CHAPTER 1

ELEMENTARY CALCULUS

Introduction

We begin our studies by reviewing the topics that form the standard core of a first-year college calculus or an AP calculus (AB or BC) class. Precalculus topics such as the real numbers and functions are assumed to be well known. Our plan in this chapter is to briefly describe the concept, placing the idea historically, and then immediately move to definitions and results, followed by examples and problems. The problems will be chosen to illustrate both examples and counterexamples and to lead to further thought.

1.1 PRELIMINARY CONCEPTS

We will take a number of concepts as previously understood. We are familiar with the properties of the standard number systems, the natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, and the real numbers $\mathbb{R}$. For two real numbers $a < b$, the *open interval* (a, b) is the set $\{x \in \mathbb{R} \mid a < x < b\}$. Similarly, the *closed interval* is

Introduction to Real Analysis. By William C. Bauldry

$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$. We do not use [or] for infinite intervals since ∞ is not a real number. An interval may be half-open by including only one endpoint.

A *function* f is a set of ordered pairs such that no two different pairs have the same first coordinate. For a function f, the set of first coordinates of the ordered pairs is called the *domain*; we write $\operatorname{dom}(f)$. The set of second coordinates of the function is called the *image*. The *range* of a function is a superset of the image. If X is the domain of the function f and Y is the range, we write $f : X \to Y$ and say "f maps X into Y." For $f : X \to Y$ with $A \subseteq X$ and $B \subseteq Y$, we set $f(A) = \{f(x) \in Y \mid x \in A\}$ and $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$. Note that writing $f^{-1}(B)$ does not imply that f^{-1} is a function. A function is *injective* or *one-to-one* if and only if whenever $f(x_1) = f(x_2)$ then we must have $x_1 = x_2$. A function is *surjective* or *onto* if and only if for each $y \in Y$ there must be an $x \in X$ with $f(x) = y$. A *bijective* function is both one-to-one and onto. If f is bijective, then f^{-1} is a function. If f is not one-to-one, then f^{-1} cannot be a function.

The absolute value

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$

gives the distance from x to 0; thus $|x - y|$ gives the distance from x to y. The following inequalities hold for any real numbers a, a_i, b, and b_i.

- $|a| \ge 0$
- $|a| \le b$ if and only if $-b \le a \le b$
- $|a \cdot b| = |a| \cdot |b|$
- The *triangle inequality* $|a + b| \le |a| + |b|$
- $\big||a| - |b|\big| \le |a - b|$
- The *Cauchy-Bunyakovsky-Schwarz inequality*

$$\left[\sum_{i=1}^{n} a_i b_i\right]^2 \le \left[\sum_{i=1}^{n} a_i^2\right] \cdot \left[\sum_{i=1}^{n} b_i^2\right]$$

- The *Minkowski inequality*

$$\sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \le \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$$

An *elementary function* is built from a finite combination of operations ($+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$, $\circ$) applied to polynomials, exponentials, logarithms, or trigonometric or inverse trigonometric functions. If we allow complex numbers, then the trigonometric functions and their inverses are included as exponentials and logarithms.

For proofs and further information, refer to any advanced calculus text or outline; e.g., Wrede & Spiegel (2002) or Farand & Poxon (1984). We will revisit the properties of the real numbers more formally in Chapter 2.

1.2 LIMITS AND CONTINUITY

Even though studied first in calculus classes, historically, limits and continuity were developed formally as concepts well after integrals and derivatives were established (see the calculus timeline in Appendix B). The need to put rigorous foundations under the calculus led to the modern definitions of limit first given by Bolzano and Cauchy. Both Newton and Leibniz wrestled with justifying limits in their computations without solving the problem. Newton, following the lead of Fermat, used what we would call infinitesimals and let them be zero at the end of a computation, interpreting all constructions geometrically. Leibniz used equivalent differentials but worked from an arithmetic point of view. [See, e.g., Boyer (1959, Chapter V) or Burton (2007, Chapter 8).] The dispute over credit for inventing calculus between Newton, Leibniz, and their followers was very unfortunate. Bardi gives an account of the controversy in *The Calculus Wars* (Bardi, 2006) with a thorough historical perspective.

Limits of Functions

We'll begin by looking at intuitive forms of the definitions. Our first version will be taken from Cauchy's seminal 1821 text, *Cours d'Analyse* (Cauchy, 1821). These statements would be encountered in an elementary calculus course. Through problems, we move from an elementary definition to difficulties and challenges that will, in turn, lead us to Weierstrass's ϵ-δ definition of the limit.

Our first experience in calculus was to see limits numerically and graphically, often with the heuristics "are the function values coming together" or "do the two sides of the graph appear to connect?" These intuitive approaches work well for elementary calculus but are not sufficient for a deeper understanding and may even lead to misperceptions.

Definition 1.1 (Cauchy) *When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the* limit *of all the others.*

Or, in modern terms, we write

$$\lim_{x \to a} f(x) = L$$

if and only if we can make values of $f(x)$ arbitrarily close to L by taking x sufficiently close but not equal to a.

Before we try to compute limits, we need to know that there is at most one answer.

Theorem 1.1 *If $f(x)$ has a limit at $x = a$, that limit is unique.*

Now it's time to do some computations.

■ **EXAMPLE 1.1**

1. Determine the value of $\lim_{x \to 1} 5x - 2$.
   Either a table of values or a graph quickly leads us to the answer 3. Do both!

2. Let U be the *unit step function*

$$U(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Find $\lim_{x\to 0} U(x)$.
Again using a table or graph, we see this limit does not exist. Verify this! ■

The simple examples above help us to begin to gain intuition on limits. We can see whether function values "approach indefinitely a fixed value" with a table or with graphs. However, we quickly run into trouble with expressions that are a little more complex. Try generating tables and graphs for the limits in Example 1.2. No algebraic simplifications allowed!

■ **EXAMPLE 1.2**

1. What is the value of
$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$
2. What is the value of
$$\lim_{x\to 2} \frac{x+2}{x^2-4}$$
3. Evaluate
$$\lim_{y\to 0} \frac{\sqrt{y^2+16}-4}{y^2}$$
4. Calculate
$$\lim_{z\to 0} \sqrt[z]{1+z}$$

■

Several of the limits above present difficulties that can be easily overcome with algebra. The natural reduction in the first two limits in Example 1.2 reduces the expressions to simple forms. Rationalizing the numerator of the third limit again leaves an easier form. We need theorems for the algebra of limits. The last limit can be tamed by applying logarithms, but we'll need *continuity* to handle that calculation; we'll return to this limit later.

Theorem 1.2 (Algebra of Limits) *Suppose that $f(x)$ and $g(x)$ both have finite limits at $x = a$ and let $c \in \mathbb{R}$. Then*

- $\lim_{x\to a} c\, f(x) = c \lim_{x\to a} f(x)$
- $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

- $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
- *if* $\lim_{x \to a} g(x) \neq 0$, *then* $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

Our challenge now is to understand how to verify that the answers in the examples are correct. The algebra of limits lets us reduce to simpler forms but doesn't establish particular instances. We could formally show that $\lim_{x \to a} x = a$ and then apply the algebraic rules to handle rational functions at any point in their domain. Nevertheless, this technique would leave a lot of functions unresolved; i.e., all *transcendental functions*. A transcendental function is one that does not satisfy a polynomial equation whose coefficients are themselves polynomials. For example, sines and logarithms are transcendental.

A very useful theorem for computing limits is based on controlling an expression by finding upper and lower bounding functions and sandwiching the desired limit between.

Theorem 1.3 (Sandwich Theorem) *Suppose that* $g(x) \le f(x) \le h(x)$ *for all* x *in an open interval containing* a. *If both* $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, *then* $\lim_{x \to a} f(x) = L$.

Let's use the sandwich theorem to find an interesting limit.

■ **EXAMPLE 1.3**

- Determine

$$\lim_{\phi \to 0} \frac{1 - \cos(\phi)}{\phi^2}$$

Using graphs, we are led to the inequality

$$\frac{1}{2} \quad \phi^2 < \frac{1 - \cos(\phi)}{\phi^2} < \frac{1}{2}$$

The sandwich theorem now easily gives the value of the limit as $1/2$. *Draw the graphs!* ■

Even though the sandwich theorem is very powerful, the last limit of Example 1.2 resists algebra and it's not easy to find bounding functions for $\sqrt[z]{1+z}$. Our intuitive definition doesn't offer guidance for the task of computing limits. What we really need is to formalize what the phrases "arbitrarily close" and "sufficiently close" mean. The "ϵ-δ definition" we use today is attributed to Weierstrass; we state it in a form appropriate to a basic calculus course.

Definition 1.2 (Weierstrass) *Let* f *be defined on an open interval* $\mathcal{I}$ *containing* a, *except possibly at* a *itself. Then*

$$\lim_{x \to a} f(x) = L$$

if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every x in the interval $\mathcal{I}$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Another advantage of this definition is that it easily splits into one sided limits. Often, it is easier to analyze a function from one side of a point than both sides simultaneously.

Definition 1.3 *Let f be defined on an open interval $\mathcal{I}$ containing a, except possibly at a itself. Then the* limit from the left *of f is M and the* limit from the right *is N,*

$$\lim_{x \to a^-} f(x) = M \quad \text{and} \quad \lim_{x \to a^+} f(x) = N$$

respectively, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every x in the interval $\mathcal{I}$, if $0 < a - x < \delta$, then $|f(x) - M| < \epsilon$ from the left and if $0 < x - a < \delta$, then $|f(x) - N| < \epsilon$ from the right.

The limits from the left and right must match for the limit to exist.

Theorem 1.4 *The limit of f as x approaches a exists and equals L if and only if the limits from both the left and the right exist and are each equal to L.*

We can now give a demonstration that the unit step function has no limit at $a = 0$. Choose $\epsilon = 1/2$. Showing that no δ suffices is left to the exercises. Compare with showing that the limit doesn't exist since the left- and right-hand limits are different.

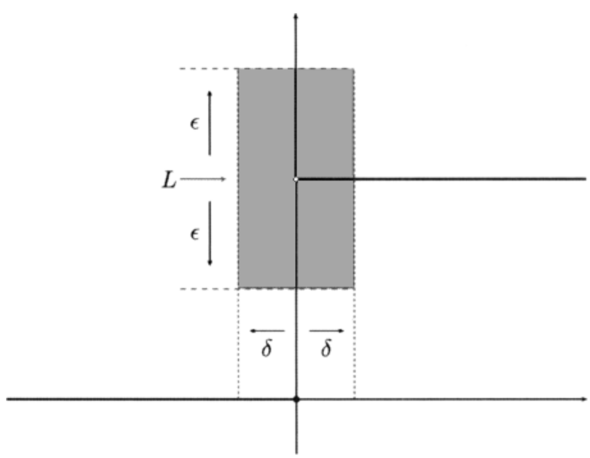

Figure 1.1 An ϵ-δ Box for the Unit Step Function

Verifying the last limit of Example 1.2 by the definition would be quite hard. We need to combine the concepts of limit and continuity to find that limit.

