1

Defining the Signal

... the distance of the invisible background [is] so immense that no ray from it has yet been able to reach us at all.

–Edgar Allan Poe in *Eureka*, 1848

1.1 The power of light – luminosity and spectral power

The *luminosity*, L, of an object is the rate at which the object radiates away its energy (cgs units of erg s^{-1} or SI units of watts),

$$dE = L\, dt \tag{1.1}$$

This quantity has the same units as *power* and is simply the radiative power output from the object. It is an intrinsic quantity for a given object and does not depend on the observer's distance or viewing angle. If a star's luminosity is L_* at its surface, then at a distance r away, its luminosity is still L_*.

Any object that radiates, be it spherical or irregularly shaped, can be described by its luminosity. The Sun, for example, has a luminosity of $L_\odot = 3.85 \times 10^{33}$ erg s^{-1} (Table G.3), most of which is lost to space and not intercepted by the Earth (Example 1.1).

Example 1.1

Determine the fraction of the Sun's luminosity that is intercepted by the Earth. What luminosity does this correspond to?

At the distance of the Earth, the Sun's luminosity, $L_\odot$, is passing through the imaginary surface of a sphere of radius, $r_\oplus = 1$ AU. The Earth will be intercepting photons over only

Astrophysics: Decoding the Cosmos Judith A. Irwin

ISBN: 978-0-470-01305-2(HB) 978-0-470-01306-9(PB)

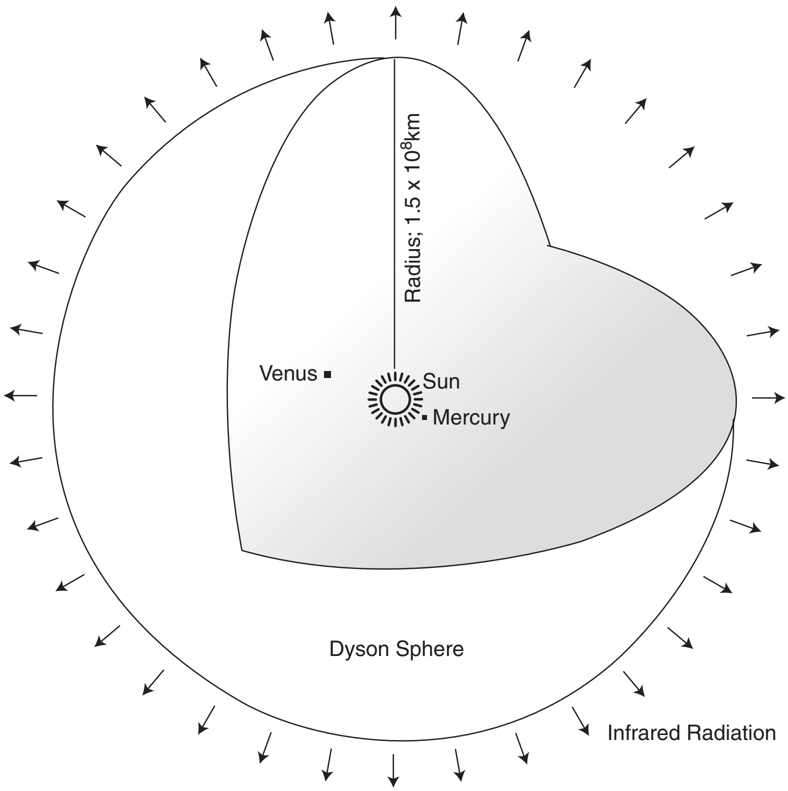

Figure 1.1. Illustration of a Dyson Sphere that could capture the entire luminous ouput from the Sun. Some have suggested that advanced civilizations, if they exist, would have discovered ways to build such spheres to harness all of the energy of their parent stars.

the cross-sectional area that is facing the Sun. This is because the Sun is so far away that incoming light rays are parallel. Thus, the fraction will be

$$\mathbf{f} = \frac{\pi R_{\oplus}{}^2}{4\pi r_{\oplus}^2} \tag{1.2}$$

where $R_{\oplus}$ is the radius of the Earth. Using the values of Table G.3, the fraction is $\mathbf{f} = 4.5 \times 10^{-10}$ and the intercepted luminosity is therefore $L_{\text{int}} = \mathbf{f}\, L_{\odot} = 1.73 \times 10^{24}\ \text{erg s}^{-1}$. A hypothetical shell around a star that would allow a civilization to intercept *all* of its luminosity is called a *Dyson Sphere* (Figure 1.1).

When one refers to the luminosity of an object, it is the *bolometric* luminosity that is understood, i.e. the luminosity over all wavebands. However, it is not possible to determine this quantity easily since observations at different wavelengths require different techniques, different kinds of telescopes and, in some wavebands, the

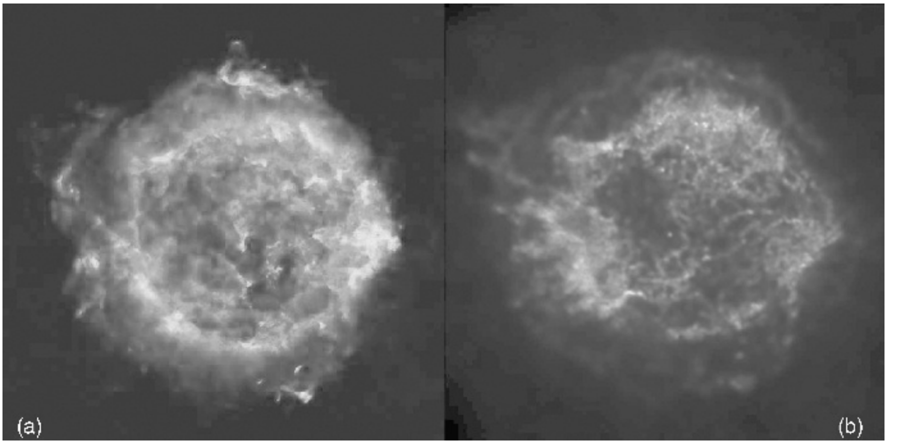

Figure 1.2. The supernova remnant, Cas A, at a distance of 3.4 kpc and with a linear diameter of $\approx$4 pc, was produced when a massive star exploded in the year AD 1680. It is currently expanding at a rate of 4000 km s^{-1} (Ref. [181]) and the proper motion (angular motion in the plane of the sky, see Sect. 7.2.1.1) of individual filaments have been observed. One side of the bipolar jet, emanating from the central object, can be seen at approximately 10 o'clock. **(a)** Radio image at λ20 cm shown in *false colour* (see Sect. 2.6) from Ref. [7]. Image courtesy of NRAO/AUI/NSF. **(b)** X-ray emission, with red, green and blue colours showing, respectively, the intensity of low, medium and high energy X-ray emission. (Reproduced courtesy of NASA/CXC/SAO) (see colour plate)

necessity of making measurements above the obscuring atmosphere of the Earth. Thus, it is common to specify the luminosity of an object for a given waveband (see Table G.6). For example, the *supernova remnant*, Cas A (Figure 1.2), has a radio luminosity (from $\nu_1 = 2 \times 10^7$ Hz to $\nu_2 = 2 \times 10^{10}$ Hz) of $L_{\rm radio} = 3 \times 10^{35}$ erg s^{-1} (Ref. [6]) and an X-ray luminosity (from 0.3 to 10 keV) of $L_{\rm X\text{-}ray} = 3 \times 10^{37}$ erg s^{-1} (Ref. [37]). Its bolometric luminosity is the sum of these values plus the luminosities from all other bands over which it emits. It can be seen that the radio luminosity might justifiably be neglected when computing the total power output of Cas A. Clearly, the source *spectrum* (the emission as a function of wavelength) is of some importance in understanding which wavebands, and which processes, are most important in terms of energy output. The spectrum may be represented mostly by *continuum emission* as implied here for Cas A (that is, emission that is continuous over some spectral region), or may include *spectral lines* (emission at discrete wavelengths, see Chapter 3, 5, or 9). Even very weak lines and weak continuum emission, however, can provide important clues about the processes that are occurring within an astronomical object, and must not be neglected if a full understanding of the source is to be achieved.

In the optical region of the spectrum, various *passbands* have been defined (Figure 1.3). The Sun's luminosity in V-band, for example, represents 93 per cent of its bolometric luminosity.

