9 - 0 = 9$
10. $1 - 1 = 0$
11. $2 - 1 = 1$
12. $2 - 2 = 0$
13. $3 - 1 = 2$
14. $3 - 2 = 1$
15. $3 - 3 = 0$
16. $4 - 1 = 3$
17. $4 - 2 = 2$
18. $4 - 3 = 1$
19. $4 - 4 = 0$
20. $5 - 1 = 4$
21. $5 - 2 = 3$
22. $5 - 3 = 2$
23. $5 - 4 = 1$
24. $5 - 5 = 0$
25. $6 - 1 = 5$
26. $6 - 2 = 4$
27. $6 - 3 = 3$
28. $6 - 4 = 2$
29. $6 - 5 = 1$
30. $6 - 6 = 0$
31. $7 - 1 = 6$
32. $7 - 2 = 5$
33. $7 - 3 = 4$
34. $7 - 4 = 3$
35. $7 - 5 = 2$
36. $7 - 6 = 1$
37. $7 - 7 = 0$
38. $8 - 1 = 7$
39. $8 - 2 = 6$
40. $8 - 3 = 5$
41. $8 - 4 = 4$
42. $8 - 5 = 3$
43. $8 - 6 = 2$
44. $8 - 7 = 1$
45. $8 - 8 = 0$

46. 9 – 1 = 8
47. 9 – 2 = 7
48. 9 – 3 = 6
49. 9 – 4 = 5
50. 9 – 5 = 4
51. 9 – 6 = 3
52. 9 – 7 = 2
53. 9 – 8 = 1
54. 9 – 9 = 0
55. 10 – 1 = 9
56. 10 – 2 = 8
57. 10 – 3 = 7
58. 10 – 4 = 6
59. 10 – 5 = 5
60. 10 – 6 = 4
61. 10 – 7 = 3
62. 10 – 8 = 2
63. 10 – 9 = 1
64. 10 – 10 = 0

Just to break up the monotony, let me interject the hope that you've noticed that "0" is the identity element in subtraction as well as in addition. That is to say, subtracting 0 does not affect any number's value.

When larger numbers are subtracted, the place value property will kick in, and you'll never be required to subtract from anything greater than 9, but I'm getting ahead of myself. Just remember that it will be necessary to consider the sums that combine to make 10. That consideration will help to guide you when subtracting.

The answer to a subtraction is known as a **difference** or a **remainder.** Like addition, subtraction is binary. You'll recall that means you can operate with only two numbers at a time.

Unlike addition, order does matter. No commutative property exists for subtraction: 5 – 3 does not yield the same result as 3 – 5. There is also no associative property for subtraction.

One final thought on subtraction facts: Learn your subtraction facts, and you'll always make a difference (pun intended)!

LESSON 4

Multiplication Facts

As already noted in the Introduction, multiplication is a shortcut for repeated addition of the same number. You and/or your student may learn the multiplication facts by rote, or you can learn just some of them and use those to build others. For example, suppose that you need to know what 7×8 is, but you just happen to know that $5 \times 8 = 40$ and $2 \times 8 = 16$. Well, here's a thought for you. Build the multiplication facts you don't know by adding those that you do:

$$\begin{array}{r} 5 \times 8 = 40 \\ +\underline{2 \times 8 = 16} \\ 7 \times 8 = 56 \end{array}$$

There really are 100 multiplication facts. I wouldn't lie to you (unless it served a purpose, of course).

1. $1 \times 1 = 1$
2. $1 \times 2 = 2$
3. $1 \times 3 = 3$
4. $1 \times 4 = 4$
5. $1 \times 5 = 5$
6. $1 \times 6 = 6$
7. $1 \times 7 = 7$
8. $1 \times 8 = 8$
9. $1 \times 9 = 9$
10. $1 \times 10 = 10$
11. $2 \times 1 = 2$
12. $2 \times 2 = 4$
13. $2 \times 3 = 6$
14. $2 \times 4 = 8$
15. $2 \times 5 = 10$
16. $2 \times 6 = 12$
17. $2 \times 7 = 14$
18. $2 \times 8 = 16$
19. $2 \times 9 = 18$
20. $2 \times 10 = 20$
21. $3 \times 1 = 3$
22. $3 \times 2 = 6$
23. $3 \times 3 = 9$
24. $3 \times 4 = 12$
25. $3 \times 5 = 15$
26. $3 \times 6 = 18$
27. $3 \times 7 = 21$
28. $3 \times 8 = 24$
29. $3 \times 9 = 27$
30. $3 \times 10 = 30$
31. $4 \times 1 = 4$
32. $4 \times 2 = 8$
33. $4 \times 3 = 12$
34. $4 \times 4 = 16$
35. $4 \times 5 = 20$
36. $4 \times 6 = 24$
37. $4 \times 7 = 28$
38. $4 \times 8 = 32$
39. $4 \times 9 = 36$
40. $4 \times 10 = 40$
41. $5 \times 1 = 5$
42. $5 \times 2 = 10$
43. $5 \times 3 = 15$
44. $5 \times 4 = 20$
45. $5 \times 5 = 25$
46. $5 \times 6 = 30$
47. $5 \times 7 = 35$
48. $5 \times 8 = 40$
49. $5 \times 9 = 45$
50. $5 \times 10 = 50$

Before going on, I want to call your attention to multiplication by or of "1". For reasons that should be apparent, "1" is known as the identity element for multiplication. Your multiplying "1" and another number results in a **product** (the name we give the answer in a multiplication) that is identical to what you began with. Also, multiplication by "0" results in a product of "0." All right. Let's finish the job we started.

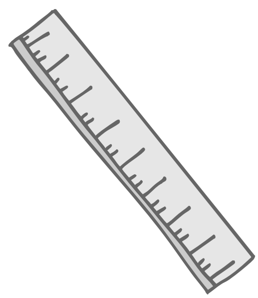
Templates for multiplication flash cards may be found in the Appendix.

51. $6 \times 1 = 6$
52. $6 \times 2 = 12$
53. $6 \times 3 = 18$
54. $6 \times 4 = 24$
55. $6 \times 5 = 30$
56. $6 \times 6 = 36$
57. $6 \times 7 = 42$
58. $6 \times 8 = 48$
59. $6 \times 9 = 54$
60. $6 \times 10 = 60$
61. $7 \times 1 = 7$
62. $7 \times 2 = 14$
63. $7 \times 3 = 21$
64. $7 \times 4 = 28$
65. $7 \times 5 = 35$
66. $7 \times 6 = 42$
67. $7 \times 7 = 49$
68. $7 \times 8 = 56$
69. $7 \times 9 = 63$
70. $7 \times 10 = 70$
71. $8 \times 1 = 8$
72. $8 \times 2 = 16$
73. $8 \times 3 = 24$
74. $8 \times 4 = 32$
75. $8 \times 5 = 40$
76. $8 \times 6 = 48$
77. $8 \times 7 = 56$
78. $8 \times 8 = 64$
79. $8 \times 9 = 72$
80. $8 \times 10 = 80$
81. $9 \times 1 = 9$
82. $9 \times 2 = 18$
83. $9 \times 3 = 27$
84. $9 \times 4 = 36$
85. $9 \times 5 = 45$
86. $9 \times 6 = 54$
87. $9 \times 7 = 63$
88. $9 \times 8 = 72$
89. $9 \times 9 = 81$
90. $9 \times 10 = 90$
91. $10 \times 1 = 10$
92. $10 \times 2 = 20$
93. $10 \times 3 = 30$
94. $10 \times 4 = 40$
95. $10 \times 5 = 50$
96. $10 \times 6 = 60$
97. $10 \times 7 = 70$
98. $10 \times 8 = 80$
99. $10 \times 9 = 90$
100. $10 \times 10 = 100$

Multiplication is binary. It is only possible to multiply 2 numbers at a time.

The **commutative property for multiplication** says that no matter the order in which you multiply two numbers, the product is always the same. This is usually abbreviated as $5 \times 3 = 3 \times 5 = 15$, or some similar example.

Finally, there's the **associative property for multiplication.** This says if you're going to multiply $3 \times 4 \times 5$, since you have to group them in pairs, you can group $(3 \times 4) \times 5$, meaning you'll multiply the 3×4 first, and then multiply 5 by the resulting 12, or you can group $3 \times (4 \times 5)$ and then multiply 3 by the resulting 20. Or, you could rewrite it as $(3 \times 5) \times 4$ and get 15×4. The associative property says however you group the numbers for multiplication, the answer is not affected. The groups above all total to 60. So, as it says in the Bible, be fruitful and . . . Oh, I just can't do it!



LESSON 5

Division Facts

We already commented in the Introduction that division is the most complicated operation, since it is both a shortcut for repeated subtraction of the same number and the undoing (or inverse) operation for multiplication.

$$12 - 3 - 3 - 3 - 3 = 12 \div 3 = 4$$

You and/or your student do not need to learn the division facts by rote, as long as you can relate them to the corresponding multiplication facts, since ultimately that is the relationship that is most critical.

There are 100 division facts, and that does not happen to be a coincidence. They are

1. $1 \div 1 = 1$
2. $2 \div 1 = 2$
3. $3 \div 1 = 3$
4. $4 \div 1 = 4$
5. $5 \div 1 = 5$
6. $6 \div 1 = 6$
7. $7 \div 1 = 7$
8. $8 \div 1 = 8$
9. $9 \div 1 = 9$
10. $10 \div 10 = 1$
11. $2 \div 2 = 1$
12. $4 \div 2 = 2$
13. $6 \div 2 = 3$
14. $8 \div 2 = 4$
15. $10 \div 2 = 5$
16. $12 \div 2 = 6$
17. $14 \div 2 = 7$
18. $16 \div 2 = 8$
19. $18 \div 2 = 9$
20. $20 \div 2 = 10$

Note that since any odd numbers divided by 2 would result in answers that are not whole numbers, they are excluded from division facts for 2. You'll see similar patterns in the following set. You should also have noticed that 1 is the identity element for division. Anything divided by 1 is itself.

21. $3 \div 3 = 1$
22. $6 \div 3 = 2$
23. $9 \div 3 = 3$
24. $12 \div 3 = 4$
25. $15 \div 3 = 5$
26. $18 \div 3 = 6$
27. $21 \div 3 = 7$
28. $24 \div 3 = 8$
29. $27 \div 3 = 9$
30. $30 \div 3 = 10$
31. $4 \div 4 = 1$
32. $8 \div 4 = 2$
33. $12 \div 4 = 3$
34. $16 \div 4 = 4$
35. $20 \div 4 = 5$
36. $24 \div 4 = 6$
37. $28 \div 4 = 7$
38. $32 \div 4 = 8$
39. $36 \div 4 = 9$
40. $40 \div 4 = 10$
41. $5 \div 5 = 1$
42. $10 \div 5 = 2$
43. $15 \div 5 = 3$
44. $20 \div 5 = 4$
45. $25 \div 5 = 5$
46. $30 \div 5 = 6$
47. $35 \div 5 = 7$
48. $40 \div 5 = 8$
49. $45 \div 5 = 9$
50. $50 \div 5 = 10$

Templates for division flash cards may be found in the Appendix.

You might have noticed that there have not been any divisions by "0". That's because division by "0" results in a **quotient** (yes, that's the name for the answer in a division) that is undefined. Of course, you can divide "0" by anything and get a quotient of 0. Let's get back to the job at hand.

51. $6 \div 6 = 1$
52. $12 \div 6 = 2$
53. $18 \div 6 = 3$
54. $24 \div 6 = 4$
55. $30 \div 6 = 5$
56. $36 \div 6 = 6$
57. $42 \div 6 = 7$
58. $48 \div 6 = 8$
59. $54 \div 6 = 9$
60. $60 \div 6 = 10$
61. $7 \div 7 = 1$
62. $14 \div 7 = 2$
63. $21 \div 7 = 3$
64. $28 \div 7 = 4$
65. $35 \div 7 = 5$
66. $42 \div 7 = 6$
67. $49 \div 7 = 7$
68. $56 \div 7 = 8$
69. $63 \div 7 = 9$
70. $70 \div 7 = 10$
71. $8 \div 8 = 1$
72. $16 \div 8 = 2$
73. $24 \div 8 = 3$
74. $32 \div 8 = 4$
75. $40 \div 8 = 5$
76. $48 \div 8 = 6$
77. $56 \div 8 = 7$
78. $64 \div 8 = 8$
79. $72 \div 8 = 9$
80. $80 \div 8 = 10$
81. $9 \div 9 = 1$
82. $18 \div 9 = 2$
83. $27 \div 9 = 3$
84. $36 \div 9 = 4$
85. $45 \div 9 = 5$
86. $54 \div 9 = 6$
87. $63 \div 9 = 7$
88. $72 \div 9 = 8$
89. $81 \div 9 = 9$
90. $90 \div 9 = 10$
91. $10 \div 10 = 1$
92. $20 \div 10 = 2$
93. $30 \div 10 = 3$
94. $40 \div 10 = 4$
95. $50 \div 10 = 5$
96. $60 \div 10 = 6$
97. $70 \div 10 = 7$
98. $80 \div 10 = 8$
99. $90 \div 10 = 9$
100. $100 \div 10 = 10$

Did you notice that these facts are the multiplication facts backward? Speaking of multiplication facts, there are some helpful hints you might or might not have noticed. Did you notice that every multiple of "5" ends in a "0" or a "5"?

Did it occur to you that every multiple of 9 has a first digit one lower than what 9 is being multiplied by? So, 6×9 is in the 50s, 7×9 is in the 60s, and so on. Additionally, the second digit is what you have to add to the first one to make 9: $5 + 4 = 9$, so 54; $6 + 3 = 9$, so 63; and so forth. You might also notice that in every multiple of 3, the digits add up to a multiple of 3. 12: $1 + 2 = 3$. 24: $2 + 4 = 6$, which is 2×3, and so on.

