

APPLIED FUNCTIONAL
ANALYSIS

APPLIED FUNCTIONAL ANALYSIS

Second Edition

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To my children,

Anne Laure,

who studied the first edition of this book when she was a student;

Henri-Jean and Marc,

who escaped this chore;

and to Pierre-Cyril,

who may regard 20 years this new edition as an historical document.

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PREFACE

Yet another book on functional analysis! Yabfa!, would exclaim a computer scientist in his or her exotic language.

Why, 20 years after the first edition of *Applied Functional Analysis*, after so many other monographs on this basic topic, do I propose a second edition of this text devoted to an introduction—an induction?—to this seductive field?

The mathematicians of my generation were lucky enough to receive as a dowry the tools of Functional Analysis created at the dawn of our finishing century by David Hilbert and Stefan Banach, to name just those two visionaries. Along with many other mathematicians, they offered us a formidable unifying framework and an array of tools for solving problems stemming from many different areas of knowledge, making a universe of a “multiverse” of motivating applications: *It is this universality of mathematical results, having their origin in one discipline and finding applications in others, that makes functional analysis in particular, and mathematics in general, so fascinating.*

The success of this machinery allowed thousands of mathematicians to use it in so many different areas that it is impossible to pursue the early Dunford-Schwartz or the Bourbaki attempts to present an exhaustive overview of the state of the art. Many other books then evolved in a Darwinian way: exploring many specific and diverse directions, reflecting the experiences as well as the views of the purpose of mathematics of each author, eventually finding an adequate niche through the natural selection created by the readership.

The first edition of this book reflected my personal experience at the time, derived from numerical analysis of partial differential equations, and later, from mathematical economics. After two decades my views have evolved and my experience has broadened, my teaching of functional analysis to the students of Université Paris-Dauphine evolving year after year. I could not resist both the pleasure and the pain of divulging to the young students what was continuously going on, at their level, on the research front. I then felt it was time to write down an account of some of the recent discoveries that have helped me revise some of the perspectives I had formed earlier.

However, several pedagogical choices remain invariant: (1) convey the feeling of the variety of applications; (2) keep the length of the exposition within

reasonable limits—about 120 teaching hours—(3)—restrict the initiation to functional analysis to the linear framework; (4) keep to the simple Hilbertian structure, and (5) present distributions as elements of Sobolev spaces.

I shall thus be able to take a quick look at boundary-value problems for elliptic and parabolic partial differential equations. I added a short introduction to set-valued analysis and presented the Nagumo theorem on the viability of closed subsets under differential equations. It is not only interesting by itself, but allows us to forge efficient tools for rapidly and easily solving other problems, such as boundary-value problems for systems of first-order partial differential equations, or minimal and stopping-time problems, or building Lyapunov functions. I removed the first edition's chapter on nonlinear analysis, as well as occasional sections or paragraphs that are no longer essential.

In order to illustrate the abstract exposition as soon as possible, I chose applications derived from numerical analysis, systems theory, the calculus of variations, control theory, optimization of allocations of scarce resources, demography (McKendrick boundary-value problems), convex and nonsmooth analysis, and set-valued analysis. This selection is partial and may not be to everyone's taste. In order to keep the time and space allocated to these examples short, I had to go so far as to sacrifice the use of weak topologies and to deprive the reader of the grace of the weak compactness of the unit ball of the dual of a Banach space. However, as long as the linear theory is concerned, one can survive without it. This allows us to provide a *larger number of results in the simplest way*, at the price, of course, of generality.

I hope that by doing so, I may persuade the readers of the advantages of an abstract approach to theories motivated by concrete problems, and to attract them to applied and motivated mathematics.

Naturally, the nature and the deep meaning of mathematical concepts and statements evolve with time. This was the case during the course of the century of the views on differential calculus, inherited from Pierre de Fermat, Isaac Newton and Gottfried Leibniz three centuries ago, and formalized when a little more than a century ago Augustin-Louis Cauchy defined rigorously the concept of limit. The consensus on the formalization of derivatives as limits of difference quotients for the pointwise convergence was so strong that the concept of derivative became a permanent reality, protected from any dissenting view. This could have been the case in this kind of paradise in which one is free to choose the assumptions and the rules of the game. The overwhelming curiosity and the concern for interpreting the environment with the help of mathematical metaphors was Eve's apple. Are all problems arising outside pure mathematics "well-posed" in the Hadamard sense? Should the nondifferentiable functions popping up in so many fields be deprived forever of the benefits of some properties of the derivatives?

Since then, the history of the derivatives of functions and maps has been a kind of *mathematical striptease*, the modern version of what Parmenides and the pre-Socratic Greeks called *a-letheia*, the dis-covering, un-veiling of the world that surrounds us. This is nothing else than the drive to "abstraction,"

isolating, in a given perspective, the relevant information in each concept and investigating the interplay between them.