The *spectral luminosity* or *spectral power* is the luminosity per unit bandwidth and can be specified per unit wavelength, L_λ (cgs units of erg s^{-1} cm^{-1}) or per unit

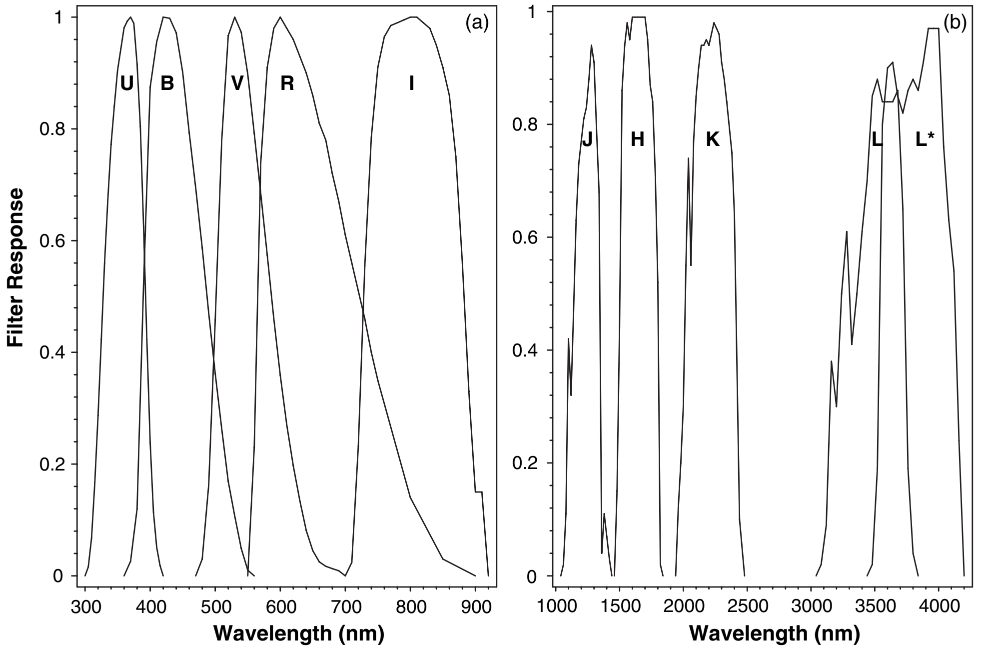

Figure 1.3. Filter bandpass responses for **(a)** the UBVRI bands (Ref. [17]) and **(b)** the JHKLL* bands (Ref. [19]). (The U and B bands correspond to UX and BX of Ref. [17].) Corresponding data can be found in Table 1.1

frequency, L_ν (erg s^{-1} Hz^{-1}),

$$dL = L_\lambda \, d\lambda = L_\nu \, d\nu \tag{1.3}$$

$$\text{so} \qquad L = \int L_\lambda \, d\lambda = \int L_\nu \, d\nu \tag{1.4}$$

Note that, since $\lambda = \frac{c}{\nu}$,

$$d\lambda = -\frac{c}{\nu^2} \, d\nu \tag{1.5}$$

so the magnitudes of L_λ and L_ν will not be equal (Prob. 1.1). The negative sign in Eq. (1.5) serves to indicate that, as wavelength increases, frequency decreases. In equations like Eq. (1.4) in which the wavelength and frequency versions of a function are related to each other, this negative is already taken into account by ensuring that the lower limit to the integral is always the lower wavelength or frequency. Note that the cgs units of L_λ (erg s^{-1} cm^{-1}) are rarely used since 1 cm of bandwidth is exceedingly large (Table G.6). Non-cgs units, such as erg s^{-1} Å^{-1} are sometimes used instead.

Luminosity is a very important quantity because it is a basic parameter of the source and is directly related to energetics. Integrated over time, it provides a measure of the energy required to make the object shine over that timescale. However, it is not a quantity that can be measured directly and must instead be derived from other measurable quantities that will shortly be described.

1.2 Light through a surface – flux and flux density

The *flux* of a source, f (erg s^{-1} cm^{-2}), is the radiative energy per unit time passing through unit area,

$$dL = f\, dA \tag{1.6}$$

As with luminosity, we can define a flux in a given waveband or we can define it per unit spectral bandwidth. For example, the *spectral flux density*, or just *flux density* (erg s^{-1} cm^{-2} Hz^{-1} or erg s^{-1} cm^{-2} cm^{-1})[1] is the flux per unit spectral bandwidth, either frequency or wavelength, respectively,

$$\begin{aligned} dL_\nu &= f_\nu\, dA \qquad & dL_\lambda &= f_\lambda\, dA \\ df &= f_\nu\, d\nu \qquad & df &= f_\lambda\, d\lambda \end{aligned} \tag{1.7}$$

A special unit for flux density, called the *Jansky* (Jy) is utilized in astronomy, most often in the infrared and radio parts of the spectrum,

$$1\ \text{Jy} = 10^{-26} \quad \text{W m}^{-2}\,\text{Hz}^{-1} = 10^{-23} \quad \text{erg s}^{-1}\,\text{cm}^{-2}\,\text{Hz}^{-1} \tag{1.8}$$

Radio sources that are greater than 1 Jy are considered to be strong sources by astronomical standards (Prob. 1.3).

The spectral response is independent of other quantities such as area or time so Eq. (1.6) and the first line of Eq. (1.7) show the same relationships except for the subscripts. To avoid repitition, then, we will now give the relationships for the bolometric quantities and it will be understood that these relationships apply to the subscripted 'per unit bandwidth' quantities as well.

The luminosity, L, of a source can be found from its flux via,

$$L = \int f\, dA = 4\,\pi\, r^2 f \tag{1.9}$$

where r is the distance from the centre of the source to the position at which the flux has been determined. The $4\,\pi\, r^2$ on the right hand side (RHS) of Eq. (1.9) is strictly

[1]The two 'cm' designations should remain separate. See the Appendix at the end of the Introduction.

Figure 1.4. An image of the Centaurus A jet emanating from an *active galactic nucleus* (AGN) at the centre of this galaxy and at lower right of this image. Radio emission is shown in red and X-rays in blue. (Reproduced by permission of Hardcastle M.J., *et al.*, 2003 ApJ, **593**, 169.) Even though gaseous material may be moving along the jet in a highly directional fashion, the RHS of Eq. (1.9) may still be used, provided that photons generated within the jet (such as at the knot shown) escape in all directions. (see colour plate)

only true for sources in which the photons that are generated can escape in all directions, or *isotropically*. This is usually assumed to be true, even if the source itself is irregular in shape (Figure 1.4). These photons pass through the imaginary surfaces of spheres as they travel outwards. The $\frac{1}{r^2}$ fall-off of flux is just due to the geometry of a sphere (Figure 1.5.a). In principle, however, one could imagine other geometries. For example, the flux of a man-made laser beam would be constant with r if all emitted light rays are parallel and without losses (Figure 1.5.b). Light that is beamed into a narrow cone, such as may be occurring in pulsars[2] is an example of an intermediate case (Prob. 1.4).

For *stars*, we now define the *astrophysical flux*, F, to be the flux at the surface of the star,

$$L_* = 4\pi R_*^2 F = 4\pi r^2 f \qquad \Rightarrow f = \left(\frac{R_*}{r}\right)^2 F \tag{1.10}$$

where L_* is the star's luminosity and R_* is its radius.

Using values from Table G.3, astrophysical flux of the Sun is $F_\odot = 6.33 \times 10^{10}\,\mathrm{erg\,s^{-1}\,cm^{-2}}$ and the *Solar Constant*, which is the flux of the Sun at the distance

[2]Pulsars are rapidly spinning *neutron stars* with strong magnetic fields that emit their radiation in beamed cones. Neutron stars typically have about the mass of the Sun in a diameter only tens of km across.

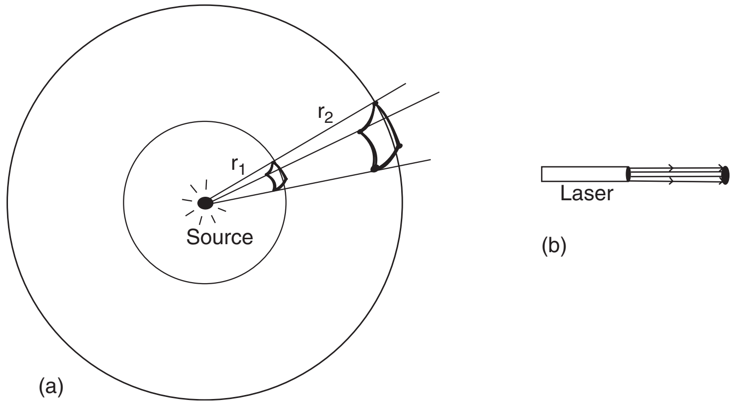

Figure 1.5. **(a)** Geometry illustrating the $\frac{1}{r^2}$ fall-off of flux with distance, r, from the source. The two spheres shown are imaginary surfaces. The same amount of energy per unit time is going through the two surface areas shown. Since the area at r_2 is greater than the one at r_1, the energy per unit time per unit cm^2 is smaller at r_2 than r_1. Since measurements are made over size scales so much smaller than astronomical distances, the detector need not be curved. **(b)** Geometry of an artificial laser. For a beam with no divergence, the flux does not change with distance.

of the Earth[3], denoted, $S_\odot$, is 1.367×10^6 erg s^{-1} cm^{-2}. The Solar Constant is of great importance since it is this flux that governs climate and life on Earth. Modern satellite data reveal that the solar 'constant' actually varies in magnitude, showing that our Sun is a variable star (Figure 1.6). Earth-bound measurements failed to detect this variation since it is quite small and corrections for the atmosphere and other effects are large in comparison (e.g. Prob. 1.5).