What makes it so easy to learn the 2 table is that it's the same as counting by 2s. (Of course, every other number's multiplication table is the same as counting by that number.)

Neither the commutative property nor the associative property is applicable to the operation of division. Just remember, in the immortal words of Groucho Marx, "Divide and conquer!"

LESSON 6

Place Value

A **digit** is a finger or a toe. Interestingly, we have 10 of each. A digit is also a single-place numeral, such as 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Count those, and you'll see that there are, not coincidentally, 10 of them. Although the 7 Roman numerals were fine for many hundreds of years, the 10 digits of the modern place value system—especially with the Arab invention of the zero—lets us get a lot more done with less effort. We are able to represent any number with just 10 digits with an economy that the Romans were unable to, because our system of numeration relies not only on the value of the digits, but on their positions within numerals, also known as their **places.**

Number is the idea of quantity, or the actual countable quantity itself. Numeral is the representation of number. If we count five people, that is the number five. We represent that number with the numeral, 5. In the **place value** system, a digit has value based not only upon its magnitude, but also upon its position or place within the numeral.

	hundreds (h)	tens (t)	units (u)
one			1
ten		1	
one hundred	1		

Places are arranged in periods, each of which contains three places, which are arranged from right to left as units (u), tens (t), and hundreds (h). To find the value of a place, read the digit in that place times the name of the place it's in. On the top line of the preceding table is 1 in the units place, or one. On the second line is 1 in the tens place or 1 ten, or as we more commonly call it, ten. Name the three quantities in the following table:

h	t	u
	3	
		4
6		

In this table we have first 3 tens, or 30, then 4 units, or 4, and finally, 6 hundreds, or six hundred. We have looked so far, at the ones period only. Just to the left of the ones period is the thousands period. Check this out.

Thousands			Ones		
h	t	u	h	t	u
	3				
		4			
6					

After we've moved beyond the ones period, it becomes necessary to say the name of the period after reading the column heading, so we have here 3 times 10 or thirty thousand on the first line, 4 thousand on the second line (remember, we don't say units), and 6 hundred thousand.

But there's not always just one digit on a line. Try reading these multiple digit entries:

Thousands			Ones		
h	t	u	h	t	u
	3	5		7	
2		4		6	1
5	2	6	9	3	

We read the first two digits of the top numeral the way we would if they had been in the ones period, but then we add the name of the period before continuing, so we have thirty-five thousand, seventy. Can you see why? Having read the thirty-five thousand in the second period, we continue reading what's in the units period. There were only three digits in that place value numeral. Let's go for four. The middle row's numeral is read two hundred four thousand, sixty-one. Notice that the name of the period is not used until all digits in that period have been read. We don't say two hundred thousand, four thousand, seventy. And, once more, we never say the name of the ones period. Are you ready for the big one? Okay, this last numeral is five hundred twenty-six thousand, nine hundred thirty.

Now let's add one more period. Left of the thousands is the millions period.

Millions			Thousands			Ones		
h	t	u	h	t	u	h	t	u
		4		7				9
	3	5	2	1	5	6	8	
9	2	8	4	3	7	1	6	5

Notice that just like the two periods we've been working with, adding the millions period means another h, t, and u place are added. Have you tried reading these numbers yet? If you haven't, I'll wait while you do.

By now, you should be getting good at this. Try to not be intimidated by the length of the numerals. As long as you're methodical in your approach, longer doesn't mean harder; it just means a few extra words. The first numeral is four million, seventy thousand, nine. Did you notice the commas in the last sentence? It is often customary to separate periods by commas when writing numerals larger than a thousand. More about that in a bit.

The second numeral starts in the ten millions place. It's 3 ten millions, read thirty million, but there's also a 5 in the units column in the millions period, so we read it thirty-five million, two hundred fifteen thousand, six hundred eighty. Finally, the numeral on the bottom line is read nine hundred twenty-eight million, four hundred thirty-seven thousand, one hundred sixty-five.

The next period to the left of the millions is the billions, but we're not going to go there—yet. Hopefully, you have absorbed the pattern of units-tens-hundreds, units-tens-hundreds, repeated as many times as needed, as it would be with billions if needed, trillions if needed, and so forth. The time has come to stop writing those column headings and find some other way of distinguishing one number from another in place value notation.

The problem with writing numbers without column headings is that there is no apparent way to know whether a 3 standing alone means three, thirty, three hundred, or, for that matter, three million. When writing the numerals in columns, we could look at the top of the column and read h, t, or u, and then read which period it was in, so there was no doubt as to what the value was. In order to serve the equivalent purpose, we introduce something called a place-holder. We use "0" as a place-holder, since it has no intrinsic value of its own but will hold any places where there is no other value indicated.

For purposes of writing place value numerals without column headings, we consider all numerals to be right justified. That is to say, the rightmost digit in any place value numeral is considered to be in the units column in the ones period:

Look at 3. Three is in the units place in the ones period. This numeral is three.

Consider 30. Zero is in the units place in the ones period. That forces three into the tens place. This numeral must have a value of thirty.

Think about the meaning of 300. Zeroes occupy the units and tens places. That means that the 3 is in the hundreds place. 300 must be worth three hundred.

Let's look at the last three large numbers one more time.

Millions			**Thousands**			**Ones**		
h	**t**	**u**	**h**	**t**	**u**	**h**	**t**	**u**
		4		7				9
	3	5	2	1	5	6	8	
9	2	8	4	3	7	1	6	5

Without the column headings and using place-holders, the top numeral is 4,070,009. Notice the commas separating the different periods. Some books and some teachers use a comma for every numeral with more than three digits. In other words, three thousand would be written 3,000. Other books use commas with numerals greater than four digits in length. This book follows the second convention.

The second numeral written in **standard notation** is 35,215,680. The commas should help you to distinguish between periods, but it is now up to you to recall the names and orders of the periods, as well as whether a specific digit is an "h," a "t," or a "u".

The last of the three large numbers, in standard (also known as decimal, since it's based on 10 digits) notation is 928,437,165.

As a general rule, elementary school students are expected to be able to read numbers of as many as 12 digits. In the United States it's the billions period. (In the U.K., a million million is a billion. In the United States, it's a thousand million.)

Essential to understanding place value is the concept of renaming, or regrouping. If you consider counting with individual Popsicle sticks or another type of counter, it is possible to go as high as 9 units, but no higher. In order to add another 1, that 1 must be bundled together with the other 9 units and renamed as 1 group of 10 and 0 units. This is written as 10. Having that group of 10, units can be added to that one or more at a time up to and including 19, but at 9, the units column is full. Adding one more will require a renaming as 2 bundles of 10 and 0 units.

It is possible to continue in the same manner, regrouping 9 units + 1 unit as a bundle of 10 until both the units and the tens column are full. At that time there will be 9 tens and 9 units, or 99. Now, in order to add 1 more, exchange the 9 + 1 units for a bundle of 10, but there's no room for that in the tens place. You must take the 9 bundles of 10 along with the new bundle and exchange those 10 tens for a single bundle of one hundred, making a total of one hundred, no tens and no ones, or 100.

Since no place can ever hold a quantity greater than 9, you can see how the idea of renaming carries on from each column to the next as hundreds are renamed as thousands, thousands as ten thousands, ten thousands as hundred thousands, hundred thousands as millions, and so forth.

EXERCISES

Questions 1–7 refer to the following number: 38,409,056,217

1. What digit is in the ten billions place?
2. What digit is in the millions place?
3. What digit is in the ten thousands place?
4. What digit is in the tens place?
5. What digit is in the hundred millions place?
6. What digit is in the hundreds place?
7. What digit is in the billions place?

8. Write the number four million, eight hundred thousand, seventy-two.
9. Write the number three hundred twenty-nine thousand, five hundred sixty-three.
10. Write the number eighty billion, seventy-one thousand, four.
11. What is the meaning of 38,297?
12. What is the meaning of 321,459,068?
13. What is the meaning of 567,000,814,309?

ANSWERS

1. 3
2. 9
3. 5
4. 1
5. 4
6. 2
7. 8
8. 4,800,072
9. 329,563
10. 80,000,071,004
11. thirty-eight thousand, two hundred ninety-seven
12. three hundred twenty-one million, four hundred fifty-nine thousand, sixty-eight
13. five hundred sixty-seven billion, eight hundred fourteen thousand, three hundred nine

LESSON 7

Place Value Addition

Let's assume that you have your addition facts down pat. There is never a need to add any two numbers that sum to more than 18. You might think that I'm making that up, but I'm really not. There might be occasions when you're adding multiple numbers and you find it convenient to add numbers that sum to more than 18, but you never need to. Remember, addition is binary. You can add only two numbers at a time. Well, the largest two digits you ever need to add are 9 + 9, which sums to 18.

Do you need to add 36 and 47? Okay, stack them; that is put one above the other.

$$\begin{array}{r} 36 \\ +\ 47 \\ \hline \end{array}$$

Now consider the digits in the units (or ones) column. Add them together:

$$\begin{array}{r} 36 \\ +\ 47 \\ \hline 13 \end{array}$$

But we don't want to put a sum into the tens column since we haven't added the tens yet, so we'll move that ten into the tens column as something still to be added:

$$\begin{array}{r} {}^{1}36 \\ +\ 47 \\ \hline 3 \end{array}$$

Now we add the 1 + 3 in the tens column to make 4 tens, and finally add those 4 tens to the 4 tens already there: 4 + 4 = 8:

$$\begin{array}{r} {}^{1}36 \\ +\ 47 \\ \hline 83 \end{array}$$

Now let's recap. To add 36 + 47 we put them into column addition form (stacked), added the ones and when the answer exceeded 10, we renamed the 10 ones as 1 ten. Then, finally, we added the tens. The largest sum we made in a single addition was 13, yet we found that the sum of 36 and 47 is 83. That's pretty cool, don't you think?

Let's try another one. How about adding 65 + 53?

First, stack them:

$$\begin{array}{r} 65 \\ +\ 53 \\ \hline \end{array}$$

We start adding in the ones column. 5 + 3 = 8, so . . .

$$\begin{array}{r} 65 \\ +\ 53 \\ \hline 8 \end{array}$$

That was easy enough. The sum of the ones is less than 10, so there's nothing in need of renaming. Now add the tens. 6 + 5 = 11. Let's show that:

$$\begin{array}{r} 65 \\ +\ 53 \\ \hline 118 \end{array}$$

Now 11 is greater than 9, so there's a "1" that falls into the hundreds column, but since there are no other hundreds to add, we can just leave it there. If there had been hundreds to be added, we would move it up top as a number to be added in the hundreds column. We found: 65 + 53 = 118.

Just to prove the point, let's consider the largest two-digit addition of two numbers that it's possible to have: 99 + 99 = __.

To start, put one above the other.

$$\begin{array}{r} 99 \\ +99 \\ \hline \end{array}$$

Next, add the digits in the units column.

$$\begin{array}{r} 99 \\ +99 \\ \hline 18 \end{array}$$

They sum to 18, which is a ten and 8 ones, but we don't want to put a sum into the tens column since we haven't added the tens yet, so we'll move that ten to the top of the tens column as one ten still to be added:

$$\begin{array}{r} {}^{1}99 \\ +99 \\ \hline 8 \end{array}$$

Now we add the 1 + 9 in the tens column to make 10 tens, and finally add those 10 tens to the 9 tens already there: 10 + 9 = 19:

$$\begin{array}{r} {}^{1} \\ 99 \\ +\ 99 \\ \hline 198 \end{array}$$

Alternately, we could have stopped after we added the 1 to the 9, renaming the 10 tens as one hundred:

$$\begin{array}{r} {}^{1} \\ 199 \\ {}_{0} \\ +\ 99 \\ \hline 8 \end{array}$$

Finally, add the 0 to the bottom 9 and bring down that 1 hundred:

$$\begin{array}{r} 1\overset{1}{9}9 \\ +\ \overset{0}{9}9 \\ \hline 198 \end{array}$$

Addition is unique among the operations of arithmetic, in that while it is binary and you can add only two numbers at a time, an addition might contain several numbers to be added. Here's an example of that: $5 + 17 + 135 + 72 + 86 = \_\_$

We already noted in our discussion of place value that numerals are right justified when put into place value columns. That is just as true when aligning numbers in columns for addition. The preceding numbers get arranged as follows:

$$\begin{array}{r} 5 \\ 17 \\ 135 \\ 72 \\ 86 \\ \hline \end{array} \qquad\qquad \begin{array}{rl} 5 & \leftarrow \\ 17 & \leftarrow \\ 135 & \\ 72 & \\ 86 & \\ \hline \end{array}$$

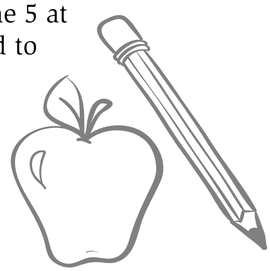
In order to add, we start with the first two digits in the units column, 5 and 7. Add them together, and we get 12, which we write as a 2 in the ones column and rename the 10 ones as 1 ten.

$$\begin{array}{rl} \overset{1}{}5 & \\ 17 & \\ 13\overset{2}{5} & \overset{\leftarrow}{\leftarrow} \\ 72 & \\ 86 & \\ \hline \end{array}$$

Next we move to the 2 and 5, which make 7; then the 7 and 2:

$$\begin{array}{rl} \overset{1}{}5 & \\ 17 & \\ 13\overset{2}{5} & \\ 7\overset{7}{2} & \overset{\leftarrow}{\leftarrow} \\ 8\overset{9}{6} & \\ \hline \end{array}$$

Finally, in the units column, we add 9 to 6 and get 15, from which we write the 5 at the bottom of the column, while renaming the 10 ones as 1 ten, which we add to the already renamed ten at the top of the tens column.