Continuity

The main impetus for studying limits in the beginning of a calculus course is to set the stage for continuity and for computations giving derivatives and integrals. Newton and Leibniz only worked with expressions that were essentially continuous functions. While Leibniz was the first to use the term *function*, mathematicians up through Euler's time considered a function as something that could be written as a single expression, thus eliminating even simple types such as step functions (Burton, 2007, p. 611). Today we have a broader conception of function but still use a narrower view in beginning classes. Current calculus texts follow the lead, either explicitly or implicitly, of Granville, Smith, and Longley (1911, p. 16): "In this book we shall only deal with functions which are in general continuous, that is, continuous for all values of x, with the possible exception of certain isolated values." Since most "calculus-style" functions are quite simple, the common heuristic for describing a continuous function is "graphing without lifting the pencil." Though intuitively appealing, this rubric quickly breaks down when considering more general functions and, as we'll see in Chapter 2, is actually misleading.

Let's start with a standard definition.

Definition 1.4 *The function f is* continuous *at $x = a$ if and only if*

$$\lim_{x \to a} f(x) = f(a)$$

Definition 1.4 combines three conditions into a single limit statement.

1. "$= f(a)$" requires that a belongs to the domain of f; i.e., f has a value at a.
2. "$\lim_{x \to a} f(x) =$" requires that the limit exists.
3. "$\lim_{x \to a} f(x) = f(a)$" requires that the value of the limit match the value of the function at a.

Creating examples violating each condition listed above is left to the exercises.

Since the ϵ-δ definition of limit specifies that f must be defined on an interval containing the point in question, we have eliminated isolated points of f's domain from our discussion of limits and hence from consideration as points of continuity.

■ EXAMPLE 1.4

1. Show that linear functions $f(x) = mx + b$ are continuous everywhere.
   Simple application of the definition.

2. Verify that the rational functions $r(x) = (ax^2 + bx + c)/(a'x^2 + b'x + c')$ are continuous wherever defined.
   Another simple application of the definition.

3. Let

$$g(\theta) = \begin{cases} \sin(\theta)/\theta & x \neq 0 \\ \alpha & x = 0 \end{cases}$$

Determine a value for α that makes g continuous at $\theta = 0$, if possible.
We need to find $\lim_{\theta\to 0} g(\theta)$. *First, graph it!* ■

A function that is continuous at each point of a set D is said to be *continuous on D.* What is the largest set where the functions of the example above are continuous? The functions studied in elementary calculus are typically continuous or continuous on contiguous intervals.

Definition 1.5 *If a function f is continuous on an interval $\mathcal{I}$ except at a finite number of points, then f is called* piecewise continuous on $\mathcal{I}$.

Classifying points of discontinuity is very useful in understanding a function's behavior. We categorize these points into four kinds of discontinuity. The classifications are made on the behavior of the one-sided limits. If both exist, the discontinuity is *simple*, otherwise the discontinuity is *essential*. Subclassification of a simple discontinuity is based on whether the one-sided limits are equal. If the limit from the left equals that from the right, the discontinuity is *removable*, otherwise it is a *jump*. Subclassification of an essential discontinuity is based on whether the one-sided limits are unbounded, called an *infinite discontinuity*, or bounded, called an *oscillatory discontinuity*. These classifications are summarized in Table 1.1 and shown in Figure 1.2.

Table 1.1 Four Principal Types of Discontinuity

Simple or First Type Discontinuity

Removable:	$\lim_{x\to a} f(x) = L \neq f(a)$
Jump:	$\lim_{x\to a+} f(x) \neq \lim_{x\to a-} f(x)$

Essential or Second Type Discontinuity

Infinite:	$\lim_{x\to a+} f(x) = \pm\infty$ and/or $\lim_{x\to a-} f(x) = \pm\infty$
Oscillating:	$\lim_{x\to a+} f(x)$ and/or $\lim_{x\to a-} f(x)$ doesn't exist but is bounded.

Specific examples of each kind of discontinuity are left to the exercises.

One of the first result-oriented theorems that follows easily from the definition of continuity by using the algebra of limits describes polynomials and quotients of polynomials.

Theorem 1.5 *Polynomials are continuous everywhere. Rational functions are continuous wherever defined; the discontinuities of rational functions are either removable or infinite.*

Which class of discontinuity could contain the vertical asymptotes or poles of a rational function?

Having just used the algebra of limits and realizing that continuity is defined in terms of limits, we are led naturally to the algebra of continuity.

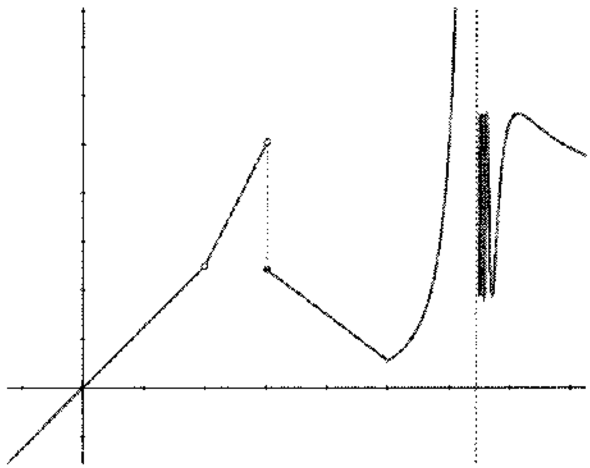

Figure 1.2 Four Types of Discontinuity

Theorem 1.6 (Algebra of Continuity) *Suppose that f and g are both continuous at $x = a$ and that $c \in \mathbb{R}$. Then*

1. *cf is continuous at a*
2. *$f + g$ is continuous at a*
3. *$f \cdot g$ is continuous at a*
4. *if $g(a) \neq 0$, then f/g is continuous at a*

The question of composition of functions is more subtle. The first step is to see how continuity affects limits. Since the change in the output of a continuous function is small if the change in input is small, we expect that near a point of continuity a limit would be preserved.

Theorem 1.7 *Suppose that f is continuous at $x = a$ and that $\lim_{x \to x_0} \phi(x) = a$ with the range of ϕ contained in the domain of f. Then*

$$\lim_{x \to x_0} f(\phi(x)) = f\left(\lim_{x \to x_0} \phi(x)\right)$$

Let's return to the fourth limit from Example 1.2.

■ **EXAMPLE 1.5**

- Calculate

$$\lim_{z \to 0} \sqrt[3]{1 + z}$$

Rewrite the expression as $y = (1 + z)^{(1/z)}$ and take logs of both sides to have $\ln(y) = (1/z) \cdot \ln(1 + z)$. With a little work, it can be shown that

$$z - z^2 \leq \ln(1 + z) \leq z$$

for $|z| < 1/2$. Divide by z, assuming that $z > 0$, and recognizing that, since $z \to 0$, we know that z is never equal to 0. Thus

$$1 - z < \frac{\ln(1 + z)}{z} \leq 1$$

The sandwich theorem tells us that $\ln(y) \to 1$ as $z \to 0+$. Similarly, $\ln(y) \to 1$ as $z \to 0$ from below. *Show it!* Hence $\lim_{z \to 0} \ln(y) = 1$. Now exponentiate this relation and apply Theorem 1.7. We arrive at

$$\lim_{z \to 0} \sqrt[z]{1 + z} = e^{\lim \ln(y)} = e^1$$

■

An immediate consequence of the "continuity preserves limits" theorem is that composition preserves continuity.

Theorem 1.8 *The composition of continuous functions is a continuous function.*

Continuous functions have many useful features. One of the most important is the *intermediate value property*: A continuous function must take on all intermediate values. This property is so geometrically natural that mathematicians such as Gauss, Euler, and Lagrange thought it obvious and used the result without proving it. Bolzano was the first to publish a rigorous argument. [See Hairer & Wanner (1996).]

Theorem 1.9 (Intermediate Value Theorem) *Let f be continuous on the closed interval $[a, b]$. If c is any value between $f(a)$ and $f(b)$, there is at least one $x \in [a, b]$ for which $f(x) = c$.*

The existence of zeros or roots is a direct application of the intermediate value theorem. If a continuous function changes sign in an interval, then it must have a zero in that interval. *Prove it!*

Another extremely valuable feature of continuous functions is that on a closed interval the function must have both a largest and a smallest value.

Theorem 1.10 (Extreme Value Theorem) *Let f be continuous on the closed interval $[a, b]$. Then f is bounded on $[a, b]$ and achieves a maximum and a minimum. That is, there are (at least) two values x_m and x_M in $[a, b]$ such that for any $x \in [a, b]$*

$$\min_{x \in [a,b]} f(x) = f(x_m) \leq f(x) \leq f(x_M) = \max_{x \in [a,b]} f(x)$$

The search for extreme values, maxima and minima, of a function was one of the themes that led to derivatives, our next topic. It's fascinating to realize that in the 1630s Fermat developed a method to find extreme values essentially using the derivative in the same way we do in calculus classes today—and that was before either Newton or Leibniz was born! [See Boyer (1959, pg. 155).]

1.3 DIFFERENTIATION

Just as the formal concept of limit was investigated after the mechanics of calculus were already well developed, differentiation followed long after integration as a focus of mathematical research. The derivative was originally designed to handle the "tangent problem," finding a line tangent to a curve at a point. The ancient Greeks defined a tangent as a line touching a curve only once. This definition was broadened as algebra advanced. Oresme noted in passing in the fifteenth century that "the rate of change is least ... at the point corresponding to the maximum ..." Boyer (1959, p. 85). Oresme's work was very influential and paved the way for the mathematicians of the 1600s. Fermat and others used algebraic methods for finding extrema and for mathematical modeling. In Section 1.2, we noted that Fermat's method for finding extreme values is equivalent to the process, based on derivatives, used in calculus classes today. Fermat used E in his calculations, Newton o, and Leibniz dx. The intuitive appeal of Leibniz's notation led us to use dx and Δx in current books. Newton's kinesthetic approach led to the "rate of change" motivation of modern calculus texts. Both Newton and Leibniz knew the differential triangle from Barrow's *Lectiones Opticæ et Geometricæ* (1669) shown in Figure 1.3.

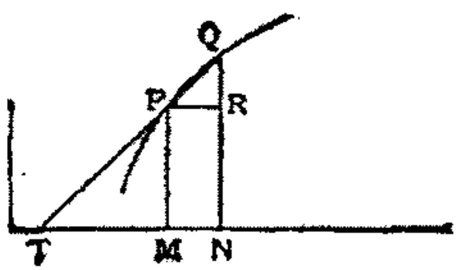

Figure 1.3 Barrow's Differential Triangle.

Typically, calculus classes study average rates of change, relating them to the slope of secant lines, and pass to the limit to obtain the instantaneous rate of change. We'll define the derivative directly using the *difference quotient* $\Delta y/\Delta x$ written in two different ways.

Definition 1.6 (Derivative) *The* derivative *of a function f defined on an interval $\mathcal{I}$ at a point $a \in \mathcal{I}$ is given, if the limit exists, by*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The derivative function of f is given, when the limit exists, by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If the limit exists, f is said to be differentiable at a.

Standard notations for the derivative of f also include $df(x)/dx$, dy/dx, and $D_x y$. Physicists often write the derivative with respect to time as $\dot{y}$; this notation is due to Newton.