The flux of a source in a given waveband is a quantity that is measurable, provided corrections are made for atmospheric and telescopic responses, as required (see Sects. 2.2, 2.3). If the distance to the source is known, its luminosity can then be calculated from Eq. (1.9).

1.3 The brightness of light – intensity and specific intensity

The *intensity*, I (erg s^{-1} cm^{-2} sr^{-1}), is the radiative energy per unit time per unit solid angle passing through a unit area that is perpendicular to the direction of the emission. The *specific intensity* (erg s^{-1} cm^{-2} Hz^{-1} sr^{-1} or erg s^{-1} cm^{-2} cm^{-1} sr^{-1}) is the radiative energy per unit time per unit solid angle per unit spectral bandwidth (either frequency or wavelength, respectively) passing through unit area perpendicular to the direction of the emission. The intensity is related to the flux via,

$$df = I \cos\theta \, d\Omega \tag{1.11}$$

[3]This is taken to be above the Earth's atmosphere.

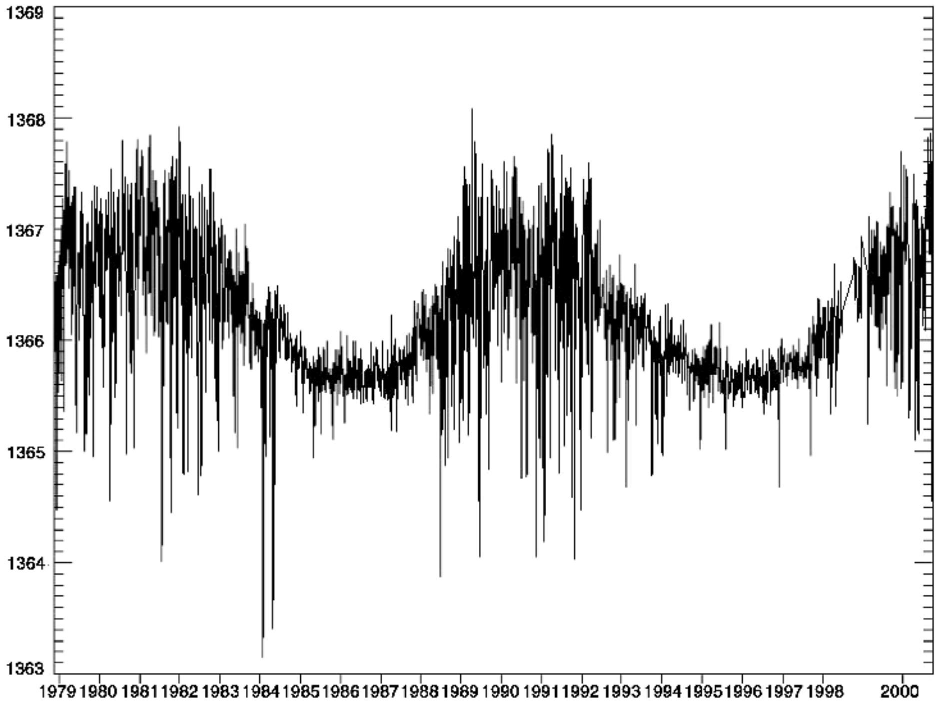

Figure 1.6. Plot of the Solar Constant (in W m^{-2}) as a function of time from satellite data. The variation follows the 11-year Sunspot cycle such that when there are more sunspots, the Sun, on average, is brighter. The peak to peak variation is less than 0.1 per cent. This plot provides definitive evidence that our Sun is a variable star. (Reproduced by permission of www.answers.com/topic/solar-variation)

As before, the same kind of relation could be written between the quantities per unit bandwidth, i.e. between the specific intensity and the flux density.

The specific intensity, I_ν, is the most basic of radiative quantities. Its formal definition is written,

$$dE = I_\nu \cos\theta \, d\nu \, d\Omega \, dA \, dt \tag{1.12}$$

Note that each elemental quantity is independent of the others so, when integrating, it doesn't matter in which order the integration is done.

The intensity isolates the emission that is within a given solid angle and at some angle from the perpendicular. The geometry is shown in Figure 1.7 for a situation in which a detector is receiving emission from a source in the sky and for a situation in which an imaginary detector is placed on the surface of a star. In the first case, the source subtends some solid angle in the sky in a direction, θ, from the zenith. The factor, cos θ accounts for the foreshortening of the detector area as emission falls on it.

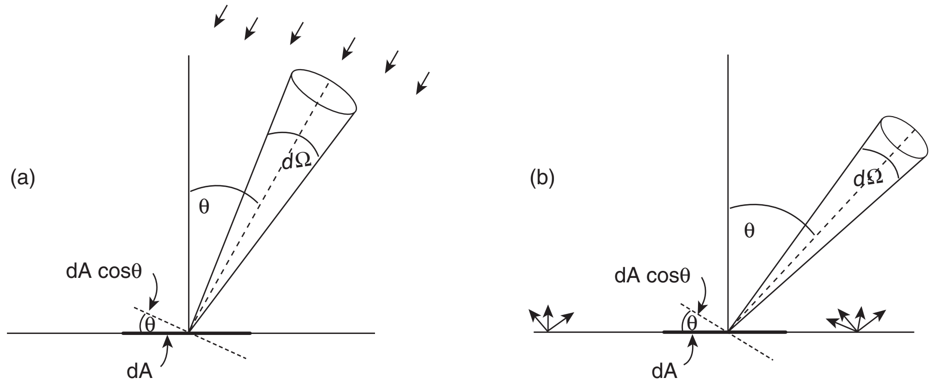

Figure 1.7. Diagrams showing intensity and its dependence on direction and solid angle, using a spherical coordinate system such as described in Appendix B. **(a)** Here dA would be an element of area of a detector on the Earth, the perpendicular upwards direction is towards the zenith, a source is in the sky in the direction, θ, and $d\Omega$ is an elemental solid angle on the source. The arrows show incoming rays from the *centre* of the source that flood the detector. **(b)** In this example, an imaginary detector is placed at the surface of a star. At each point on the surface, photons leave in all directions away from the surface. The intensity would be a measure of only those photons which pass through a given solid angle at a given angle, θ from the vertical

Usually, a detector would be pointed directly at the source of interest in which case $\cos\theta = 1$. In the second case, the coordinate system has been placed at the surface of a star. At any position on the star's surface, radiation is emitted over all directions away from the surface. The intensity refers to the emission in the direction, θ radiating *into* solid angle, $d\Omega$. Example 1.2 indicates how the intensity relates to the flux for these two examples. Figure 1.7 also helps to illustrate the generality of these quantities. One could place the coordinate system at the centre of a star, in interstellar space, or wherever we wish to determine these radiative properties of a source (Probs. 1.6, 1.7).

Example 1.2

(a) A detector pointed directly at a uniform intensity source in the sky of small solid angle, Ω, would measure a flux,

$$f = \int_{\Omega} I \cos\theta \, d\Omega \approx I\,\Omega \tag{1.13}$$

(b) The astrophysical flux at the surface of an object (e.g. a star) whose radiation is escaping freely at all angles outwards (i.e. over 2π sr), can be calculated by integrating in spherical coordinates (see Appendix B),

$$F = \int I \cos\theta \, d\Omega = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} I \cos\theta \sin\theta d\theta \, d\phi = \pi I \tag{1.14}$$

Figure 1.8 shows a practical example as to how one might calculate the flux of a source for a case corresponding to Example 1.2a, but for which the intensity varies with position. The intensity in a given waveband is a measurable quantity, provided a solid angle can also be measured. If a source is so small or so far away that its angular size cannot be discerned (i.e. it is *unresolved*, see Sects. 2.2.3, 2.2.4, 2.3.2), then the intensity cannot be determined. In such cases, it is the flux that is measured, as shown in Figure 1.9. All stars other than the Sun would fall into this category[4].