$$\begin{array}{r} \overset{2}{}5 \\ 17 \\ 13\overset{2}{5} \\ 7\overset{7}{2} \\ 8\overset{9}{6} \leftarrow \\ \hline 5 \end{array}$$

We can speed through the tens column by noting that 2 + 1 = 3, then 3 + 3 = 6, which brings us to 6 + 7:

$$\begin{array}{r} \overset{2}{}5 \\ 17 \\ 1\overset{3}{3}\overset{2}{5} \\ \rightarrow \overset{6}{7}\overset{7}{2} \\ 8\overset{9}{6} \\ \hline 5 \end{array}$$

If you don't recognize at once that the sum is 13, you should realize that it makes 10 and 3, which is really the more important fact, since we'll write the 3 tens in the tens column and rename the 10 tens as 1 hundred:

$$\begin{array}{r} \overset{1}{}\overset{2}{}5 \\ 17 \\ 1\overset{3}{3}\overset{2}{5} \\ \overset{6}{7}\overset{7}{2} \\ \rightarrow \overset{3}{8}\overset{9}{6} \\ \hline 5 \end{array}$$

Finish off the tens column by adding 3 tens + 8 tens to get 11 tens, or 1 in the tens column and 10 tens to rename into the hundreds column, adding it to the 1 already at the top to make 2 renamed hundreds:

$$\begin{array}{r} \overset{2}{}\overset{2}{}5 \\ 17 \\ 1\overset{3}{3}\overset{2}{5} \\ \overset{6}{7}\overset{7}{2} \\ \rightarrow \overset{3}{8}\overset{9}{6} \\ \hline 15 \end{array}$$

Finally, 2 hundreds + 1 hundred makes 3 hundreds, which we write below the line, for a total sum of 315.

$$\begin{array}{r}
{\scriptstyle \rightarrow\ 2\ 2}\ \ \\
5 \\
17 \\
{\scriptstyle 3\ 2} \\
\rightarrow 135 \\
{\scriptstyle 6\ 7} \\
72 \\
{\scriptstyle 3\ 9} \\
86 \\
\hline
315
\end{array}$$

If any part of that did not make sense to you, go back and follow the pairs of arrows through each step. Hopefully, that will clear it up.

Now, all that was to show you that you can do column addition keeping the largest partial sum you will ever have to deal with at 18 or less. Although that is in the realm of the possible, you would have to devote more time than you're likely to be able to spare if you were given a page of additions like that. In reality, after having gained some proficiency with addition, you would be much more likely to solve the same addition in the following manner:

$$\textbf{(a)}\ \begin{array}{r} 5 \\ 17 \\ 135 \\ 72 \\ 86 \\ \hline \end{array} \qquad \textbf{(b)}\ \begin{array}{r} {\scriptstyle 2}\ \ \\ 5 \\ 17 \\ 135 \\ 72 \\ 86 \\ \hline 5 \end{array} \qquad \textbf{(c)}\ \begin{array}{r} {\scriptstyle 2\ 2}\ \ \\ 5 \\ 17 \\ 135 \\ 72 \\ 86 \\ \hline 15 \end{array} \qquad \textbf{(d)}\ \begin{array}{r} {\scriptstyle 2\ 2}\ \ \\ 5 \\ 17 \\ 135 \\ 72 \\ 86 \\ \hline 315 \end{array}$$

Starting with the ones column in (a), you would go "5 + 7 = 12, + 5 makes 17, + 2 makes 19, + 6 makes 25 [go to (b)] put down 5 and carry the 2." (*Carry* is an old-fashioned word that means rename as whatever the tens digit of what you added was in the next column to the left. That's how I learned it, and by gum I ain't gonna unlearn it. But I digress.)

Moving to the tens column in (b), you'll say to yourself or out loud, "2 + 1 makes 3, + another 3 makes 6, + 7 makes 13, + 8 makes 21 [go to (c)], put down 1 and rename 2 tens at the top of the hundreds column.

Finally, move to the hundreds and add 2 + 1 = 3, [go to (d)], and write the 3, and you're done. The sum is three hundred fifteen. Note, it's not three hundred "and" fifteen. If your student says "and" try to dissuade her. We save "and" for work involving fractions mixed with whole numbers—a subject for which we're not quite ready.

By the way, did you notice that there was no " + " sign in that addition? Since no other operation permits working with more than two numbers at a time, when you see three or more numbers stacked up, you should automatically recognize it as addition.

Let's try one more example before you do some practice exercises. Let's add 737 + 895 + 604. These stack easily:

$$\textbf{(a)}\quad \begin{array}{r} 7\;3\;7\\ 8\;9\;5\\ 6\;0\;4\\ \hline \end{array}$$

These happen to all be three-digit numerals, so the columns line up well. Be sure to remember that when they don't line up so neatly, numerals must be right justified.

Since we're now on a new page, I'll redraw the column addition diagram:

$$\textbf{(a)}\quad \begin{array}{r} 7\;3\;7\\ 8\;9\;5\\ 6\;0\;4\\ \hline \end{array} \qquad \textbf{(b)}\quad \begin{array}{r} {\scriptstyle 1}\;\\ 7\;3\;7\\ 8\;9\;5\\ 6\;0\;4\\ \hline 6 \end{array} \qquad \textbf{(c)}\quad \begin{array}{r} {\scriptstyle 1}\;{\scriptstyle 1}\;\\ 7\;3\;7\\ 8\;9\;5\\ 6\;0\;4\\ \hline 3\;6 \end{array} \qquad \textbf{(d)}\quad \begin{array}{r} {\scriptstyle 2}\;{\scriptstyle 1}\;{\scriptstyle 1}\;\\ 7\;3\;7\\ 8\;9\;5\\ 6\;0\;4\\ \hline 2\;2\;3\;6 \end{array}$$

Starting with (a), you're going to add 7 + 5 to get 12, and if you must, write the partial sum, renaming the 10 ones as 1 ten. Or, continue and add 4 to the 12, writing the 6 below the line, while renaming as in (b).

Continuing with (b), we move to the tens column, adding 1 + 3 to make 4, 4 +9 to make 13, and 13 + 0 is still 13 (thought you would never add a zero, didn't you?). Move on to (c) to write the three below the line and rename the 10 tens as 1 hundred.

Continuing with (c), move to the hundreds column. 1 + 7 makes 8, certainly no need to write that. 8 + 8 makes 16; your choice of whether to write before continuing or just keep it in your head, 16 + 6 (you can tell which way I went) makes 22. Since there are no more places to the left to be added, or for the sake of form, write the 2 and regroup the two hundreds. In either case you're covered in (d), where 2 thousands have been renamed at the top and then brought down, since there's nothing to add it to (literally).

EXERCISES

1. 27 + 54 + 67 + 95 + 76 = __

2. 85 + 69 + 94 + 35 = __

3. 28 + 9 + 47 + 6 + 93 = __

4. 687 + 49 + 86 + 394 + 7 = __

5. 81 + 539 + 62 + 94 + 188 = __

6. 534 + 671 + 483 + 32 + 8 = __

7. 95 + 612 + 837 + 86 + 456 = __

8. 437 + 69 + 320 + 4300 = __

9. 526 + 3157 + 694 + 14 = __

10. 6234 + 5893 + 475 + 872 = __

11.
$$\begin{array}{r} 95 \\ 48 \\ 687 \\ \underline{908} \end{array}$$

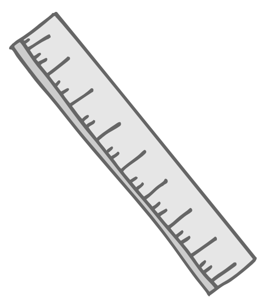
12.
$$\begin{array}{r} 7649 \\ 450 \\ 847 \\ \underline{3574} \end{array}$$

13.
$$\begin{array}{r} 908 \\ 8564 \\ 478 \\ \underline{6853} \end{array}$$

14.
$$\begin{array}{r} 9576 \\ 8684 \\ 7987 \\ \underline{4675} \end{array}$$

ANSWERS

1. 319
2. 283
3. 183
4. 1223
5. 964
6. 1728
7. 2086
8. 5126
9. 4391
10. 13,474
11. 1738
12. 12,520
13. 16,803
14. 30,922



LESSON 8

Place Value Subtraction

The operation of subtraction is often known as "take away" by virtue of the fact that it is an undoing operation in which one is often heard to say "What do you get if you take away [some amount] from [a larger amount]?" The subtraction 58 – 26 = __ would be displayed in place value form as:

$$\begin{array}{r} 58 \\ -\ 26 \\ \hline \end{array}$$

The subtraction is accomplished by taking the bottom number away from the top number, moving from right to left, and writing down the **remainder.**

Take 6 from 8 and 2 remain:

$$\begin{array}{r} 58 \\ -\ 26 \\ \hline 2 \end{array}$$

Next, take 2 from 5 and 3 remain:

$$\begin{array}{r} 58 \\ -\ 26 \\ \hline 32 \end{array}$$

The remainder, 32, is also known as the **difference** between 58 and 26.

In a subtraction, the top number, or the number being subtracted from, is known as the **minuend.** The amount being subtracted, or the bottom number, is known as the **subtrahend.** Your student might or might not be required to know these terms, so don't compel him to. First find out whether his teacher requires it.

Let's try that with three digit numerals. Find the difference between 657 and 345.

First write the subtraction:

$$\begin{array}{r} 657 \\ -\ 345 \\ \hline \end{array}$$

How did we know which number was the minuend and which the subtrahend? For all math at the level of the audience for this book, the larger number in a subtraction is always on the top. Again, we'll start subtracting on the right. You might very well ask, "What difference does it make whether we start on the right or the left?"

Well, in the case of this particular subtraction it does not matter, but soon it is going to, so it's better to form good habits than to have to unlearn bad ones later.

Take 5 from 7:

$$\begin{array}{r} 657 \\ -\ 345 \\ \hline 2 \end{array}$$

Next, 4 tens from 5 tens:

$$\begin{array}{r} 657 \\ -\ 345 \\ \hline 12 \end{array}$$

Finally, 3 hundreds from 6 hundreds:

$$\begin{array}{r} 657 \\ -\ 345 \\ \hline 312 \end{array}$$

So the difference is 312. To check a subtraction to make sure that it is correct, add up from below the line. In the preceding subtraction, $2 + 5 = 7$, $1 + 4 = 5$, and $3 + 3 = 6$, so you know the difference is correct.

And now for something completely different, er, well, a little different. Consider this subtraction: $74 - 38 = \_\_$.

First write the subtraction:

$$\begin{array}{r} 74 \\ -\ 38 \\ \hline \end{array}$$

Now subtract 8 from 4. Huh? How do I subtract 8 from 4? The answer, of course, is without introducing the concept of negative numbers, I can't!

But don't despair; there's always renaming. In subtraction, we use renaming the opposite way of how we used it in addition. We're going to take 1 ten from the 7 tens and rename it as 10 ones:

$$\begin{array}{rr} \overset{6}{\cancel{7}} & {}^{1}4 \\ -\ 3 & 8 \\ \hline \end{array}$$

Now we can subtract 8 from 14:

$$\begin{array}{rr} \overset{6}{\cancel{7}} & {}^{1}4 \\ -\ 3 & 8 \\ \hline & 6 \end{array}$$

Then on to the tens column:

$$\begin{array}{rr} \overset{6}{\cancel{7}} & {}^{1}4 \\ -\ 3 & 8 \\ \hline 3 & 6 \end{array}$$

Now let's check that by adding up. Beginning below the line in the ones column, $6 + 8 = 14$. Rename the 10 ones as 1 ten and add it to the 3 below the line in the tens column, to make it a 4. Adding, we get $4 + 3 = 7$. Yep, it checks.

Do you see now why we work subtraction from right to left? Let's add a new wrinkle, by trying the following subtraction:

$$\begin{array}{r} 756 \\ -\ 468 \\ \hline \end{array}$$

Note that 8 cannot be subtracted from 6, so you're going to the tens column to get a ten to **exchange** for ten ones:

$$\begin{array}{r} 7\overset{4}{\cancel{5}}{}^{1}6 \\ -\ 4\,6\,8 \\ \hline \end{array}$$

Now you can take 8 from 16:

$$\begin{array}{r} 7\overset{4}{\cancel{5}}{}^{1}6 \\ -\ 4\,6\,8 \\ \hline 8 \end{array}$$

Uh oh! There's a similar problem in the tens column. Do you know what to do?

It's time to rename a hundred as 10 tens:

$$\begin{array}{r} \overset{6}{\cancel{7}}\overset{14}{\cancel{5}}{}^{1}6 \\ -\ 4\,6\,8 \\ \hline 8 \end{array}$$

Now you can take 6 from 14:

$$\begin{array}{r} \overset{6}{\cancel{7}}\overset{14}{\cancel{5}}{}^{1}6 \\ -\ 4\,6\,8 \\ \hline 8\,8 \end{array}$$

And finally, subtract hundreds:

$$\begin{array}{r} \overset{6}{\cancel{7}}\overset{14}{\cancel{5}}{}^{1}6 \\ -\ 4\,6\,8 \\ \hline 2\,8\,8 \end{array}$$

I leave it to you to check the difference by adding up.