The definitions of left and right hand derivatives are left to the exercises. Considering a few examples of using the limit of the difference quotient to find a derivative is our next task.

■ **EXAMPLE 1.6**

Use the definition of the derivative in the following.

1. Find $f'(0)$ when $f(x) = x^2$.
   Simple limit calculation. Do it!

2. Calculate $g'(0)$ when

$$g(x) = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0 \end{cases}$$

   Another simple limit, but we need to split into two cases. Do this one, too!

3. What is $U'(0)$ for the unit step function U?
   This one is just a little more subtle; slopes arbitrarily close to $a = 0$ on the left and on the right are both equal to 0. Let's look carefully.
   As x approaches 0 from above,

$$\lim_{x \to 0+} \frac{U(x) - U(0)}{x - 0} = \lim_{x \to 0+} \frac{1}{x} = \infty$$

   However, for x approaching 0 from below, we see that

$$\lim_{x \to 0-} \frac{U(x) - U(0)}{x - 0} = \lim_{x \to 0-} \frac{0}{x} = 0$$

   Hence, the limit as x approaches 0 does not exist, and therefore, the derivative $U'(0)$ does not exist.

■

An immediate consequence of the definition is the theorem that relates differentiability to continuity.

Theorem 1.11 *If a function f is differentiable at a, then f is continuous at a.*

The converse is not true, although Cauchy thought it was. Cauchy's was a more restrictive notion of function than ours today. Bolzano found a nondifferentiable continuous function in 1834, but his example was relatively unknown. Riemann also gave an example in his 1854 thesis. However, in 1872 Weierstrass presented an example of an everywhere continuous, nowhere differentiable function that "profoundly surprised" mathematicians. [See Kleiner (1989).] Today, the absolute value function is the canonical example in current calculus texts showing that continuity at a point does not give differentiability there.

Just as with limits and with continuity, the algebra of derivatives extends our toolbox. The proofs in the following result are relatively easy applications of the definition, appear in many elementary calculus textbooks, and here are left to the reader as an exercise.

Theorem 1.12 (Algebra of Derivatives) *Suppose that f and g are differentiable at $x = a$ and that $c \in \mathbb{R}$. Then*

1. $(c \cdot f)'(a) = c \cdot f'(a)$
2. $(f \pm g)'(a) = f'(a) \pm g'(a)$
3. $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$
4. *if* $g(a) \neq 0$, *then* $\left(\dfrac{f}{g}\right)'(a) = \dfrac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{g^2(a)}$

With these techniques in hand, we can easily differentiate any polynomial or rational function. For transcendental functions, we use the limit definition of the derivative or trigonometric identities to develop formulas. The three basic results follow.

Theorem 1.13 *For any real number x*

1. $\dfrac{d}{dx}\sin(x) = \cos(x)$
2. $\dfrac{d}{dx}e^x = e^x$
3. $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ *for* $x > 0$

Many texts first establish the *generalized power rule*

$$\frac{d}{dx}f^r(x) = rf^{r-1}(x)f'(x)$$

for $r \in \mathbb{Q}$ as part of the development of the *chain rule*. Since the power rule is a direct corollary, we'll proceed directly to the chain rule.

Theorem 1.14 (Chain Rule) *Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ be differentiable functions at $x = a$ and $x = f(a)$, respectively. Then $g \circ f : [a, b] \to \mathbb{R}$ is differentiable at $x = a$ and its derivative is given by*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

The name of the result comes from the form

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

appearing to chain derivatives together.

The chain rule is an extremely powerful tool but has a nontrivial proof which causes most elementary calculus texts to focus on motivation, heuristics, and usage. We'll defer the proof until later. One of the nicer "mathematical applications" of the chain rule is to develop a formula for the derivative of an inverse function. Since $f \circ f^{-1}$ is the identity function, we see that

$$(f \circ f^{-1})'(x) = 1$$

Applying the chain rule tells us that

$$f'(f^{-1}(x)) \cdot f^{-1\prime}(x) = 1$$

Employing just a little algebra, assuming proper assumptions preventing dividing by zero, yields a general result.

Theorem 1.15 *If $f : [a,b] \to \mathbb{R}$ is differentiable and $f^{-1}(x) \neq 0$ on $[a,b]$, then, for $y = f(x)$,*

$$f^{-1\prime}(y) = \frac{1}{f'(x)}$$

Now that we have the basic mechanics of differentiation in hand, there are two directions to go. We can study the geometry of derivatives and higher order derivatives, and we can investigate applications. First, let's consider geometry.

Since the derivative describes the instantaneous rate of change of a function, it tells us whether a function is momentarily increasing, decreasing, or constant. Values where the derivative is zero, where the function has a horizontal tangent, are called *critical values* or *stationary points*. [Note: some calculus authors, e.g., Stewart (2009), add values where the derivative fails to exist to the definition of critical value; others, e.g., Thomas (1968), also add endpoints of the domain.] A stationary point where a function does not change from increasing to decreasing or vice versa is called a *terrace point*. We can further analyze the change in the derivative, the change in that change, etc. Let's define higher order derivatives recursively.

Definition 1.7 *Let f be a differentiable function and $n \in \mathbb{N}$. Then for $n > 1$ the* nth derivative of f, *denoted by $f^{(n)}(x)$, is given, when it exists, by*

$$f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x) \qquad or \qquad \frac{d^n y}{dx^n} = \frac{d}{dx}\frac{d^{n-1} y}{dx^{n-1}}$$

Some authors (and calculators like the TI-Nspire) adopt the convention that the zeroth derivative is the original function; i.e., $f^{(0)}(x) = f(x)$. *Rewrite the definition above using this convention.* A little classroom fun can be had by asking calculus students, "What is the 'leap year'th derivative of the sine?"

Geometrically interpreting an increasing rate of change as a graph curving up, we call it *concave up*. A decreasing rate of change is called *concave down*. Our definitions follow Granville, Smith, and Longley (1911) and use "positive" versus "nonnegative," etc., to eliminate linear functions being simultaneously both concave up and down.

Definition 1.8 *Let f be a twice-differentiable function on the interval $\mathcal{I}$. Then:*

- *If $f''(x)$ is positive on $\mathcal{I}$, then $f'(x)$ is increasing and f is* concave up *on $\mathcal{I}$.*
- *If $f''(x)$ is negative on $\mathcal{I}$, then $f'(x)$ is decreasing and f is* concave down *on $\mathcal{I}$.*
- *If f changes concavity about a point x_0, then x_0 is an* inflection point *of f.*

In economics, an inflection point of an increasing function changing from concave up to down is often called a *point of diminishing return* since a corresponding change in input produces a smaller change in output as x passes the inflection point. *Explain why this happens.*

Write this geometric information algebraically to define procedures equivalent to Fermat's method, but more general, for finding local extreme values of a function.

Theorem 1.16 (First Derivative Test for Extrema) *Suppose that f is differentiable on an interval containing the critical value a.*

- *If f' changes from positive below a to negative above a, then $f(a)$ is a local maximum of f.*
- *If f' changes from negative below a to positive above a, then $f(a)$ is a local minimum of f.*
- *If f' does not change sign about a, then $(a, f(a))$ is a* terrace point *and not a local extreme value.*

Using the information on concavity provided by the second derivative yields a simpler test for local extrema.

Theorem 1.17 (Second Derivative Test for Extrema) *Suppose that f is twice differentiable on an interval containing the critical value a.*

- *If $f''(a) < 0$, then $f(a)$ is a local maximum of f.*
- *If $f''(a) > 0$, then $f(a)$ is a local minimum of f.*
- *If $f''(a) = 0$, then the test fails.*

Let's use the famous *Norman window problem* as an illustration.

■ EXAMPLE 1.7 The Norman Window Problem

A Norman window is a window in the shape of a rectangle surmounted by a semicircle. See Figure 1.4. If the perimeter of the window is P, find the dimensions allowing the greatest amount of light to be admitted.

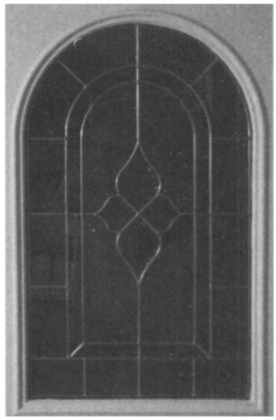

Figure 1.4 A Norman Window

Solution. Let the rectangle have height h and width d. Then the diameter of the semicircle is also d. Thus, the perimeter is

$$P = (2h + d) + \frac{\pi}{2}d$$

Since the light admitted is directly proportional to the area A, we need to maximize A. That is, we need to maximize

$$A = hd + \frac{\pi}{8}d^2$$

Solve the perimeter equation for h and substitute the result into the area equation:

$$A = \frac{P}{2}d - \left(\frac{\pi}{8} + \frac{1}{2}\right)d^2$$

Set the derivative of A with respect to d equal to 0 :

$$A' = \frac{P}{2} - \left(\frac{\pi}{4} + 1\right)d = 0$$

The critical value is seen to be

$$d_c = \frac{2P}{\pi + 4}.$$

Since $A'' = -1 - \pi/4 < 0$, we must have a maximum when the diameter and width are $2P/(\pi + 4)$ and the height is $P/(\pi - 4)$. ■

The Norman window problem has been a favorite of calculus authors for many, many years. See, for example, Granville et al. (1911, p. 57), Thomas (1968, p. 129), Ellis et al. (1999, p. 189), Hughes-Hallett et al. (2009, p. 263), Stewart (2009, p. 208) (tunnel), etc.

Continuing with our geometric theme, we consider what happens to a differentiable function between points having equal functional values. If the function rises, then it must smoothly turn and come back down. If the function sinks, it must smoothly turn and come back up. A function that neither increases nor decreases must be constant. Whatever the case, there must be at least one point with a horizontal tangent on the graph. This result is known as Rolle's theorem.

Theorem 1.18 (Rolle's Theorem) *Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is a value $c \in (a, b)$ such that $f'(c) = 0$.*

The natural generalization of Rolle's theorem is to let the endpoints be at different levels. The generalized theorem was first stated by Lagrange.

Theorem 1.19 (Mean Value Theorem) *Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there is a value $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Both Rolle's theorem and Lagrange's mean value theorem are *existence theorems*; that is, these theorems specify that there is some value satisfying the conclusion but give no direction on how to find that value. Nevertheless, existence theorems often have very practical ramifications. Two important corollaries of the mean value theorem concern properties of differentiable functions.

Corollary 1.20 *If $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant function on (a, b).*

The unit step function has a zero derivative everywhere except at $a = 0$ yet is not a constant function. How does this not violate the corollary? Now apply this idea to two functions.

Corollary 1.21 *If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + c$ for all $x \in (a, b)$ for some constant c.*

The last application of the mean value theorem we will consider is known as the *speed limit law;* cf. Ostebee & Zorn (2002, p. 42). This law is the first step on the road to Taylor series, as we'll see later in Section 1.6.