Specific intensity is also referred to as *brightness* which has its intuitive meaning. A faint source has a lower value of specific intensity than a bright source. Note that it is possible for a source that is faint to have a larger flux density than a source that is bright if it subtends a larger solid angle in the sky (Prob. 1.9).

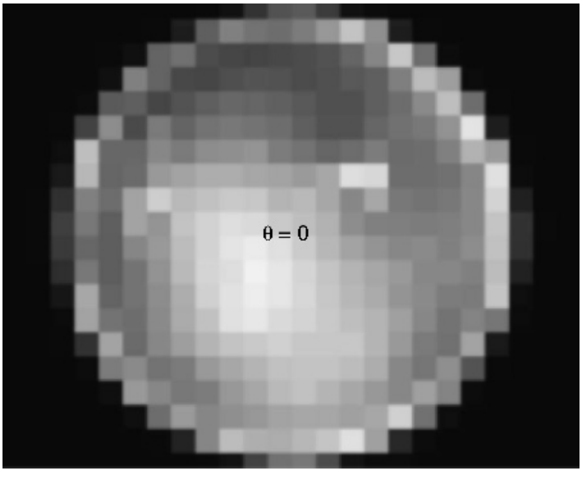

Figure 1.8. Looking directly at a hypothetical object in the sky corresponding to the situation shown in Figure 1.7.a but for $\theta = 0$ (i.e. the detector pointing directly at the source). The object subtends a total solid angle, Ω, which is small and therefore $\theta \approx 0$ at any location on the source. In this example, the object is of non-uniform brightness and Ω is split up into many small square solid angles, each of size, Ω_i and within which the intensity is I_i. Then we can approximate $f = \int I \cos\theta \, d\Omega$ using $f \approx \sum I_i \Omega_i$. Basically, to find the flux, we add up the individual fluxes of all elements.

[4]The exception is a few nearby stars for which special observing techniques are required.

Figure 1.9. In this case, a star has a very small angular size (left) and so, when detected in a square solid angle, Ω_p (right), which is determined by the properties of the detector, its light is 'smeared out' to fill that solid angle. In such a case, it is impossible to determine the intensity of the surface of the star. However, the flux of the star, f_*, is preserved, i.e. $f_* = I_* \Omega_* = \bar{I}\,\Omega_p$ (Eq. 1.13) where I_* is the true intensity of the star, Ω_* is the true solid angle subtended by the star, and $\bar{I}$ is the mean intensity in the square. Thus, for an object of angular size smaller than can be resolved by the available instruments (see Sects. 2.2.3, 2.2.4, and 2.3.2), we measure the flux (or flux density), but not intensity (or specific intensity) of the object

The intensity and specific intensity are *independent of distance* (constant with distance) in the absence of any intervening matter[5]. The easiest way to see this is via Eq. (1.13). Both f and Ω decline as $\frac{1}{r^2}$ (Eq. (1.11), Eq. (B.2), respectively) and therefore I is constant with distance. The Sun, for example, has $I_\odot = F_\odot/\pi = 2.01 \times 10^{10}$ erg s^{-1} cm^{-2} sr^{-1} as viewed from any source at which the Sun subtends a small, measurable solid angle. The constancy of I with distance is general, however, applying to large angles as well. This is a very important result, since a measurement of I allows the determination of some properties of the source without having to know its distance (e.g. Sect. 4.1).

1.4 Light from all angles – energy density and mean intensity

The *energy density*, u (erg cm^{-3}), is the radiative energy per unit volume. It describes the energy content of radiation in a unit volume of space,

$$du = \frac{dE}{dV} \tag{1.15}$$

The *specific energy density* is the energy density per unit bandwidth and, as usual, $u = \int u_\nu\, d\nu = \int u_\lambda\, d\lambda$. The energy density is related to the intensity (see Figure 1.10, Eq. 1.12) by,

$$u = \frac{1}{c}\int I\, d\Omega = \frac{4\,\pi}{c} J \tag{1.16}$$

[5]More accurately, I/n^2 is independent of distance along a ray path, where n is the index of refraction but the difference is negligible for our purposes.

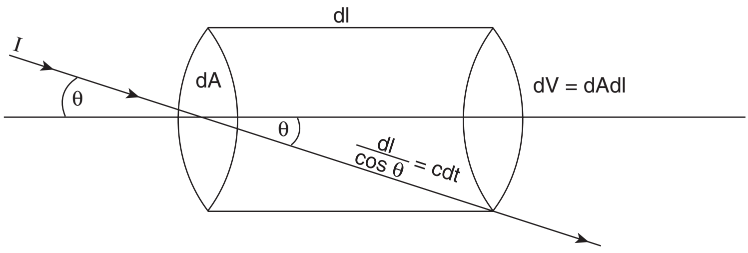

Figure 1.10. This diagram is helpful in relating the energy density (the radiative energy per unit volume) to the light intensity. An individual ray spends a time, $dt = dl/(c\ \cos\theta)$ in an infinitesimal cylindrical volume of size, $dV = dl\,dA$. Combined with Eq. (1.12), the result is Eq. (1.16).

where J is the *mean intensity*, defined by,

$$J \equiv \frac{1}{4\,\pi}\int I\,d\Omega \tag{1.17}$$

The mean intensity is therefore the intensity averaged over all directions. In an isotropic radiation field, $J = I$. In reality, radiation fields are generally not isotropic, but some are close to it or can be approximated as isotropic, for example, in the centres of stars or when considering the 2.7 K cosmic microwave background radiation (Sect. 3.1). In a non-isotropic radiation field, J is not constant with distance, even though I is. Example 1.3 provides a sample computation.

Example 1.3

Compute the mean intensity and the energy density at the distance of Mars. Assume that the only important source is the Sun.

$$\begin{aligned} J &= \frac{1}{4\,\pi}\int_0^{4\pi} I\,d\Omega \\ &= \frac{1}{4\,\pi}\int_{\Omega_\odot} I_\odot\,d\Omega \approx \frac{I_\odot\,\Omega_\odot}{4\,\pi} = \frac{I_\odot}{4\,\pi}\,\frac{\pi\,{\theta_\odot}^2}{4} = \frac{I_\odot}{16}\left(\frac{2\,R_\odot}{r_{\mathrm{Mars}}}\right)^2 \end{aligned} \tag{1.18}$$

where we have used Eq. (B.3) to express the solid angle in terms of the linear angle, and Eq. (B.1) to express the linear angle in terms of the size of the Sun and the distance of Mars. Inserting $I_\odot = 2.01 \times 10^{10}\ \mathrm{erg\,s^{-1}\,cm^{-2}\ sr^{-1}}$ (Sect. 1.3), $R_\odot = 6.96 \times 10^{10}$ cm, and $r_{\mathrm{Mars}} = 2.28 \times 10^{13}$ cm (Tables G.3, G.4), we find, $J = 4.7 \times 10^4\ \mathrm{erg\,s^{-1}\,cm^{-2}\ sr^{-1}}$. Then $u = \frac{4\pi}{c}(4.7 \times 10^4) = 2.0 \times 10^{-5}\ \mathrm{erg\ cm^{-3}}$.

The radiation field (u or J) in interstellar space due to randomly distributed stars (Prob. 1.10) must be computed over a solid angle of 4π steradians, given that

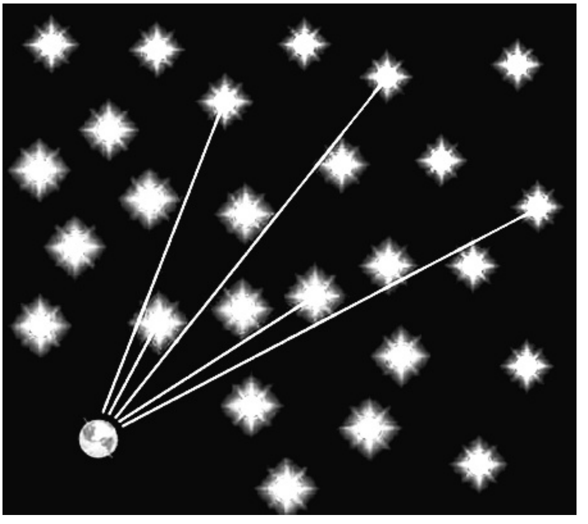

Figure 1.11. Why is the night sky dark? If the Universe is infinite and populated in all directions by stars, then eventually every sight line should intersect the surface of a star. Since I is constant with distance, the night sky should be as bright as the surface of a typical star. This is known as Olbers' Paradox, though Olbers was not the first to note this discrepancy. See Sect. 1.4.

starlight contributes from many directions in the sky. However, in this case, $J \neq I$ because there is no emission from directions between the stars. If there were so many stars that every line of sight eventually intersected the surface of a star of brightness, I_*, then $J = I_*$ and the entire sky would appear as bright as I_*. This would be true even if the stars were at great distances since I_*, being an intensity, is independent of distance. If this is the case, we would say that the stellar *covering factor* is unity.