Do you think you've seen all there is to see about subtracting? Well, you haven't. Cast your eyes upon this beautiful subtraction:

$$\begin{array}{r} 804 \\ -\ 789 \\ \hline \end{array}$$

All right; in the units column we have 4 – 9, but you just saw how to deal with that. Whoa! There are no tens to rename as 10 ones. Now what? Well, the answer is go to the hundreds. After all, it's the only place where there is anything to rename. Start by exchanging 1 hundred for 10 tens:

1 hundred makes 10 tens:

$$\begin{array}{r} \overset{7}{\cancel{8}}{}^{1}0\,4 \\ -\ 7\,8\,9 \\ \hline \end{array}$$

Now there are some tens to rename:

$$\begin{array}{r} \overset{7}{\cancel{8}}\,{}^{1}\overset{9}{\cancel{0}}\,{}^{1}4 \\ -\ 7\ 8\ 9 \\ \hline \end{array}$$

Now subtract the ones:

$$\begin{array}{r} \overset{7}{\cancel{8}}\,{}^{1}\overset{9}{\cancel{0}}\,{}^{1}4 \\ -\ 7\ 8\ 9 \\ \hline 5 \end{array}$$

And subtract the tens:

$$\begin{array}{r} \overset{7}{\cancel{8}}\,{}^{1}\overset{9}{\cancel{0}}\,{}^{1}4 \\ -\ 7\ 8\ 9 \\ \hline 1\ 5 \end{array}$$

The hundreds subtract to 0, which we never write as the left-most digit of a whole number representation. The difference is 15.

There's yet one more form of subtraction that we need to deal with in this lesson, but after the problem just worked, this probably won't be much of a problem for you.

$$\begin{array}{r} 6000 \\ -\ 2354 \\ \hline \end{array}$$

There are an awful lot of zeroes there before we reach the only place from which we can start to regroup, namely that 6 in the thousands place. So, start by exchanging 1 of those thousands for 10 hundreds.

$$\begin{array}{r} \overset{5}{\cancel{6}}\,{}^{1}000 \\ -\ 2354 \\ \hline \end{array}$$

Now that we have 10 hundreds, we need to rename one of them as 10 tens, in order to have something to subtract the 5 tens from:

$$\begin{array}{r} \overset{5}{\cancel{6}}\,{}^{1}\overset{9}{\cancel{0}}\,{}^{1}0\ 0 \\ -\ 2\ 3\ 5\ 4 \\ \hline \end{array}$$

Now that we have tens, we need to rename one of them in order to get some ones to subtract the 4 from:

$$\begin{array}{r} \overset{5}{\cancel{6}}\,{}^{1}\overset{9}{\cancel{0}}\,{}^{1}\overset{9}{\cancel{0}}\,{}^{1}0 \\ -\ 2\ 3\ 5\ 4 \\ \hline \end{array}$$

And there you have it. Every column is ready for its subtraction to be completed, so it's time to do it.

$$\begin{array}{r} \overset{5}{\not{6}}\,{}^{1}\overset{9}{\not{0}}\,{}^{1}\overset{9}{\not{0}}\,{}^{1}0 \\ -\ 2\ 3\ 5\ 4 \\ \hline 6 \end{array} \qquad \begin{array}{r} \overset{5}{\not{6}}\,{}^{1}\overset{9}{\not{0}}\,{}^{1}\overset{9}{\not{0}}\,{}^{1}0 \\ -\ 2\ 3\ 5\ 4 \\ \hline 4\ 6 \end{array} \qquad \begin{array}{r} \overset{5}{\not{6}}\,{}^{1}\overset{9}{\not{0}}\,{}^{1}\overset{9}{\not{0}}\,{}^{1}0 \\ -\ 2\ 3\ 5\ 4 \\ \hline 6\ 4\ 6 \end{array} \qquad \begin{array}{r} \overset{5}{\not{6}}\,{}^{1}\overset{9}{\not{0}}\,{}^{1}\overset{9}{\not{0}}\,{}^{1}0 \\ -\ 2\ 3\ 5\ 4 \\ \hline 3\ 6\ 4\ 6 \end{array}$$

(a) **(b)** **(c)** **(d)**

Starting on the right, in (a), 10 – 4 = 6; in the tens, (b), 9 – 5 = 4; in the hundreds place, (c), 9 – 3 = 6; finally, in the thousands, (d), 5 – 2 = 3. Now let's check that answer of 3646 by adding up. Starting on the extreme right, 6 + 4 = 10, put down 0 and rename 1 ten. 1 + 4 = 5; 5 + 5 = 10, put down 0 and rename 1 ten (really a hundred, but we're going to disregard that). 1 + 6 = 7 + 3 = 10, put down 0 and rename 1 into the next column. 1 + 3 = 4, to which we add the 2 to get the top number, 6.

Did you notice that it's a routine we are following? After awhile, it's no longer necessary to think in terms of hundreds, thousands, tens, or ones. What we are doing when subtracting is, if necessary, renaming 1 whatever as 10 in the next column to the right. When adding, if necessary, we are renaming 10 whatevers as 1 in the next column to the left. It is important that you understand what you are doing as it reflects operating within the laws of place value, but, having said that, with a little practice, it should become a routine in which the rationale for the mechanics of what you are doing is less important than the actual operating of the mechanism.

Here are some exercises to allow you to practice and develop those necessary routines. And remember, after you've mastered subtraction, you have a skill that nobody can take away.

EXERCISES

1. 87 – 69 = __

2. 78 – 48 = __

3. 817 – 694 = __

4. 572 – 469 = __

5. 368 – 247 = __

6. 534 – 349 = __

7. 344 – 96 = __

8. 634 – 561 = __

9. 354 – 265 = __

10. 836 – 678 = __

11. 737 – 356 = __

12. 807 – 653 = __

13. $\begin{array}{r} 8007 \\ -\ 6354 \\ \hline \end{array}$

14. $\begin{array}{r} 7403 \\ -\ 4528 \\ \hline \end{array}$

15. $\begin{array}{r} 5060 \\ -\ 3421 \\ \hline \end{array}$

16. $\begin{array}{r} 5374 \\ -\ 4466 \\ \hline \end{array}$

17. $\begin{array}{r} 3005 \\ -\ 2968 \\ \hline \end{array}$

18. $\begin{array}{r} 6040 \\ -\ 3508 \\ \hline \end{array}$

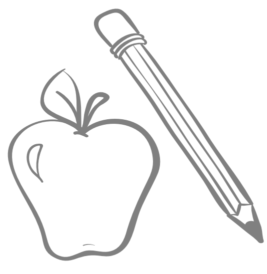
19. $\begin{array}{r} 8000 \\ -\ 7265 \\ \hline \end{array}$

20. $\begin{array}{r} 7000 \\ -\ 4356 \\ \hline \end{array}$

21. $\begin{array}{r} 6000 \\ -\ 2857 \\ \hline \end{array}$

ANSWERS

1. 18
2. 30
3. 123
4. 103
5. 121
6. 185
7. 248
8. 73
9. 89
10. 158
11. 381
12. 154
13. 1653
14. 2875
15. 1639
16. 908
17. 37
18. 2532
19. 735
20. 2644
21. 3143



LESSON 9

Place Value Multiplication

Place value multiplication at first glance looks pretty much like place value subtraction or two-numeral addition, except for the times sign. It is a pretty good idea, however, to have a good command of the multiplication facts before embarking upon multiplying in place value format.

One-Digit by Two-Digit

The simplest form of place value multiplication is of the one-digit by 2-digit variety:

$$\begin{array}{r} 24 \\ \times\ 6 \\ \hline \end{array}$$

There are traditional names given to the parts of a multiplication. We already mentioned the name of the answer, product, in Lesson 3. The number being multiplied by (the 6 in the previous multiplication) is known as the **multiplier,** while the number being multiplied (the 24) is called the **multiplicand.** Your student's teacher might or might not use these names. The two numbers above the line are also collectively known as **factors.**

Like addition and subtraction, multiplication is performed from right to left, and for a reason. To solve the previous multiplication, first multiply the 4 by 6. The units digit of the product is written in the units column below the line, while the tens digit is renamed into the tens column, as shown here.

$$\begin{array}{r} {\scriptstyle 2} \\ 24 \\ \times\ 6 \\ \hline 4 \end{array}$$

Next, the 2 is multiplied by the 6 (the 2 that was there initially) to make 12, and the renamed number is added to the product to make 14. Since there are no additional numbers to be multiplied, the complete result of the foregoing is written below the line, like so:

$$\begin{array}{r} {\scriptstyle 2} \\ 24 \\ \times\ 6 \\ \hline 144 \end{array}$$

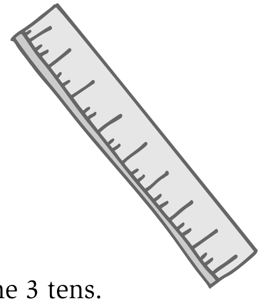
So the product of 6 and 24 is 144.

Try these on your own:

1. $\begin{array}{r} 35 \\ \times\ 7 \\ \hline \end{array}$

2. $\begin{array}{r} 69 \\ \times\ 5 \\ \hline \end{array}$

3. $$\begin{array}{r} 75 \\ \times\ 9 \\ \hline \end{array}$$



Here are the worked-out solutions, step by step:

1. **(a)** $$\begin{array}{r} 35 \\ \times\ 7 \\ \hline \end{array}$$ **(b)** $$\begin{array}{r} {\scriptstyle 3} \\ 35 \\ \times\ 7 \\ \hline 5 \end{array}$$ **(c)** $$\begin{array}{r} {\scriptstyle 3} \\ 35 \\ \times\ 7 \\ \hline 245 \end{array}$$

(a) states the multiplication. **(b)** $7 \times 5 = 35$, put down the 5 and rename 3 tens. **(c)** $7 \times 3 = 21 +$ the renamed $3 = 24$. The product is 245.

2. **(a)** $$\begin{array}{r} 69 \\ \times\ 5 \\ \hline \end{array}$$ **(b)** $$\begin{array}{r} {\scriptstyle 4} \\ 69 \\ \times\ 5 \\ \hline 5 \end{array}$$ **(c)** $$\begin{array}{r} {\scriptstyle 4} \\ 69 \\ \times\ 5 \\ \hline 345 \end{array}$$

(a) states the multiplication. **(b)** $5 \times 9 = 45$, put down the 5 and rename 4 tens. **(c)** $5 \times 6 = 30 +$ the renamed $4 = 34$. The product is 345.

3. **(a)** $$\begin{array}{r} 75 \\ \times\ 9 \\ \hline \end{array}$$ **(b)** $$\begin{array}{r} {\scriptstyle 4} \\ 75 \\ \times\ 9 \\ \hline 5 \end{array}$$ **(c)** $$\begin{array}{r} {\scriptstyle 4} \\ 75 \\ \times\ 9 \\ \hline 675 \end{array}$$

(a) states the multiplication. **(b)** $9 \times 5 = 45$, put down the 5 and rename 4 tens. **(c)** $9 \times 7 = 63 +$ the renamed $4 = 67$. The product is 675.

Two-Digit by Two-Digit

Are you ready to get a bit more complex? Let's get to 2-d × 2-d (that's two-digit by two-digit). For openers, let's try 46×68.

$$\begin{array}{r} 68 \\ \times\ 46 \\ \hline \end{array}$$

Remember, multiplication is commutative. If we'd put 68 on the bottom and 46 on the top, the product would still be the same. Once more, we begin on the right, multiplying 6×8.

$6 \times 8 = 48$, so write the 8 below the line and rename the 40 as 4 in the tens place.

$$\begin{array}{r} {\scriptstyle 4} \\ 68 \\ \times\ 46 \\ \hline 8 \end{array}$$

Next, multiply the 6 times the 6 to get 36, to which you'll add the renamed 4 tens to get a total of 40 tens, also known as 400. Since there are no more digits to the left, we simply write the total into the product:

$$\begin{array}{r} {\scriptstyle 4} \\ 68 \\ \times\ 46 \\ \hline 408 \end{array}$$

Next, we're going to multiply by the 4, but bear in mind that it is 4 tens. That means there will be no ones in the answer, since 40 times something is not going to produce any units. To account for this, place a 0 in the ones column beneath the 8:

$$\begin{array}{r} 68 \\ \times\ 46 \\ \hline 408 \\ 0 \end{array}$$

You might have noticed that we cleared the renamed 4 from over the 6. That's because we could. Since you're using paper (I hope), you can just put a line through that 4, so that it doesn't get in your way with the multiplication by 40. Now we're going to multiply by 40, but since its proper place has already been assured by placing that 0, we can act as if we're simply multiplying by 4.

$4 \times 8 = 32$, so we place the 2 in the next available place and rename the 3 to the top of the next column:

$$\begin{array}{r} {}^{3} \\ 68 \\ \times\ 46 \\ \hline 408 \\ 20 \end{array}$$

Now, we multiply 4×6 to get 24. Add the 3 to make 27. With no more digits to be multiplied, we can write the whole 27 (really 2700) in place, and add an addition line:

$$\begin{array}{r} {}^{3} \\ 68 \\ \times\ 46 \\ \hline 408 \\ \underline{2720} \end{array}$$

Next, add the two partial products together:

$$\begin{array}{r} {}^{3} \\ 68 \\ \times\ 46 \\ \hline 408 \\ 2720 \\ \hline 3128 \end{array}$$

And you'll get the product three thousand, one hundred twenty-eight.