Theorem 1.22 (Speed Limit Law) *Suppose that f is differentiable on (a, b) and continuous on $[a, b]$ and that*

$$m \le f'(x) \le M$$

for all $x \in (a, b)$ for two constants m and M. Then

$$m(b - a) \le f(b) - f(a) \le M(b - a)$$

Further, if $x \in [a, b]$, then

$$f(a) + m(x - a) \leq f(x) \leq f(a) + M(b - a)$$

The speed limit law defines a cone containing the function as shown in Figure 1.5. We are able to find upper and lower bounds for the values of f between a and b.

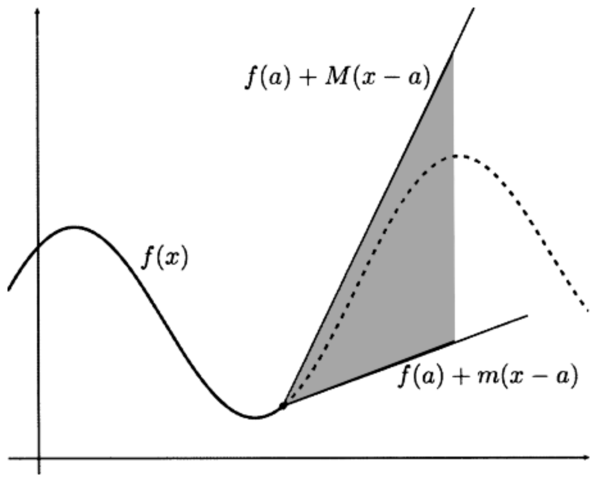

Figure 1.5 The "Speed Limit Cone"

A two-function variant of the speed limit law, named by Jerry Uhl, Professor Emeritus of Mathematics at the University of Illinois, is known as the *race track principle* (Davis et al., 1994).

Theorem 1.23 (Race Track Principle) *Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) and that $f'(x) \leq g'(x)$ for $a < x < b$:*

1. *If $f(a) = g(a)$, then $f(x) \leq g(x)$ for $a \leq x \leq b$.*
2. *If $f(b) = g(b)$, then $f(x) \geq g(x)$ for $a \leq x \leq b$.*

The name comes from interpreting the derivative as velocity. The principle then says that if two horses start a race together, the faster horse is always in the lead. The second statement says that if two horses finish together, the faster horse was always behind.

Derivatives are incredibly useful in real life. Since change is much easier to measure than quantity, mathematical models of continuous phenomena are often specified as differential equations. The simplest models have the form $y' = f(x)$. The exponential growth of Malthusian population dynamics comes from the model $p' = r \cdot p$, where p represents a population at time t and r is a constant. These models ask the question, "What function has the specified derivative?" The connection between differentiation and integration then becomes the focus of our study.

1.4 INTEGRATION

Integration is the oldest part of calculus and developed from the need for formulas for areas and volumes of geometric figures. From Hippocrates of Chios' "Quadrature of the Lune" [see Dunham (1990)] onward, the search to understand area and volume has led to deeper mathematical discoveries. The Greek mathematician Eudoxus of Cnidus' *method of exhaustion* (about 370 B.C.) is the progenitor of the definite integral as a limit of approximating sums. The method of exhaustion was *the* technique used to calculate area and volume formulas for 2000 years. The word *quadrature* originally meant constructing, with only a straightedge and compass, a square having exactly the same area as a given figure but later came to mean determining an area or volume. Modern usage of the term denotes "numerical integration."

A number of basic results for definite integrals had been determined before either Newton or Leibniz studied mathematics. For example, the formula

$$\int_0^a x^n \, dx = \frac{a^{n+1}}{n+1},$$

appears (in very different notation) in the work of Cavalieri (1635), Torricelli (1635), Roberval (1636), Fermat [c. 1644, Boyer (1945)], Pascal [1654, Boyer (1959, p. 149)], and Wallis [1665, Burton (2007, p. 386)]. It was the genius of Newton and Leibniz that they saw the significance of the connection between integration and differentiation, that these are inverse operations—the *fundamental theorem of calculus*. Isaac Barrow, Newton's teacher and predecessor in the Lucasian Chair at Cambridge, had formulated the fundamental theorem (published in his 1670 monograph *Lectiones Geometricæ*, or *Lectures on Geometry*, after he had left Cambridge) but had not understood the theorem's importance since his focus was very geometrical, not analytic.

Most calculus texts motivate the definite integral via calculating the area enclosed between a curve and the x-axis. We'll begin our look at integration by directly defining the definite integral as a limit and then move to the fundamental theorem and applications. However, texts diverge in their definitions. Some use equipartitions [e.g., Stewart (2009)], while others use more general subdivisions [e.g., Thomas (1968)]. Since we'll be studying the integral in much more generality later, we'll begin with Riemann sums. First, we define a *partition* and *sample points* or *tags*.

Definition 1.9 *A* partition *of the interval* $[a, b]$ *is a set of* $n + 1$ *values*

$$\mathcal{P} = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$$

that divides $[a, b]$ *into* n *subintervals with widths* $\Delta x_k = x_k - x_{k-1}$. Sample points *or* tags *are values* c_k *chosen in each subinterval of* $\mathcal{P}$; *that is, for each* $k = 1, \ldots, n$, *there is a* $c_k \in [x_{k-1}, x_k]$. *A partition together with a set of tags will be denoted by* $\mathcal{P}_c$.

Now that we've partitioned the interval and chosen sample points, we can build a Riemann sum.

Definition 1.10 (Riemann Sum) *Let f be a function defined on $[a,b]$. The* Riemann sum *of f over $\mathcal{P}_c$, the partition and chosen sample points, is*

$$\mathcal{R}_{\mathcal{P}_c}(f) = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

Passing to the limit as the largest Δx_k goes to 0 gives us the definite integral of f over $[a,b]$. Set $\|\mathcal{P}_c\| = \max_k \Delta x_k$.

Definition 1.11 (Riemann Integral) *Let f be a function defined on $[a,b]$. The* Riemann integral *of f over $[a,b]$ is, if the limit exists,*

$$\int_a^b f(x)\,dx = \lim_{\|\mathcal{P}_c\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

There are subtleties to this limit that many elementary calculus texts finesse by using only partitions with uniform widths so that $\Delta x_k = \Delta x = (b-a)/n$. Example calculations are also much simpler with uniform partitions as the limit reduces to

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

as shown in Figure 1.6.

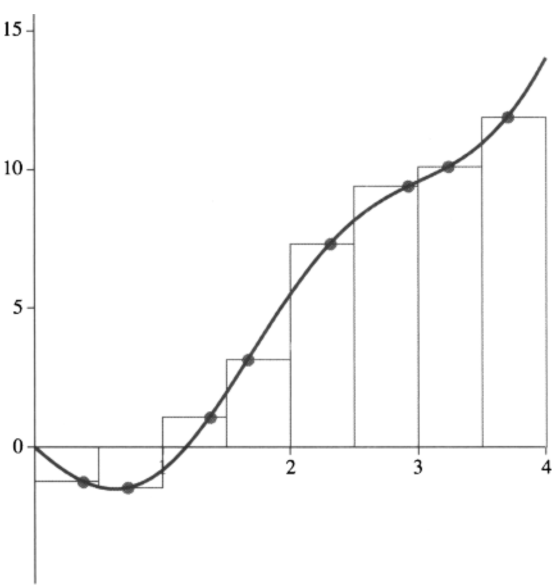

Figure 1.6 A Uniform Width Riemann Partition

A typical example follows.

■ **EXAMPLE 1.8**

Calculate $\int_0^1 x^3\,dx$.

Choose the partition $\{x_k = k/n \mid k = 0, \ldots, n\}$ with the sample points $c_k = k/n$ for $k = 1, \ldots, n$. Then $\Delta x = 1/n$, and thus

$$\begin{aligned}
\int_0^1 x^3\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \cdot \frac{1}{n} \\
&= \lim_{n\to\infty} \frac{1}{n^4} \cdot \sum_{k=1}^{n} k^3 \\
&= \lim_{n\to\infty} \frac{1}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
&= \frac{1}{4}
\end{aligned}$$

■

Just as we did for limits, continuity, and differentiation, we extend our toolbox by showing that integration is a linear operation.

Theorem 1.24 (Algebra of Integrals) *Suppose that f and g are integrable on $[a, b]$ and that $c \in \mathbb{R}$. Then*

1. $\displaystyle\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$
2. $\displaystyle\int_a^b (f \pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

The proofs are easy verifications using the definition.

An image of a function inscribed in one rectangle and circumscribing another gives a "proof by picture" for our next result. *Draw the image!* The proof is left to the exercises.

Theorem 1.25 *Let f be integrable on $[a, b]$. If there are two constants m and M such that $m \le f(x) \le M$ for all $x \in [a, b]$, then*

$$m \cdot (b - a) \le \int_a^b f(x)\,dx \le M \cdot (b - a)$$

A third rectangle can be drawn with the rectangle's area equal to the function's. This rectangle's height gives the *average value* of the function.

Definition 1.12 (Average Value of a Function) *If f is integrable on $[a, b]$, then the* average value *of f is given by*

$$\bar{f} = \frac{1}{b - a} \cdot \int_a^b f(x)\,dx$$

■ EXAMPLE 1.9

The average value depends on the interval. The average value of the sine function over $[0, 2\pi]$ is

$$\overline{\sin} = \frac{1}{2\pi} \cdot \int_0^{2\pi} \sin(x)\, dx = 0$$

while the average over $[0, \pi]$ is

$$\overline{\sin} = \frac{1}{\pi} \cdot \int_0^{\pi} \sin(x)\, dx = \frac{2}{\pi} \approx 0.63662$$

If f is continuous, then, just as with derivatives, the function takes on its average value. This result is proven by showing that the function must intersect that middle rectangle.

Theorem 1.26 (Mean Value Theorem for Integrals) *If f is continuous on $[a, b]$, then there is at least one point $c \in (a, b)$ where $f(c) = \bar{f}$.*

At this point it is natural to ask, "What type of function has a definite integral?" An easy to prove answer is continuous.

Theorem 1.27 *If f is continuous on $[a, b]$, then $\int_a^b f(x)\, dx$ exists.*

In 1823, Cauchy used uniform continuity to show that given $\epsilon > 0$ there is a $\delta > 0$ so that any two partitions with $\max_k \Delta x_k < \delta$ have to have Riemann sums within ϵ of each other. This argument proved the limit existed without finding the value of the limit. We will use this method again later when we consider sequences in Section 1.5. Cauchy's technique is based strongly on the structure of the real numbers; it does not hold if we only allow ourselves to use rational numbers.

The title *fundamental theorem* indicates that this result forms the core of calculus, linking the two main concepts, derivative and integral. While special cases were known previously, Newton and Leibniz were the first to realize that the theorem provided a new, general form of analysis connecting the tangent problem to quadrature and vice versa. The theorem has two statements: The integral is differentiable and the antiderivative gives the quadrature.

Theorem 1.28 (Fundamental Theorem of Calculus) *Let f be a continuous function on the interval $[a, b]$.*

1. *Define $F(x) = \int_a^x f(x)\, dx$. Then F is continuous and differentiable on $[a, b]$ with $F'(x) = f(x)$.*

2. *If F is any antiderivative of f, then $\int_a^b f(x)\, dx = F(b) - F(a)$.*

Boyer (1959, p. 280) tells us that Cauchy was the first to give a rigorous proof of the fundamental theorem.