A variant of this concept is called *Olbers' Paradox* after the German astronomer, Heinrich Wilhelm Olbers who popularized it in the 19th century. It was discussed as early as 1610, though, by the German astronomer, Johannes Kepler, and was based on the idea of an infinite starry Universe which had been propounded by the English astronomer and mathematician, Thomas Digges, around 1576. If the Universe is infinite and populated throughout with stars, then every line of sight should eventually intersect a star and the night sky as seen from Earth should be as bright as a typical stellar surface. Why, then, is the night sky dark?[6] Kepler took the simple observation of a dark night sky as an argument for the finite extent of the Universe, or at least of its stars. The modern explanation, however, lies with the

[6]The earlier form of the question was posed somewhat differently, referring to increasing numbers of stars on increasingly larger shells with distance from the Earth.

intimate relation between time and space on cosmological scales (Sect. 7.1). Since the speed of light is constant, as we look farther into space, we also look farther back in time. The Universe, though, is not infinitely old but rather had a beginning (Sect. 3.1) and the formation of stars occurred afterwards. The required number of stars for a bright night sky is $\approx 10^{60}$ and the volume needed to contain this quantity of stars implies a distance of 10^{23} light years (Ref. [74]). This means that we need to see stars at an epoch corresponding to 10^{23} years ago for the night sky to be bright. The Universe, however, is younger than this by 13 orders of magnitude (Sect. 3.1)! Thus, as we look out into space and back in time, our sight lines eventually reach an epoch prior to the formation of the first stars when the covering factor is still much less than unity. (Today, we refer to this epoch as *the dark ages*.) Remarkably, this solution was hinted at by Edgar Allan Poe in his prose-poem, *Eureka* in 1848 (see the prologue to this chapter).

1.5. How light pushes – radiation pressure

Radiation pressure is the momentum flux of radiation (the rate of momentum transfer due to photons, per unit area). It can also be thought of as the force per unit area exerted by radiation and, since force is a vector, we will treat radiation pressure in this way as well[7]. Thus, the pressure can be separated into its normal, $\mathrm{P}_{\perp}$, and tangential, $\mathrm{P}_{\parallel}$, components with respect to the surface of a wall. The normal radiation pressure will be,

$$d\mathrm{P}_{\perp} = \frac{d\mathrm{F}_{\perp}}{dA} = \frac{dp}{dt\,dA}\cos\theta = \frac{dE}{c\,dt\,dA}\cos\theta \tag{1.19}$$

where we have expressed the momentum of a photon in terms of its energy (Table I.1). Using Eq. (1.12) we obtain,

$$d\mathrm{P}_{\perp} = \left(\frac{1}{c}\right) I\,\cos^2\theta\,d\Omega \tag{1.20}$$

For the tangential pressure, we use the same development but take the sine of the incident angle, yielding,

$$d\mathrm{P}_{\parallel} = \left(\frac{1}{c}\right) I\,\cos\theta\sin\theta d\Omega \tag{1.21}$$

[7]Pressure is actually a *tensor* which is a mathematical quantity described by a matrix (a vector is a specific kind of tensor). We do not need a full mathematical treatment of pressure as a tensor, however, to appreciate the meaning of radiation pressure.

Then for isotropic radiation,

$$P_{\perp} = \left(\frac{1}{c}\right) \int_{4\pi} I \cos^2\theta \, d\Omega = \frac{4\pi}{3c} I$$

$$P_{\parallel} = \left(\frac{1}{c}\right) \int_{4\pi} I \cos\theta \sin\theta d\Omega = 0$$

$$\text{therefore} \qquad P = \sqrt{{P_{\perp}}^2 + {P_{\parallel}}^2} = \frac{4\pi}{3c} I = \frac{1}{3} u \tag{1.22}$$

where we have used a spherical coordinate system for the integration (Appendix B), Eq. (1.16), and the fact that $J = I$ in an isotropic radiation field. Note that the units of pressure are equivalent to the units of energy density, as indicated in Table 0.A.2. Since photons carry momentum, the pressure is not zero in an isotropic radiation field. A surface placed within an isotropic radiation field will not experience a *net* force, however. This is similar to the pressure of particles in a thermal gas. There is no net force in one direction or another, but there is still a pressure associated with such a gas (Sect. 3.4.2).

We can also consider a case in which the incoming radiation is from a fixed angle, θ and the solid angle subtended by the radiation source, Ω, is small. This would result in an acceleration of the wall, but the result depends on the kind of surface the photons are hitting. We consider two cases, illustrated in Figure 1.12: that in which the photon loses all of its energy to the wall (perfect absorption) and that in which the photon loses none of its energy to the wall (perfect reflection).

For perfect absorption, integrating Eqs. (1.20), (1.21) with θ, Ω, constant, yields,

$$P_{\perp} = \left(\frac{1}{c}\right) I\,\Omega \cos^2\theta = \frac{f}{c} \cos^2\theta$$

$$P_{\parallel} = \left(\frac{1}{c}\right) I\,\Omega \cos\theta \, \sin\theta = \frac{f}{c} \cos\theta \, \sin\theta$$

$$P = \sqrt{{P_{\perp}}^2 + {P_{\parallel}}^2} = \frac{f}{c} \cos\theta \tag{1.23}$$

where we have used Eq. (1.13) with f the flux along the directed beam[8].

For perfect reflection, only the normal component will have any effect against the wall (as if the surface were hit by a ball that bounces off). Also, because the momentum of the photon reverses direction upon reflection, the change in momentum is twice the value of the absorption case. Thus, the situation can be described by Eq. (1.20) except

[8] For a narrow beam, this is equivalent to the Poynting flux (Table I.1).

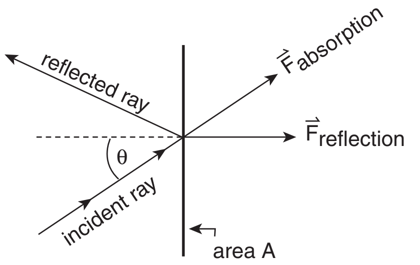

Figure 1.12. An incoming photon exerts a pressure on a surface area. For perfect absorption, the area will experience a force in the direction, $\vec{F}_{absorption}$. For perfect reflection, only the normal component of the force is effective and the resulting force will be in the direction, $\vec{F}_{reflection}$

for a factor of 2.

$$P = P_{\perp} = \left(\frac{2}{c}\right) I\,\Omega \cos^2\theta = \frac{2f}{c}\cos^2\theta \tag{1.24}$$

A comparison of Eqs. (1.23) and (1.24) shows that, provided the incident angle is not very large, a reflecting surface will experience a considerably greater radiation force than an absorbing surface. Moreover, as Figure 1.12 illustrates, the direction of the surface is *not* directly away from the source of radiation as it must be for the absorbing case. These principles are fundamental to the concept of a *Solar sail* (Figure 1.13).

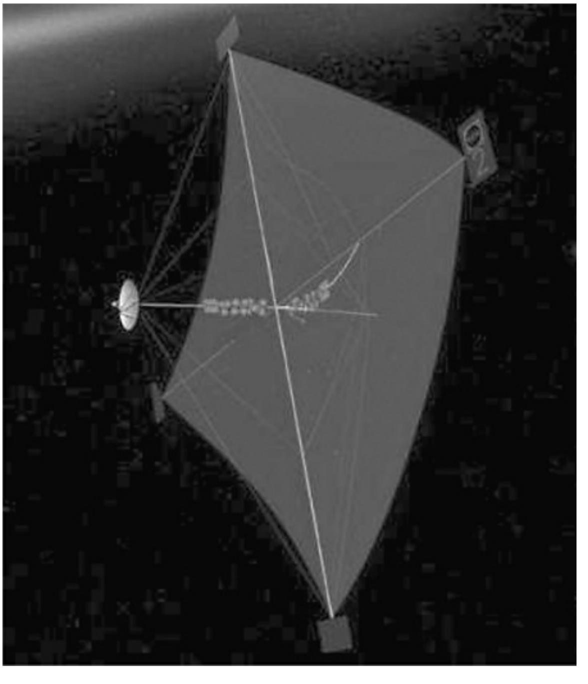

Figure 1.13. Artist's conception of a thin, square, reflective Solar sail, half a kilometre across. Reproduced by permission of NASA/MSFC

Since the direction of motion depends on the angle between the radiation source and the normal to the surface, it would be possible to 'tack' a Solar sail by altering the angle of the sail, in a fashion similar to the way in which a sailboat tacks in the wind. Moreover, even if the acceleration is initially very small, it is *continuous* and thus very high velocities could eventually be achieved for spacecraft designed with Solar sails (Prob. 1.11).