Two-Digit by Three-Digit

Let's move on to a multiplication involving a two-digit multiplier and a three-digit multiplicand, say 358×79:

$$\begin{array}{r} 358 \\ \times\ 79 \\ \hline \end{array}$$

As usual, begin multiplying on the right side, first multiplying 9 × 8. That product is 72, so:

$$\begin{array}{r} {\scriptstyle 7} \\ 358 \\ \times\ 79 \\ \hline 2 \end{array}$$

The 2 goes beneath the 9, while the 7 is renamed as 7 tens on top of the 5. Next, multiply 9 × 5 to get 45, to which we add the renamed 7, for a total of 52. Since there's another digit in the hundreds column, we write the 2 beneath the 7 and rename the 50 tens as 5 hundreds (that's the meaning of it, but in reality, we write the 2 and rename the 5 into the next column):

$$\begin{array}{r} {\scriptstyle 5\,7} \\ 358 \\ \times\ 79 \\ \hline 22 \end{array}$$

Next, complete the multiplication by 9, multiplying 9 × 3 to get 27, plus the renamed 5, which make 32. Since there are no more digits to the left, write the 32 beneath the line, (bearing in mind that it's 32 hundred).

$$\begin{array}{r} {\scriptstyle 5\,7} \\ 358 \\ \times\ \ 79 \\ \hline 3222 \end{array}$$

Before multiplying by 70, place a zero in the ones place (multiplying by 70 generates no ones).

$$\begin{array}{r} 358 \\ \times\ \ 79 \\ \hline 3222 \\ 0 \end{array}$$

I've also cleared the renamed numbers, to make room for new ones. You might want to put a line through each of those on your worksheet to avoid confusion with your forthcoming renamings. Since placing the 0 forces all parts of the next multiplications into the proper places, we can act as if we're multiplying by 7. 7 × 8 = 56; put the 6 next to the 0 and rename the 5 into the next column:

$$\begin{array}{r} {\scriptstyle 5} \\ 358 \\ \times\ \ 79 \\ \hline 3222 \\ 60 \end{array}$$

Next, 7 × 5 = 35, to which you'll add the renamed 5 to get 40. Put down the 0 next to the 6, and rename the 4 into the next column:

$$\begin{array}{r} {\scriptstyle 4\,5} \\ 358 \\ \times\ \ 79 \\ \hline 3222 \\ 060 \end{array}$$

The last multiplication called for is 7 × 3, which makes 21, to which the renamed 4 is added. Since this is the last multiplication, the whole 25 may be written next to the 0 in the second partial product:

$$\begin{array}{r} {\scriptstyle 4\,5} \\ 358 \\ \times \quad 79 \\ \hline 3222 \\ 25060 \\ \hline \end{array}$$

Finally, we add the two partial products together to get the final product:

$$\begin{array}{r} {\scriptstyle 4\,5} \\ 358 \\ \times \quad 79 \\ \hline 3222 \\ 25060 \\ \hline 28{,}282 \end{array}$$

The final product is 28,282. Notice that since the product is 5 digits in length, a comma has been placed between the ones and thousands periods.

Three-Digit by Three-Digit

The most complex multiplication we'll do in this chapter is 3-d × 3-d, as in this one.

$$\begin{array}{r} 576 \\ \times\ 648 \\ \hline \end{array}$$

I'll spare you the walk-through of the multiplication by 8. It looks like this:

$$\begin{array}{r} {\scriptstyle 6\,4} \\ 576 \\ \times\ 648 \\ \hline 4608 \\ 0 \end{array}$$

Feel free to go through it step by step, just to make sure that I didn't screw it up. I've also placed the "0" for the multiplication by 40. I'm going to spare you the step-by-step multiplication by 40, although, again, you are more than welcome to check it out. I've replaced the renaming numbers from the ones multiplication with the numbers from the tens multiplication.

$$\begin{array}{r} {\scriptstyle 3\,2} \\ 576 \\ \times\ 648 \\ \hline 4608 \\ 23040 \end{array}$$

What do you think comes next? Should there be a zero at the right end of the next line before multiplying by the 6? What is the smallest amount the answer to the next multiplication could possibly be?

$$\begin{array}{r} 576 \\ \times\ 648 \\ \hline 4608 \\ 23040 \\ 00 \end{array}$$

That's right. There can be no ones or tens, since the next multiplication is by a figure in the hundreds column. Two zeroes are, therefore, called for to push the first partial product into the hundreds place. I've also removed the old renamed numbers to make room for the new ones.

$$\text{(a)}\quad \begin{array}{r} {}^{3} \\ 576 \\ \times\ \ 648 \\ \hline 4608 \\ 23040 \\ 600 \end{array} \qquad\qquad \text{(b)}\quad \begin{array}{r} {}^{4\,3} \\ 576 \\ \times\ \ 648 \\ \hline 4608 \\ 23040 \\ 5600 \end{array} \qquad\qquad \text{(c)}\quad \begin{array}{r} {}^{4\,3} \\ 576 \\ \times\ \ 648 \\ \hline 4608 \\ 23040 \\ 345600 \end{array}$$

(a) $6 \times 6 = 36$; write the 6 and rename the 3. (b) $6 \times 7 = 42 +$ the renamed $3 = 45$; write the 5 next to the 6, and rename the 4 into the next column. (c) $6 \times 5 = 30 +$ the renamed $4 = 34$; since there are no more numbers to multiply, write it down next to the 5.

Now we're ready to add the partial products, which is done below. Feel free to check the addition. I never was very good at it.

$$\begin{array}{r} 576 \\ \times\ \ 648 \\ \hline 4\,608 \\ 23\,040 \\ 345\,600 \\ \hline 373{,}248 \end{array}$$

That comma separates the periods and makes that big numeral easier to read. Now you get to try some on your own. How lucky can you get!

EXERCISES

1. 68 × 35	**6.** 348 × 96	**11.** 381 × 567	**16.** 693 × 465
2. 89 × 64	**7.** 657 × 48	**12.** 476 × 598	**17.** 830 × 490
3. 73 × 87	**8.** 782 × 63	**13.** 259 × 436	**18.** 387 × 645
4. 92 × 59	**9.** 854 × 64	**14.** 800 × 706	**19.** 897 × 900
5. 48 × 76	**10.** 509 × 23	**15.** 579 × 687	**20.** 749 × 976

ANSWERS

1. 2380	**6.** 33,408	**11.** 216,027	**16.** 322,245
2. 5696	**7.** 31,536	**12.** 284,648	**17.** 406,700
3. 6351	**8.** 49,266	**13.** 112,924	**18.** 249,615
4. 5428	**9.** 54,656	**14.** 564,800	**19.** 807,300
5. 3648	**10.** 11,707	**15.** 397,773	**20.** 731,024

LESSON 10

Place Value Division

Division is a bit different from everything we've done until now, first of all because it's done from left to right, and second because it can be handled by approaching it from the vantage point of either of the two related combining operations. Consider the following:

$$12\overline{)156}$$

The bracket indicates division, but unlike the other notation for division, $156 \div 12 = \_\_\_$, which reads "156 divided by 12," this reads "12 divided into 156," in other words, the divisor comes first.

Considering division as a shortcut for repeated subtraction, we can ask, "How many 12s can be subtracted from 156?" There are certainly more than 2 of them, so let's start with 2:

$$\begin{array}{rr} 12\overline{)156} & \\ 24 & 2 \\ \hline 132 & \end{array}$$

Subtracting those 2 12s leaves 132 still to be divided out. Do you know that 10 12s make 120? They do, so I'm going to subtract that out.

$$\begin{array}{rr} 12\overline{)156} & \\ 24 & 2 \\ \hline 132 & \\ 120 & 10 \\ \hline 12 & \end{array}$$

Subtracting those 10 12s leaves 12 still to be uncombined. I know how many 12s there are in 12. (I'll bet you do, too):

$$\begin{array}{rr} 12\overline{)156} & \\ 24 & 2 \\ \hline 132 & \\ 120 & 10 \\ \hline 12 & \\ 12 & 1 \\ \hline & 13 \end{array}$$

By repeated subtraction, we have managed to find that 12 goes into 156 13 times. Some children repeat "goes into" so many times that they think another name for division is "the gazintas." Notice that with this form of division, called **ladder division** after the steps on which the partial quotients sit, is not especially structured. I could have, if I had not recognized 120 as being 10 12s, removed 5 12s and another 5 12s in an additional step, or 4 12s, 4 12s, and 2 12s, or 3 12s, 3 12s, 3 12s, and 1 12.

Far more structured than ladder division, but totally depending upon your knowledge of multiplication facts is what is known as **long division,** but can actually be much shorter than the ladder. Let's start with the same division:

$$12\overline{)156}$$

First ask yourself "Self, how many 12s are there in 15?" Hopefully, you conclude the answer is 1 and write that partial quotient above the ones digit of the "15." (Realize that it's actually one hundred fifty we're dividing by 12, and so the answer is going into the tens place, since there are 10 of them.)

$$\begin{array}{r} 1 \\ 12\overline{)156} \end{array}$$

Next, multiply the 1 from the partial quotient times the divisor and write the product beneath the first 2 digits of the dividend:

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \end{array}$$

Next, subtract to find the amount we still need to divide:

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \\ 3 \end{array}$$

And bring down the next digit. We've placed an "x" under it to show that it has been brought down. Although that might not seem much of a big deal in this division, it becomes a big deal in larger dividends:

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \text{x} \\ \underline{12} \\ 36 \end{array}$$

The new partial dividend is the 36, so ask yourself, "How many 12s are there in 36?" or "How many times 12 'gazinta' 36?"

$$\begin{array}{r} 13 \\ 12\overline{)156} \\ \text{x} \\ \underline{12} \\ 36 \end{array}$$

Hopefully, we've both reached the same conclusion. Now multiply 3 times 12, write the answer beneath the dividend, and subtract:

```
     13
12)156
      x
    12
     36
     36
```

Since the difference between the two numbers is nothing, we write "nothing" down. Do you see the repeating pattern of seven steps? It goes like this:

1. Divide.
2. Place the partial quotient.
3. Multiply the partial quotient just written by the divisor.
4. Write the product below the dividend.
5. Subtract.
6. Bring down the next digit (if there is one).
7. IF there are digits not yet dealt with in the dividend,
   THEN go to Step 1.
   ELSE Go to Ending.
8. Ending to be dealt with next!

A couple of safety checks are built into the 7-step method. First off, when the divisor is multiplied by the partial quotient (4), the result must be smaller than the number above it. If it's not, then the partial quotient is too large. When you subtract (5) the difference must be smaller than the divisor. If it is not, then the partial quotient was too small.

Remainders

The preceding division was a perfect one, in that the divisor times the quotient equals the dividend.

That usually won't be the case. That brings us to the question of remainders. "Remainders in division?" you ask; "I thought that was subtraction." Well, I've been telling you all along that division is a form of subtraction. Now, I'm going to prove it to you. Here's a relatively easy division:

```
9)85
```

9 doesn't go into 8, since it's larger than 8, so we must divide the full 85 by the 9. If you know your 9s table, then you know:

```
   9
9)85
  81
   4
```

We put in the partial quotient of 9, multiplied that times the divisor, and got 81, which we subtract from the divisor to get a remainder of 4. What do we do with it? There are no more digits in the dividend to "bring down." We didn't divide the 4; we arrived at it by subtraction, so the answer to this division is:

$$\begin{array}{r} 9\text{r}4 \\ 9\overline{)85} \\ \underline{81} \\ 4 \end{array}$$

That is read, "9 remainder 4." It's really a combination obtained by division and subtraction. The 9 was obtained by division; the 4 by subtraction. The quotient could have been written another way, as well. The remainder could have been put over the divisor to form the common fraction, $\frac{4}{9}$:

$$\begin{array}{r} 9\frac{4}{9} \\ 9\overline{)85} \\ \underline{81} \\ 4 \end{array}$$

Estimating Quotients

Suppose that you have a division like 8896 ÷ 27. It is obvious that you're not going to divide 27 into 8, so you will divide it into 88. It is not so obvious how many 27s there are in 88. To make it easier to estimate how many 27s there are in 88, it helps to create a **trial divisor** and a **trial dividend.** A trial divisor is created by rounding the divisor to the nearer 10 and then using only the tens digit. If the ones digit in a two-digit numeral is 5 or greater, round up to the next 10, so 27 would round up to 30. If the divisor had been 23, we would have rounded down to 20. The trial divisor for the preceding division would be 3 (the tens digit from rounding 27 up to 30). The trial dividend is created in the same way, in this case from rounding 88 to 90, and then using the 9. How many 3s are there in 9? Okay, then try 3 27s in 88:

$$\begin{array}{r} 3 \\ 27\overline{)8896} \\ \underline{81} \end{array}$$

That worked. The possibility exists that you might get a partial quotient a little too big or a little too small. If that happens, adjust your answer accordingly. Next, subtract and bring down:

$$\begin{array}{r} 3 \\ 27\overline{)8896} \\ \text{x} \\ \underline{81} \\ 79 \end{array}$$

Now the trial divisor is still 3, but the trial dividend is 8 (from rounding 79 to 80). 8 divided by 3 is 2 and change:

$$\begin{array}{r} 32 \\ 27\overline{)8896} \\ \text{x} \\ \underline{81} \\ 79 \end{array}$$

Next multiply and subtract:

```
      32
27 )8896
       x
    81
     79
     54
     25
```

Since the difference is less than the divisor, the 2 partial quotient was correct. So, the final step is to bring down the 6:

```
      32
27 )8896
       xx
    81
     79
     54
     256
```

The trial divisor, 3, goes into the trial dividend, 25, 8 times. Multiply 8×27 and you'll get 216, which is more than 27 too small, so the last digit of the quotient should be 9.

```
      329
27 )8896
       xx
    81
     79
     54
     256
     243
      13
```

The quotient is 329, r(remainder)13, or $329\frac{13}{27}$.

EXERCISES

1. $16\overline{)285}$
2. $24\overline{)372}$
3. $35\overline{)280}$
4. $47\overline{)593}$
5. $43\overline{)649}$
6. $58\overline{)784}$
7. $69\overline{)852}$
8. $84\overline{)798}$
9. $37\overline{)1569}$
10. $54\overline{)2836}$
11. $37\overline{)2483}$
12. $46\overline{)3910}$
13. $64\overline{)3648}$
14. $24\overline{)4896}$
15. $73\overline{)6815}$
16. $62\overline{)7750}$

ANSWERS

1. 17, r13
2. 15, r12
3. 8
4. 12, r29
5. 15, r4
6. 13, r30
7. 12, r24
8. 9, r42
9. 42, r15
10. 52, r28
11. 67, r4
12. 85
13. 57
14. 204
15. 93r26
16. 125

LESSON 11

Adding and Subtracting Larger Numbers

You might think that adding and subtracting larger numbers works the same way as adding and subtracting smaller numbers, and to a certain extent you're correct. When you magnify the numbers, however, you also magnify the possibility that you'll make an error. Because of that, it is helpful to estimate the sum or difference before actually performing the operation.

