Chapter 1

Introduction

Effective Mathematics Instruction

A problem is not necessarily solved because
the correct answer has been made.
A problem is not truly solved unless the learner
understands what he has done and knows
why his actions were appropriate.

—William A. Brownell,
The Measurement of Understanding (1946)

Teaching mathematics is a complex enterprise. Mathematics involves a language, a body of knowledge, a way of thinking, and a set of skills. Having mathematical proficiency involves acquiring several skills and dispositions. According to the National Research Council (2002), these include computing, applying, understanding, reasoning, and engaging in mathematics. Mathematical proficiency also involves mastery of different content strands. The National Council of Teachers of Mathematics Standards classifies these strands as numbers and operations, algebra, geometry, measurement, and data analysis and probability. Alongside the content strands are also the process strands of problem solving, reasoning and proof, communication, connections, and representation. Whereas there are multiple frameworks for classifying and organizing mathematics, there is little disagreement about the competencies and attitudes that effective mathematics thinkers and users display.

Students who know and use mathematics well are able to understand it. They can comprehend mathematical concepts, operations, and relations—knowing what mathematics symbols, diagrams, and procedures mean. They can also interpret the mathematical principles in a problem and translate those ideas into a coherent mathematical representation using the important facts of the problem.

Proficient students can compute well. They can carry out mathematical procedures—such as adding, subtracting, multiplying, and dividing numbers—in a way that is accurate, flexible, efficient, and appropriate. Computation and execution require that students learn skills so that they can remember them, apply them when needed, and adjust them to solve new problems.

Proficient students can formulate problems mathematically and can devise strategies for solving them using concepts and procedures appropriately. They can reason well by using logic to explain and justify a solution to a problem or to extend it from something known to something unknown.

Requiring students to present their reasoning processes and, where appropriate, to justify them, can help them reflect on their thinking, identify mistakes, and improve their strategies. It also enhances their communication and discussion skills, while enabling teachers to identify the emergence of powerful mathematical ideas among students, so that these can be clarified and nurtured. Reasoning and communication go hand in hand.

Effective mathematical communication requires that students see the connections and relationships in the things they know. It involves using the language of mathematics to express ideas as well as to organize and consolidate mathematical thinking through communication.

Proficient students develop mathematical insights. They can recognize the significance of a problem and its relationship to other problems, other disciplines, or "real world" applications. They understand how mathematics ideas connect and build on each other to produce a coherent whole.

Despite an apparent agreement on the knowledge and skills that mathematics requires, teachers' own background and experience often lead them to adopt a particular view of mathematics and an accompanying approach to teaching it. According to Ernst (1988), teachers tend to have one of the following three conceptions of mathematics:

1. *Problem-solving view:* mathematics as an expanding field of human creation and invention, in which patterns are generated and then distilled into knowledge. Mathematics is seen as a process of inquiry and coming to know, with its results open to revision.

2. *Platonic view:* mathematics as a static but unified body of knowledge, with interconnecting structures and truths. From this perspective, we discover mathematics rather than create it.
3. *Instrumentalist view:* mathematics as a bag of tools made up of an accumulation of facts, rules, and skills used for utilitarian purposes.

Teachers' beliefs and understanding of mathematics have a significant impact on students' beliefs about the nature of mathematics. Many teachers believe that mathematics is the discipline of the "right answer." This belief prevents them from helping students express their mathematical thinking, learn from their mistakes, experiment effectively, and pursue their mathematical interests. Teachers need to help transform the student question *Am I right?* into the questions *How can I develop confidence and judgment that I am on the right track when working on a problem?* and *How can I know that I am improving my mathematical problem-solving and communication skills?* (Math Forum Bridging Research and Practice Group).

Most students in the United States believe that doing mathematics requires a lot of practice in following rules. According to a 2003 study of K–12 mathematics education, only 15 percent of the K–12 mathematics lessons in the United States would be considered high in quality. The factors that distinguish effective lessons from ineffective ones are teachers' ability to do the following:

- Engage students with mathematics content by relating mathematics problems to the real world, connecting to students' interests, and promoting their active involvement.
- Create an environment conducive to learning, through the use of interesting and engaging problems whereby the teacher encourages multiple solution methods, supports students' conceptual understanding, and encourages mathematical thinking.
- Ensure access for all students by allowing them different entry points to problems that are challenging and that require persistence. Good problems are nonroutine, unfamiliar, and just beyond the student's skill level so that the student does not automatically know how to solve them.
- Use questioning to monitor and promote understanding through the diversified and strategic use of content and of processing, leading, nonleading, and clarifying questions.
- Help students make sense of the mathematics by using content that connects to other problems and mathematical concepts, aligning their teaching with current mathematics curriculum and standards and integrating mathematics with other subject areas.