1.6 The human perception of light – magnitudes

Magnitudes are used to characterize light in the optical part of the spectrum, including the near IR and near UV. This is a logarithmic system for light, similar to decibels for sound, and is based on the fact that the response of the eye is logarithmic. It was first introduced in a rudimentary form by Hipparchus of Nicaea in about 150 B.C. who labelled the brightest stars he could see by eye as 'first magnitude', the second brightest as 'second magnitude', and so on. Thus began a system in which brighter stars have *lower* numerical magnitudes, a sometimes confusing fact. As the human eye has been the dominant astronomical detector throughout most of history, a logarithmic system has been quite appropriate. Today, the need for such a system is less obvious since the detector of choice is the CCD (charge coupled device, Sect. 2.2.2) whose response is linear. However, since magnitudes are entrenched in the astronomical literature, still widely used today, and well-characterized and calibrated, it is very important to understand this system.

1.6.1 Apparent magnitude

The *apparent magnitude* and its corresponding flux density are values as measured above the Earth's atmosphere or, equivalently, as measured from the Earth's surface, corrected for the effects of the atmosphere,

$$\begin{aligned} m_\lambda - m_{\lambda_0} &= -2.5 \log\left(\frac{f_\lambda}{f_{\lambda_0}}\right) \\ m_\nu - m_{\nu_0} &= -2.5 \log\left(\frac{f_\nu}{f_{\nu_0}}\right) \end{aligned} \tag{1.25}$$

where the subscript, 0, refers to a standard calibrator used as a reference, m_λ and m_ν are apparent magnitudes in some waveband and f_λ, f_ν are flux densities in the same band. Note that this equation could also be written as a ratio of fluxes, since this would only require multiplying the flux density (numerator and denominator) by an effective bandwidth to make this conversion. This system is a *relative* one, such that the magnitude of the star of interest can be related to that of *any* other star in the same waveband via Eq. (1.25). For example, if a star has a flux density that is 100 times greater

Table 1.1. Standard filters and magnitude calibration[a]

	U	B	V	R	I	J	H	K	L	L*
λ_{eff}^{b}	0.366	0.438	0.545	0.641	0.798	1.22	1.63	2.19	3.45	3.80
$\Delta\lambda^{c}$	0.065	0.098	0.085	0.156	0.154	0.206	0.298	0.396	0.495	0.588
$f_{\nu_0}^{d}$	1.790	4.063	3.636	3.064	2.416	1.589	1.021	0.640	0.285	0.238
$f_{\lambda_0}^{e}$	417.5	632	363.1	217.7	112.6	31.47	11.38	3.961	0.708	0.489
ZP_{ν}	0.770	−0.120	0.000	0.186	0.444	0.899	1.379	1.886	2.765	2.961
ZP_{λ}	−0.152	−0.601	0.000	0.555	1.271	2.655	3.760	4.906	6.775	7.177

[a]UBVRIJHKL Cousins–Glass–Johnson system (Ref. [18]). The table values are for a fictitious A0 star which has 0 magnitude in all bands. A star of flux density, f_ν in units of 10^{-20} erg s^{-1} cm^{-2} Hz^{-1} or f_λ in units of 10^{-11} erg s^{-1} cm^{-2} Å^{-1}, will have a magnitude, $m_\nu = -2.5\log(f_\nu) - 48.598 - ZP_\nu$ or $m_\lambda = -2.5\log(f_\lambda) -21.100 - ZP_\lambda$, respectively. [b]The effective wavelength, in μm, is defined by $\lambda_{\text{eff}} = [\int \lambda f(\lambda)\, R_W(\lambda)\, d\lambda]/[\int f(\lambda)\, R_W(\lambda)\, d\lambda]$, where $f(\lambda)$ is the flux of the star at wavelength, λ, and $R_W(\lambda)$ is the response function of the filter in band W (see Figure 1.3). Thus, the effective wavelength varies with the spectrum of the star considered. [c]Full width at half-maximum (FWHM) of the filters in μm. [d]Units of 10^{-20} erg s^{-1} cm^{-2} Hz^{-1}. [e]Units of 10^{-11} erg s^{-1} cm^{-2} Å^{-1}.

than a second star, then its magnitude will be 5 less than the second star. However, in order to assign a specific magnitude to a specific star, it is necessary to identify certain standard stars with known flux densities to which all others can be compared.

Several slightly different calibration systems have evolved over the years so, for careful and precise work, it is necessary to specify which system is being used when measuring or stating a magnitude. An example of such a system is the UBVRIJHKL Cousins–Glass–Johnson system for which parameters are provided in Table 1.1. The corresponding wavebands, U, B, V, etc., are illustrated in Figure 1.3. The apparent magnitude is commonly written in such a way as to specify these wavebands directly, e.g.

$$\begin{aligned} V - V_0 &= -2.5\log\left(\frac{f_V}{f_{V_0}}\right) \\ B - B_0 &= -2.5\log\left(\frac{f_B}{f_{B_0}}\right) \text{etc.} \end{aligned} \tag{1.26}$$

where the flux densities can be expressed in either their λ-dependent or ν-dependent forms. The V-band, especially, since it corresponds to the waveband in which the eye is most sensitive (cf. Table G.5), has been widely and extensively used. Some examples of apparent magnitudes are provided in Table 1.2.

The standard calibrator in most systems has historically been the star, Vega. Thus, Vega would have a magnitude of 0 in all wavebands (i.e. $U_0 = 0$, $B_0 = 0$, etc) and its flux density in these bands would be tabulated. However, concerns over possible variability of this star, its possible IR excess, and the fact that it is not observable from the Southern hemisphere, has led to modified approaches in which the star Sirius is also taken as a calibrator and/or in which a model star is used instead. The latter approach has been taken in Table 1.1 which lists the reference flux densities for

Table 1.2. Examples of apparent visual magnitudes[a]

Object or item	Visual magnitude	Comments
Sun	−26.8	
Approx. maximum of a supernova	−15	assuming V = 0 precursor
Full Moon	−12.7	
Venus	−3.8 to −4.5[b]	brightest planet
Jupiter	−1.7 to −2.5[b]	
Sirius	−1.44	brightest nighttime star
Vega	0.03	star in constellation Lyra
Betelgeuse	0.45	star in Orion
Spica	0.98	star in Virgo
Deneb	1.23	star in Cygnus
Aldebaran	1.54	star in Taurus
Polaris	1.97	the North Star[c]
Limiting magnitude[d]	3.0	major city
Ganymede	4.6	brightest moon of Jupiter
Uranus	5.7[e]	
Limiting magnitude[d]	6.5	dark clear sky
Ceres	6.8[e]	brightest asteroid
Pluto	13.8[e]	
Jupiter-like planet	26.5	at a distance of 10 pc
Limiting magnitude of HST[f]	28.8	1 h on A0V star
Limiting magnitude of OWL[g]	38	future 100 m telescope

[a]From Ref. [71] (probable error at most 0.03 mag) and on-line sources. [b]Typical range over a year. [c]A variable star. [d]This is the faintest star that could be observed by eye without a telescope. It will vary with the individual and conditions. [e]At or close to 'opposition' (180° from the Sun as seen from the Earth). [f]'Hubble Space Telescope', from Space Telescope Science Institute on-line documentation. The limiting magnitude varies with instrument used. The quoted value is a best case. [g]OWL (the 'Overwhelmingly Large Telescope') refers to the European Southern Observatory's concept for a 100 m diameter telescope with possible completion in 2020.

reference magnitudes of zero in all filters. The flux density and reference flux density must be in the same units. The above equations can be rewritten as,

$$m_\lambda = -2.5\log(f_\lambda) - 21.100 - \mathrm{ZP}_\lambda$$
$$m_\nu = -2.5\log(f_\nu) - 48.598 - \mathrm{ZP}_\nu \qquad (1.27)$$

where f_λ is in units of erg $\mathrm{cm^{-2}\,s^{-1}\,Å^{-1}}$, f_ν is in units of erg $\mathrm{cm^{-2}\,s^{-1}\,Hz^{-1}}$, and ZP_λ, ZP_ν are called *zero point* values (Ref. [18]). Example 1.4 provides a sample calculation.

Example 1.4

An apparent magnitude of $B = 1.95$ *is measured for the star, Betelgeuse. Determine its flux density in units of* erg $\mathrm{cm^{-2}\,s^{-1}\,Å^{-1}}$.