The next three results form the basis of a good portion of the second half of an elementary calculus course: substitution, integration by parts, and partial fraction integration. All three address the question of how to find an antiderivative; all three are reductions to simpler forms, not results. Substitution is the inverse of the chain rule.

Theorem 1.29 (Integration by Substitution) *Let f be a continuous function of u and u be a continuously differentiable function of x on $[a,b]$. Then*

$$\int f(u)\,du = \int f(u(x))u'(x)\,dx$$

Integration by parts is the inverse of the product rule.

Theorem 1.30 (Integration by Parts) *Let f and g be continuously differentiable functions on $[a,b]$. Then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

This rule is often written with differentials in the form $\int u\,dv = uv - \int v\,du$.

The last of the three rules is really an algebraic technique for rational functions. Any rational function $R(x)$ can be decomposed into a sum of a polynomial p and terms having the form $a/(x-b)^k$ where b is a root of the denominator and a and k are constants. The partial fraction expansion gives the form

$$R(x) = p(x) + \sum_{k=1}^{n} \left(\frac{a_{k1}}{x-b_k} + \frac{a_{k2}}{(x-b_k)^2} + \cdots + \frac{a_{km_k}}{(x-b_k)^{m_k}} \right)$$

for a rational function $R(x)$ with denominator roots b_k having multiplicity m_k. *Integration by partial fractions* is just applying the integral to both sides of the equation above. Beginning with his 1833 paper on integrals that are algebraic, which heavily used partial fractions, Liouville analyzed the process of integration to find which functions could or could not have antiderivatives with finite elementary forms. [See Kasper (1980) for a nice discussion of Liouville's work and where it led others.] Liouville's approach was very influential in the development of computer algebra systems. We now can use software like Maple and Mathematica and calculators such as Texas Instruments' Voyage 200, TI-89, and TI-Nspire CAS to calculate integrals that would be very difficult by hand.

Calculus texts usually include tables having over a hundred formulas for antiderivatives. The *CRC Standard Mathematical Tables and Formulae* (Zwillinger, 2002) contains over 600 antiderivative forms, Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products* (Gradshteyn & Ryzhik, 2007) countains thousands. Even though techniques of integration form a major focus of second-semester calculus and there are texts like the *Handbook of Integration* (Zwillinger, 1992), most functions do not have elementary antiderivatives. Extremely powerful computers and calculators have become ubiquitous, leading to a resurgence of approximation theory. These observations provide motivation for us to study approximate methods of integration called *numerical quadratures*. We will consider three of the standard techniques that are used in elementary courses. The methods use constant, linear, and quadratic approximations, respectively.

For the constant approximation, we will use the left endpoint to measure the function's height, effectively giving us Cauchy's definition of the definite integral; the associated error is proportional to the first derivative. Other choices of where to measure the height include the midpoint and right endpoint. The trapezoid rule uses the linear approximation through the left and right endpoints with an error proportional to the second derivative. Simpson's rule uses quadratic approximation over the endpoints of two consecutive intervals; the error bound is proportional to the fourth derivative.

Theorem 1.31 (Numerical Integration Methods) *Suppose that $\mathcal{I} = \int_a^b f(x)\,dx$ exists. Given $n \in \mathbb{N}$, set $\Delta x = (b-a)/n$ and $x_k = a + k\Delta x$ with $k = 0, \ldots, n$. Also, let M_k be an upper bound for $|f^{(k)}(x)|$ on $[a, b]$. Then:*

Left Sum *The left sum approximation L_n to $\mathcal{I}$ with error bound ε_n is*

$$L_n = \sum_{k=1}^{n} f(x_{k-1})\Delta x \qquad \varepsilon_n = |L_n - \mathcal{I}| \leq M_1 \cdot \frac{(b-a)^2}{2} \cdot \frac{1}{n}$$

Trapeziod Rule *The trapezoid rule approximation T_n to $\mathcal{I}$ with error bound ε_n is*

$$T_n = \sum_{k=1}^{n} \frac{1}{2}\left(f(x_{k-1}) + f(x_k)\right)\Delta x \qquad \varepsilon_n = |T_n - \mathcal{I}| \leq M_2 \cdot \frac{(b-a)^3}{12} \cdot \frac{1}{n^2}$$

Simpson's Rule *The Simpson's rule approximation S_n (n must be even) to $\mathcal{I}$ with error bound ε_n is*

$$S_n = \sum_{k=1}^{n/2} \frac{1}{3}\left(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\right)\Delta x$$

$$\varepsilon_n = |S_n - \mathcal{I}| \leq M_4 \cdot \frac{(b-a)^5}{180} \cdot \frac{1}{n^4}$$

In calculus classes, bounds for M_k are usually found graphically. Table 1.2 summarizes the results of applying the quadrature rules to $\int_0^3 e^{-x^2}\,dx$.

Finding an antiderivative is much more challenging than calculating a derivative. It's not easy to know a priori whether or not a given function has an elementary antiderivative. Sometimes it's easy to prove as with $f(x) = e^{-x^2}$. Other times, it's very hard to determine if an elementary antiderivative exists. This difficulty makes integration quite a challenge. One of the questions we ask is if the values of the Riemann sums from the definition of the definite integral converge. This question leads to our next topic, sequences and series of constants.

Table 1.2 Numerical Integration of $\int_0^3 e^{-x^2}\,dx$

n	L_n	ε_n	T_n	ε_n	S_n	ε_n
4	1.26113	0.37493	0.886180	0.000027	0.886206	$1.7297 \cdot 10^{-6}$
10	1.07368	0.18747	0.886199	0.000008	0.886207	$1.398 \cdot 10^{-7}$
20	0.96120	0.07499	0.886206	0.000001	0.886207	$4.0 \cdot 10^{-9}$

1.5 SEQUENCES AND SERIES OF CONSTANTS

Some students argue that $1 \neq 0.9$. They are unwittingly reprising Zeno's paradox from the fifth century B.C. The most familiar version of the paradox states that Achilles can never catch a tortoise as first he must cover half the distance to the tortoise, then half the remaining distance, then again half the remaining distance, ad infinitum. What Zeno objected to was infinite divisibility. We overcome the problem with the concept of *convergence*, basing it on our formal definition of limit. In this section, we will consider sequences and series of constants and allow Achilles ultimately to catch the tortoise.

Sequences

A real-valued sequence is a real-valued function that has a special domain.

Definition 1.13 *A* sequence *is a function from the natural numbers* $\mathbb{N}$ *to the real numbers. For a sequence* $a : \mathbb{N} \to \mathbb{R}$, *we denote* $a(n)$ *by* a_n *and the sequence by* $\{a_n\}$.

Zeno's paradox concerns the sequence $\{1/2, 1/4, 1/8, \dots\}$. The paradox arises from the fact that every term is greater than zero, but we know that the terms eventually get smaller than any given positive number. We need to define the limit of a sequence in order to let Achilles catch the tortoise.

Definition 1.14 *Let* $\{a_n\}$ *be a sequence. Then* L *is the limit of* $\{a_n\}$ *or*

$$\lim_{n\to\infty} a_n = L$$

if and only if for every $\epsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that whenever* $n > N$ *it must follow that*

$$|a_n - L| < \epsilon$$

If $\lim_{n\to\infty} a_n = L$, then we say the sequence *converges to* L; a sequence that does not converge is said to *diverge*. It's not hard to prove that Zeno's sequence converges to zero. *Do it!*

In 1202, Leonardo of Pisa's book *Liber Abaci* introduced Hindu-Arabic numbers and positional notation to Europe. However, the text is better known for a famous sequence arising from an example:

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is

supposed that every month each pair begets a new pair which from the second month on becomes productive?

The solution to the problem is the sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$, which is now called, using Leonardo's nickname, the *Fibonacci sequence*. We show the Fibonacci sequence diverges by combining two facts: first, all terms are positive; second, each term beyond the second is larger than the previous by at least one.

Let's look at two sequences that may not obviously converge or diverge at first glance.

■ EXAMPLE 1.10

1. Does the sequence with general term $a_n = (\sqrt{n^2+n} - n)/n$ converge or diverge?

   First, rationalize the numerator:

   $$a_n = \frac{\sqrt{n^2+n}-n}{n} \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{n^2+n}+n}$$

   Now, it is easy to see that a_n converges to 0.

2. Does the sequence with general term $b_n = (c+1)(-1)^n - 1$ converge or diverge for an arbitrary $c \in \mathbb{R}$?

   Write several terms. What do you see? (Be careful—there are two cases.) ■

Once again, it's time for an algebra, now of sequences. The proofs are all straightforward, essentially restatements of earlier limit theorems for functions, and are left to the exercises.

Theorem 1.32 (Algebra of Sequence Limits) *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and let $c \in \mathbb{R}$. Then*

1. $\lim\limits_{n\to\infty} (c \cdot a_n) = c \cdot \lim\limits_{n\to\infty} a_n$
2. $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
3. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
4. $\lim\limits_{n\to\infty} (a_n/b_n) = \lim\limits_{n\to\infty} a_n / \lim\limits_{n\to\infty} b_n$ *provided* $\lim b_n \neq 0$

The sandwich theorem can also be restated for sequences.

Theorem 1.33 (The Sandwich Theorem for Sequences) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences for which there is an $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n \le b_n \le c_n$. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

The sandwich theorem limits the growth of a sequence. At times, it is useful to see if a sequence can describe its own action. We can classify sequences as increasing or decreasing.

Definition 1.15 *A sequence $\{a_n\}$ is*

- increasing *if $a_n < a_{n+1}$ for all n,*
- decreasing *if $a_n > a_{n+1}$ for all n,*
- bounded *if $|a_n| < M$ for all n for some real number M.*

A monotone sequence *is either increasing or decreasing.*

The properties of bounded and monotone are very strong and lead to important results, such as the following due to Bolzano and Weierstrass.

Theorem 1.34 (Bolzano-Weierstrass Theorem for Sequences) *If $\{a_n\}$ is a bounded sequence of real numbers, then $\{a_n\}$ has a subsequence that converges.*

Bolzano discovered the theorem first, but his work was relatively unknown; some of his manuscripts were not even published until the 1960s (Burton, 2007, p. 671). Weierstrass independently proved the result in the 1860s (Burton, 2007, p. 688). An immediate corollary of the theorem is that *a bounded, monotone sequence converges.*

Series

A *series* $\{s_n\}$ is a special sequence in which the elements are formed from a sequence $\{a_n\}$ by

$$s_n = \sum_{k=1}^{n} a_k.$$

Each s_n is called a *partial sum*. We often write $\sum a_n$ to represent the series. A series converges if its sequence of partial sums converges. We can further classify convergence as follows.

Definition 1.16 (Absolute and Conditional Convergence) *A series $\sum a_n$ is* absolutely convergent *if the series $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is called* conditionally convergent.

The series $\sum(-1/2)^k$ is absolutely convergent while $\sum(-1)^k/k$ is conditionally convergent. We'll develop tests to verify these statements in a moment.