Some Strategies for Improving Mathematics Instruction

To improve student engagement with mathematics content, teachers can develop or adapt problems so that these have a real-life context or purpose. They can also remind students of problem situations that are conceptually similar to the ones they are solving, and they can provide them with needed background knowledge on the problems they have to face. Another strategy for promoting engagement is to lead students through instant replays of a problem situation and encourage them to request the assistance of the teacher or other students.

Among the strategies that teachers can use to create a classroom environment that is conducive to learning is the use of problems that can generate many solutions. Teachers can maximize the use of such problems by waiting for, and listening to, students' descriptions of solution methods, as well as encouraging students to elaborate on their problem-solving strategies and solutions. When teachers use students' explanations as a basis for the lesson's content, they convey an attitude of acceptance of students' errors and efforts.

Mathematics instruction can be greatly enhanced through the use of diversified questioning skills. Such skills include knowing how to use leading, nonleading, clarifying, and processing questions that tap students' different levels of thinking. Leading questions can be used to begin a lesson, probe the depth of students' understanding, elicit content, help a student clarify or extend his or her thinking, or provide a focus; for example, *How did we get the length of this desk? Why is this a circle?*

Nonleading questions are most effective when responding to students' ideas. The context defines whether a question is leading or not. Teachers ask nonleading questions when we are trying to facilitate students' thinking; for example, *What happened? What did you observe? How did you get it? Why are you asking that question? Why does it work?* Chapter Four provides teachers with different strategies for enhancing their use of questions in mathematics.

An effective use of wait time can enhance teachers' questioning skills. Wait time includes providing time for students to find the words for an explanation and listening patiently as students try to put their questions into words. It is also a time for teachers to ponder options before acting. Much can happen as a result of acting as though we believe that students have much to say.

Teachers can improve their instruction in mathematics through the development of their communication skills. For example, paraphrasing students' answers to questions shows students that they are being listened to; it can also help introduce mathematical terms to students. Prompts that denote paraphrasing include: *What I hear you saying is . . .* or *Do you mean that . . . ?* Another effective communication skill involves summarizing. Summaries provide students with a compact record of their thinking that they can revisit later. Finally, listening is also part of the overall effort to diagnose students' strengths and needs in order to determine appropriate interventions. It involves listening to understand by recognizing that students' mathematical actions and explanations are reasonable from their point of view, even if the reason is not immediately apparent to us. (Cobb et al., 1991).

In many classrooms, students are expected neither to explain their thinking about mathematics nor to justify it. If students are taught and supported, they can present their reasoning processes clearly and can justify them as well. Even young students can present their thinking processes quite clearly once they are encouraged to do so and know that their thinking is valued. The use of explicit criteria in the form of rubrics and checklists as instructional devices, along with many of the process questions included in Chapter Three, will help students articulate and develop their reasoning skills.

The chapters in this book model the development and use of effective instruction. Chapter Two introduces teachers to diverse assessment measures and processes. Through the use of concrete problems and situations, teachers learn about diagnostic, formative, and summative assessment, and explore the uses of recall-based, performance, product, and process assessment. The chapter ends with an overview of portfolio assessment and the role it can play in the overall evaluation of students' growth and achievement.

Chapter Three introduces teachers to performance criteria related to different content and processes in mathematics and enables them to discover many different ways of articulating their expectations for student achievement. It also explores ways of involving students in the assessment process and helps teachers identify their own preferences with respect to rubrics and checklists.

Chapter Four introduces teachers to diverse questioning practices and helps them identify questions that tap different thinking and reasoning processes. Teachers learn about convergent and divergent questions, questions that tap different levels of Bloom's Taxonomy (Bloom, 1956), and the use of process questions to help students develop conceptual understanding, reasoning, and other mathematics competencies and dispositions.

Chapter Five showcases the use of problems as a means of contextualizing the teaching of mathematical skills. This chapter includes a number of annotated, open-ended problems so that teachers can see the possible range of students' approaches or answers to each of them. The primary intent of this chapter is to help teachers see the use and value of problems as learning and assessment opportunities.

Chapters Six through Nine highlight the use of engaging mathematics lessons centered around patterns, measurement, money, and fractions, many of which embed problems like the ones included in Chapter Five. These chapters are structured so that teachers can compare teacher-directed lessons with student-centered lessons that address the same content and skills. These lessons include explicit differentiation strategies as well as formal assessment opportunities.

The appendices include an annotated list of Web-based and other resources for teaching and assessing in mathematics.