Estimating for Addition

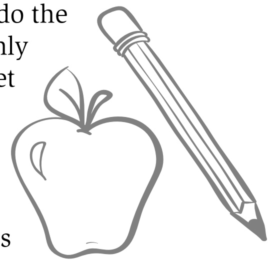
Suppose that you have 50,285 persons attending a Mets game, 39,417 persons attending a Tigers game, 41,817 persons attending a Phillies game, and 47,283 persons attending a Cardinals game all on the same July evening. Without adding the totals, would it be reasonable for you to **estimate** that there are around 140,000 persons (give or take 5000) attending all four of those games? Think about it before answering. All ready? The answer is *absolutely not.* An estimate is an approximation, and it is often as good as, if not better than, an exact count. If the same food-service outfit provided refreshments for all four of those ballgames, they would need to know about how many hot dogs they would need to order from their frankfurter supplier in order to have enough on hand. Suppose that each attendee averages one hot dog apiece at each game, and our pretend food-service operator had used the preceding estimate; would they have ordered too few or too many hot dogs?

How can I be sure that the estimate of 140,000 was way off? Look at the first two figures: one was more than 50,000, and the second was nearly 40,000. That makes a sum of about 90,000. Now look at the Phillies and Cardinals figures: almost 42,000 and more than 47,000, for a total there of almost 90,000. That's 90,000 + 90,000, or a total of about 180,000. It's actually going to be a little less than 180,000; say 178,000. Now add the figures and see what you get:

$$
\textbf{(a)}\ \begin{array}{r} {\scriptstyle 2} \\ 50,285 \\ 39,417 \\ 41,817 \\ \underline{47,283} \\ 2 \end{array}
\qquad
\textbf{(b)}\ \begin{array}{r} {\scriptstyle 2\,2} \\ 50,285 \\ 39,417 \\ 41,817 \\ \underline{47,283} \\ 02 \end{array}
\qquad
\textbf{(c)}\ \begin{array}{r} {\scriptstyle 1\ \ 2\,2} \\ 50,285 \\ 39,417 \\ 41,817 \\ \underline{47,283} \\ 802 \end{array}
\qquad
\textbf{(d)}\ \begin{array}{r} {\scriptstyle 1\,1\ \ 2\,2} \\ 50,285 \\ 39,417 \\ 41,817 \\ \underline{47,283} \\ 8\ 802 \end{array}
\qquad
\textbf{(e)}\ \begin{array}{r} {\scriptstyle 1\,1\ \ 2\,2} \\ 50,285 \\ 39,417 \\ 41,817 \\ \underline{47,283} \\ 178,802 \end{array}
$$

In (a), I've added the units column, gotten 22, and you know what to do with that. In (b), the tens have been added to get a column total of 20. The addition of the hundreds is shown in (c), totaling to 18. Next come the thousands, shown in (d), and I have deliberately not placed a comma, but your student's teacher might want one there. Finally, (e) shows the addition of the ten thousands, and the comma is placed.

Hey, I'm rather pleased with the closeness of my estimate. And I really did *not* do the addition in advance. An awful lot of people would have been disappointed if only 140,000 hot dogs had been distributed—sellers as well as fans. We have not yet discussed percent, but the error in the estimate of 178,000 was off by less than half a percentage point (see Lesson 33 for more on percentages). Most important: If after estimating 178,000, my sum had been 200,000 or 150,000, I would have suspected that I had done something wrong and would have gone back and checked my work. The easiest way to check an addition of this magnitude is with a calculator, and don't think I won't.



Yep! That's the sum.

Estimating for Subtraction

The notion of estimating the difference when subtracting larger numbers might scare you at first, but it is really much easier than doing the same for addition. After all, only two numbers are involved, no matter what their magnitudes. Suppose that 17,283,724 persons live in the state of New York, and of those, 7,938,953 live in New York City. From that, I estimate that about 10,456,271 live in the rest of New York State (the numbers are made up, so don't use them if you're ever on *Jeopardy*). How good an estimate is that? Don't read any further until you've had a chance to decide whether my estimate was a good one or a bad one.

Well, it's my objective to not play favorites so, just as in addition, my estimate was a lousy one. The first figure is almost 17,300,000, while the second is nearly 8,000,000. Just from those two figures, knock 5 "0s" off of each to get 173 vs. 80. That's a difference of 93. Now put the 5 "0s" back and estimate the difference as 9,300,000. Did you follow that? Removing the 5 "0s" from each made for an easier subtraction (use the maximum number of zeroes present in both numerals). Then by putting the 5 zeroes back, you can see the actual magnitude of the number with which you're dealing. Now let's see how close the new estimate is to the actual difference. Keep in mind the fact that the whole purpose of estimating the difference before subtracting is to recognize when your computed answer doesn't make sense, so the more accurate the estimate, the better:

(a)
$$\begin{array}{r} 17,283,724 \\ -\ 7,938,953 \\ \hline 1 \end{array}$$

(b)
$$\begin{array}{r} 17,283,\overset{6}{\cancel{7}}{}^{1}24 \\ -\ 7,938,953 \\ \hline 71 \end{array}$$

Step (a) is easy enough, since 4 – 3 = 1. In step (b), we can't subtract 5 tens from 2 tens, so we go to the hundreds column and rename 1 of the 7 hundreds as 10 tens. That leaves 6 in the hundreds place and makes 12 tens. Now subtract: 12 – 5 = 7.

(c)
$$\begin{array}{r} \scriptstyle 2\ \ 16 \quad \\ 17,28\not{3},\not{7}\,{}^{1}24 \\ -\ \ 7,938,953 \\ \hline 771 \end{array}$$

(d)
$$\begin{array}{r} \scriptstyle 7\,12\ \ 16 \quad \\ 17,2\not{8}\not{3},\not{7}\,{}^{1}24 \\ -\ \ 7,938,953 \\ \hline 4\ 771 \end{array}$$

Step (c) requires taking 9 from 6, which obviously can't be done, so it's necessary to rename the 3 in the thousands place. Then subtract: 16 – 9 = 7.

Let's not even worry about the names of the places any more. We can pretty much mechanically move from right to left one place at a time. (d) requires the impossible job of taking 8 from 2, so we go left one column to rename 1 from there as 10 in the column where we're subtracting, and adding it to the 2 to make 12. Then, we find that 12 – 8 = 4.

(e)
$$\begin{array}{r} \scriptstyle 7\,12\ \ 16 \quad \\ 17,2\not{8}\not{3},\not{7}\,{}^{1}24 \\ -\ \ 7,938,953 \\ \hline 44,771 \end{array}$$

(f)
$$\begin{array}{r} \scriptstyle 6 \qquad 7\,12\ \ 16 \quad \\ 1\not{7},{}^{1}2\not{8}\not{3},\not{7}\,{}^{1}24 \\ -\ \ 7,938,953 \\ \hline 344,771 \end{array}$$

Step (e) is the easiest one we've had since (a). 7 – 3 = 4. Use this opportunity to place a comma between the ones and thousands periods. In (f) it'll take another renaming before 9 can be taken from 2. Moving to the left we'll rename that 7 as a 6 and regroup ten, which gets added to the 2 to make 12, Then subtract 9 from the 12: 12 – 9 = 3.

(g)
$$\begin{array}{r} \scriptstyle 6 \qquad 7\,12\ \ 16 \quad \\ 1\not{7},{}^{1}2\not{8}\not{3},\not{7}\,{}^{1}24 \\ -\ \ 7,938,953 \\ \hline 9,344,771 \end{array}$$

Finally, in (g), bundle that 1 together with the 6 in your head, so as to be able to subtract 7 from 16: 16 – 7 = 9, and, of course, place that comma for a difference of 9 million, 3 hundred 44 thousand, 7 hundred 71. Wow! That's some difference!

Now I have to look back a page to see what the estimate was: It was 9 million 3 hundred thousand. It was certainly in the ballpark. Again, you can check this difference by adding up.

By the way, that's the way to check an addition, too, but in addition you don't add the sum to what's above it. You just add up each column to see whether you get the same sum going up as you did going down. With one this size, however, I'd much rather check my addition on a calculator. For goodness sake, why do you think calculators were created?!

It checks.

Second Chances

Now that you've had one chance at estimating the results in an addition and a subtraction involving large numbers, I think you deserve a second chance, or more, at each.

Here are two additions to estimate to the nearest 1,000. (To round to thousands, drop everything to the right of thousands.)

1. 472,671 + 627,832 + 1,545,439 + 372,218
2. 837,158 + 467,281 + 591,347 + 914,643

To solve number one, round to thousands 473,000 + 628,000 + 1,545,000 + 372,000. Next, get rid of 3 zeroes: 473 + 628 + 1545 + 372. Next, separate the hundreds just with the hundreds: There are 4, 6, 15 and 3 hundreds. Conveniently, 4 hundred and 6 hundred make 10 hundred (yes, I know that's a thousand, but I only want to think in terms of hundreds right now). 15 hundred and 3 hundred make 18 hundred, which I'll add to the 10 to get a total of 10 + 18 = 28 hundreds. Now look at the remaining parts, 73, 28, 45, and 72. 72 + 28 = 100. That makes 30 hundreds. Take 27 from the 45 to add to the remaining 73 to make 30 hundreds. Finally, subtracting the 27 from the 45 left leaves 18 remaining. Let's recap: there are 30 hundreds and 18. That's written 3018. Finally, stick those three zeroes back on to get the excellent estimate of 3,018,000.

I'm now going to put the original numbers into my calculator, but you can feel perfectly free to add them on paper if you would like the practice. I got 3,018,060. That's within 1 thousand!

I have a confession to make. When I first did that solution, I was off by a hundred thousand. That was because I accidentally copied "1545" as "1646." It wasn't until my third time through it that I caught the error. The reason I'm confessing this is because there's a lesson to be learned. That lesson is *be careful!*

Now for number two: Round to thousands and strip three zeroes at the same time: 837 + 467 + 591 + 915. (Remember, if the digit to the right of the one you're rounding to is 5 or greater, round up, so "9146" becomes "915.") Collect the hundreds: 8, 4, 5, 9 hundreds: 8 + 4 = 12 + 5 = 17 + 9 = 26 hundreds. Remaining are 37 + 67 + 91 + 15: 65 + 35 = 100, so 37 + 67 = 4 more than 100, or 104. 91 needs 9 more to be 100, which you'll take from the 15 making 106. 104 + 106 = 210. Combine the 2 hundreds with the 26 hundreds you found before and get 28 hundred ten. That's written 2810. Now put back the 3 zeroes to make 2,810,000, and there's the estimate. How close is it? Do the addition and find out. Just between you, me, and the wall, I'll tell you that it's off the mark by only 429, but if you want to know what the answer is, you'll have to do the addition.

Here are two subtractions to estimate to the nearest 10,000. (To round to ten-thousands, drop everything to the right of ten-thousands.)

1. 875,734 – 457,372
2. 2,785,632 – 1,586,729

To do number 1, round to the ten thousands: That gives you 880,000 – 460,000. Drop the 4 zeroes and subtract: 88 – 46 = 42, so with the zeroes that's 420,000. That means the difference will be no less than 410,000 nor more than 430,000. If you want to try the subtraction before I tell you what the answer is, don't look past this point. Using my trusty scientific calculator, I get a difference of 418,362. That's certainly well **within tolerance.**

Rounding number 2 to the ten thousands, we get 2,790,000 – 1,590,000. Drop the 4 zeroes and subtract: 279 – 159 = 120. That makes the estimated quotient 1,200,000. That means the difference must be no smaller than 1,190,000 or larger than 1,210,000. As a teaser, I'll tell you that the actual difference is 1097 away from the estimated value, but if you want to know what it is, you'll have to do the subtraction.

EXERCISES

Estimate the answer to the nearest 10,000; then solve.

1.
$$\begin{array}{r} 539,734 \\ 2,347,586 \\ 382,967 \\ \underline{81,263} \end{array}$$

2.
$$\begin{array}{r} 1,725,619 \\ \underline{-\ 286,574} \end{array}$$

3.
$$\begin{array}{r} 4,317,503 \\ \underline{-\ 634,871} \end{array}$$

4.
$$\begin{array}{r} 6,459,738 \\ 87,563 \\ 453,891 \\ \underline{2,564,917} \end{array}$$

5.
$$\begin{array}{r} 472,634 \\ \underline{-254,975} \end{array}$$

6.
$$\begin{array}{r} 45,387 \\ 263,546 \\ 538,295 \\ 67,809 \\ \underline{438,752} \end{array}$$

7.
$$\begin{array}{r} 2,186,493 \\ \underline{-\ 798,674} \end{array}$$

8.
$$\begin{array}{r} 759,826 \\ 97,870 \\ 438,561 \\ 583,452 \\ \underline{645,343} \end{array}$$

9.
$$\begin{array}{r} 5,479,253 \\ 6,587,580 \\ 8,365,397 \\ \underline{4,843,674} \end{array}$$

Estimate the answer to the nearest 1,000; then solve.