Zeno's claim rephrased in terms of series is

$$\sum_{k=1}^{n} \frac{1}{2^k} < 1 \quad \text{for all } n \in \mathbb{N}$$

But what happens to Zeno's series as n goes to infinity? Does the series converge? Before looking at different tests for convergence, let's consider examples of convergent and divergent series. Explore these series numerically and graphically using different values for their parameters. *Zeno's series belongs to which type below?*

■ EXAMPLE 1.11 Special Series of Elementary Calculus

1. A *geometric series* $\sum ar^k$ converges when $|r| < 1$ with

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

2. The *harmonic series* $\sum 1/k$ diverges since

$$\sum_{k=1}^{n} \frac{1}{k} \approx \ln(n)$$

3. A *p-series* $\sum 1/k^p$ converges if $p > 1$ and diverges for $p \leq 1$ since

$$\sum_{k=1}^{n} \frac{1}{k^p} \approx \int_1^n \frac{1}{x^p}\,dx = \frac{n^{1-p}-1}{1-p}$$

4. The *telescoping series* $\sum 1/(k(k+1))$ and $\sum 1/((2k-1)(2k+1))$ converge to 1 and $1/2$, respectively, since

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1}$$

and

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

■

Oresme showed, in the fourteenth century, that the harmonic series diverges with a technique that is still used in classes today (Struik, 1986, p. 320). Collect the terms of the series as

$$\begin{aligned} 1 &+ \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\ &> 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \cdots \end{aligned}$$

taking groups of size 2^n. Since the last series is growing infinitely large, the harmonic series, being larger, must diverge.

If a series we are investigating isn't one of the standards, how do we determine convergence? If the general term does not go to zero, then the series must diverge. What about positive answers? We need a collection of criteria. The easiest procedure to apply is the ratio test.

Theorem 1.35 (D'Alembert's Ratio Test) *Let $\sum a_n$ be a series with positive terms. Set*

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Then:

- *If $r < 1$, the series converges.*
- *If $r > 1$, the series diverges.*
- *If $r = 1$, the test fails.*

The proof of the ratio test is an application of geometric series convergence. While the ratio test is usually easy to apply, it doesn't always prove conclusive. Try the ratio test on $\sum 1/n^3$. The root test is more sensitive but can be harder to use.

Theorem 1.36 (Cauchy's Root Test) *Let $\sum a_n$ be a series with positive terms. Set*

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$$

Then:

- *If $\rho < 1$, the series converges.*
- *If $\rho > 1$, the series diverges.*
- *If $\rho = 1$, the test fails.*

The comparison test is easier to use than the root test. The difficulty is that we must use a series with known convergence or divergence to compare to the series we're analyzing.

Theorem 1.37 (Comparison Test) *Let $\sum a_n$ and $\sum b_n$ be series of positive terms with $a_n \leq b_n$ for all n.*

- *If $\sum b_n$ converges, then so does $\sum a_n$.*
- *If $\sum a_n$ diverges, then so does $\sum b_n$.*

Another sort of comparison test is based on Riemann sums. If we partition $[1, \infty)$ with the natural numbers, then $\Delta x = 1$ and the Riemann sum becomes the summation we're considering. Hence, the integral and the sum either both converge or both diverge.

Theorem 1.38 (Integral Test) *Suppose that f is continuous, positive, and decreasing for $x \geq 1$ and that $a_n = f(n)$ for all n.*

- *If* $\displaystyle\int_1^{\infty} f(x)\,dx$ *converges, then so does* $\sum a_n$.
- *If* $\displaystyle\int_1^{\infty} f(x)\,dx$ *diverges, then so does* $\sum a_n$.

The last test we consider, due to Leibniz, is for a special type of series. An *alternating series* has terms that alternate in sign and monotonically go to zero.

Theorem 1.39 (Alternating Series Test) *Let $\{a_n\}$ be a positive, decreasing sequence that converges to zero. Then the alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

It's interesting to note that the *tail* of an alternating series $\sum_{k=n+1}^{\infty} (-1)^{k-1} a_k$ is bounded in absolute value by a_{n+1}.

Each of the convergence tests is an analog of an existence theorem, not providing a limit value, just indicating whether or not a series converges. For more delicate tests, see, for example, Wrede & Spiegel (2002) or, for detailed explanations, see Kosmala (2004).

We have focused on sequences and series of constants in this section, getting ready to study sequences and series of functions. Part of Newton's success was his facility with expressing functions as series and manipulating them with his new methods of calculus. As Newton knew and we'll see, series of functions are a powerful tool.

1.6 POWER SERIES AND TAYLOR SERIES

Geometric series had been widely used before the calculus methods of Newton and Leibniz were invented. Nicolas Mercator used a geometric series to develop a power series representation for $\ln(1+x)$ in 1668. Both Newton (1665) and Gregory (1670) independently discovered the general binomial formula

$$(1+x)^r = 1 + rx + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots$$

and made significant use of it, Newton in developing integration formulas and Gregory in series expansions. Taylor's theorem on the expansion of a function in a power series was first discovered by Gregory in 1671, although in a different form using differences. [See Stillwell (1989, Chapter 9).]

Power Series

Generalizing the geometric series $\sum ar^n$ by replacing r with a variable and a with a sequence $\{a_n\}$ is a natural step.

Definition 1.17 (Power Series) *A* power series centered at c *is a series having the form*

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

where the a_n *are constants.*

By convention, $a_0(x-c)^0 = a_0$ for all values of x even though 0^0 is indeterminate. Suppose a power series converges for some x that is R units from c; then the comparison test tells us that the series converges for any x closer to c, i.e., for all x with $|x-c|<R$. *Prove it!* Similarly, if a power series diverges for some x that is R units from c, then it diverges for all x with $|x-c|>R$. This observation leads us to define the *radius of convergence* of a power series.

Definition 1.18 (Radius of Convergence) *For a power series* $\sum a_n(x-c)^n$, *exactly one of the three following statements must be true.*

1. *The series converges only at* $x=c$. *The radius of convergence is* $R=0$.
2. *The series converges for all* $|x-c|<R$ *and diverges for all* $|x-c|>R$ *for some positive value* R. *The radius of convergence is* R.
3. *The series converges for all* x. *The radius of convergence is* $R=\infty$.

Given a positive radius of convergence R for the power series $\sum a_n(x-c)^n$, the *interval of convergence* is the interval from $c-R$ to $c+R$ that may or may not contain the endpoints—they must both be checked. Often, the radius of convergence can be found using the ratio or root tests. However, both tests fail at an endpoint of the interval of convergence, and other methods are required.

Mercator's expansion of $\ln(1+x)$ comes from integrating the geometric series expansion for $1/(1+x)$. Integrating

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

yields

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Newton investigated integrating series expansions for functions like $\sqrt{1+y'^2}$ to find arc lengths without questioning convergence. Fortunately, power series convergence is preserved by differentiation and integration.

Theorem 1.40 *If* $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *converges for* $|x-c| < R < \infty$, *then*

1. $f'(x) = \sum_{n=1}^{\infty} n\, a_n (x-c)^{n-1}$ *converges for* $|x-c| < R$ *and*

2. $\int_c^x f(t)\, dt = \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$ *converges for* $|x-c| < R$.

Let's explore techniques for finding power series expansions.

■ EXAMPLE 1.12

Given that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

find power series expansions for the following functions.

1. $f(x) = \dfrac{1}{1+x^2}$

   Replace x in the geometric power series with $-x^2$ to have

$$f(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad |x| < 1$$

2. $g(x) = x^{-1}$

   Rewrite x^{-1} as $1/(1+(x-1))$. As above, replace x with $-(x-1)$ in the geometric power series. Note the shift to center $c = 1$:

$$g(x) = \sum_{n=0}^{\infty} (-(x-1))^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \qquad |x-1| < 1$$

3. $h(x) = \dfrac{1}{(1-x)^2}$

   Since h is the derivative of $1/(1-x)$, differentiate the geometric power series termwise:

$$h(x) = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n x^{n-1} \qquad |x| < 1$$

4. $j(x) = \tan^{-1}(x)$

   The arctangent is the integral of $1/(1+x^2)$, the first series above:

$$j(x) = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad |x| < 1$$

How does the constant of integration come into this computation? ■

In 1772, Lagrange attempted to fix the "ghost of departed quantities" problem in the definition of derivatives and integrals by assuming that all functions could be represented as power series. [See Boyer (1959, p. 252) and Burton (2007, p. 525).] Unfortunately, not all functions have power series representations. The classic example of a function without a power series that has a nonzero radius of convergence is $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(0) = 0$. Taylor and Maclaurin studied the questions of when does a function have a power series representation and how to compute one.

Taylor Series

In 1715, Taylor published *Methodus Incramentorum Directa et Inversa* [see Struik (1986, p. 328)], which contained the result we now call Taylor's theorem. Maclaurin's 1742 text *Treatise on Fluxions* used the approach that we call "order of contact" to develop these power series. Since Maclaurin concentrated on series centered at $c = 0$, we call them Maclaurin series. The order of contact method matches derivatives of the function to derivatives of the power series in order to find the power series' coefficients. [See Burton (2007, p. 526).]

Theorem 1.41 (Taylor's Theorem) *Let f be a function that is continuous together with its first $n+1$ derivatives on an interval containing c and x. Then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(c, x)$$

where

$$R_n(c, x) = f^{(n+1)}(a)\frac{(x-c)^{n+1}}{(n+1)!}$$

for some a between c and x.

We can rephrase the theorem in terms of power series.

Theorem 1.42 (Taylor Series) *If f has a power series expansion that is valid on $|x-c| < R$, that is, if*

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k \qquad |x-c| < R$$

then

$$a_k = \frac{f^{(k)}(c)}{k!}$$

■ EXAMPLE 1.13 Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x| < 1$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad x \in \mathbb{R}$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad x \in \mathbb{R}$$

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \qquad |x| < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad x \in \mathbb{R}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \qquad |x| < 1$$

■

To find the radius of convergence of a Taylor series, elementary calculus students usually use the ratio or root test. Typical examples of the computations are shown for e^x and $\ln(1+x)$.

■ EXAMPLE 1.14

1. Find the interval of convergence of the Maclaurin series for e^x.
   The general term of the series is $x^n/n!$, so the limit of the ratios is

   $$r = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

   Hence, the series converges for all x and the interval of convergence is $\mathbb{R}$.