Indeed, one by one, and very shyly, the required properties of the derivative of a function or of a functional were taken away. We shall go quite far to leave the derivatives with the bare minimum.

This is quite natural, though, because each problem demands its own amount of properties that the derivative should enjoy (i.e., its own degree of regularity). Without going too far by always requiring minimal assumptions, some problems could not be solved by sticking to the richest structure. The right balance between generality and readability is naturally a subjective choice.

The concepts of the derivatives of functionals go back to Volterra in 1887. Then Gâteaux, in a note written in 1913 and published in 1919 after his death during the First World War, introduced the concept of *first variation*: If $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ denotes a map from one finite dimensional vector space to another, and

$$\nabla_h f(x)(v) := \frac{f(x + hv) - f(x)}{h}$$

denotes its differential quotients, the first variation of f at x in the direction v is the limit $Df(x)(v)$ of these differential quotients when it exists. In defining the *Gâteaux derivative* $Df(x)$, Fréchet added the requirement that the map $v \mapsto Df(x)(v)$ is linear and continuous! He proposed his own concept of the derivatives (with the mandatory linearity) of a function as early as in 1912 in the case of functions, and in 1925 for maps from one normed space to another. Mathematicians of this period still insisted that the derivatives of functionals have many properties, and were not ready to give away linearity.

These definitions were too restrictive, so that they were weakened in several ways, and led to a *ménagerie* of concepts: strong or weak Fréchet and Gâteaux derivatives; Hadamard, bounded (Suchomlinov), locally uniform (Vainberg) derivatives; Dini directional semiderivatives; or derivatives from the right, to give a few.

This was not enough, however, as the topologies used to define the limits of the difference quotients were still too strong to allow more maps to retain some kind of differentiability. But weakening the topologies allows us to get more limits at the price of obtaining these limits outside the set of single-valued maps. This was even worse than loosening the linearity condition for the directional derivatives.

However, in the 1940s, Serge Sobolev and Laurent Schwartz dared to introduce weak derivatives and *distributions* to obtain solutions to partial differential equations; just as in the 1960s, Jean-Jacques Moreau and Terry Rockafellar defined *set-valued subdifferential* of convex functions to implement the Fermat rule in optimization; while the 1980s witnessed the emergence of *graphical derivatives* of set-valued maps and set-valued analysis for dealing, for instance, with control systems and differential games; and the 1990s saw the appearance of *mutations of set-valued maps* for grasping kind of differential

equation—called a mutational equation—that govern the evolution of sets and devise a differential calculus in metric spaces. This process of differentiating “less and less differentiable maps,” so to speak, continues its random course in a nonteleological way.

To briefly elaborate this point: the strong requirement of pointwise convergence of differential quotients can be weakened in (at least) two ways, each sacrificing different groups of properties of the usual derivatives:

- Fix the direction v and take the limit of the function $x \mapsto \nabla_h f(x)(v)$ in the weaker sense of distributions, to be defined later in this book. The limit $D_v f$ may then be a distribution, and no longer a single-value map. However, it coincides with the usual limit when f is Gâteaux differentiable. Moreover, one can define the difference quotients of distributions, take their limit, and thus differentiate distributions.

Distributions, as we shall see, are no longer functions or maps defined on \mathbb{R}^n , so they lose the pointwise character of functions and maps, while retaining the linearity of the operator $f \mapsto D_v f$, which is mandatory for using the theory of linear operators for solving partial differential equations.

- Fix the direction x and take the limit of the function $v \mapsto \nabla_h f(x)(v)$ in the weaker sense of “graphical convergence,” to be defined later in this book. The limit $Df(x)$ may then be a set-valued map, and no longer a single-valued map. However, it coincides with the usual limit when f is Gâteaux differentiable. Moreover, one can define the difference quotients of set-valued maps, take their limit, and thus differentiate them. These graphical derivatives retain the pointwise character of functions and maps, which is mandatory for implementing the Fermat Rule, proving inverse function theorems under constraints, or using Lyapunov functions, for instance, but lose the linearity of the map $f \mapsto Df(x)$.

In both cases, the approaches are similar: they use (different) convergences *weaker than the pointwise convergence* to increase the possibility of the difference-quotients to converge. But the price to pay is the loss of some properties by passing to these weaker limits (the pointwise character for distributional derivatives, the linearity of the differential operator for graphical derivatives).

We shall use both of them to study boundary value-problems for partial differential equations, the second approach being for instance involved in the definition of set-valued solutions (with shocks) of systems of first-order partial differential equations and of the *viscosity solutions* to Hamilton-Jacobi variational equations and inequalities.

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