From Eq. (1.26) and the values from Table 1.1, we have,

$$B - B_0 = 1.95 - 0 = -2.5 \log\left(\frac{f_B}{632 \times 10^{-11}}\right) \tag{1.28}$$

Solving, this gives $f_B = 1.0 \times 10^{-9}$ erg cm^{-2} s^{-1} Å^{-1} for Betelgeuse.
Eq. (1.27) can also be used,

$$B = 1.95 = -2.5 \log(f_B) - 21.100 + 0.601 \tag{1.29}$$

which, on solving, gives the same result.

1.6.2 Absolute magnitude

Since flux densities fall off as $\frac{1}{r^2}$, measurements of apparent magnitude between stars do not provide a useful comparison of the intrinsic properties of stars without taking into account their various distances. Thus the *absolute magnitude* has been introduced, either as a bolometric quantity, M_{bol}, or in some waveband (e.g. M_V, M_B, etc). The absolute magnitude of a star is the magnitude that would be measured if the star were placed at a distance of 10 pc. Since the magnitude scale is relative, we can let the reference star in Eq. (1.25) be the *same* star as is being measured but placed at a distance of 10 pc,

$$m - M = -2.5 \log\left(\frac{f}{f_{10\,\mathrm{pc}}}\right) = -5 + 5 \log\left(\frac{d}{\mathrm{pc}}\right) \tag{1.30}$$

where we have dropped the subscripts for simplicity and used Eq. (1.9). Here d is the distance to the star in pc. Eq. (1.30) provides the relationship between the apparent and absolute magnitudes for any given star. The quantity, $m - M$, is called the *distance modulus*. Since this quantity is directly related to the distance, it is sometimes quoted as a proxy for distance. Writing a similar equation for a reference star and combining with Eq. (1.30) (e.g. Prob. 1.12) we find,

$$M - M_{b\odot} = -2.5 \log\left(\frac{L}{L_\odot}\right) \tag{1.31}$$

where we have used the Sun for the reference star. Eq. (1.31) has been explicitly written with bolometric quantities (Table G.3) but one could also isolate specific bands, as before, provided the correct reference values are used.

1.6.3 The colour index, bolometric correction, and HR diagram

The *colour index* is the difference between two magnitudes in different bandpasses for the same star, for example,

$$B - V = -2.5 \log\left(\frac{f_B}{f_V}\frac{f_{V_0}}{f_{B_0}}\right) = -2.5 \log\left(\frac{f_B}{f_V}\right) - (ZP_B - ZP_V) \qquad (1.32)$$

or between any other two bands. Eq. (1.32) is derivable from Eqs. (1.26) or (1.27). Various colour indices are provided for different kinds of stars[9] in Table G.7. Since this quantity is basically a measure of the ratio of flux densities at two different wavelengths (with a correction for zero point), it is an indication of the *colour* of the star. A positive value for $B - V$, for example, means that the flux density in the V band is *higher* than that in the B band and hence the star will appear more 'yellow' than 'blue' (see Table G.5).

The colour index, since it applies to a single star, is *independent of distance*. (To see this, note that converting the flux density to a distance-corrected luminosity would require the same factors in the numerator and denominator of Eq. (1.32)). Consequently, the colour index can be compared directly, star to star, without concern for the star's distance. We will see in Sect. 4.1.3 that colour indices are a measure of the surface temperature of a star. This means that stellar temperatures can be determined without having to know their distances.

Since a colour index could be written between any two bands, one can also define an index between one band and *all* bands. This is called the *bolometric correction*, usually defined for the V band,

$$BC = m_{bol} - V = M_{bol} - M_V \qquad (1.33)$$

For any given star, this quantity is a correction factor that allows one to convert from a V band measurement to the bolometric magnitude (Prob. 1.16). Values of BC are provided for various stellar types in Table G.7.

Figure 1.14 shows a plot of absolute magnitude as a function of colour index for over 5000 stars in the Galaxy near the Sun. Such a plot is called a *Hertzsprung–Russell (HR) diagram* or a *colour–magnitude diagram (CMD)*. The absolute magnitude can be converted into luminosity (see Eq. 1.31) and the colour index can be converted to a temperature (see the calibration of Figure G.1) which are more physically meaningful parameters for stars. The HR Diagram is an essential tool for the study of stars and stellar evolution and shows that stars do *not* have arbitrary temperatures and luminosities but rather fall along well-defined regions in $L - T$ parameter space. The positions of stars in this diagram provide important information about stellar parameters and stellar evolution, as described in Sect. 3.3.2. It is important to note that the distance to each star must be measured in order to obtain absolute magnitudes, a feat that has been accomplished to unprecedented accuracy by Hipparcos, a European satellite launched

[9]Stellar spectral types will be discussed in Sect. 3.4.7.

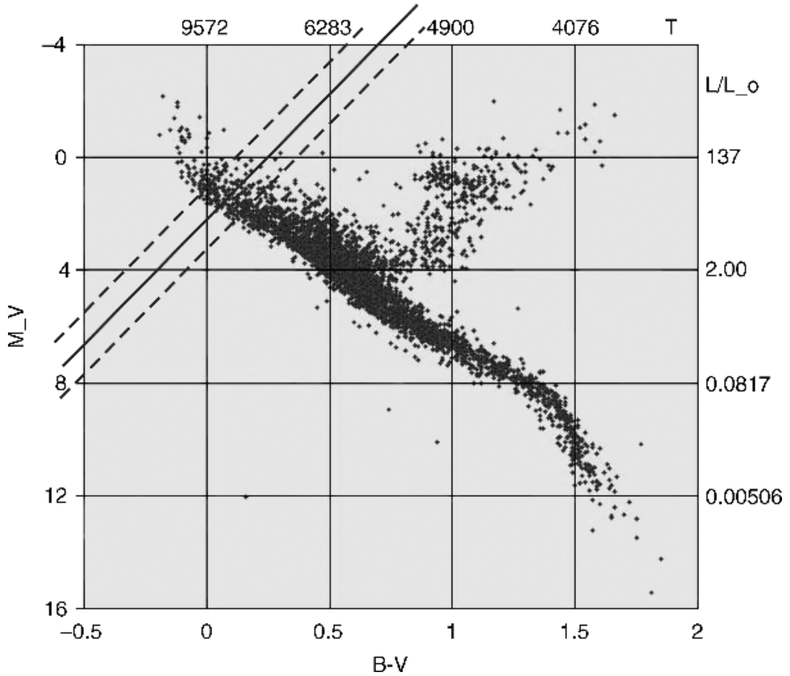

Figure 1.14. Hertzsprung–Russell (HR) diagram for approximately 5000 stars taken from the Hipparcos catalogue. Hipparcos determined the distances to stars, allowing absolute magnitude to be determined. The M_V data have errors of about ± 0.1 magnitudes and the $B - V$ data have errors of < 0.025 magnitudes. Most stars fall on a region passing from upper left (hot, luminous stars) to lower right (dim, cool stars), called the *main sequence*. The main sequence is defined by stars that are burning hydrogen into helium in their cores (see Sect. 3.3.2). The temperature, shown at the top and applicable to the main sequence, was determined from the Temperature $-B - V$ calibration of Table G.7 and Figure G.1. The luminosity, shown at the right, was determined from Eqs. (1.31) and (1.33), using the bolometric correction from Table G.7. The straight line, defined by the equation, $M_V = -8.58(B - V) + 2.27$, locates the central ridge of the instability strip for Galactic Cepheid variable stars and the dashed lines show its boundaries (see Ref. [166a] and Sect. 5.4.2).

in 1989. The future GAIA (Global Astrometric Interferometer for Astrophysics) satellite, also a European project with an estimated launch date of 2011, promises to make spectacular improvements. This satellite will provide a census of 10^9 stars and it is estimated that it will discover 100 new asteroids and 30 new stars that have planetary systems *per day.*

1.6.4 Magnitudes beyond stars

Magnitudes are widely used in optical astronomy and, though the system developed to describe stars (for which specific intensities cannot be measured, Sect. 1.3), it can be applied to any object, extended or point-like, as an alternate description of the flux density. One could quote an apparent magnitude for other point-like sources such as distant QSOs ('quasi-stellar objects')[10], or of extended sources like galaxies. Since the specific intensity of an extended object is a flux density per unit solid angle, it is also common to express this quantity in terms of magnitudes per unit solid angle (Prob. 1.17).