2. Find the interval of convergence of the Maclaurin series for $\ln(1+x)$.
   The general term of the series is $(-1)^{n-1}x^n/n$, so the limit of the ratios is

   $$r = \lim_{n\to\infty} \left| \frac{x^{n-1}/(n-1)}{x^n/n} \right| = \lim_{n\to\infty} \left| \frac{n}{n-1} \cdot x \right| = |x|$$

   Hence, the series converges for all $|x| < 1$. Since the function is undefined at $x = -1$, the series cannot converge there. By the alternating series test, the series does converge at $x = 1$. Thus the interval of convergence is $(-1, 1]$. ■

The ratio and root tests work for a large number of cases. However, there is more subtlety than is readily apparent. Consider the function $f(x) = \tan(\sin(x))$. It is quite difficult to find a simple expression for the general term of f's Maclaurin series. Naive reasoning suggests that since the sine is bounded by one and the tangent behaves quite well on $[-1, 1]$, there should be no problem for any x, no matter how large. Unfortunately, it's not that simple. Try as we might, we can never get convergence for $|x| > 1.9$. See Figure 1.7. What's happening here? The answer requires complex variables. When $z = \pi/2 - i\ln\left((\pi + (\pi^2 - 4)^{1/2})/2\right)$, we have $\sin(z) = \pi/2$, and hence $f(z) = \tan(\sin(z))$ is undefined. But that value of z is a complex number and we're only using real values! That doesn't matter. The radius of convergence is the distance to the nearest singularity, real or complex. And so, for $f(z) = \tan(\sin(z))$, we can never reach beyond $|z| \approx 1.87$ with a Maclaurin series, even on the real axis.

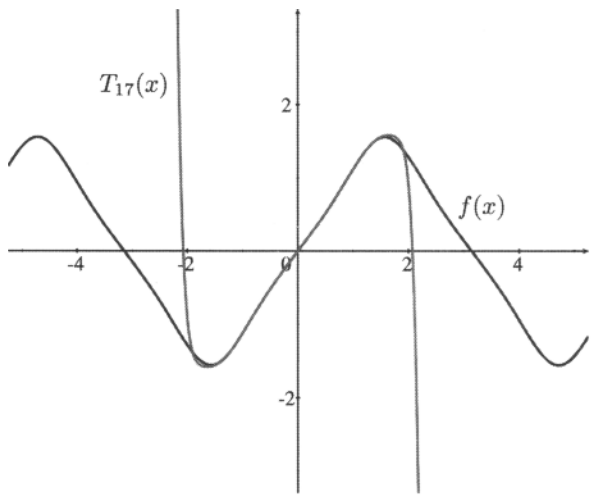

Figure 1.7 Graph of $f(x) = \tan(\sin(x))$ and $T_{17}(x)$

The delicateness of the radius of convergence question notwithstanding, Taylor and Maclaurin series provide powerful tools for analysis. Not only can we use the series very effectively for numerical approximation of functions that are difficult to compute, but the series also provide important theoretical tools. Taylor series are used in proofs throughout the theory of numerical analysis. Maclaurin series can be used as an integral part of the proof of the Weierstrass approximation theorem: *Let f be a continuous function on the closed interval $\mathcal{I}$. Given $\epsilon > 0$, there is a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in \mathcal{I}$.*

Summary

We have recapitulated the standard topics of an elementary calculus course in this chapter. The order that concepts are presented here and in current courses is nearly opposite to their historical development. It took over two hundred years from the time that Newton and Leibniz made the crucial connection between derivative and integral to the limit-based definitions that are taught today. Nevertheless, it is remarkable that the origins of the limit concept can be traced to Eudoxus and even further back to Antiphon's bounds for π from the fifth century BC. What is equally surprising to students is that calculus is not fossilized and moribund but is still an active area of research.

EXERCISES

1.1 Let $f: X \to Y$ with $A, B \subseteq X$ and $C, D \subseteq Y$. Prove:

- **a)** $f(A \cap B) \subseteq f(A) \cap f(B)$
- **b)** $f(A \cup B) = f(A) \cup f(B)$
- **c)** $f\left(f^{-1}(C)\right) \subseteq C$
- **d)** $A \subseteq f^{-1}(f(A))$

1.2 Give examples of functions and sets for which

- **a)** $f\left(f^{-1}(C)\right) \neq C$
- **b)** $A \neq f^{-1}(f(A))$

1.3 Who was the first to write $f(x)$ to denote a function?

1.4 Describe the behavior of the two functions $f(x) = |ax + b|$ and $g(x) = |a|x + |b|$ for various values of a and b.

1.5 Compare $y = \sin(\arcsin(x))$ and $z = \arcsin(\sin(x))$.

1.6 Prove the *triangle inequality:* For $x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|$$

1.7 Prove the *inverse triangle inequality:* For $x, y \in \mathbb{R}$,

$$\big||x| - |y|\big| \leq |x - y|$$

1.8 Who first used the notation

$$\lim_{x \to a} f(x)$$

1.9 Determine the value of

$$\lim_{x \to 0} \frac{\sqrt{x^2 + a^2} - a}{x^2}$$

- **a)** for any real number $a > 0$,
- **b)** for any real number $a < 0$.

1.10 Find the value of

$$\lim_{x \to 0} (1 + nx)^{(1/x)}$$

for $n > 0$.

1.11 Using $\epsilon = \frac{1}{2}$, show that there is no $\delta > 0$ satisfying the definition of the limit for the unit step function at $a = 0$. Conclude that $\lim_{x \to 0} U(x)$ does not exist. (*Hint:* Suppose the limit is L. For any $\delta > 0$, choose $x_0 \in (0, \delta)$. Consider both $|U(x_0) - L|$ and $|U(-x_0) - L|$.)

1.12 The limit

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

is important in calculus.

- **a)** Use the sandwich theorem to prove this limit.
- **b)** Use a geometric argument based on areas and the unit circle to establish this limit.

1.13 Calculate the limit

$$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta}$$

1.14 Compute

$$\lim_{y \to 1} \frac{y^4 - 1}{y - 1}$$

1.15 Find

$$\lim_{x \to 3} \frac{\sqrt{x + 22} - 5}{x - 3}$$

a) by graphing.

b) by rationalizing the numerator.

1.16 Create a set of limit problems for an elementary calculus class illustrating the main techniques.

1.17 Create a set of limit problems for an elementary calculus class illustrating the main pitfalls.

1.18 Compare and contrast the approaches used in standard calculus textbooks such Granville et al. (1911), Thomas (1968), Finney et al. (1999), Ostebee & Zorn (2002), and Stewart (2009) to the limit concept.

1.19 For each of the three conditions listed for the definition of continuity, give examples that fail to satisfy that condition.

1.20 Show that a polynomial is continuous everywhere.

1.21 Give an example for each kind of the four types of discontinuity listed in Table 1.1.

1.22 Is the function

$$f(x) = \begin{cases} e^x - x - 1 & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

continuous at $x = 0$?

1.23 Discuss the continuity of

$$g(x) = x - |x|$$

1.24 Choose a value of a so that

$$h(x) = \begin{cases} x^2 - ax - 1 & x < 1 \\ x^3 - x^2 + a & x > 1 \end{cases}$$

is continuous everywhere.

1.25 Determine the points of continuity of the function

$$k(z) = \begin{cases} e^z - 1 & z > 0 \\ 0 & \text{otherwise} \end{cases}$$

1.26 Let $r(x) = p(x)/q(x)$ be a rational function. If the degree of p is n and that of q is m, what is the largest number of discontinuities that r can have? The smallest?

1.27 If f is continuous at a and g is not, then $f + g$ is not continuous at a. Can $f + g$ be continuous at a when neither f nor g is continuous at a?

1.28 Prove that the composition of continuous functions is continuous (Theorem 1.8) by using Theorem 1.7.

1.29 Give examples showing that each hypothesis, continuous and closed interval, is necessary in the intermediate value theorem.

1.30 Compare and contrast the approaches used in standard calculus textbooks like Granville et al. (1911), Thomas (1968), Anton et al. (2009), Finney et al. (1999), Ostebee & Zorn (2002), and Stewart (2009) to continuity.

1.31 Write careful definitions for left- and right-hand derivatives at a point.

1.32 Let $f(x) = \sqrt{x+22}$.

a) Compute $f'(3)$.

b) Compute $f''(3)$.

c) Compute $f'(x)$.

d) Compute $f''(x)$.

1.33 Explain why any tangent line to a parabola intersects the curve exactly once.

1.34 Find the derivative of

$$g(x) = \ln\left(x + \ln\left(x + \ln(x)\right)\right).$$

1.35 Compute the derivative of

$$h(t) = \frac{1}{1 + \frac{1}{1+t}}$$

a) without first simplifying the complex fraction,

b) after simplifying the complex fraction.

1.36 Let $k(x) = x^3 - 3x + 2$.

a) Find the equation of the line L tangent to k at $x = -1/2$.

b) Find all intersection points of L and k.

c) Graph L and k together.

1.37 Let $k(x) = x^3 - 3x + 2$.

a) Find the equation of the line N normal to k at $x = -1$.

b) Find all intersection points of N and k.

c) Graph N and k together.

1.38 Write the derivative rules

a) for the six trigonometric functions,

b) for the six inverse trigonometric functions.

1.39 Using the convention $f^{(0)} = f$, prove

$$\cos^{(n)}(x) = \cos^{(n \bmod 4)}(x)$$

for all $n \in \mathbb{N}$.

1.40 Consider logarithmic derivatives:

a) Execute the following four Maple statements several times.

```
r := [seq(rand(-9..9)(), k=1..4)];
p := expand(
      product(x-r[k], k=1..4));
dp := diff(ln(p), x);
dp = convert(dp, parfrac, x)
```

What do you observe?

b) Let $p(x)$ be a polynomial. Compute

$$\frac{d}{dx}\ln(p(x))$$

1.41 Differentiate $y = x^x$.

1.42 What is the minimum value of the function

$$f(x) = x + \frac{1}{x}$$

1.43 The distance by bus from New York to Boston is 215 miles. A bus driver gets paid \$19.50 per hour. The cost of running the bus at a steady speed of r miles per hour is $0.80 + 0.005r$ dollars per mile. The minimum and maximum legal speeds on the roads are 40 and 55 miles per hour. What steady speed minimizes the total cost of a nonstop trip?

1.44 A farmer wishes to fence a rectangular field and to divide the field in half with another fence. The outside fence costs \$4 per foot, and the fence in the middle costs \$3 per foot. If the farmer has budgeted \$1000 for fencing, what dimensions maximize the total area?

1.45 A cylindrical grain bin of radius 10 feet and height 30 feet is being filled with corn at the rate of 30 cubic feet per minute. How fast is the depth of the corn increasing?

a) Assume the corn is spread in a uniform layer.

b) Assume the corn falls in a conical pile constrained by the bin walls.

1.46 Compare and contrast the approaches used in standard calculus textbooks like Granville et al. (1911), Thomas (1968), Finney et al. (1999), Ostebee & Zorn (2002), and Stewart (2009) to derivatives.

1.47 Use a computer algebra system to generate a random, fifth-degree polynomial $p(x)$. Define *Newton's method func-*

tion $N(x)$ by

$$N(x) = x - \frac{p(x)}{p'(x)}$$

a) Find the smallest positive root r by iterating N, starting with an initial guess taken from a graph.

b) Define a new iterating function

$$M(x) = x - \frac{p'(x)}{p''(x)}$$

What is the result of iterating M with the starting value r found in part a?

1.48 Use a geometric argument to show that

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \frac{1}{2}\pi r^2$$

for $r > 0$.