10.
$$\begin{array}{r} 815,328 \\ \underline{-\ 697,499} \end{array}$$

11.
$$\begin{array}{r} 327,486 \\ 632,598 \\ 217,843 \\ 85,617 \\ \underline{56,409} \end{array}$$

12.
$$\begin{array}{r} 243,536 \\ \underline{-\ 94,618} \end{array}$$

13.
$$\begin{array}{r} 2,507,642 \\ 95,468 \\ 635,703 \\ 4,367,580 \\ \underline{857,458} \end{array}$$

14.
$$\begin{array}{r} 567,433 \\ \underline{-\ 426,532} \end{array}$$

15.
$$\begin{array}{r} 138,495 \\ \underline{-\ 97,628} \end{array}$$

16. 891,267 + 482,658 + 2,567,460 + 9847 + 821,369

17. 654,179 + 2,367,518 + 63,678 + 840

18. 7,589,326– 4,276,458

ANSWERS

1. 3,350,000; 3,351,550
2. 1,440,000; 1,439,045
3. 3,690,000; 3,682,632
4. 9,560,000; 9,566,109
5. 220,000; 217,659
6. 1,360,000; 1,353,789
7. 1,390,000; 1,387,819
8. 2,530,000; 2,525,052
9. 25,280,000; 25,275,904
10. 118,000; 117,829
11. 1,320,000; 1,319,953
12. 149,000; 148,918
13. 8,464,000; 8,463,851
14. 140,000; 140,901
15. 40,000; 40,867
16. 4,772,000; 4,772,601
17. 3,087,000; 3,086,215
18. 3,313,000; 3,312,868

LESSON 12

Multiplying Larger Numbers

As with addition and subtraction, it is helpful to know about how big the answer is going to be in order to have a measuring stick against which to compare your worked out answer. When multiplying large numbers, the product tends to get large quickly, and you'll find the same thing is true of your estimates. Fortunately, a little trick to estimating products will make it easier than you might think. Check out the following.

Mental Multiplication

$$2 \times 10 = 20$$
$$20 \times 10 = 200$$
$$200 \times 10 = 2000$$
$$2 \times 40 = 80$$
$$20 \times 40 = 800$$
$$200 \times 40 = 8000$$

Do you see the pattern? To mentally multiply any number by 10, **append** a zero to the end of it (that means stick it on the end); to mentally multiply any number by 100, append two zeroes to the end of it. In general, to mentally multiply any number by a multiple of 10, multiply the non-zero parts together, and then append as many zeroes as there are in the multiple of 10:

Following that format, $300 \times 43 = 3 \times 43$ (which is 129), append "00" = 12,900.

Similarly, $4000 \times 21 = 4 \times 21$ (which is 84), append "000" = 84,000.

Estimating Large Products

Since you've made it to this point, I'll assume you understand the last section. When estimating large products, the same formula as before is followed, with one exception. Consider the following multiplication:

$$\begin{array}{r} 526 \\ \times\ 385 \\ \hline \end{array}$$

You'll be multiplying hundreds times hundreds. $100 \times 100 = 10{,}000$. Again, you multiplied the non-zero portions together ($1 \times 1 = 1$), but this time you have zeroes from both factors. Each hundred contributes 2 zeroes, so you have 4 zeroes to append to the partial product, 1. That's the 10,000.

Suppose that you were multiplying 900 × 900. Following the same format, you'll multiply 9 × 9, and then append 4 zeroes (2 from each factor): 9 × 9 = 81; append 4 zeroes, and get 810,000.

If you wished to estimate the product to the nearest hundred thousand, start with hundreds times hundreds. You've already seen that the range could go as low as 10,000, but it's not going to unless the multiplication is 100 × 100.

Do you remember this multiplication? It first appeared on the last page.

$$\begin{array}{r} 526 \\ \times\ 385 \\ \hline \end{array}$$

To estimate the product to the nearest 10,000, first multiply the hundreds together:

$$500 \times 300 = 15 \text{ and 4 zeroes, } = 150{,}000$$

Then multiply the tens in the bottom factor times the hundreds in the top:

$$80 \times 500 = 40 \text{ and 3 zeroes, } = 40{,}000$$

Then add them together: $150{,}000 + 40{,}000 = 190{,}000$

That's it. The product should be in the neighborhood of 190,000. Now let's actually work it out.

$$\begin{array}{r} {\scriptstyle 1\ 3\ \ } \\ 526 \\ \times\ 385 \\ \hline 2630 \end{array}$$

Starting with the ones, multiply: 5 × 6 = 30; write the 0 and rename the 3 tens; 5 × 2 = 10, + 3 = 13; write the 3 and rename the ten. (Note, after you have it down pat, you no longer have to concern yourself with the technical fact that it's really a hundred.) Next multiply 5 × 5 to get 25, + 1 = 26, which you can write below the line in its entirety.

$$\begin{array}{r} {\scriptstyle 2\ 4\ \ } \\ 526 \\ \times\ 385 \\ \hline 2630 \\ 42080 \end{array}$$

We place a zero in the ones column, since there are no ones when multiplying by tens. Then we multiply 8 × 6 = 48; write the 8 below the 3 and rename the 4 tens above the 2. 8 × 2 = 16, + 4 = 20; write the 0 below the 6 and rename the 2 above the 5. Next multiply 8 × 5 to get 40, + 2 = 42, which you can write in its entirety.

$$\begin{array}{r} {\scriptstyle 1\ \ } \\ 526 \\ \times\ 385 \\ \hline 2630 \\ 42080 \\ 157800 \end{array}$$

We place zeroes in the ones and tens places, since there will be no ones or tens when multiplying by hundreds. Then we multiply 3 × 6 = 18; write the 8 below the 0 and rename the ten above the 2. 3 × 2 = 6, + 1 = 7; write it below the 2. Next multiply 3 × 5 to get 15, which you write in its entirety.

$$\begin{array}{r} 526 \\ \times\ 385 \\ \hline 2630 \\ 42080 \\ 157800 \\ \hline 202,510 \end{array}$$

Finally, add the three partial products together (and insert a comma) to get a total product of 202,510. Is that within 10,000 of the original estimate?

The estimate was 190,000. 202,520 – 190,000 = 12,520.

Close, but no cigar! (No, I'm not encouraging smoking. It's just an expression I grew up with, which apparently originated in carnival contests in the 1800s and early 1900s, when a contestant had to hit a target to win a cigar. The intimation is "You were almost successful, but not quite." or "Even a near miss is still a miss.")

If you recall, the estimate was arrived at by multiplying the hundreds of one factor by the hundreds and tens of the other. Suppose that the top factor had been rounded to the nearest 10 (530) and multiplied by the ones digit of the other:

$$530 \times 5 = 2650$$

Adding that to the prior estimate gets: 190,000 + 2650 = 192,650, and that estimate wins the cigar by a margin of 140: (202,510 – 192,650 = 9860)

Let's try doing one more of these, estimating the product, and then multiplying a 4-digit numeral times a 3-digit one. I promise no more discussion of stogies:

$$\begin{array}{r} 654 \\ \times\ 8467 \\ \hline \end{array} \text{ becomes } \begin{array}{r} 8467 \\ \times\ 654 \\ \hline \end{array}$$

First, turn over the factors. Why would you want to multiply by four different numbers when you only need to multiply by three? Next, do the estimate, aiming for a number within 10,000 of the actual product. Although you could multiply each of the bottom factors by 8400, it's easier to do the mental multiplication by separating 8400 into 8000 and 400:

$$\begin{aligned} 600 \times 8000 &= 6 \times 8 \text{ and "00000"} = 4{,}800{,}000 \\ 600 \times 400 &= 6 \times 4 \text{ and "0000"} = 240{,}000 \\ 50 \times 8000 &= 5 \times 8 \text{ and "0000"} = 400{,}000 \\ 50 \times 400 &= 5 \times 4 \text{ and "000"} = 20{,}000 \\ 4 \times 8000 &= 4 \times 8 \text{ and "000"} = 32{,}000 \\ 4 \times 400 &= 4 \times 4 \text{ and "00"} = 1600 \end{aligned}$$

Add up all but the top number and the bottom number first: 240 + 400 = 640, + 20 = 660, + 32 = 692,000. The last sentence is an example of keeping a **running total.** Take note of the fact that since all the numbers being added were thousands, we didn't bother to add the thousands, waiting until the last step to tack them on.

Now, add the 692 thousand to the 4800 thousand and get 5,492,000. Why didn't we add the final 1600? Because it contains no ten thousands, so it's not relevant. That means the product must be within 10,000 either side of 5,492,000, or within the gap 5,392,000 and 5,592,000. Let's see whether it is:

Ones:

$$\begin{array}{r} {}^{1\,2\,2} \\ 8467 \\ \times\ 654 \\ \hline 33868 \\ 0 \end{array}$$

Tens:

$$\begin{array}{r} {}^{2\,3\,3} \\ 8467 \\ \times\ 654 \\ \hline 33868 \\ 423350 \\ 00 \end{array}$$

Hundreds:

$$\begin{array}{r} {}^{2\,4\,4} \\ 8467 \\ \times\ 654 \\ \hline 33868 \\ 423350 \\ 5080200 \end{array}$$

Note that we first multiplied by the ones and placed the 0 for the tens. Then we multiplied by the tens and placed the two 0s for the hundreds. Finally, we multiplied by the hundreds.

Add:

$$\begin{array}{r} 8467 \\ \times\ 654 \\ \hline 33868 \\ 423350 \\ 5080200 \\ \hline 5537418 \end{array}$$

Place commas:

$$\begin{array}{r} 8467 \\ \times\ 654 \\ \hline 33\,868 \\ 423\,350 \\ 5\,080\,200 \\ \hline 5{,}537{,}418 \end{array}$$

Take note of the fact that the commas were not placed in the partial products but were placed only after the complete product was found.

The estimate was that the product must be within 10,000 either side of 5,492,000, or within the gap between 5,392,000 and 5,592,000. 5,537,418 is well within that range. Remember that an estimate is just that. It won't always necessarily be as close as you would like it to be, but that's a tradeoff you make in exchange for being able to make it using a minimum amount of time.

It might occur to you, and/or your student, that the answer in the previous product was arrived at by adding, so why is it referred to as a "product" rather than a "sum?" My answer is that first of all, we have never tried to hide the fact that multiplication is a form of addition. Having said that, however, each of the partial products was arrived at by multiplication. The final answer is a sum of partial products, and as such is a product. If, after that explanation, you're still not satisfied, remember that all mathematics is the product of natural and/or physical law, or of human invention.

EXERCISES

Estimate each product to the nearest 10,000. Then solve to find the product.

1. $357 \times 468 =$ ___

2. $563 \times 732 =$ ___

3. $249 \times 385 =$ ___

4. $938 \times 765 =$ ___

5. $\begin{array}{r} 897 \\ \times\ 678 \\ \hline \end{array}$

6. $\begin{array}{r} 389 \\ \times\ 342 \\ \hline \end{array}$

7. $\begin{array}{r} 986 \\ \times\ 569 \\ \hline \end{array}$

8. $\begin{array}{r} 638 \\ \times\ 460 \\ \hline \end{array}$

9. $\begin{array}{r} 756 \\ \times\ 385 \\ \hline \end{array}$

10. $\begin{array}{r} 447 \\ \times\ 398 \\ \hline \end{array}$

Estimate each product to the nearest 100,000. Then solve to find the product.

11. $8470 \times 231 =$ ___

12. $574 \times 6572 =$ ___

13. $6643 \times 846 =$ ___

14. $279 \times 5018 =$ ___

15. $\begin{array}{r} 2891 \\ \times\ 345 \\ \hline \end{array}$

16. $\begin{array}{r} 5834 \\ \times\ 627 \\ \hline \end{array}$

17. $\begin{array}{r} 874 \\ \times\ 2695 \\ \hline \end{array}$

18. $\begin{array}{r} 597 \\ \times\ 6438 \\ \hline \end{array}$

19. $\begin{array}{r} 204 \\ \times\ 7934 \\ \hline \end{array}$

20. $\begin{array}{r} 67345 \\ \times\ \ 489 \\ \hline \end{array}$

ANSWERS

1. 170,000; 167,076
2. 410,000; 412,116
3. 100,000; 95,865
4. 720,000; 717,570
5. 600,000; 608,166
6. 130,000; 133,038
7. 560,000; 561,034
8. 290,000; 293,480
9. 282,000; 291,060
10. 180,000; 177,906
11. 1,900,000; 1,956,570
12. 3,700,000; 3,772,328
13. 5,600,000; 5,619,978
14. 1,400,000; 1,400,022
15. 900,000; 997,395
16. 3,700,000; 3,657,918
17. 2,300,000; 2,355,430
18. 3,800,000; 3,843,486
19. 1,600,000; 1,618,536
20. 32,300,000; 32,931,705

I hope you knew to turn the factors in 17, 18, and 19 over before multiplying them. It would have made your life much easier.

LESSON 13

Dividing Large Numbers

Since division is the undoing (or inverse) of multiplication, there are certain aspects of multiplication that reverse in division. I have to be careful of how far I can go with that subject at this time, since we have not yet dealt with decimal fractions, which would normally come into play here.

Mental Division

For the reason already mentioned, I'm going to confine the discussion of mental division to multiples of 10. When hundreds are divided by tens, tens result:

$$800 \div 10 = 80$$

$$800 \div 20 = 40$$

$$800 \div 40 = 20$$

$$800 \div 80 = 10$$

Do you see a pattern there? It's 8 divided by 1, 2, 4, and 8 with the zeroes from the 100 redistributed into tens and tens. In other words, if you multiply from right to left across the " = " (quotient times divisor) you'll get the dividend as the product. (Remember multiplying tens, or look back to mental multiplication in the last lesson.)

What do you suppose happens when you divide thousands by hundreds?

$$8000 \div 100 = 80$$

$$8000 \div 200 = 40$$

$$8000 \div 400 = 20$$

$$8000 \div 800 = 10$$

Count the zeroes on each side of the "divided by" sign. Do you see that there are three zeroes on each side? This time it's 8 divided by 1, 2, 4, and 8 with the zeroes from the 1000s redistributed into hundreds and tens. Check these out:

$$80{,}000 \div 100 = 800$$

$$80{,}000 \div 200 = 400$$

$$80{,}000 \div 400 = 200$$

$$80{,}000 \div 800 = 100$$

Here we've divided eight ten thousands (also known as 80 thousand) by hundreds. Did you notice that there are four zeroes on each side of the division sign? This time it's 8 divided by 1, 2, 4, and 8 with the zeroes from the 10,000s redistributed into hundreds and hundreds. The pattern should by now be clear. When dividing a multiple of ten by another, smaller one, the number of zeroes remains the same on both sides of the division. They are just redistributed between the divisor and the quotient.