1.7 Light aligned – polarization

The magnetic and electric field vectors of a wave are perpendicular to each other and to the direction of propagation (Figure I.2). A signal consists of many such waves travelling in the same direction in which case the electric field vectors are usually randomly oriented around the plane perpendicular to the propagation direction. However, if all of the electric field vectors are aligned (say all along the z axis of Figure I.2) then the signal is said to be polarized. Partial polarization occurs if some of the waves are aligned but others randomly oriented. The *degree of polarization*, D_p is defined as the fraction of total intensity that is polarized (often expressed as a percentage),

$$D_p \equiv \frac{I_{pol}}{I_{tot}} = \frac{I_{pol}}{I_{pol} + I_{unpol}} \tag{1.34}$$

Polarization can be generated internally by processes intrinsic to the energy generation mechanism (see Sect. 8.5, for example), or polarization can result from the scattering of light from particles, be they electrons, atoms, or dust particles (see Sect. 5.1 or Appendix D). When polarization is observed, D_p is usually of order only a few percent and 'strong' polarization, such as seen in radio jets (Figure 1.4), typically is $D_p \lesssim 15$ per cent (Ref. [175]). Values of D_p over 90 per cent, however, have been detected in the jets of some pulsars (Ref. [113]) and in the low frequency radio emission from planets (Sect. 8.5.1). For the Milky Way, $D_p = 2$ per cent over distances 2 to 6 kpc from us due to scattering by dust (Ref. [175]). In practical terms, this means that polarized emission is usually much fainter than unpolarized emission and requires greater effort to detect.

Problems

1.1 Assuming Cas A (see Sect. 1.1 for more data) has a spectral luminosity between $\nu_1 = 2 \times 10^7$ Hz and $\nu_2 = 2 \times 10^{10}$ Hz, of the form, $L_\nu = K\,\nu^{-0.7}$, where K is a constant. Determine the value of K, and of L_ν and L_λ at $\nu = 10^9$ Hz. What are the units of K?

1.2 Find, by comparison with exact trigonometry, the angle, θ (provide a numerical value in degrees), above which the small angle approximation, Eq. (B.1), departs from the exact

[10] A QSO is the bright active nucleus of a very distant galaxy that looks star-like at optical wavelengths. QSOs that also emit strongly at radio wavelengths are called *quasars*.

result by more than 1 per cent. How does this compare with the relatively large angle subtended by the Sun?

1.3 Determine the flux density (in Jy) of a cell phone that emits 2 mW cm^{-2} at a frequency of 1900 MHz over a bandwidth of 30 kHz, and of the Sun, as measured at the Earth, at the same frequency. (Eq. 4.6 provides an expression for calculating the specific intensity of the Sun.) Compare these to the flux density of the supernova remnant, Cas A ($\sim$ 1900 Jy as measured at the Earth at 1900 MHz) which is the strongest radio source in the sky after the Sun. Comment on the potential of cell phones to interfere with the detection of astronomical signals.

1.4 (a) Consider a pulsar with radiation that is beamed uniformly into a circular cone of solid angle, Ω. Rewrite the right hand side (RHS) of Eq. (1.9) for this case.

(b) If $\Omega = 0.02$ sr, determine the error that would result in L if the RHS of Eq. (1.9) were used rather than the correct result from part (a).

1.5 Determine the percentage variation in the solar flux incident on the Earth due to its elliptical orbit. Compare this to the variation shown in Figure 1.6.

1.6 Determine the flux in a perfectly isotropic radiation field (i.e. I constant in all directions).

1.7 (a) Determine the flux and intensity of the Sun (i) at its surface, (ii) at the mean distance of Mars, and (iii) at the mean distance of Pluto.

(b) How large (in arcmin) would the Sun appear in the sky at the distances of the two planets? Would it appear resolved or as a point source to the naked eye at these locations? That is, would the angular diameter of the Sun be larger than the resolution of the human eye (Table G.5) or smaller? (Sects. 2.2, 2.2.3, and 2.3.2 provide more information on the meaning of 'resolution'.)

1.8 The radio spectrum of Cas A, whose image is shown in Figure 1.2, is given in Figure 8.14 in a log–log plot. The plotted specific intensity can be represented by, $I_\nu = I_{\nu 0}\,(\nu/\nu_0)^\alpha$ in the part of the graph that is declining with frequency, where ν_0 is any reference frequency in this part of the plot, $I_{\nu 0}$ is the specific intensity measured at ν_0 and α is the slope. In Prob. 1.1, we assumed that $\alpha = -0.7$. Now, instead, measure this value from the graph and determine, for the radio band from $\nu_1 = 2 \times 10^7$ to $\nu_2 = 2 \times 10^{10}$ Hz,

(a) the intensity of Cas A, I,

(b) the solid angle that it subtends in the sky, Ω,

(c) its flux, f,

(d) its radio luminosity, L_{rad}. Confirm that this value is approximately equal to the value given in Sect. 1.1.

1.9 On average, the brightness of the Whirlpool Galaxy, M 51 (see Figure 3.8 or 9.8), which subtends an ellipse of major $\times$ minor axis, $11.2' \times 6.9'$ in the sky, is 2.1 times that of the Andromeda Galaxy, M 31 (subtending $190' \times 60'$). What is the ratio of their flux densities?

1.10 Where does the Solar System end? To answer this, find the distance (in AU) at which the radiation energy density from the Sun is equivalent to the ambient mean energy density of interstellar space, the latter about 10^{-12} erg cm^{-3}. After 30 years or more of space travel, how far away are the Pioneer 10 and Voyager 1 spacecraft? See `http://spaceprojects.arc.nasa.gov/Space_Projects/pioneer/PNhome.html` and `http://voyager.jpl.nasa.gov`. Are they out of the Solar System?

1.11 Consider a circular, perfectly reflecting Solar sail that is initially at rest at a distance of 1 AU from the Sun and pointing directly at it. The sail is carrying a payload of 1000 kg (which dominates its mass) and its radius is $R_s = 500\,\text{m}$.

(a) Derive an expression for the acceleration as a function of distance, $a(r)$, for this Solar sail. Include the Sun's gravity as well as its radiation pressure. (The constants may be evaluated to simplify the expression.)

(b) Manipulate and integrate this equation to find an expression for the velocity of the Solar sail as a function of distance, $v(\mathrm{r})$. Evaluate the expression to find the velocity of the Solar sail by the time it reaches the orbit of Mars.

(c) Finally, derive an expression for the time it would take for the sail to reach the orbit of Mars. Evaluate it to find the time. Express the time as seconds, months, or years, whatever is most appropriate.

1.12 Derive Eq. (1.31) (see text).

1.13 Repeat *Example 1.4* but expressing the flux density in its frequency-dependent form.

1.14 For the U band and L* band filters, verify that f_ν corresponds to f_λ in Table 1.1. Why might there be minor differences?

1.15 Refer to Table 1.2 for the following.

(a) Determine the range (ratio of maximum to minimum flux density) over which the unaided human eye can detect light from astronomical objects. Research the range of human hearing from 'barely audible' to the 'pain threshold' and compare the resulting range to the eye.

(b) Use the 'Multiparameter Search Tool' of the *Research Tools* at the web site of the Hipparcos satellite (`http://www.rssd.esa.int/Hipparcos/`) to determine what percentage of stars in the night sky one would lose by moving from a very dark country site into a nearby light polluted city.

(c) The star, Betelgeuse, is at a distance of 130 pc. Determine how far away it would have to be before it would be invisible to the unaided eye, if it were to undergo a supernova explosion.

1.16 A star at a distance of 25 pc is measured to have an apparent magnitude of $V = 7.5$. This particular type of star is known to have a bolometric correction of $BC = -0.18$. Determine the following quantities: (a) the flux density, f_V in units of erg cm^{-2} s^{-1} Å^{-1}, (b) the absolute V magnitude, M_V, (c) the distance modulus, (d) the bolometric apparent and absolute magnitudes, m_{bol}, and M_{bol}, respectively, and (e) the luminosity, L in units of $L_\odot$.

1.17 A galaxy of uniform brightness at a distance of 16 Mpc appears elliptical on the sky with major and minor axis dimensions, $7.9' \times 1.4'$. It is observed first in the radio band centred at 1.4 GHz (bandwidth = 600 MHz) to have a specific intensity of 4.8 mJy beam^{-1}, where the 'beam' is a circular solid angle of diameter, $15''$ (see also Example 2.3d). A measurement is then made in the optical B band of 22.8 magnitudes per pixel, where the pixel corresponds to a square on the sky which is one arcsecond on a side. Determine (all in cgs units) I_ν,f_ν,f, and L of the galaxy in each band. In which band is the source brighter? In which band is it more luminous?

1.18 The limiting magnitude of some instruments can be pushed fainter by taking extremely long exposures. Estimate the limiting magnitude of the Hubble Ultra-Deep Field from the information given in Figure 3.1.