1.49 Compute

a) $\displaystyle\int \frac{\sqrt{x} + 3x^3}{x}\,dx$

b) $\displaystyle\int \frac{\sin(\sqrt{z}\,)}{\sqrt{z}}\,dz$

c) $\displaystyle\int \frac{t}{\sqrt{t+1}}\,dt$

1.50 Compute

a) $\displaystyle\int_{-1}^{1} 1 - x^2\,dx$

b) $\displaystyle\int_{0}^{\pi} \frac{\sin(x)}{x}\,dx$

c) $\displaystyle\int_{0}^{1} 2\pi\left(\sqrt{x} - x^2\right)^2 dx$

1.51 Find an equation for the curve passing through the point $(2, 1)$ and having slope $y' = 2x^3 - x - 3$ at each point (x, y).

1.52 Find the geometric area between the sine and cosine curves from 0 to 2π.

1.53 Use a computer algebra system to integrate

$$\int \sin\left(\frac{\pi}{2}t^2\right) dt$$

Investigate the result.

1.54 Let f be integrable on $[a, b]$ and be bounded above by M and below by m. Prove

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b - a)$$

1.55 Consider the function

$$L(x) = \frac{x^{n+1} - 1}{n + 1}$$

a) Show that $L'(x) = x^n$ and conclude that L is an antiderivative for x^n.

b) Calculate

$$\lim_{n \to -1} L(x)$$

How does this result relate to part a?

1.56 Use a graphing utility to construct an accurate "three-rectangle diagram" showing upper and lower bounds and the average value for the integral $\int_0^3 e^{-x^2}\,dx$.

1.57 Use a computer algebra system or a calculator to make the counterpart of Table 1.2 for the integral $\int_0^\pi \sin(x)/x\,dx$.

1.58 Compare and contrast the approaches used in standard calculus textbooks like Granville et al. (1911), Thomas (1968), Anton et al. (2009), Finney et al. (1999), Ostebee & Zorn (2002), and Stewart (2009) to the concept of integration.

1.59 Prove Theorem 1.32, the algebra of sequence limits.

1.60 Determine whether the given sequence converges.

a) $a_n = (-1)^n \cdot \frac{n-1}{n}$

b) $b_n = \frac{n^2}{2^n}$

c) $c_n = \frac{n!}{2 \cdot 4 \cdot 6 \dots (2n)}$

1.61 Determine whether the given recursively defined sequence converges.

a) $p_n = \sqrt{2 + p_{n-1}}$ for $n > 1$ and $p_1 = \sqrt{2}$

b) $q_n = \frac{q_{n-1} + q_{n-2}}{2}$ for $n > 2$ with $q_1 = 0$ and $q_2 = 1$

c) $f_n = f_{n-1} + f_{n-2}$ for $n > 2$ with $f_0 = 1$ and $f_1 = 1$

1.62 Suppose that a_n converges to 0. Does the sequence

$$b_n = \sin\left(\frac{n\pi}{2}\right) \cdot a_n$$

converge or diverge?

1.63 Suppose that c_n converges to 1. Does the sequence

$$d_n = \sin\left(\frac{n\pi}{2}\right) \cdot c_n$$

converge or diverge?

1.64 Investigate the Maclaurin series for

a) $\tan(x)$,

b) $\sec(x)$.

1.65 What value of the center a gives the best Taylor fourth-degree polynomial approximation to $y = e^x$ on the interval $[-1, 1]$?

1.66 Explain why the following functions do not have Maclaurin series.

a) $f(x) = |x|$

b) $g(x) = \ln(x)$

c) $h(x) = \cot(x)$

d) $k(x) = \sqrt{x}$

1.67 Find the Taylor expansion for

$$f(x) = \frac{1}{\sqrt{1 - x^2}}$$

centered at $a = 0$.

1.68 Let T_n be the nth-degree Maclaurin polynomial of arcsin.

a) Find T_n for $n = 10$.

b) Plot T_n and arcsin together for $n = 3, 6, 10$.

1.69 Let $f(x) = \tan(\cos(x))$.

a) Produce graphs of several Taylor polynomials of f for different choices of n. What do you observe?

b) Using a 3D graphing utility, plot $z(x, y) = |f(x + iy)|$ in the window $[-2, 2] \times [-2, 2]$. How does this relate to part a?

1.70 Let i be the imaginary unit $\sqrt{-1}$.

a) Calculate all the integer powers of i.

b) Substitute ix in the Maclaurin expansion of e^x.

c) Separate the expansion of e^{ix} into terms with i and terms without.

d) Compare the results above with the Maclaurin expansions of sine and cosine.

e) Write Euler's identity

$$e^{ix} = \dots$$

1.71 Who said,

I seem to have been like a child playing on the sea shore, finding now and then a prettier shell than ordinary, whilst the great ocean of truth lay undiscovered before me.

1.72 Who said,

> *I hold that the mark of a genuine idea is that its possibility can be proved, either a priori by conceiving its cause or reason, or a posteriori when experience teaches us that it is in fact in nature.*

Now that you have finished looking back at calculus, it's time to begin analysis. There is one last assignment, however—reflect on Morris Kline's comment:

> Contrary to common belief, the calculus is not the height of the so-called "higher mathematics." It is, in fact, only the beginning.

INTERLUDE: FERMAT, DESCARTES, AND THE TANGENT PROBLEM

If we were to teach calculus following the order that topics were originally developed, we would start with integration, then study differentiation, and finally consider limits—the exact opposite of modern classes.

Eudoxus of Cnidus (408–355 B.C.) based his results concerning areas and volumes on his *method of exhaustion*. The term "exhaustion" refers to the difference in area or volume of a given object to the approximating regular figures being "exhausted" or "used up." Eudoxus would recognize the illustrations of approximating sums for a definite integral shown in modern calculus texts. The method of exhaustion was the main tool used to prove quadrature formulas (our definite integrals) up to the time of Fermat. Cavalieri had developed formulas for the integral of x^n for x from 0 to a using "indivisibles" in his 1635 text *Geometria indivisibilibus*. These indivisibles became Newton's *moments* and Leibniz's *differentials*. Cavalieri's book was extremely influential and widely read by mathematicians of the 1600s [see O'Connor & Robertson (2008)].

The tangent problem—construct a line tangent to a curve at a specified point—is the generalization arising from Euclid's definition of the tangent to a circle. Archimedes gave methods to construct tangents to spirals and other curves. Descartes, in his *Géométrie*, considered "algebraic curves," or curves with simple algebraic formulas, and rejected "mechanical curves." While Descartes used algebra heavily, his approach was really based on geometry. Fermat used algebra with infinitesimals, possibly based on Cavalieri's indivisibles, and approached the tangent line problem in a fashion similar to what we do in classes today.

Let's consider the problem of finding the tangent line to the curve $y^2 = 3x$ at the point $(3, 3)$ using both Descartes' and Fermat's methods.

Descartes' Tangent Circle

Descartes' technique was to find a circle centered on the x-axis that was tangent to the curve at the point in question and then recognize that the normal to the radius at that point forms the tangent to the original curve. See Figure I.1.

Begin by looking at the equation of a circle centered on the x-axis at $(h, 0)$ and passing through the point $(3, 3)$. The circle's radius is $r = \sqrt{(3-h)^2 + 3^2}$. Hence,

$$(x-h)^2 + y^2 = h^2 - 6h + 18$$

Substitute $y^2 = 3x$ to have the curve and circle intersect. Apply a little algebra to obtain

$$x^2 - 2xh + 3x + 6h - 18 = 0$$

A crucial observation is that the circle is tangent to the curve when the equation above has a single root, i.e., a single intersection in the neighborhood of 3. Since the roots are 3 and $2h - 6$, we take $h = 9/2$. Descartes' computation reduces the

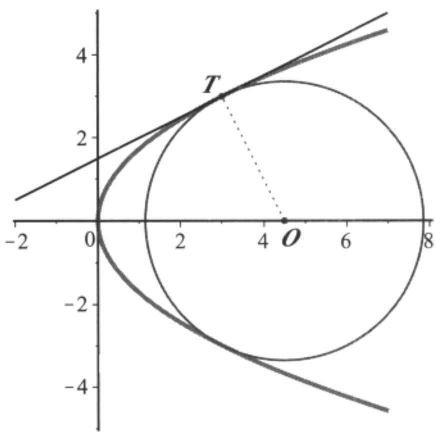

Figure 1.1 Descartes Tangent Circle

problem to the simple task of finding a line passing through $(3, 3)$ and perpendicular to the segment joining $(9/2, 0)$ to $(3, 3)$. The desired tangent is easily calculated to be $y = 1/2 \cdot (x - 3) + 3$.

Note that Descartes does not use limits—even implicitly—in his calculations.

Fermat's Similar Triangles

Fermat's technique is based on similar triangles. When the triangles coincide, the hypotenuse lies on the tangent line. See Figure 1.2.

Fermat starts by drawing the line tangent to the curve at $(3, 3)$. He then creates triangle $\triangle OAT$ by dropping a line to the x-axis. Add the length E to point A to create a new triangle $\triangle OBP$. Set the length of the segment $\overline{OA}$ to be s; then the length of $\overline{OB}$ is $s + E$. Since the two triangles are similar, the ratios of the legs are equal. At this point, Fermat substitutes $f(3 + E) = \sqrt{3(3 + E)}$ for the length of $\overline{BP}$, claiming that the error disappears when $E = 0$. Thus

$$\frac{s}{s + E} = \frac{\sqrt{3 \cdot 3}}{\sqrt{3(3 + E)}} = \frac{3}{\sqrt{3(3 + E)}}$$

Solve this expression for s. *Do the algebra!*

$$s = \frac{3}{\left(\sqrt{3(3 + E)} - 3\right) / E} = \sqrt{3(3 + E)} + 3$$

With $E = 0$, we have $s = 6$.

The critical observation is that the slope of the tangent line is equal to the length of $\overline{AT}$ divided by s. Hence the slope is $3/6 = 1/2$. The equation of the tangent line is given by $y = 1/2 \cdot (x - 3) + 3$.

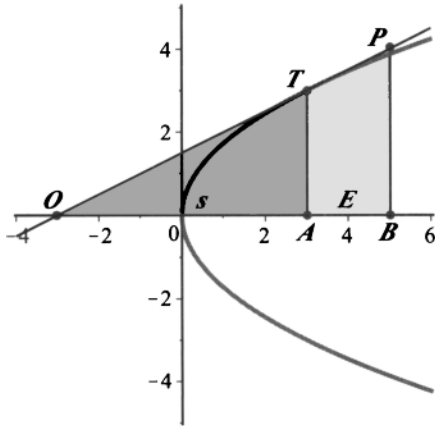

Figure I.2 Fermat's Similar Triangles

Conclusion

Descartes' technique did not involve using infinitesimals and letting values not be zero for simplification, then be zero for completion. Because he relied on standard algebraic manipulation, Descartes' approach could be rigorously defended. However, his method was limited to simple algebraic curves by the nature of the computations.

Fermat's technique implicitly uses a limiting process as E goes to zero. Even though there was no contemporary logical justification, Fermat's approach handled a wide variety of curves and produced results that were correct.

Unfortunately, Descartes and Fermat became entangled in a controversy over priority. Finally, Descartes admitted that Fermat's method was the more general, but the enmity between them was never overcome.