Short Division

Let's digress, or, if you like, take a "short" break from the main theme of the current symposium. Short division may be the first form of division you ever learned. It certainly was the first that I ever learned. It is meant for division by a single-digit divisor. It works from left to right and is 1 step removed from mental division, but completely disregards place value. To divide 832 by 4, we would proceed like this:

$$\begin{array}{r} 2 \\ 4\overline{)832} \end{array}$$

Ask yourself: "How many times does 4 go into 8?" The answer is 2 and none remaining, so write the 2 above the 8.

Next, ask yourself, "How many times does 4 go into 3?"

$$\begin{array}{r} 20 \\ 4\overline{)83\,{}^{3}2} \end{array}$$

The answer is none, so write the 0 above the 3 and regroup the 3 with the 2 to make 32. (Some teachers will not bother with the additional notation but ask the student to mentally group the 3 with the 2 to make 32.)

Next, ask yourself, "How many times does 4 go into 32?"

$$\begin{array}{r} 208 \\ 4\overline{)83\,{}^{3}2} \end{array}$$

The answer is 8 (since $8 \times 4 = 32$), so write the 8 above the 2, and you're done. It's not quite as simple as it looks, but you can get used to it after you know all the wrinkles, and it does work well.

Here's another example:

$$9\overline{)2357}$$

Ask yourself: "How many times does 9 go into 2?"

The answer, of course, is, "It doesn't," so regroup the 2 with the 3 to make 23. This you do in your head, since you can visualize the 23 by blocking out the rest of the numeral mentally or with your hand. No numeral will be written above the 2, since we're treating it as if it were the tens digit in a two-digit number.

Next, ask yourself, "How many times does 9 go into 23?"

$$\begin{array}{r} 2 \\ 9\overline{)23\,{}^{5}57} \end{array}$$

The largest number of 9s that can be crammed into 23 is 2, for a total of 18, which is 5 short of 23. Write the 2 over the 3 of the 23 you divided it into, and look what I've done with the remaining 5 (from $23 - 18 = 5$). It has been attached to the 5 to make 55. (Don't worry about the 7. We'll pay that toll when we come to the bridge—or something like that.)

So, now ask yourself, "How many times does 9 go into 55?"

$$\begin{array}{r} 2\;6\phantom{\,{}^{1}7} \\ 9\overline{)2\,3\,{}^{5}5\,{}^{1}7} \end{array}$$

If you're up on your 9s, you know that 6 of them make 54. Guess what happens to the difference between 55 and 54. Oh, you must have peeked.

Finally, ask yourself "How many 9s in 17?"

$$\begin{array}{r} 2\;6\;1r8 \\ 9\overline{)2\,3\,{}^{5}5\,{}^{1}7} \end{array}$$

There is exactly one 8 in 17. Subtracting the 9 from 17 gives a remainder of 8, which is dutifully tacked on as an "r."

Estimating Large Quotients

$$265\overline{)5839}$$

Look at the divisor in the preceding division. It's closer to 300 than it is to 200, so for purposes of estimating the quotient, we'll round it up. Now look at the dividend. Round it up to 6000. *About what* would you expect the quotient to be? Well, we're dividing thousands (3 zeroes) by hundreds (2 zeroes), so we're going to get (3 zeroes – 2 zeroes = 1 zero) an answer that's in the 10s. 6 divided by 3 is 2, so the quotient should be somewhere around 2 tens, or 20. I'd feel comfortable with a quotient somewhere between 17 or 18 and 22 or 23. In fact, the quotient is 22, r9

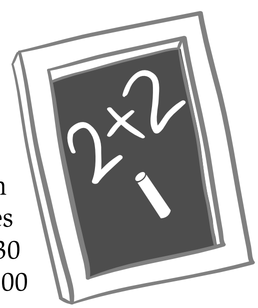
Now estimate the quotient for this one:

$$423\overline{)12,395}$$

Did you actually estimate the quotient or are you waiting for me to do it? Okay. You've had enough time. You should have rounded 423 down to 400 and 12,395 down to 12,000. Once again, it's hundreds and thousands, so the answer should be in the **decades** (tens). 12 divided by 4 is 3, so I would estimate the quotient to be around 30; would you? It's actually 29, r128. Are you wondering why we called 12,000 in the thousands instead of in the ten-thousands? The "1" is in the ten thousands, but if we're bundling it with the "2" to make "12," it's 12 thousand, which, in point of fact has its left-most digit in the 10,000's place.

Let's do one more. Try estimating the quotient for this one:

$$567\overline{)352,457}$$



Are you finished? All right then, here goes. 567 rounds up to 600; 352,457 rounds down to 350,000. That's 2 zeroes versus 4 zeroes, so the estimate should have "00" (4 – 2 = 2). 35 divided by 6 is 5 and change. Actually, that change is much closer to 1 than to 0, since there are 6 "6s" in 36. For that reason, I would go with 600 as my rough estimated quotient. Now I'm going to take a look at the quantities I discarded, because they're pretty sizeable. I increased the divisor by about 30 when rounding up, and I decreased the dividend by almost 2,500. 30 goes into 2500 about 80 times, and if both had been rounded up or down I would either increase or decrease my estimate by 80, respectively. However, since one got bigger and one got smaller, I would split the difference and say the actual quotient might be 40 either side of 600—in other words, greater than 560 and less than 640. Does that sound reasonable? Let's see what actually happens:

```
(a)         6            (b)          62            (c)          621
    567)352,457              567)352,457              567)352,457
        3402                     3402↓                    3402↓↓
         122                      1225                     1225↓
                                  1134                     1134↓
                                    91                      917
                                                            567
                                                            350
```

(a) Using 600 as a trial divisor, put it over the hundreds digit (you already estimated that your answer is going to be in the hundreds). Multiply 6×567, write the product below 3524, and subtract.

(b) Start by bringing down the 5, placing it next to the second 2 of 122, and turning that into 1225. Divide that 1225 by 600 and get 2. Place that 2 above the 5 in the dividend and multiply that 2 times the numerator to get 1134. Subtract the 1134 from the 1225 above it to get 91.

(c) Start by bringing down the 7 and placing it next to the 1 of 91 and turning that into 917. Divide that 917 by 600 and get 1. Place that 1 above the 7 in the dividend and multiply that 1 times the numerator to get 567. Subtract the 567 from the 915 above it to get 350, the remainder. The final quotient is 621, r 350.

You might recall that the estimated quotient was 600 ± 40, so this quotient certainly falls within those parameters.

EXERCISES

For 1–8, solve by short division.

1. $8\overline{)9645}$

2. $5\overline{)2769}$

3. $3\overline{)81,654}$

4. $7\overline{)144,956}$

5. $6\overline{)32,952}$

6. $9\overline{)573,481}$

7. $4\overline{)38,216}$

8. $8\overline{)329,766}$

For 9–20, estimate the quotient; then find it.

9. $241\overline{)6895}$

10. $384\overline{)9268}$

11. $857\overline{)17,586}$

12. $435\overline{)62,391}$

13. $308\overline{)81,736}$

14. $436\overline{)46,729}$

15. $684\overline{)19,435}$

16. $542\overline{)38,679}$

17. $365\overline{)148,362}$

18. $725\overline{)648,925}$

19. $234\overline{)657,532}$

20. $441\overline{)768,462}$

ANSWERS

1. 1205, r5
2. 553, r4
3. 27,208
4. 20,708
5. 5492
6. 63,720, r1
7. 9554
8. 41220, r6
9. 35*; 28, r147
10. 22; 24, r52
11. 20; 20, r446
12. 150; 143, r186
13. 270; 265, r116
14. 120; 107, r77
15. 30; 28, r283
16. 80; 71, r197
17. 400; 406, r172
18. 900; 895, r50
19. 3000; 2809, r226
20. 1900; 1742, r240

*This might seem like a strange estimate, but consider the trial divisor is 200 and the dividend is 7000. $70 \div 2 = 35$.

LESSON 14

Evens, Primes, and Divisibility

A few things might be helpful to you when dealing with whole numbers, and I thought that this might be a good place to bring them to your attention. Some things are easy to take for granted, such as everyone knows the difference between odd and even numbers. Just in case you're looking for an easy definition, even numbers are those that can be divided by 2 with no remainders. Odd ones are those that cannot be. Zero is an even number, since two goes into it exactly 0 times with no remainder. (Actually, zero divided by any number is zero.)

Prime Numbers

A **prime number** is defined as a number that has exactly two factors, itself and 1. The first prime number is 2, with factors of 1 and 2 and no others. Two is also the only even prime number, since every other even number must have 2 as a factor, as well as itself and 1. A number with more than 2 factors is known as a **composite number.**

Prime numbers show up very frequently among the first 20 natural numbers. The higher you count, however, the rarer they become. The easiest way to find the prime numbers in the first 100 is by using the ancient **Sieve of Eratosthenes.**

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Make a copy of the preceding table and then start crossing out. Start with 1 and put a line through it. 1 does not meet the criteria for being prime. It has only one factor, itself. Next circle the 2; then cross out every multiple of 2.

~~1~~	**2**	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~
51	~~52~~	53	~~54~~	55	~~56~~	57	~~58~~	59	~~60~~
61	~~62~~	63	~~64~~	65	~~66~~	67	~~68~~	69	~~70~~
71	~~72~~	73	~~74~~	75	~~76~~	77	~~78~~	79	~~80~~
81	~~82~~	83	~~84~~	85	~~86~~	87	~~88~~	89	~~90~~
91	~~92~~	93	~~94~~	95	~~96~~	97	~~98~~	99	~~100~~

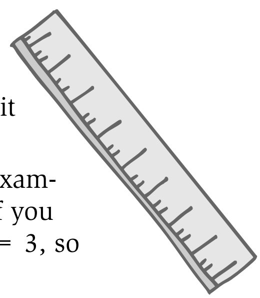
Well, that kind of narrowed the field in a hurry. To speed things up, I'm going to circle the 3, 5, and 7, since each of those has exactly two factors. Then I'm going to cross out every multiple of 3, 5, and 7 that's left in the chart. You might want to do that with the 3s first, then the 5s, and finally the 7s.

~~1~~	**2**	**3**	~~4~~	**5**	~~6~~	**7**	~~8~~	~~9~~	~~10~~
11	~~12~~	**13**	~~14~~	~~15~~	~~16~~	**17**	~~18~~	**19**	~~20~~
~~21~~	~~22~~	**23**	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	**29**	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	**37**	~~38~~	~~39~~	~~40~~
41	~~42~~	**43**	~~44~~	~~45~~	~~46~~	**47**	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	**53**	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	**59**	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	**67**	~~68~~	~~69~~	~~70~~
71	~~72~~	**73**	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	**79**	~~80~~
~~81~~	~~82~~	**83**	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	**89**	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	**97**	~~98~~	~~99~~	~~100~~

Whoa! "Why," you're asking yourself, "did he put all those other circles up on the chart?" Well, the answer is that after the multiples of 3, 5, and 7 have been eliminated, nothing but prime numbers remain on the chart. I suppose that's why Eratosthenes called it a "sieve." All the non-primes fall right through it. Of the first 100 natural numbers, 22 are prime, 77 are composite, and 1 is neither.

Divisibility Tests

Certain tests make it easier to tell whether a large number is **divisible** (can be perfectly divided) by another without having to actually divide.



1. Every whole number is divisible by 1, so that's not a number for which you need to test divisibility.
2. If a number is even, it is divisible by 2. A number is even if its ones digit is 2, 4, 6, 8, or 0.
3. A number is divisible by 3 if the sum of its digits is divisible by 3. For example, to test 18,291 for divisibility by 3, add 1 + 8 + 2 + 9 + 1 = 21. If you don't recognize 21 as being divisible by 3, simply add its digits: 2 + 1 = 3, so 18,291 is divisible by 3.

4. A number is divisible by 4 if its tens and ones digits form a number that is a multiple of 4. For instance, 217,5<u>24</u> is divisible by 4, because $6 \times 4 = 24$.
5. A number is divisible by 5 if its ones digit is a “5” or a “0.”
6. A number is divisible by 6 if it is divisible by 3 *and* is even.
7. Fuhgeddaboudit! The tests for divisibility by 7 are not worth memorizing. Just divide.
8. If the number formed by the rightmost 3 digits is divisible by 8, then the whole number is divisible by 8. Of course, if you need a calculator to compute whether that 3-digit number is divisible by 8, then there’s no point in using this method.
9. A number is divisible by 9 if adding the digits together repeatedly gets you to 9. This is the same as the 3s rule, only with 9s.
10. A number is divisible by 10 if its 1s digit is 0.

EXERCISES

1. Find the next 3 primes after 97.

Tell whether the following numbers are prime or composite:

2. 133
3. 627
4. 143
5. 286
6. 507
7. 307
8. 139
9. 301
10. 683

By which numbers between 2 and 10 are the following numbers divisible?

11. 1,567,982
12. 87,592,428
13. 789,465
14. 32,173,330
15. 8,673,256
16. 69,835,317
17. 379,831,732
18. 421,634,569
19. 834,953,826
20. 10,080

ANSWERS

1. 101, 103, 1072
2. prime
3. composite
4. composite
5. composite
6. composite
7. prime
8. prime
9. composite
10. prime
11. 2
12. 2, 3, 4, 6, 7, 8, 9
13. 3, 5
14. 2, 5, 10
15. 2, 4, 8
16. 3
17. 2, 4, 7
18. none
19. 2, 3, 6, 7
20. 2, 3, 4, 5, 6, 7, 8, 9, 10